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# SIGNATURA OF MAGIC AND LATIN INTEGER SQUARES: ISENTROPIC CLANS AND INDEXING

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**Abstract**

The 2010 study of the Shannon entropy of order nine Sudoku and Latin square matrices by Newton and DeSalvo [Proc. Roy. Soc. A 2010] is extended to natural magic and Latin squares up to order nine. We demonstrate that decimal and integer measures of the Singular Value sets, here named SV clans, are a powerful way of comparing different integer squares.

Several complete sets of magic and Latin squares are included, including the order eight Franklin subset which is of direct relevance to magic square line patterns on chess boards. While early examples suggested that lower rank specimens had lower entropy, sufficient data is presented to show that some full rank cases with low entropy possess a set of singular values separating into a dominant group with the remainder much weaker. An effective rank measure helps understand these issues.

We also introduce a new measure for integer squares based on the sum of the fourth powers of the singular values which appears to give a useful method of indexing both Latin and magic squares. This can be used to begin cataloging a "library" of magical squares.

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**Keywords:** Shannon entropy, magic square, Latin square, singular value decomposition, singular value clan.

**2010 Mathematics Subject Classification:** 15B04, 15B06.

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## 1. **Electronic files**

1. shannonData.txt: Matrix elements for squares cited in tables, where not easily found in a reference.

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