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A Note on the Theorem of Stone-Weierstrass

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Theorem of Stone-Weierstrass for C^*-algebra A is the assertion that only a sub-C^*-algebra of A which separates the union of the w*-closure of pure states and $\{0\}$ is A itself [1]. As a corollary of the theorem we have the following:

LEMMA 1. *Let T be a compact Hausdorff space, $\mathfrak{A}=((A(t))_{t\in T}, \Theta)$ the continuous field of C^*-algebras on T, and A the C^*-algebra defined by $\mathfrak{A}$. Suppose that a subalgebra B of A satisfies the following*:

1 *For any $t\in T$ and $\xi\in A(t)$, there exists $x\in B$ such that $x(t)=\xi$,*

2 *For any $t_1\neq t_2$ of T and $\xi\in A(t_1)$, there exists $x\in B$ such that $x(t_1)=\xi$, $x(t_2)=0$ in $A(t_2)$.*

Then we have $A=B$.

The proof is almost the same as that of Corollaire 11.5.3 of [1].

Now let T be a compact Hausdorff space, A a C^*-algebra, and A^T the C^*-algebra of continuous maps of T into A. Then A^T is the C^*-algebra defined by the constant field on T defined by A. For a subset S of A^T and $t\in T$, $\tilde{t}=\{t'\in T \mid f(t)=f(t')$ for all $f\in S\}$ is called the set of constancy for S. Then the sets $(\tilde{t})_{\tilde{t}\in T}$ of constancy for S constitute an upper semi-continuous decomposition of T. If we set

$$\tilde{T}(S)=\{\tilde{t} \mid t\in T\},$$

$\tilde{T}(S)$ becomes a compact Hausdorff space under the natural topology [2].

DEFINITION 1. Let S be a subset of A^T. We say that S has sufficiently many elements if for any $x\in A$ and $t\in T$, there exists $f\in S$ such that $f(t)=x$.

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DEFINITION 2. Let A be a C^*-algebra with unit element e. For any compact Hausdorff space T and a sub-C^*-algebra S of A^T which contains 1 ($1(t)=e$ for all $t \in T$) and has sufficiently many elements, assume that if S separates T, $(Ce)^T \cap S$ separates T already. Then we say that A is of C-type.

LEMMA 2. *Let A be a C^*-algebra of C-type, T a compact Hausdorff space, and S a sub-C^*-algebra which contains* 1 *and has sufficiently many elements. Then an isometric *-isomorphism of S onto $A^{\tilde{T}(S)}$ is defined canonically.*

Set $\tilde{f}(\tilde{t})=f(t)$ for $f \in S$ and $t \in T$. Then it is obvious that the map $f \to \tilde{f}$ is an isometric *-isomorphism of S into $A^{\tilde{T}(S)}$. Denote the image of S by $\tilde{S}$. Since $\tilde{S}$ separates, $\tilde{T}$ we have $(Ce)^{\tilde{T}} \subset \tilde{S}$ by the definition of C-type and the classical Stone-Weierstrass theorem. For any $x \in A$ and $\tilde{t}_1 \neq \tilde{t}_2$ of $\tilde{T}$, there exists $\alpha \in (Ce)^{\tilde{T}}$ such that $\alpha(t_1)=e \in A$ and $\alpha(t_2)=0 \in A$ by Urysohn's theorem. Since $\tilde{S}$ has sufficiently many elements, there exists $\tilde{f} \in \tilde{S}$ such that $\tilde{f}(\tilde{t}_1)=x$, hence we have $\alpha\, \tilde{f} \in S$, $(\alpha\, \tilde{f})\,(\tilde{t}_1)=x$ and $(\alpha\, \tilde{f})(\tilde{t}_2)=0$. Accordingly it follows that $\tilde{S}=A^{\tilde{T}(S)}$ by lemma 1.

DEFINITION 3. Let S be a family of functions on T, T_0 a subset of T, *and* f a function defined on T. If there exists $g \in S$ such that $g(t)=f(t)$ for all $t \in T_0$, f is said to belong to S on T_0.

DEFINITION 4. Let A be a C^*-algebra with unit element e. For any compact Hausdorff space T and a self-adjoint subalgebra S_0 of A^T which contains 1 and has sufficiently many elements, assume that if, for any closed subalgebra S containing S_0, $f \in A^T$ belongs to S on every set of constancy for S_0, f belongs to S. Then we say that A satisfies the S-condition.

THEOREM. *Let A be a C^*-algebra with unit element. Then A is of C-type if and only if A satisfies the S-condition.*

Since S is a closed subalgebra of A in definition 4, it may as well be assumed that S_0 is also closed (i. e. a sub-C^*-algebra of A). Accordingly, we can reduce the proof to the proof of G. Šilov [3] from lemma 2.

REMARK. If we take complex numbers for A, our theorem is nothing but the theorem of Šilov.

References

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[2] C. E. Rickart, General theory of Banach algebras, D. Van Nostrand, Princeton, (1960) 124-128.

[3] G. Šilov, On rings of functions with uniform convergence, Ukrain. Math. Zurnal 3 (1951) 404-411.