# ABUNDANCE FOR 3-FOLDS WITH NON-TRIVIAL ALBANESE MAPS IN POSITIVE CHARACTERISTIC 

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#### Abstract

In this paper, we prove abundance for 3 -folds with non-trivial Albanese maps, over an algebraically closed field of characteristic $p>5$.


## 1. Introduction

Over an algebraically closed field of characteristic $p>5$, existence of $\log$ minimal models of 3 -folds has been proved by Birkar, Hacon and Xu ([2] [21); and log abundance has been proved for minimal klt pairs $(X, B)$ when $K_{X}+B$ is big or $B$ is big ([2] [7] [45]), and when $X$ is non-uniruled and has non-trivial Albanese map ([48]).

This paper aims to prove abundance for 3 -folds with non-trivial Albanese maps.
Theorem 1.1. Let $X$ be a klt, $\mathbb{Q}$-factorial, projective minimal 3-fold, over an algebraically closed field $k$ with char $k=p>5$. Assume that the Albanese map $a_{X}: X \rightarrow A_{X}$ is non-trivial. Then $K_{X}$ is semi-ample.

Precisely we prove log abundance in some cases.
Theorem 1.2. Let $(X, B)$ be a klt, $\mathbb{Q}$-factorial, projective minimal pair of dimension 3, over an algebraically closed field $k$ with char $k=p>5$. Assume that the Albanese map $a_{X}: X \rightarrow A_{X}$ is non-trivial. Denote by $f: X \rightarrow Y$ the fibration arising from the Stein factorization of $a_{X}$ and by $X_{\eta}$ the generic fiber of $f$. Assume moreover that $B=0$ if
(1) $\operatorname{dim} Y=2$ and $\kappa\left(X_{\eta},\left.\left(K_{X}+B\right)\right|_{X_{\eta}}\right)=0$, or
(2) $\operatorname{dim} Y=1$ and $\kappa\left(X_{\eta},\left.\left(K_{X}+B\right)\right|_{X_{\eta}}\right)=1$.

Then $K_{X}+B$ is semi-ample.
Throughout of this paper, as $f: X \rightarrow Y$ frequently appears as a projective morphism of varieties, we denote by $\eta$ (resp. $\bar{\eta}$ ) the generic (resp. geometric generic) point of $Y$ and by $X_{\eta}$ (resp. $X_{\bar{\eta}}$ ) the generic (resp. geometric generic) fiber of $f$. Moreover we say $f: X \rightarrow Y$ is a fibration if $f_{*} \mathcal{O}_{X}=\mathcal{O}_{Y}$.

To study abundance for varieties with non-trivial Albanese maps, it is necessary to study the following conjecture on subadditivity of Kodaira dimensions.

Conjecture 1.3 (Iitaka conjecture). Let $f: X \rightarrow Y$ be a fibration between two smooth projective varieties over an algebraically closed field $k$, with $\operatorname{dim} X=n$ and $\operatorname{dim} Y=m$. Then

$$
C_{n, m}: \kappa(X) \geq \kappa(Y)+\kappa\left(X_{\eta}, K_{X_{\eta}}\right) .
$$

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The divisor $K_{X_{\eta}}=\left.K_{X}\right|_{X_{\eta}}$ is a Cartier divisor corresponding to the dualizing sheaf of $X_{\eta}$, which is invertible since $X_{\eta}$ is regular. In characteristic zero, since the geometric generic fiber $X_{\bar{\eta}}$ is smooth over $k(\bar{\eta})$, so $\kappa\left(X_{\bar{\eta}}\right)=\kappa\left(X_{\eta}, K_{X_{\eta}}\right)$. In positive characteristic, $X_{\bar{\eta}}$ is not necessarily smooth over $k(\bar{\eta})$, if $X_{\bar{\eta}}$ has a smooth projective birational model $\tilde{X}_{\bar{\eta}}, 1$ then by [8, Corollary 2.5]

$$
\kappa\left(X_{\eta}, K_{X_{\eta}}\right)=\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}\right) \geq \kappa\left(\tilde{X}_{\bar{\eta}}\right) .
$$

We refer to [46] for more discussions on this conjecture. In this paper we prove the following theorem, which implies Theorem 1.2 after combined with the results of 488.

Theorem 1.4 (= Theorem 4.1). Let $f: X \rightarrow Y$ be a fibration from a $\mathbb{Q}$-factorial projective 3-fold to a smooth projective variety of dimension 1 or 2, over an algebraically closed field $k$ with char $k=p>5$. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is klt. Assume that $Y$ is of maximal Albanese dimension, and assume moreover that
© if $\kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)=\operatorname{dim} X / Y-1$, then $B$ does not intersect the generic fiber $X_{\xi}$ of the relative Iitaka fibration $I: X \rightarrow Z$ induced by $K_{X}+B$ on $X$ over $Y$.
Then

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)+\kappa(Y) .
$$

Remark 1.5. As char $k=p>5$, for a fibration $h: X \rightarrow Z$, if the generic fiber is a curve with arithmetic genus one, then the geometric generic fiber must be a smooth elliptic curve (Proposition 2.11). So in case $\kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)=\operatorname{dim} X / Y-1$, the assumption guarantees that the relative Iitaka fibration $I: X \rightarrow Z$ is fibred by elliptic curves. The advantage of an elliptic fibration is the canonical bundle formula: if $h: W \rightarrow Z$ is a flat relative minimal elliptic fibration then there exists an effective divisor $B_{Z}$ on $Z$ such that $K_{W} \sim_{\mathbb{Q}} h^{*}\left(K_{Z}+B_{Z}\right)([8,3.2])$. Canonical bundle formula is the key technique to study $\log$ abundance ([25]). Here we mention that in positive characteristic, only under some very strong conditions, $B_{Z}$ has been proved to be effective ([7, Lemma 6.7], [11, Theorem 3.18], [10, Theorem B]).

The result above has been proved when the geometric generic fiber $X_{\bar{\eta}}$ is smooth and $B=0([8,12])$, which was used in [48] to prove log abundance for non-uniruled 3 -folds with non-trivial Albanese maps. To study the uniruled case, we have to treat fibrations with singular geometric generic fibers. For a separable fibration with possibly singular geometric generic fiber, $C_{3, m}$ has been proved by the author [46] when $K_{X}+B$ is $f$-big, $K_{Y}$ is big and $Y$ is of maximal Albanese dimension. To prove Theorem [1.4, the essentially new results are the following two theorems.

Theorem 1.6 (= Theorem 4.2). Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair of dimension 3, over an algebraically closed field $k$ with char $k=p>5$. Let $f: X \rightarrow Y=A$ be a fibration to an elliptic curve or a simple abelian surface. Assume that $K_{X}+B$ is $f$-big. Then

$$
\kappa\left(K_{X}+B\right) \geq \kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)
$$

[^0]Theorem 1.7 (= Theorem 4.4). Let $(X, B)$ be a $\mathbb{Q}$-factorial klt pair of dimension 3, over an algebraically closed field $k$ with char $k=p>5$. Let $f: X \rightarrow Y$ be a fibration to a normal curve $Y$ of genus $g(Y) \geq 1$. Assume $\kappa\left(X_{\eta},\left.\left(K_{X}+B\right)\right|_{X_{\eta}}\right)=0$. Then

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa(Y)
$$

If moreover $K_{X}+B$ is nef then it is semi-ample.
We summarize the techniques to study $C_{3, m}$ which appeared in the papers [37, 11, 46, 12], then explain the ideas of the proof of Theorem [1.6 and 1.7 ,
(1) Positivity results. Let $f: X \rightarrow Y$ be a separable fibration of normal projective varieties. In positive characteristic, Parakfalvi [37] first proved that for sufficiently divisible $n$, the sheaf $f_{*} \mathcal{O}_{X}\left(n\left(K_{X / Y}+B\right)\right)$ is weakly positive under the condition that $\left(X_{\bar{\eta}}, B_{\bar{\eta}}\right)$ is strongly $F$-regular and $K_{X / Y}+B$ is $f$-ample, then Ejiri [11] reproved it and made some generalizations using different method. This result may fail when $X_{\bar{\eta}}$ is singular. But in dimension three and over an algebraically closed field $k$ with char $k=p>5$, we can take advantage of minimal model theory. Under the condition that $K_{X}+B$ is nef, relatively big and semi-ample over $Y$, the author 46] (or [38] under stronger conditions) proved that for sufficiently divisible $n, g>0$, the sheaf $F_{Y}^{g *}\left(f_{*} \mathcal{O}_{X}\left(n\left(K_{X / Y}+B\right)\right) \otimes \omega_{Y}^{n-1}\right)$ contains a non-zero weakly positive sub-sheaf $V_{n}$, which plays a similar role as $f_{*} \mathcal{O}_{X}\left(n\left(K_{X / Y}+B\right)\right)$ does in studying subadditivity of Kodaira dimensions.

If $f: X \rightarrow Y$ is a fibration from a 3 -fold to a smooth curve, when $\kappa\left(X_{\bar{\eta}}\right)=1$ the relative Iitaka fibration of $X$ over $Y$ is fibred by elliptic curves since $p>5$ (Proposition[2.11). By canonical bundle formula ([8, 3.2]) and minimal model theory, one can reduce to a pair $\left(Z, B_{Z}\right)$ of dimension two with $K_{Z}+B_{Z}$ being relatively big over $Y$. Then one can show that $f_{*} \omega_{X / Y}^{n}$ contains a non-zero nef sub-sheaf $V_{n}$ ([12, Theorem 2.8]).

Using these positivity results above, one can usually treat the case when $K_{Y}$ is big or the case when det $V_{n}$ is big. Note that this approach only requires $K_{X}+B$ to be nef (not necessarily klt). This paper also treats inseparable fibrations, which can be reduced to a separable fibration of a pair $\left(X^{\prime}, B^{\prime}\right)$ not necessarily klt by applying foliation theory (Proposition [2.10).

To treat the case when $Y$ is an elliptic curve and $\operatorname{deg} V_{n}=0$, we have the following two approaches.
(2.1) Trace maps of relative Frobenius. Ejiri [11, Theorem 1.7] introduced a clever trick as follows. First there exists an isogeny $\pi: Y^{\prime} \rightarrow Y$ such that $\pi^{*} V_{n}=\bigoplus L_{i}$ where $L_{i} \in \operatorname{Pic}^{0}\left(Y^{\prime}\right)$ by [27, 1.4. Satz] and [34, Corollary 2.10]. Then by applying trace maps of relative Frobenius, one gets many relations of $L_{i}$, from which one can prove that every $L_{i}$ is torsion in $\operatorname{Pic}^{0}\left(Y^{\prime}\right)$. This indicates that for sufficiently divisible $N$, the line bundle $\mathcal{O}\left(N K_{X}\right)$ has many global sections.

This method heavily relies on the smoothness of the geometric generic fiber $X_{\bar{\eta}}$, it was applied to prove $C_{3,1}$ when $X_{\bar{\eta}}$ is smooth and either $K_{X}$ is $f$-big (11, Thm. $1.7]$ ) or is $f$ - $\mathbb{Q}$-trivial ([12, Sec. 3]).
(2.2) Adjunction formula. Granted the isogeny $\pi: Y^{\prime} \rightarrow Y$ and the splitting $\pi^{*} V_{n}=\bigoplus L_{i}$, first by applying covering theorem (Theorem 2.3) one can construct effective divisors $D_{i} \sim N K_{X}+P_{i}$ for some $P_{i} \in f^{*} \operatorname{Pic}^{0}(Y)$ and an integer $N$. Then
applying adjunction formula to different components of $D_{i}$ and $\log$ abundance for surfaces (41), one can get some relations of $P_{i}$ which conclude that $P_{i}$ are torsion.

This approach was used in [12, Sec. 4] and [47]. In fact it applies once granted positivity results as in (1) without requiring $X_{\bar{\eta}}$ to be smooth.

Now we explain the ideas of the proof of Theorem 1.6 and 1.7 .
(3) To prove Theorem 1.6, we consider the cohomological locus

$$
V^{0}\left(f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)\right)=\left\{L \in \operatorname{Pic}^{0}(A) \mid h^{0}\left(A, f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right) \otimes L\right)>0\right\}
$$

If $\operatorname{dim}\left(V^{0}\left(f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)\right)\right)>0$, then $V^{0}\left(f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)\right)$ generates $\operatorname{Pic}^{0}(A)$ since $\operatorname{Pic}^{0}(A)$ is simple, and it is easy to show that $\kappa\left(X, K_{X}+B\right) \geq \operatorname{dim} A$. For the remaining case $\operatorname{dim} V^{0}\left(f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)\right) \leq 0$, we follow the approach in (2.2). The key point is to find at least two effective divisors $D_{i} \sim n\left(K_{X}+B\right)+P_{i}$ for some $P_{i} \in f^{*} \operatorname{Pic}^{0}(A)$. We try to find a subsheaf $\mathcal{F}$ of $f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)$ and an isogeny $\pi: A^{\prime} \rightarrow A$ such that $\pi^{*} \mathcal{F}=\bigoplus L_{i}$ where $L_{i} \in \operatorname{Pic}^{0}\left(A^{\prime}\right)$. The sheaf $\mathcal{F}$ is obtained as the image of the trace map

$$
\operatorname{Tr}_{X, B}^{e, n}: f_{*}\left(F_{X *}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}\right)\left(K_{X}+B\right)\right) \otimes \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right)\right) \rightarrow f_{*} \mathcal{O}_{X}\left(n\left(K_{X}+B\right)\right) .
$$

We apply Frobenius amplitude to show that $\mathcal{F}$ satisfies a property slightly weaker than generic vanishing (Theorem 3.7), then apply "killing cohomology" (40, Prop. 12 and Sec. 9] or [20, Lemma 1.3]) to get an isogeny $\pi: A^{\prime} \rightarrow A$, some $P_{i} \in \operatorname{Pic}^{0}\left(A^{\prime}\right)$ and a generically surjective homomorphism $\bigoplus_{i} P_{i} \rightarrow \pi^{*} \mathcal{F}$.
(4) To prove Theorem [1.7, we can assume $K_{X}+B$ to be $f$-nef by replacing $(X, B)$ with a relative minimal model, then for a sufficiently divisible integer $k>0$, $k\left(K_{X}+B\right)=f^{*} L$ for some nef $L \in \operatorname{Pic}(Y)$ ([43]). It suffices to show that either $\operatorname{deg} L>0$ or $L$ is torsion. For the case $\operatorname{deg} L=0$, we use the strategy of [1, Theorem 8.10]. First we construct a nef divisor $D=a H-b F$ on $X$ with $\nu(D)=2$ where $H$ is an ample divisor and $F$ is a general fiber of $f$, second we prove that there exists a semi-ample divisor $D^{\prime} \equiv D$, which induces a fibration $g: X \rightarrow Z$ to a surface, finally we can show the restriction on the generic fiber $\left.k\left(K_{X}+B\right)\right|_{G}=k\left(K_{G}+\left.B\right|_{G}\right) \sim 0$, which implies that $L$ is torsion.

Remark 1.8. The crucial results to be used include minimal model theory of 3folds [2, 4, 21, 45], abundance for surfaces [41, 43] and results on positivity and subadditivity of Kodaira dimensions in [46]. We try to make the proof self-contained. But for some cases we only sketch the proof and refer to [12] and [48] for details.

This paper is organized as follows. In Section 2, we collect some useful results and study inseparable fibrations as preparations. Section 3 is devoted to studying sheaves on abelian varieties. In Section 4, we study subadditivity of Kodaira dimensions. Finally in Section 5 we finish the proof of abundance in Theorem 1.2.

Conventions: Sometimes we do not distinguish between line bundles and Cartier divisors, and abuse the notation additivity and tensor product if no confusion occurs.

Let $X$ be a normal projective variety. Denote by $\mathrm{W} \operatorname{div}(X)$ the set of Weil divisors and by $\operatorname{Cdiv}(X)$ the set of Cartier divisors on $X$. Assume $\mathbb{K}=\mathbb{Q}$ or $\mathbb{R}$. The divisors in $\mathrm{W} \operatorname{div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ are called $\mathbb{K}$-divisors, and the ones in $\operatorname{Cdiv}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ are called $\mathbb{K}$ Cartier $\mathbb{K}$-divisors. We use $\equiv$ for the numerical equivalence relation in $\operatorname{Cdiv}(X) \otimes_{\mathbb{Z}} \mathbb{K}$.

Let $D$ be a $\mathbb{Q}$-divisor on a normal variety $X$. The Weil index of $D$ is the minimal positive integer $l$ such that $l D$ is integral. If $D$ is $\mathbb{Q}$-Cartier, the Cartier index is defined similarly. We use $\sim\left(\right.$ resp. $\sim_{\mathbb{Q}}$ ) for linear (resp. $\mathbb{Q}$-linear) equivalence between Cartier (resp. $\mathbb{Q}$-Cartier) divisors and line bundles. For two ( $\mathbb{Q}$-)divisors $D_{1}, D_{2}$ on $X$, if $\left.\left.D_{1}\right|_{X^{s m}} \sim D_{2}\right|_{X^{s m}}$ (resp. $\left.\left.D_{1}\right|_{X^{s m}} \sim_{\mathbb{Q}} D_{2}\right|_{X^{s m}}$ ), we also denote $D_{1} \sim D_{2}\left(\right.$ resp. $\left.D_{1} \sim_{\mathbb{Q}} D_{2}\right)$.

Let $X$ be a normal variety and $D$ a Weil divisor on $X$. Then $\mathcal{O}_{X}(D)$ is a subsheaf of the constant sheaf $K(X)$ of rational functions, with the stalk at a point $x$ being defined by

$$
\mathcal{O}_{X}(D)_{x}:=\left\{f \in K(X)|((f)+D)|_{U} \geq 0 \text { for some open set } U \text { containing } x\right\}
$$

For notions in minimal model theory such as lc, klt dlt pairs, flip and divisorial contraction and so on, we refer to [2].

For a variety $X$, we usually use $F_{X}^{e}: X \rightarrow X$ to denote the $e^{\text {th }}$ iteration of absolute Frobenius, we sometimes use $X^{e}$ for the origin scheme of $F_{X}^{e}$ to avoid confusions.

Let $\varphi: X \rightarrow T$ be a morphism of schemes and let $T^{\prime}$ be a $T$-scheme. Then we denote by $X_{T^{\prime}}$ the fiber product $X \times_{T} T^{\prime}$. For a divisor $D$ on $X$ (resp. an $\mathcal{O}_{X}$-module $\mathcal{G})$, the pullback of $D$ (resp. $\mathcal{G}$ ) to $X_{T^{\prime}}$ is denoted by $D_{T^{\prime}}$ or $\left.D\right|_{X_{T^{\prime}}}$ (resp. $\mathcal{G}_{T^{\prime}}$ or $\left.\mathcal{G}\right|_{X_{T^{\prime}}}$ ) if it is well-defined.

If $X$ is a projective variety in dimension $\leq 3$, it always has a smooth birational model $\tilde{X}$, then $\kappa(X):=\kappa(\tilde{X})$ which is birational invariant.

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## 2. Preliminaries

In this section we collect some technical results which will be used in the paper.

### 2.1. Separability of fibrations.

Proposition 2.1. Let $f: X \rightarrow Y$ be a fibration of normal varieties over an algebraically closed field $k$ of characteristic $p>0$. Then
(1) $f$ is separable if and only if $X_{\bar{\eta}}$ is reduced, and if and only if $X_{\bar{\eta}}$ is integral;
(2) if $\operatorname{dim} Y=1$ then $f$ is separable.

Proof. Since $f$ is a fibration we have $H^{0}\left(\mathcal{O}_{X_{\eta}}\right)=K(Y)$, and since $X_{\eta}$ is normal we can show $K(Y)$ is algebraically closed in $K(X)$. Then the assertion (1) follows from applying [28, Sec. 3.2.2 Cor. 2.14 and Prop. 2.15]. The assertion (2) is [1, Lemma 7.2].
2.2. Relative Fujita Vanishing. The following result is [26, Theorem 1.5].

Lemma 2.2. Let $f: X \rightarrow Y$ be a projective morphism over a Noetherian scheme, $H$ an $f$-ample line bundle and $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a positive integer $N$ such that, for every $n>N$ and every nef line bundle $L$,

$$
R^{i} f_{*}\left(\mathcal{F} \otimes H^{n} \otimes L\right)=0, \text { if } i>0 .
$$

2.3. Covering Theorem. The result below is [[22], Theorem 10.5] when $X$ and $Y$ are both smooth, and the proof also applies when varieties are normal.

Theorem 2.3. ([22, Theorem 10.5]) Let $f: X \rightarrow Y$ be a proper surjective morphism between complete normal varieties. If $D$ is a Cartier divisor on $Y$ and $E$ an effective $f$-exceptional divisor on $X$, then

$$
\kappa\left(X, f^{*} D+E\right)=\kappa(Y, D)
$$

As a corollary we get the following useful result, which also appeared in [12].
Lemma 2.4. ([12, Lemma 2.3]) Let $g: W \rightarrow Y$ be a surjective projective morphism between projective varieties. Assume $Y$ is normal and let $L_{1}, L_{2} \in \operatorname{Pic}(Y)$ be two line bundles on $Y$. If $g^{*} L_{1} \sim_{\mathbb{Q}} g^{*} L_{2}$ then $L_{1} \sim_{\mathbb{Q}} L_{2}$.
Proof. Let $L=L_{1} \otimes L_{2}^{-1}$. Denote by $\sigma: W^{\prime} \rightarrow W$ the normalization and let $g^{\prime}=g \circ \sigma: W^{\prime} \rightarrow Y$. Then $g^{\prime *} L \sim_{\mathbb{Q}} 0$. Applying Theorem 2.3 to $g^{\prime}: W^{\prime} \rightarrow Y$ shows that $L \sim_{\mathbb{Q}} 0$, which is equivalent to that $L_{1} \sim_{\mathbb{Q}} L_{2}$.
2.4. Minimal model theory of 3 -folds. The following theorem includes some recent results of minimal model theory for 3 -folds in positive characteristic.

Theorem 2.5. Assume char $k=p>5$. Let $(X, B)$ be a $\mathbb{Q}$-factorial projective pair of dimension three and $f: X \rightarrow Y$ a projective surjective morphism.
(1) If either $(X, B)$ is klt and $K_{X}+B$ is pseudo-effective over $Y$, or $(X, B)$ is lc and $K_{X}+B$ has a weak Zariski decomposition over $Y$, then $(X, B)$ has a log minimal model over $Y$.
(2) If $(X, B)$ is dlt and $K_{X}+B$ is not pseudo-effective over $Y$, then $(X, B)$ has a Mori fibre space over $Y$.
(3) Assume that $(X, B)$ is klt in (3.1) and is dlt in other cases, and that $K_{X}+B$ is nef over $Y$.
(3.1) If $K_{X}+B$ or $B$ is big over $Y$, then $K_{X}+B$ is semi-ample over $Y$.
(3.2) If $\operatorname{dim} Y \geq 1$, then $\left(K_{X}+B\right)_{\eta}$ is semi-ample on $X_{\eta}$.
(3.3) If $Y$ is a smooth curve, $X_{\eta}$ is integral and $\kappa\left(X_{\eta},\left(K_{X}+B\right)_{\eta}\right)=0$ or 2 , then $K_{X}+B$ is semi-ample over $Y$.
(3.4) If $Y$ contains no rational curves, then $K_{X}+B$ is nef.
(4) Assume $(X, B)$ is klt. If $Y$ is a non-uniruled surface and $K_{X}+B$ is pseudoeffective over $Y$, then $K_{X}+B$ is pseudo-effective, and there exists a map $\sigma: X \rightarrow \bar{X}$ to a minimal model $(\bar{X}, \bar{B})$ of $(X, B)$ such that, the restriction map $\left.\sigma\right|_{X_{\eta}}$ is an isomorphism from $X_{\eta}$ to its image.

Proof. Assertion (1) is from [2, Theorem 1.2 and Proposition 7.3]; (2) is [4, Theorem $1.7] ;(3.1)$ is [2, Theorem 1.4]; (3.2) is from [43, Theorem 1.1]; and (3.3) is from [3, Theorem 1.5 and 1.6 and the remark below 1.6].

Assertion (3.4) follows from the cone theorem [4, Theorem 1.1]. Indeed, otherwise we can find an extremal ray $R$ generated by a rational curve $\Gamma$, so $\Gamma$ is contained in a fiber of $f$ since $Y$ contains no rational curves, this contradicts that $K_{X}+B$ is $f$-nef.

For (4), $K_{X}+B$ is obviously pseudo-effective because otherwise, $X$ will be ruled by horizontal (w.r.t. $f$ ) rational curves by (2), which contradicts that $Y$ is nonuniruled. The exceptional locus of a flip contraction is of dimension one, so it does not intersect $X_{\eta}$, neither does that of an extremal divisorial contraction because it is uniruled (see the proof of [4, Lemma 3.2]). Running an LMMP for $K_{X}+B$, by induction we get a map $\sigma: X \rightarrow \bar{X}$ as required.

### 2.5. Numerical dimension.

Definition 2.6. Let $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a smooth projective variety $X$ of dimension $n$. The numerical dimension $\kappa_{\sigma}(D)$ is defined as the biggest natural number $k$ such that

$$
\liminf _{m \rightarrow \infty} \frac{h^{0}(\llcorner m D\lrcorner+A)}{m^{k}}>0 \text { for some ample divisor } A \text { on } X .
$$

If such a $k$ does not exist then we define $\kappa_{\sigma}(D)=-\infty$.
Remark 2.7. In arbitrary characteristics, since smooth alterations exist due to de Jong [23], this invariant can be defined for $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on normal varieties by pulling back to a smooth variety, which does not depend on the choices of smooth alterations by [5, 2.5-2.7].

Proposition 2.8. Let $X$ be a normal projective variety and $D$ an $\mathbb{R}$-Cartier $\mathbb{R}$ divisor on $X$.
(1) When $D$ is nef, then $\kappa_{\sigma}(D)$ coincides with

$$
\nu(D)=\max \left\{k \in \mathbb{N} \mid D^{k} \cdot A^{n-k}>0 \text { for an ample divisor } A \text { on } X\right\}
$$

If moreover $D$ is effective and $S$ is a normal component of $D$, then

$$
\nu\left(\left.D\right|_{S}\right) \leq \nu(D)-1
$$

(2) Let $\mu: W \rightarrow X$ be a generically finite surjective morphism between two normal projective varieties and $E$ an effective $\mu$-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $W$. Then

$$
\kappa_{\sigma}(D)=\kappa_{\sigma}\left(\mu^{*} D+E\right)
$$

(3) If $(X, \Delta)$ is a $\mathbb{Q}$-factorial log canonical 3-fold, and $\left(X^{\prime}, \Delta^{\prime}\right)$ is a minimal model of $(X, \Delta)$, then

$$
\kappa_{\sigma}\left(K_{X}+\Delta\right)=\nu\left(K_{X^{\prime}}+\Delta^{\prime}\right)
$$

Proof. For (1), the first assertion is [33, V, 2.7 (6)] in characteristic zero, and is [5, Porposition 4.5] in characteristic $p>0$. The second assertion follows from the relation [9, Sec. 1.2]

$$
\left(\left.D\right|_{S}\right)^{k-1} \cdot\left(\left.A\right|_{S}\right)^{n-k}=D^{k-1} \cdot A^{n-k} \cdot S \leq D^{k} \cdot A^{n-k}
$$

(2) is from applying Theorem [2.3 (cf. [5, 2.7]).

Finally for (3), taking a common log resolution of $(X, \Delta)$ and $\left(X^{\prime}, \Delta^{\prime}\right)$, this assertion follows from applying (2) and some standard arguments of minimal model theory.
2.6. Inseparable fibrations. In this section we work over an algebraically closed field $k$ of characteristic $p>0$. Let $X$ be a smooth variety. Recall that a (1-)foliation is a saturated subsheaf $\mathcal{F} \subset T_{X}$ which is involutive (i.e., $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$ ) and $p$-closed (i.e., $\xi^{p} \in \mathcal{F}, \forall \xi \in \mathcal{F}$ ). A foliation $\mathcal{F}$ is called smooth if it is a subbundle of $T_{X}$.

Proposition 2.9. Let $X$ be a smooth variety and $\mathcal{F}$ a foliation on $X$.
(1) We get a normal variety $Y=X / \mathcal{F}=\operatorname{Spec} A n n(\mathcal{F})$, and there exist natural morphisms $\pi: X \rightarrow Y$ and $\pi^{\prime}: Y \rightarrow X^{(1)}$ fitting into the following commutative diagram


Moreover $\operatorname{deg} \pi=p^{r}$ where $r=\operatorname{rk} \mathcal{F}$.
(2) There is a one-to-one correspondence between foliations and normal varieties between $X$ and $X^{(1)}$, by the correspondence $\mathcal{F} \mapsto X / \mathcal{F}$ and the inverse correspondence $Y \mapsto \operatorname{Ann}\left(\mathcal{O}_{Y}\right):=\left\{\xi \in T_{X} \mid \xi(a)=0, \forall a \in \mathcal{O}_{Y}\right\}$.
(3) The variety $Y$ is regular if and only if $\mathcal{F}$ is smooth.
(4) If $Y_{0}$ denotes the regular locus of $Y$ and $X_{0}=\pi^{-1} Y_{0}$, then

$$
K_{X_{0}} \sim \pi^{*} K_{Y_{0}}+\left.(p-1) \operatorname{det} \mathcal{F}\right|_{X_{0}} .
$$

Proof. We refer to [29, p.56-58] or [13].
The following result helps us to reduce an inseparable fibration to a separable one.
Proposition 2.10. Let $f: X \rightarrow Y$ be a fibration from a normal projective 3-fold to a normal surface of maximal Albanese dimension, and let $B$ be an effective $\mathbb{Q}$ divisor on $X$. Then there exist a purely inseparable morphism $\sigma: X \rightarrow X^{\prime}$, a separable fibration $f^{\prime}: X^{\prime} \rightarrow Y$, a rational number $t>0$ and an effective $\mathbb{Q}$-divisor $B^{\prime}$ on $X^{\prime}$ such that

$$
K_{X}+B \sim_{\mathbb{Q}} t\left(\sigma^{*}\left(K_{X^{\prime}}+B^{\prime}\right)\right) .
$$

Proof. Let $a_{Y}: Y \rightarrow A$ be the Albanese map and let $a_{X}=a_{Y} \circ f$. If $f$ is a separable morphism, then we are done.

Assume that $f$ is inseparable. Let $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$ denote the saturation of the image of the natural homomorphisms $a_{X}^{*} \Omega_{A_{X}}^{1} \rightarrow \Omega_{X}^{1}$ and $f^{*} \Omega_{Y}^{1} \rightarrow \Omega_{X}^{1}$ respectively. Then $\mathcal{L}_{1} \subseteq \mathcal{L}_{2}$ and rk $\mathcal{L}_{2} \leq 1$ since $f$ is inseparable ( $[17$, Prop. 8.6A]). And by Igusa's result ([39, Theorem 4]), we have $\mathcal{L}_{1}$ is generically globally generated, and $h^{0}\left(X, \mathcal{L}_{1}\right) \geq h^{0}\left(A_{X}, \Omega_{A_{X}}^{1}\right) \geq 2>$ rk $\mathcal{L}_{1}$. Therefore $\mathcal{L}_{1}=\mathcal{L}_{2}=\mathcal{L}$ is of rank one and $h^{0}\left(X, \mathcal{L}^{* *}\right) \geq 2([48$, Lemma 4.2]).

We get a natural foliation $\mathcal{F}=\mathcal{L}^{\perp} \subset T_{X}$ of rank 2 , and a quotient map $\rho$ : $X \rightarrow X_{1}=X / \mathcal{F}$, which is a factor of $f$ by the construction above. Denote by $X^{0}$
the maximal smooth open subset of $X$ such that $\left.\mathcal{F}\right|_{X_{0}}$ is smooth, and let $X_{1}^{0}=$ $X^{0} /\left(\left.\mathcal{F}\right|_{X^{0}}\right)$. Then

$$
\begin{equation*}
K_{X^{0}} \sim \rho^{*} K_{X_{1}^{0}}+\left.(p-1) \operatorname{det} \mathcal{F}\right|_{X^{0}} \tag{1}
\end{equation*}
$$

On the other hand, we have the following exact sequence

$$
\left.\left.0 \rightarrow \mathcal{L}\right|_{X^{0}} \rightarrow \Omega_{X^{0}}^{1} \rightarrow \mathcal{F}^{*}\right|_{X^{0}} \rightarrow 0
$$

which gives

$$
\left.\left.\operatorname{det} \mathcal{F}\right|_{X^{0}} \sim \mathcal{L}\right|_{X^{0}}-K_{X^{0}}
$$

Combining with Equation (1), we get

$$
\begin{equation*}
K_{X^{0}} \sim_{\mathbb{Q}} \frac{1}{p}\left(\rho^{*} K_{X_{1}^{0}}+\left.(p-1) \mathcal{L}\right|_{X^{0}}\right) \tag{2}
\end{equation*}
$$

Since $\mathcal{L}$ is generically globally generated, there exists an effective Weil divisor $B^{\prime}$ on $X$ such that $B^{\prime} \sim \mathcal{L}$. And since $\rho$ is purely inseparable, there exist $\mathbb{Q}$-divisors $B_{1}, B_{1}^{\prime}$ on $X_{1}$ such that $\rho^{*} B_{1}=B$ and $\rho^{*} B_{1}^{\prime}=B^{\prime}$. Let $B_{1}=p B_{1}+(p-1) B_{1}^{\prime}$. Then since $X \backslash X^{0}$ is of codimension $\geq 2$ in $X$, by Equation (2) we have that

$$
K_{X}+B \sim_{\mathbb{Q}} \frac{1}{p}\left(\rho^{*}\left(K_{X_{1}}+B_{1}\right)\right) .
$$

If the natural fibration $f_{1}: X_{1} \rightarrow Y$ is separable then we are done. If not, we consider the pair $\left(X_{1}, B_{1}\right)$ instead and repeat the process above. We can prove that $\operatorname{mult}\left(\left(X_{1}\right)_{\bar{\eta}}\right)<\operatorname{mult}\left(X_{\bar{\eta}}\right)$ by the argument of [48, the latter case of Thm. 4.3]. So this process will terminate, and we can show the assertion by induction.

For a fibration fibred by curves of arithmetic genus one, we have the following result if char $k=p \geq 5$.

Proposition 2.11. Assume char $k=p \geq 5$. Let $g: X \rightarrow Z$ be a fibration of normal varieties of relative dimension one. Assume that the generic fiber $X_{\xi}$ of $g$ is a curve with arithmetic genus $p_{a}\left(X_{\xi}\right)=1$. Then the geometric generic fiber $X_{\bar{\xi}}$ of $g$ is a smooth elliptic curve over $\overline{K(Z)}$.

Proof. Assume by contrary that $X_{\bar{\xi}}$ is singular. Applying [42, Lemma 2.3 and 2.4], we get the following commutative diagram

where $Z_{1} \rightarrow Z$ is a purely inseparable base change (or identity), $Z_{2} \rightarrow Z_{1}$ is a degree $p$ purely inseparable extension, $X_{1}=X \times_{Z} Z_{1}$ and $X_{2}=X \times_{Z} Z_{2}$, such that $X_{\xi_{1}}$ is regular and that $X_{\xi_{2}}$ is reduced but not normal, here $\xi_{1}, \xi_{2}$ denote the generic point of $Z_{1}, Z_{2}$ respectively. The normalization $\tilde{X}_{\xi_{2}}$ of $X_{\xi_{2}}$ has smaller arithmetic genus hence must be a smooth curve of genus zero. If necessary by shrinking $Z_{i}$, we can assume both $\tilde{X}_{2}$ and $X_{1}$ are smooth, so the natural morphism $\pi: \tilde{X}_{2} \rightarrow X_{1}$ is the
quotient induced by a smooth foliation $\mathcal{F}$ on $\tilde{X}_{2}$, which is a subbundle of $T_{\tilde{X}_{2}}$ of rank one. Since

$$
K_{\tilde{X}_{2}}=\pi^{*} K_{X_{1}}+(p-1) \mathcal{F}
$$

we get a contradiction by $(p-1) \operatorname{deg} \mathcal{F}_{\xi}=\operatorname{deg} K_{\tilde{X}_{\xi_{2}}}=-2$ and $p \geq 5$.
2.7. Trace maps of absolute Frobenius morphisms. Let $X$ be a projective variety over an algebraically closed field $k$ of characteristic $p>0$. We will consider the trace maps in the following two settings.

Notation 2.12. Assume $X$ is normal. Denote by $X_{0}$ the smooth open subset of $X$. Let $B$ be an effective $\mathbb{Q}$-Weil divisor with Weil index not divisible by $p$. There exists a positive integer $g$ such that $\left(p^{g}-1\right) B$ is integral, thus $\left(p^{e g}-1\right) B$ is integral for every integer $e>0$. The composition map of the natural inclusion

$$
\left.F_{X *}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)\right)\right|_{X_{0}} \hookrightarrow F_{X_{0} *}^{e g} \mathcal{O}_{X_{0}}\left(\left(1-p^{e g}\right) K_{X_{0}}\right)
$$

and the trace map $\operatorname{Tr}_{F_{X_{0}}^{e g}}: F_{X_{0} *}^{e g} \mathcal{O}_{X_{0}}\left(\left(1-p^{e g}\right) K_{X_{0}}\right) \rightarrow \mathcal{O}_{X_{0}}$ extends to a map on $X$ :

$$
\operatorname{Tr}_{X, B}^{e g}: F_{X *}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)\right) \rightarrow \mathcal{O}_{X}
$$

Let $D$ be a Cartier divisor on $X$. Twisting $\operatorname{Tr}_{X, B}^{e g}$ with $\mathcal{O}_{X}(D)$ induces

$$
\begin{aligned}
\operatorname{Tr}_{X, B}^{e g}(D): & F_{X *}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)\right) \otimes \mathcal{O}_{X}(D) \\
& \cong F_{X}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)+p^{e g} D\right) \rightarrow \mathcal{O}_{X}(D)
\end{aligned}
$$

Then taking global sections induces a trace map

$$
\Phi_{e g}: H^{0}\left(X, F_{X *}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+\Delta\right)+p^{e g} D\right)\right) \rightarrow H^{0}(X, D)
$$

Denote $S_{B}^{e g}(X, D)=\operatorname{Im} \Phi_{e g}$ and $S_{B}^{0}(X, D)=\cap_{e \geq 0} S_{B}^{e g}(X, D)$. If $B=0$, we usually use the notation $S^{0}(X, D)$ instead of $S_{0}^{0}(X, D)$. Note that for $e^{\prime}>e$, we have $S_{B}^{e^{\prime} g}(X, D) \subseteq S_{B}^{e g}(X, D)$ by the factorization

$$
\begin{gathered}
\operatorname{Tr}_{X, B}^{e^{\prime} g}(D): F_{X *}^{e g} F_{X *}^{\left(e^{\prime}-e\right) g} \mathcal{O}_{X}\left(\left(1-p^{e^{\prime} g}\right)\left(K_{X}+B\right)+p^{e^{\prime} g} D\right) \xrightarrow{F_{X *}^{e g} T r_{X, \Delta}^{\left(e^{\prime}-e\right) g}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)+p^{e g} D\right)} \\
\\
F_{X *}^{e g} \mathcal{O}_{X}\left(\left(1-p^{e g}\right)\left(K_{X}+B\right)+p^{e g} D\right) \xrightarrow{T r_{X, B}^{e g}(D)} \mathcal{O}_{X}(D) .
\end{gathered}
$$

Notation 2.13. Assume $X$ is Gorenstein in codimension one and satisfies Serre condition $S_{2}$. Let $B$ be an effective $\mathbb{Q}$-AC divisor such that $K_{X}+B$ is $\mathbb{Q}$-Cartier ([46, Sec. 2.1 and 2.3], namely, $B=\frac{M-n K_{X}}{n}$ for some $n>0$, where $M$ is a Cartier divisor and $M-n K_{X}$ is effective in codimension one). Assume moreover that the Cartier index of $K_{X}+B$ is not divisible by $p$. Let $g>0$ be an integer such that $\left(1-p^{g}\right)\left(K_{X}+B\right)$ is Cartier. Then we can define trace maps $\operatorname{Tr}_{X, B}^{e g}, \operatorname{Tr}_{X, B}^{e g}(D)$ as in 2.12 (see [46, Sec. 2.3] for details). By [14, Lemma 13.1], there exists an ideal $\sigma(B)$, namely, the non-F-pure ideal of $(X, B)$, such that for some sufficiently divisible $g^{\prime}>0$ and any $e>0$,

$$
\operatorname{Im} \operatorname{Tr}_{X, B}^{e g^{\prime}}=\sigma(B)=\operatorname{Tr}_{X, B}^{e g^{\prime}}\left(F_{X *}^{e g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{e g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)\right)
$$

Borrowing the idea of the proof of [37, Lemma 2.20], we prove the following lemma.

Lemma 2.14. Using Notation 2.13, let $A, D$ be two Cartier divisors on $X$. If $A$ is ample, then there exists $M>0$ such that for any $m>M$ and $e>0$, the trace map

$$
\begin{aligned}
\Phi_{e, m}: H^{0}\left(X, F_{X *}^{e g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{e g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)\right. & \left.\otimes \mathcal{O}_{X}(m A+D)\right) \\
& \rightarrow H^{0}\left(X, \sigma(B) \cdot \mathcal{O}_{X}(m A+D)\right)
\end{aligned}
$$

is surjective. In particular, there exists some $c>0$ such that for any $m>0$, $\operatorname{dim} S_{B}^{0}(X, m A+D) \geq c m^{\operatorname{dim} X}$.

Proof. We only need to prove the first assertion, which implies the second one. Let $K_{e g^{\prime}}=\operatorname{ker}\left(\operatorname{Tr}_{X, B}^{e g^{\prime}}: F_{X *}^{e g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{e g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \rightarrow \sigma(B)\right)$. Then we have the following commutative diagram of short exact sequences

where $\gamma_{2}$ is obtained by applying $F_{X *}^{(e-1) g^{\prime}}$ to the trace map

$$
\begin{aligned}
& F_{X *}^{g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \otimes \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right) \\
& \cong F_{X *}^{g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{e g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \rightarrow \sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right),
\end{aligned}
$$

and $\gamma_{1}$ arises naturally. Since $F_{X}^{(e-1) g^{\prime}}$ is an affine morphism, $\gamma_{2}$ is surjective and

$$
\operatorname{ker}\left(\gamma_{2}\right)=F_{X *}^{(e-1) g^{\prime}}\left(K_{g^{\prime}} \otimes \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)
$$

Let $K^{\prime}=\operatorname{ker}\left(\gamma_{1}\right)$. Since $\gamma_{2}$ is surjective, applying Snake Lemma we obtain that $\gamma_{1}$ is surjective and $K^{\prime} \cong F_{X *}^{(e-1) g^{\prime}}\left(K_{g^{\prime}} \otimes \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)$. It follows an exact sequence

$$
\begin{equation*}
0 \rightarrow F_{X *}^{(e-1) g^{\prime}}\left(K_{g^{\prime}} \otimes \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \rightarrow K_{e g^{\prime}} \rightarrow K_{(e-1) g^{\prime}} \rightarrow 0 \tag{3}
\end{equation*}
$$

And since $A$ is ample, we have
(a) there exists $M_{0}>0$ such that for any $t \in[0,1]$, the divisor $M_{0} A+D-$ $t\left(K_{X}+B\right)$ is nef; and
(b) applying Fujita vanishing (Lemma 2.2), there exists an integer $M_{1}$ such that for any $l>M_{1}$ and any nef Cartier divisor $P, H^{1}\left(X, K_{g^{\prime}} \otimes \mathcal{O}_{X}(l A+P)\right)=0$.
Let $M=M_{0}+M_{1}$. Fix an integer $m>M$. We aim to show that the trace map $\Phi_{e, m}$ is surjective. Tensoring the following exact sequence with $\mathcal{O}_{X}(m A+D)$

$$
0 \rightarrow K_{e g^{\prime}} \rightarrow F_{X *}^{e g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{e g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \rightarrow \sigma(B) \rightarrow 0
$$

and taking cohomology, it is sufficient to show that for every $e>0$,

$$
H^{1}\left(X, K_{e g^{\prime}} \otimes \mathcal{O}_{X}(m A+D)\right)=0
$$

By $m A+D=\left(m-M_{0}\right) A+\left(M_{0} A+D\right)$, applying (a) and (b) we can show the case $e=1$. Assume by induction that $H^{1}\left(X, K_{(e-1) g^{\prime}} \otimes \mathcal{O}_{X}(m A+D)\right)=0$ for some
$e \geq 2$. The condition (a) implies that $p^{(e-1) g^{\prime}}\left(M_{0} A+D\right)+\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)$ is nef. Then applying the condition (b), it follows that

$$
\begin{aligned}
& H^{1}\left(X, F_{X *}^{(e-1) g^{\prime}}\left(K_{g^{\prime}} \otimes \mathcal{O}_{X}\left(\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right)\right) \otimes \mathcal{O}_{X}(m A+D)\right) \\
& \cong H^{1}\left(X, K_{g^{\prime}} \otimes \mathcal{O}_{X}\left(p^{(e-1) g^{\prime}}\left(m-M_{0}\right) A+p^{(e-1) g^{\prime}}\left(M_{0} A+D\right)+\left(1-p^{(e-1) g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)=0
\end{aligned}
$$

Tensoring the exact sequence (3) with $\mathcal{O}_{X}(m A+D)$ and taking cohomology, we deduce that $H^{1}\left(X, K_{e g^{\prime}} \otimes \mathcal{O}_{X}(m A+D)\right)=0$.
2.8. Derived categories. Let $X$ be a projective variety, we denote by $D^{b}(X)$ the bounded derived category of coherent sheaves on $X$.

For an object $\mathcal{F} \in D^{b}(X)$ represented by the complex

$$
\mathcal{K}^{\bullet}: \cdots \rightarrow K^{n-1} \rightarrow K^{n} \rightarrow K^{n+1} \rightarrow \cdots
$$

we have truncations ([16, p. 69])
$\sigma_{\leq n}\left(\mathcal{K}^{\bullet}\right): \cdots \rightarrow K^{n-1} \rightarrow$ ker $d^{n} \rightarrow 0 \rightarrow \cdots$ with $\mathcal{H}^{i}\left(\sigma_{\leq n}\left(\mathcal{K}^{\bullet}\right)\right) \cong \mathcal{H}^{i}\left(\mathcal{K}^{\bullet}\right)$ for $i \leq n$ and

$$
\sigma_{>n}\left(\mathcal{K}^{\bullet}\right): \cdots \rightarrow 0 \rightarrow \operatorname{im} d^{n} \rightarrow K^{n+1} \rightarrow \cdots \text { with } \mathcal{H}^{i}\left(\sigma_{>n}\left(\mathcal{K}^{\bullet}\right)\right) \cong \mathcal{H}^{i}\left(\mathcal{K}^{\bullet}\right) \text { for } i>n
$$

The exact sequence below

$$
0 \rightarrow \sigma_{\leq n}\left(\mathcal{K}^{\bullet}\right) \rightarrow \mathcal{K}^{\bullet} \rightarrow \sigma_{>n}\left(\mathcal{K}^{\bullet}\right) \rightarrow 0
$$

descends to a triangle in $D^{b}(X)$ ([16, p. 63 Remark after Prop. 6.1])

$$
\sigma_{\leq n}(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \sigma_{>n}(\mathcal{F}) \rightarrow \sigma_{\leq n}(\mathcal{F})[1] .
$$

Let $f: X \rightarrow Y$ be a projective morphism of projective varieties. Assume $Y$ is smooth. We have derived functors $R f_{*}: D^{b}(X) \rightarrow D^{b}(Y)$ and $L f^{*}: D^{b}(Y) \rightarrow$ $D^{b}(X)$ of $f_{*}$ and $f^{*}$ respectively ([16, Chap. II Sec. 2,4]. By Grothendieck duality ([16, Chap. III Sec. 11]) there exists a functor $f^{!}$such that

$$
R \mathcal{H o m}{ }_{Y}\left(R f_{*} \mathcal{E}, \mathcal{F}\right) \cong R f_{*} R \mathcal{H o m}{ }_{X}\left(\mathcal{E}, f^{!} \mathcal{F}\right)
$$

where $\mathcal{E} \in D^{b}(X), \mathcal{F} \in D^{b}(Y)$. In particular if both $X$ and $Y$ are smooth, then $f^{!} \mathcal{F} \cong L f^{*} \mathcal{F} \otimes \mathcal{O}_{X}\left(K_{X / Y}\right)[\operatorname{dim} X / Y]$ ([16, Chap. VI Sec. 4]).
2.9. Cohomology of flat complexes under base changes. Let's recall the following result, which is an adaption of [17, Chap. III Cor. 12.11] to flat bounded complexes. Though this is known to experts ([36, Remark of 3.6]), we explain the modifications of the proof for the convenience of the reader.

Theorem 2.15. Let $f: X \rightarrow Y$ be a projective morphism of varieties. Let $\mathcal{K} \bullet$ be a bounded complex of coherent sheaves on $X$ such that every $\mathcal{K}^{i}$ is flat over $Y$. For a closed point $y \in Y$,
(1) if the natural map $\varphi_{y}^{i}: R^{i} f_{*} \mathcal{K}^{\bullet} \otimes k(y) \rightarrow R^{i} \Gamma\left(X_{y}, \mathcal{K}_{y}^{\bullet}\right)$ is surjective, then $\varphi_{y}^{i}$ is an isomorphism, and there exists a neighborhood $U$ of $y$ such that for any $y^{\prime} \in U$, the map $\varphi_{y^{\prime}}^{i}$ is surjective;
(2) if $\varphi_{y}^{i}$ is surjective then the following two conditions are equivalent to each other
(2.1) $\varphi_{y}^{i-1}: R^{i-1} f_{*} \mathcal{K} \bullet \otimes k(y) \rightarrow R^{i-1} \Gamma\left(X_{y}, \mathcal{K}_{y}^{\bullet}\right)$ is surjective;
(2.2) $R^{i} f_{*} \mathcal{K}^{\bullet}$ is locally free at $y$.

Proof. We assume $Y=\operatorname{Spec} A$ is affine. For an $A$-module $M$, define the functor

$$
T^{i}(M)=R^{i} \Gamma\left(X, \mathcal{K}^{\bullet} \otimes_{A} M\right)
$$

To adapt the arguments of [17, Chap. III, Sec. 12] to flat complexes, we only need to verify the analogue of [17, Chap. III, Prop. 12.1 and 12.2].
(a) Tensoring $\mathcal{K}^{\bullet}$ with a short exact exact sequence $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime \prime} \rightarrow 0$ of $A$-modules, since $\mathcal{K}^{\bullet}$ is flat, we get a short exact sequence of complexes of sheaves on $X$

$$
0 \rightarrow \mathcal{K}^{\bullet} \otimes_{A} M^{\prime} \rightarrow \mathcal{K}^{\bullet} \otimes_{A} M \rightarrow \mathcal{K}^{\bullet} \otimes_{A} M^{\prime \prime} \rightarrow 0
$$

Taking cohomology shows that $T^{i}$ is exact in the middle. So the analogue of [17, Chap. III, Prop. 12.1] holds.
(b) Fix an open affine cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$. Consider the double complex

$$
C^{\bullet}\left(\mathcal{U}, \mathcal{K}^{\bullet}\right)=\left(C^{q}\left(\mathcal{U}, \mathcal{K}^{p}\right), d^{\prime p q}, d^{\prime \prime p q}\right)
$$

where $d^{\prime p q}$ come from C Cech complexes $C^{\bullet}\left(\mathcal{U}, \mathcal{K}^{p}\right)$ and $d^{\prime \prime p q}$ are induced by the differentials of $\mathcal{K}^{\bullet}$. Consider the total complex $C^{\bullet}$

$$
C^{i}=\bigoplus_{p+q=i} C^{q}\left(\mathcal{U}, \mathcal{K}^{p}\right), d^{i}=\sum_{p+q=i}(-1)^{p} d^{\prime p q}+d^{\prime \prime p q}: C^{i} \rightarrow C^{i+1}
$$

Note that $C^{i}$ is a flat $A$-module, the total complex of $C^{\bullet}\left(\mathcal{U}, \mathcal{K}^{\bullet} \otimes_{A} M\right)$ coincides with $C^{\bullet} \otimes_{A} M$, thus $H^{i}\left(C^{\bullet} \otimes_{A} M\right) \cong R^{i} \Gamma\left(X, \mathcal{K}^{\bullet} \otimes_{A} M\right)$. Then we can show the analogue of [17, Chap. III, Prop. 12.2] by similar arguments.
2.10. Abelian subvarieties generated by subschemes. Let $A$ be an abelian variety of dimension $d$ and $V$ a closed subscheme of $A$. Let $W_{n}$ be the reduced subscheme supported on the $n$-fold sum $V+V+\cdots+V$. Let $N$ be an integer (say, $N=\operatorname{dim} A$ ) such that $d^{\prime}=\operatorname{dim} W_{N}$ attains the maximum of $\operatorname{dim} W_{n}$. Let $Z$ be an irreducible component of $W_{N}$ with $\operatorname{dim} Z=d^{\prime}$.

Lemma 2.16. With the above notation, take a closed point $z \in Z$ and let $Z_{0}=Z-z$. Then
(1) $Z_{0}$ is an abelian subvariety of $A$; and
(2) every component of $W_{n}$ is contained in a translate of $Z_{0}$.

Proof. (1) Note that $Z_{0}+Z_{0}$ is irreducible because it is the image of $m: Z_{0} \times Z_{0} \rightarrow A$ via $m\left(z_{1}, z_{2}\right)=z_{1}+z_{1}$. Since $0 \in Z_{0}$, we have $\operatorname{Supp} Z_{0} \subseteq \operatorname{Supp}\left(Z_{0}+Z_{0}\right)$, then $\operatorname{dim} Z_{0}=\operatorname{dim}\left(Z_{0}+Z_{0}\right)$ implies that $\operatorname{Supp} Z_{0}=\operatorname{Supp}\left(Z_{0}+Z_{0}\right)$. Applying [31, p. 44, Theorem of Appendix to Sec. 4], we see that $Z_{0}$ is an abelian subvariety of $A$.
(2) For a component $Z^{\prime}$, take $z^{\prime} \in Z^{\prime}$ and let $Z_{0}^{\prime}=Z^{\prime}-z^{\prime}$. Since $0 \in Z_{0}^{\prime}, Z_{0}^{\prime}+Z_{0}$ is irreducible and contains both $Z_{0}^{\prime}$ and $Z_{0}$. And since $\operatorname{dim} Z_{0}$ attains maximum, $Z_{0}^{\prime}+Z_{0}=Z_{0}$, which shows $Z_{0}^{\prime} \subseteq Z_{0}$. Then (2) follows easily.

## 3. Sheaves on abelian varieties

In this section, we will study sheaves on abelian varieties. The main result is Theorem 3.7, which, as a generalization of [12, Proposition 2.7], is used in the present paper to study fibrations over abelian varieties. The idea of the proof is stimulated by [19, Theorem 3.1.1].

Notation 3.1. We work over an algebraically closed field $k$. Let $A$ be an abelian variety of dimension $d, \hat{A}=\operatorname{Pic}^{0}(A)$ and $\mathcal{P}$ the Poincaré line bundle on $A \times \hat{A}$. Let $p, q$ denote the projections from $A \times \hat{A}$ to $A, \hat{A}$ respectively. The Fourier-Mukai transform $R \Phi_{\mathcal{P}}: D^{b}(A) \rightarrow D^{b}(\hat{A})$ w.r.t. $\mathcal{P}$ is defined as

$$
R \Phi_{\mathcal{P}}(-):=R q_{*}\left(L p^{*}(-) \otimes \mathcal{P}\right)
$$

which is a right derived functor. Similarly $R \Psi_{\mathcal{P}}: D^{b}(\hat{A}) \rightarrow D^{b}(A)$ is defined as

$$
R \Psi_{\mathcal{P}}(-):=R p_{*}\left(L q^{*}(-) \otimes \mathcal{P}\right)
$$

If $L$ is an ample line bundle on $\hat{A}$, then $H^{i}\left(A, L \otimes \mathcal{P}_{t}\right)=0$ for $i>0$ and every $t \in A\left(\right.$ 31, Sec. 13]), thus $\hat{L}:=R^{0} \Psi_{\mathcal{P}} L \cong R \Psi_{\mathcal{P}} L$ is a locally free sheaf of rank $h^{0}(\hat{A}, L)$ by Theorem 2.15.

Note that since $p, q$ are smooth morphisms, we have $L p^{*} \cong p^{*}, L q^{*} \cong q^{*}$. In what follows, for an isogeny $\pi: A_{1} \rightarrow A$ of abelian varieties, by abuse of the notation of the pull-back map of sheaves, we use $\pi^{*}: \hat{A} \rightarrow \hat{A}_{1}$ for the dual map of $\pi$.

For a coherent sheaf $\mathcal{F}$ on $A$, we define

$$
D_{A}(\mathcal{F}):=R \mathcal{H} m_{X}\left(\mathcal{F}, \omega_{A}\right)[d],
$$

then applying Grothendieck duality we have

$$
D_{k}(R \Gamma(\mathcal{F})) \cong R \Gamma\left(D_{A}(\mathcal{F})\right)
$$

For a closed point $t_{0} \in A$, the translating morphism $T_{t_{0}}: A \rightarrow A$ is defined via $t \mapsto t+t_{0}$. For $\hat{t}_{0} \in \hat{A}, T_{\hat{t}_{0}}$ is similarly defined.

Theorem 3.2 ([30]). Using Notation 3.1, we have
(1) $R \Psi_{\mathcal{P}} \circ R \Phi_{\mathcal{P}} \cong(-1)_{A}^{*}[-d], R \Phi_{\mathcal{P}} \circ R \Psi_{\mathcal{P}} \cong(-1)_{\hat{A}}^{*}[-d]$;
(2) $R \Phi_{\mathcal{P}} \circ(-1)_{A}^{*} \cong(-1)_{\hat{A}}^{*} \circ R \Phi_{\mathcal{P}}, R \Psi_{\mathcal{P}} \circ(-1)_{\hat{A}}^{*} \cong(-1)_{A}^{*} \circ R \Psi_{\mathcal{P}}$; and
(3) for $\hat{t}_{0} \in \hat{A}, R \Psi_{\mathcal{P}} \circ T_{\hat{t}_{0}}^{*} \cong \mathcal{P}_{-\hat{t}_{0}} \otimes R \Psi_{\mathcal{P}}$.

Definition 3.3. ([36, Def. 3.1]) Given a coherent sheaf $\mathcal{F}$ on an abelian variety $A$, its $i^{\text {th }}$ cohomological support locus is defined as

$$
V^{i}(\mathcal{F}):=\left\{\alpha \in \operatorname{Pic}^{0}(A) \mid h^{i}(\mathcal{F} \otimes \alpha)>0\right\}
$$

which are Zariski closed by semi-continuity. If

$$
g v(\mathcal{F}):=\min _{i>0}\left\{\operatorname{codim}_{\operatorname{Pic}^{0}(A)} V^{i}(\mathcal{F})-i\right\} \geq 0
$$

we say $\mathcal{F}$ is a $G V$-sheaf.
Proposition 3.4. Using Notation 3.1, let $\mathcal{F}$ be a coherent sheaf on $A$.
(1) For an ample line bundle $H$ on $\hat{A}$, there is a natural isomorphism

$$
D_{k}\left(R \Gamma\left(\mathcal{F} \otimes \hat{H}^{*}\right)\right) \cong R \Gamma\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right),
$$

in particular, $H^{i}\left(A, \mathcal{F} \otimes \hat{H}^{*}\right)^{*} \cong R^{-i} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right)$ for every integer $i$.
(2) The Fourier-Mukai transform

$$
R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \in D^{[-d, 0]}(A) \text { and } \operatorname{Supp}(-1)_{\hat{A}}^{*} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})=V^{0}(\mathcal{F})
$$

(3) The following three conditions are equivalent to each other
(i) the sheaf $\mathcal{F}$ is a $G V$-sheaf;
(ii) $R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \cong R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})$;
(iii) for any sufficiently ample line bundle $L$ on $\hat{A}$,

$$
H^{i}\left(A, \mathcal{F} \otimes \hat{L}^{*}\right)=0 \text { for } i>0
$$

Proof. (1) is contained in the proof of [18, Theorem 1.2], which follows from applying Grothendieck duality and projection formula.
(2) Take an ample line bundle $H$ on $\hat{A}$. By (1) we have that
(a) $R^{-i} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right)=0$ unless $0 \leq i \leq d$.

Since $R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \in D^{b}(\hat{A})$, we can assume $H$ is sufficiently ample such that, for every $q$,
(b) $R^{q} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H$ is globally generated, and
(c) $R^{p} \Gamma\left(R^{q} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right)=0$ whenever $p \neq 0$.

Since $H$ is a line bundle, we have the spectral sequence
$E_{2}^{p, q}:=R^{p} \Gamma\left(R^{q} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right) \cong R^{p} \Gamma\left(\mathcal{H}^{q}\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes^{L} H\right)\right) \Rightarrow R^{p+q} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right)$.
By (c) we can show

$$
E_{\infty}^{0, q}=E_{2}^{0, q}=\Gamma\left(R^{q} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right) \cong R^{q} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes H\right)
$$

Then by (a) and (b), we conclude that $R^{i} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})=0$ unless $-d \leq i \leq 0$, that is, $R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \in D^{[-d, 0]}(A)$.

For $\alpha \in \hat{A}$, applying Grothendieck duality we have

$$
H^{j}\left(\mathcal{F} \otimes \mathcal{P}_{\alpha}\right)^{*} \cong R^{-j} \Gamma\left(D_{A}(\mathcal{F}) \otimes \mathcal{P}_{-\alpha}\right)
$$

In particular, $R^{i} \Gamma\left(D_{A}(\mathcal{F}) \otimes \mathcal{P}_{-\alpha}\right)=0$ for $i>0$. Applying $R q_{*}$ to $p^{*} D_{A}(\mathcal{F}) \otimes \mathcal{P}$ on $A \times \hat{A}$, since $R^{1} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})=0$, by Theorem 2.15, we conclude

$$
H^{0}\left(\mathcal{F} \otimes \mathcal{P}_{\alpha}\right)^{*} \cong R^{0} \Gamma\left(D_{A}(\mathcal{F}) \otimes \mathcal{P}_{-\alpha}\right) \cong R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \otimes k(-\alpha)
$$

thus $\operatorname{Supp}(-1)_{A}^{*} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})=V^{0}(\mathcal{F})$.
(3) follows from applying [18, Theorem 1.2] and [36, Lemma 3.6].

Part of the following theorem is known to experts, in particular assertion (4) also appeared in [20, Sec. 1.2], which states a special reature of positive characteristic.

Theorem 3.5. Using Notation 3.1, then
(1) If $\tau$ is a coherent sheaf supported at finitely many closed points on $\hat{A}$ of length $r$, then $U=R \Psi_{\mathcal{P}} \tau=R^{0} \Psi_{\mathcal{P}} \tau$ is a vector bundle of rank $r$ (homogenous vector bundle); moreover if $\operatorname{Supp} \tau=\{\hat{0}\}$ then $U$ is a unipotent vector bundle, that is, $U$ admits a filtration of vector bundles

$$
0=U_{0} \subset U_{1} \subset U_{2} \subset \cdots \subset U_{r}=U
$$

such that $U_{i} / U_{i-1} \cong \mathcal{O}_{X}$.
(2) If $\pi: A_{1} \rightarrow A$ is an isogeny of abelian varieties, then $\pi_{*} \mathcal{O}_{A_{1}} \cong \bigoplus_{i} U_{\hat{t}_{i}}$ where

- $\hat{t}_{i}$ are finitely many closed points on $\hat{A}$ such that $\pi^{*} \mathcal{P}_{\hat{t}_{i}} \cong \mathcal{O}_{A_{1}}$, and
- $U_{\hat{t}_{i}}=R \Psi_{\mathcal{P}} \tau_{i}$ where $\tau_{i}$ is a skyscraper sheaf supported at $\hat{t}_{i}$.
(3) Let $\mathcal{F}$ be a coherent sheaf on $A$. Set $V^{\geq m}(\mathcal{F})=\cup_{j \geq m} V^{j}(\mathcal{F})$. If $\pi: A_{1} \rightarrow A$ is an isogeny of abelian varieties, then

$$
V^{m}\left(\pi^{*} \mathcal{F}\right) \subseteq \pi^{*} V^{m}(\mathcal{F}) \text { and } V^{\geq m}\left(\pi^{*} \mathcal{F}\right)=\pi^{*} V^{\geq m}(\mathcal{F})
$$

In particular, if for every $j>0, h^{j}\left(A_{1}, \pi^{*} \mathcal{F}\right)=0$, then $h^{j}(A, \mathcal{F})=0$ for $j>0$.
Moreover, we have $V^{0}\left(\pi^{*} \mathcal{F}\right)=\pi^{*} V^{0}(\mathcal{F})$.
(4) Assume char $k=p>0$. Let $\tau$ be a coherent sheaf supported on the union of finitely many closed points on $\hat{A}$ and let $U=R \Psi_{\mathcal{P}} \tau$. Then there exists an isogeny $\mu: B \rightarrow A$ of abelian varieties such that $\mu^{*} U \cong \bigoplus_{i} \mathcal{P}_{\hat{s}_{i}}^{\prime}$ for some closed points $\hat{s}_{i}$ on $\hat{B}=\operatorname{Pic}^{0}(B)$, where $\mathcal{P}^{\prime}$ denotes the Poincaré line bundle on $B \times \hat{B}$.

Proof. (1) Since $\operatorname{dim} \operatorname{Supp} \tau=0$, for any closed point $t \in A$ we have

$$
h^{i}\left(\hat{A}, \tau \otimes \mathcal{P}_{t}\right)=0 \text { if } i \neq 0 \text { and } h^{0}\left(\hat{A}, \tau \otimes \mathcal{P}_{t}\right)=r
$$

The first assertion of (1) follows from applying Theorem 2.15,
If moreover $\tau$ is a skyscraper sheaf supported at $\hat{0}$, then we have a filtration

$$
0=\tau_{0} \subset \tau_{1} \subset \tau_{2} \subset \cdots \subset \tau_{r}=\tau
$$

such that $\tau_{i} / \tau_{i-1} \cong k(\hat{0})$, which, by applying Fourier-Mukai transform $R \Psi_{\mathcal{P}}$, induces a filtration of $U$ as wanted.
(2) For $\hat{t} \in \hat{A}$, we have $H^{i}\left(A_{1}, \pi^{*} \mathcal{P}_{\hat{t}}\right) \cong H^{i}\left(A, \pi_{*} \mathcal{O}_{A_{1}} \otimes \mathcal{P}_{\hat{t}}\right)$. It follows that

$$
V^{0}\left(\pi_{*} \mathcal{O}_{A_{1}}\right)=V^{1}\left(\pi_{*} \mathcal{O}_{A_{1}}\right)=\cdots=V^{d}\left(\pi_{*} \mathcal{O}_{A_{1}}\right)=S_{\pi}:=\left\{\hat{t}^{\prime} \in \hat{A} \mid \pi^{*} \mathcal{P}_{\hat{t}^{\prime}} \cong \mathcal{O}_{A_{1}}\right\}
$$

We have $S_{\pi}=\operatorname{Supp}\left(\operatorname{ker} \pi^{*}\right)$, thus $\operatorname{dim} V^{k}\left(\pi_{*} \mathcal{O}_{A_{1}}\right)=0$ for $0 \leq k \leq d$, and $\pi_{*} \mathcal{O}_{A_{1}}$ is a GV-sheaf. By Grothendieck duality

$$
D_{A}\left(\pi_{*} \mathcal{O}_{A_{1}}\right) \cong \pi_{*} D_{A_{1}}\left(\mathcal{O}_{A_{1}}\right) \cong \pi_{*} \mathcal{O}_{A_{1}}[d]
$$

applying Proposition 3.4 (3) we have

$$
R \Phi_{\mathcal{P}} \pi_{*} \mathcal{O}_{A_{1}}[d] \cong R^{d} \Phi_{\mathcal{P}} \pi_{*} \mathcal{O}_{A_{1}} \text { and } \operatorname{Supp} R^{d} \Phi_{\mathcal{P}} \pi_{*} \mathcal{O}_{A_{1}}=S_{\pi}
$$

We can assume that $R^{d} \Phi_{\mathcal{P}} \pi_{*} \mathcal{O}_{A_{1}}=\bigoplus_{i=1}^{i=n} \tau_{i}^{\prime}$ where every $\tau_{i}^{\prime}$ is a skyscraper sheaf supported at some $\hat{t}_{i}^{\prime} \in S_{\pi}$. Applying Theorem 3.2, we have that

$$
\pi_{*} \mathcal{O}_{A_{1}} \cong(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} \bigoplus_{i=1}^{i=n} \tau_{i}^{\prime} \cong \bigoplus_{i=1}^{i=n} R^{0} \Psi_{\mathcal{P}}(-1)_{\hat{A}}^{*} \tau_{i}^{\prime}
$$

So we only need to set $\tau_{i}=(-1)_{\hat{A}}^{*} \tau_{i}^{\prime}, \hat{t}_{i}=(-1)_{\hat{A}}^{*} \hat{t}_{i}^{\prime}$ and $U_{\hat{t}_{i}}=R^{0} \Psi_{\mathcal{P}} \tau_{i}$.
(3) We use the notation of (2). For $\hat{t} \in \hat{A}$, we have

$$
H^{j}\left(A_{1}, \pi^{*}\left(\mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right)\right) \cong H^{j}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes \pi_{*} \mathcal{O}_{A_{1}}\right) \cong H^{j}\left(A, \bigoplus_{i} \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes U_{\hat{t}_{i}}\right)
$$

Let $\tau_{i}^{0}=T_{\hat{t}_{i}}^{*} \tau_{i}$ and $U_{i}^{0}=R^{0} \Psi_{\mathcal{P}} \tau_{i}^{0}$. Then $\tau_{i}^{0}$ is supported at $\hat{0}, U_{\hat{t}_{i}} \cong \mathcal{P}_{\hat{t}_{i}} \otimes U_{i}^{0}$ and

$$
H^{j}\left(A_{1}, \pi^{*}\left(\mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right)\right) \cong \bigoplus_{i} H^{j}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_{i}} \otimes U_{i}^{0}\right)
$$

Claim: For a coherent sheaf $\mathcal{G}$ and a unipotent vector bundle $U$ on $A$,
(3.1') if $H^{l}(A, \mathcal{G})=0$ then $H^{l}(A, \mathcal{G} \otimes U)=0$; and
(3.2') if $H^{j}(A, \mathcal{G} \otimes U)=0$ for every $j \geq m$, then $H^{j}(A, \mathcal{G})=0$ for $j \geq m$.

Granted the claim above, we prove assertion (3) by the following arguments.
(3.1) For $\hat{t} \in \hat{A}$, if $\pi^{*} \hat{t} \in V^{m}\left(\pi^{*} \mathcal{F}\right)$, then there exists some $\hat{t}_{i} \in S_{\pi}$ such that $H^{m}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_{i}} \otimes U_{i}^{0}\right) \neq 0$. Applying (3.1') shows that $H^{m}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_{i}}\right) \neq 0$, i.e., $\hat{t}+\hat{t}_{i} \in V^{m}(\mathcal{F})$. Since $\pi^{*} \hat{t}_{i}=\hat{0}$ and $\pi^{*}: \hat{A} \rightarrow \hat{A}_{1}$ is an epimorphism, we see that

$$
V^{m}\left(\pi^{*} \mathcal{F}\right) \subseteq \pi^{*} V^{m}(\mathcal{F})
$$

(3.2) For $\hat{t} \in \hat{A}$, if $\hat{t} \in V^{\geq m}(\mathcal{F})$ then $H^{j}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right) \neq 0$ for some $j \geq m$. Since $\hat{0} \in S_{\pi}, \pi_{*} \mathcal{O}_{A_{1}}$ has a unipotent direct summand $U$. Applying (3.2') shows that there exists some $j^{\prime} \geq m$ such that $H^{j^{\prime}}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes U\right) \neq 0$, thus $H^{j^{\prime}}\left(A_{1}, \pi^{*}\left(\mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right)\right) \neq 0$. Therefore, $\pi^{*} V^{\geq m}(\mathcal{F}) \subseteq V^{\geq m}\left(\pi^{*} \mathcal{F}\right)$, and the equality holds by combining with (3.1).
(3.3) To show $V^{0}\left(\pi^{*} \mathcal{F}\right)=\pi^{*} V^{0}(\mathcal{F})$, by (3.1) it suffices to show that $\pi^{*} V^{0}(\mathcal{F}) \subseteq$ $V^{0}\left(\pi^{*} \mathcal{F}\right)$, which follows from the fact that the natural map $\pi^{*}: H^{0}\left(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right) \rightarrow$ $H^{0}\left(A_{1}, \pi^{*}\left(\mathcal{F} \otimes \mathcal{P}_{\hat{t}}\right)\right)$ is injective since $\pi$ is flat.

Proof of Claim. Let $r=\operatorname{rk} U$ and $U_{r}=U$. By (1) we have a filtration of vector bundles

$$
0=U_{0} \subset U_{1} \subset U_{2} \subset \cdots \subset U_{r}=U
$$

in turn we get the following short exact sequences

$$
\begin{array}{rll}
0 \rightarrow \mathcal{G} \otimes U_{r-1} \rightarrow \mathcal{G} \otimes U_{r} \rightarrow \mathcal{G} \rightarrow 0 & (r) \\
0 \rightarrow \mathcal{G} \otimes U_{r-2} \rightarrow \mathcal{G} \otimes U_{r-1} \rightarrow \mathcal{G} \rightarrow 0 & (r-1) \\
& \ldots & \ldots  \tag{2}\\
0 \rightarrow \mathcal{G} \rightarrow \mathcal{G} \otimes U_{2} \rightarrow \mathcal{G} \rightarrow 0 & (2)
\end{array}
$$

(3.1') Taking cohomology of those short exact sequences $(2,3, \cdots, r)$, since $H^{l}(A, \mathcal{G})=0$ we can show that

$$
H^{l}\left(A, \mathcal{G} \otimes U_{2}\right)=H^{l}\left(A, \mathcal{G} \otimes U_{3}\right)=\cdots=H^{l}\left(A, \mathcal{G} \otimes U_{r}\right)=0
$$

(3.2') We will prove $H^{j}(A, \mathcal{G})=0$ for $j \geq m$ by induction on $j$. This is trivial if $j>d$. Assume that we have proved for some fixed $l>m$,

$$
H^{i}(A, \mathcal{G})=0 \text { whenever } i \geq l
$$

which, by (3.1'), implies that for $i \geq l$

$$
H^{i}\left(A, \mathcal{G} \otimes U_{2}\right)=H^{i}\left(A, \mathcal{G} \otimes U_{3}\right)=\cdots=H^{i}\left(A, \mathcal{G} \otimes U_{r}\right)=0
$$

Taking cohomology of the short exact sequence (r), by the vanishing $H^{l-1}(A, \mathcal{G} \otimes$ $\left.U_{r}\right)=0$ and $H^{l}\left(A, \mathcal{G} \otimes U_{r-1}\right)=0$, we show that

$$
H^{l-1}(A, \mathcal{G})=0
$$

By induction, we finish the proof of this claim.
(4) We can write that $\tau=\bigoplus_{i} \tau_{i}$ where every $\tau_{i}$ is a sheaf supported at one certain closed point $\hat{t}_{i}$. Let $U_{i}=R^{0} \Psi_{\mathcal{P}} \tau_{i}$. Then $U=\bigoplus_{i} U_{i}$. We only need to show assertion (4) for a single $U_{i}$.

We can assume $\tau$ is supported at exactly one closed point $\hat{t} \in \hat{A}$. Let

$$
\bar{\tau}=T_{\hat{t}}^{*} \tau \text { and } \bar{U}=R \Psi_{\mathcal{P}} \bar{\tau}
$$

Then $\operatorname{Supp} \bar{\tau}=\{\hat{0}\}, U \cong \bar{U} \otimes \mathcal{P}_{\hat{t}}$, and $\bar{U}$ is a unipotent vector bundle. We can consider $\bar{U}$ instead and do induction on the rank. By (1) there is a filtration of vector bundles

$$
0=\bar{U}_{0} \subset \bar{U}_{1} \subset \bar{U}_{2} \subset \cdots \bar{U}_{i-1} \subset \bar{U}_{i} \subset \cdots \subset \bar{U}_{r}=\bar{U}
$$

Assume by induction that for some $i \leq r$ there exists an isogeny $\mu_{i-1}: B_{i-1} \rightarrow A$ such that $\mu_{i-1}^{*} \bar{U}_{i-1} \cong \bigoplus^{i-1} \mathcal{O}_{B_{i-1}}$. Then we have the extension

$$
0 \rightarrow \bigoplus^{i-1} \mathcal{O}_{B_{i-1}} \rightarrow \mu_{i-1}^{*} \bar{U}_{i} \rightarrow \mathcal{O}_{B_{i-1}} \rightarrow 0
$$

which corresponds to

$$
\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{i-1}\right) \in \bigoplus^{i-1} H^{1}\left(\mathcal{O}_{B_{i-1}}\right) \cong \operatorname{Ext}^{1}\left(\mathcal{O}_{B_{i-1}}, \bigoplus^{i-1} \mathcal{O}_{B_{i-1}}\right)
$$

Recall "killing cohomology", which says that for a projective variety $X$ and $\alpha \in$ $H^{1}\left(\mathcal{O}_{X}\right)$, there exists a morphism $\pi: Y \rightarrow X$ composed with some Frobenius iterations and étale $\mathbb{Z} /(p)$-covers such that $\pi^{*} \alpha=0$ in $H^{1}\left(\mathcal{O}_{Y}\right)$ ([40, Prop. 12 and Sec. 9]). So applying "killing cohomology", we get a base change $\nu_{i}: B_{i} \rightarrow B_{i-1}$, which is an isogeny of abelian varieties such that

$$
\nu_{i}^{*} \alpha_{1}=\nu_{i}^{*} \alpha_{2}=\cdots \nu_{i}^{*} \alpha_{i-1}=0 \in H^{1}\left(\mathcal{O}_{B_{i}}\right) .
$$

Let $\mu_{i}=\mu_{i-1} \circ \nu_{i}: B_{i} \rightarrow A$. Then

$$
\mu_{i}^{*} \bar{U}_{i} \cong \nu_{i}^{*} \mu_{i-1}^{*} \bar{U}_{i} \cong \bigoplus^{i} \mathcal{O}_{B_{i}}
$$

We finish the proof.
Lemma 3.6. Using Notation 3.1, for a coherent sheaf $\mathcal{G}$ on $A$, there exists a natural homomorphism

$$
\alpha_{\mathcal{G}}: \mathcal{G}^{*}=\mathcal{E} x t^{0}\left(\mathcal{G}, \mathcal{O}_{A}\right) \rightarrow(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{G})
$$

with the kernel $\mathcal{K}_{\mathcal{G}} \cong(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}}\left(\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G})\right)$.
Proof. Apply $(-1)_{A}^{*} R \Psi_{\mathcal{P}}$ to the following triangle

$$
\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \rightarrow R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \rightarrow \sigma_{>-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \rightarrow \sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G})[1]
$$

and take cohomology. Since $\sigma_{>-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \cong R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{G})$ (Proposition 3.4 (2)), we get the following exact sequence

$$
\begin{aligned}
(-1)_{A}^{*} R^{-1} \Psi_{\mathcal{P}} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \rightarrow & (-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}}\left(\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G})\right) \\
& \rightarrow(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G}) \xrightarrow{\alpha_{\mathcal{G}}^{\prime}}(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{G})
\end{aligned}
$$

Applying Theorem 3.2, we have an isomorphism

$$
\mathcal{E} x t^{0}\left(\mathcal{G}, \mathcal{O}_{A}\right) \cong(-1)_{A}^{*} \mathcal{H}^{0}\left(R \Psi_{\mathcal{P}} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G})\right)
$$

then we get the homomorphism $\alpha_{\mathcal{G}}$ by composing $\alpha_{\mathcal{G}}^{\prime}$ with this isomorphism. Since $R^{-1} \Psi_{\mathcal{P}} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{G})=0$, it follows that the kernel of $\alpha_{\mathcal{G}}$

$$
\mathcal{K}_{\mathcal{G}} \cong(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}}\left(\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{G})\right)
$$

Let us prove the main theorem of this section.
Theorem 3.7. Using Notation 3.1, let $\mathcal{F}=\mathcal{F}_{0}, \mathcal{F}_{1}, \cdots, \mathcal{F}_{d}$ be torsion free coherent sheaves on A equipped with homomorphisms $\phi_{e}: \mathcal{F}_{e} \rightarrow \mathcal{F}_{e-1}, e=1,2, \cdots, d$. Let $\varphi_{e}=\phi_{e} \circ \phi_{e-1} \circ \cdots \circ \phi_{1}: \mathcal{F}_{e} \rightarrow \mathcal{F}$. And let $H_{l}, l=0,1, \cdots, d-1$ be ample line bundles on $\hat{A}$. Assume that
(a) for $0 \leq l \leq d-1$ and every $i$, the sheaf $R^{i} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}$ is globally generated, and if $j>0$ then $H^{j}\left(\hat{A}, R^{i} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right)=0$;
(b) for $0 \leq l<m \leq d$, if $j>0$ then $H^{j}\left(A, \mathcal{F}_{m} \otimes \hat{H}_{l}{ }^{*}\right)=0$; and
(c) the dual homomorphism $\mathcal{F}^{*} \rightarrow \mathcal{F}_{d}^{*}$ of $\varphi_{d}$ is injective.

Then
(i) the homomorphism $\alpha_{\mathcal{F}}: \mathcal{F}^{*} \rightarrow(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})$ (introduced in Lemma (3.6) is injective; and
(ii) if moreover char $k=p>0$ and $R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})=\tau$ is supported at finitely many closed points, then there exist an isogeny $\pi: A_{1} \rightarrow A$ of abelian varieties, some $P_{i} \in \operatorname{Pic}^{0}\left(A_{1}\right)$ and a generically surjective homomorphism

$$
\beta_{\mathcal{F}}: \bigoplus_{i} P_{i} \rightarrow \pi^{*} \mathcal{F}
$$

Proof. The maps $\varphi_{e}: \mathcal{F}_{e} \rightarrow \mathcal{F}$ induce $\varphi_{e}^{*}: D_{A}(\mathcal{F}) \rightarrow D_{A}\left(\mathcal{F}_{e}\right)$ by taking dual, then applying $R \Phi_{\mathcal{P}}$ induces natural homomorphisms

$$
R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \rightarrow R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{e}\right)
$$

We have the following lemma with the proof postponed.
Lemma 3.8. The natural homomorphism

$$
R^{0} \Psi_{\mathcal{P}}\left(\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}(\mathcal{F})\right) \rightarrow R^{0} \Psi_{\mathcal{P}}\left(\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{d}\right)\right)
$$

is zero.
(i) Applying Lemma 3.6, we have the following commutative diagram

and the vertical map $\mathcal{K}_{\mathcal{F}} \rightarrow \mathcal{K}_{\mathcal{F}_{d}}$ is zero by Lemma 3.8. Then from the condition (c), we conclude $\mathcal{K}_{\mathcal{F}}=0$, thus $\alpha_{\mathcal{F}}$ is injective.
(ii) Let $U=(-1)_{A}^{*} R^{0} \Psi_{\mathcal{P}} \tau$. Then by Theorem 3.5 (1) and (4), the sheaf $U$ is locally free, and there exists an isogeny $\pi: A_{1} \rightarrow A$ of abelian varieties such that

$$
\pi^{*} U \cong \bigoplus_{i} Q_{i} \text { for some } Q_{i} \in \operatorname{Pic}^{0}\left(A_{1}\right)
$$

Applying $(-1)_{A}^{*} R \Psi_{\mathcal{P}}$ to the natural homomorphism $R \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \rightarrow \tau$ induces a homomorphism

$$
\gamma: R \mathcal{H o m}\left(\mathcal{F}, \mathcal{O}_{A}\right) \cong D_{A}(\mathcal{F})[-d] \rightarrow U
$$

then applying $R \mathcal{H} \operatorname{om}\left(\cdot, \mathcal{O}_{A}\right)$ induces

$$
\gamma^{*}: U^{*} \rightarrow \mathcal{F}
$$

Since by (i) the homomorphism $\alpha_{\mathcal{F}}: \mathcal{F}^{*} \rightarrow U$ is injective, $\gamma^{*}$ is generically surjective. It follows that the pull-back homomorphism via $\pi$

$$
\beta_{\mathcal{F}}: \pi^{*} U^{*} \rightarrow \pi^{*} \mathcal{F}
$$

is surjective over the generic point of $A_{1}$. So we are done by setting $P_{i}=Q_{i}^{*}$.
Proof of Lemma 3.8. We divide the proof into two steps.
Step 1: We prove that for fixed $0 \leq l<m \leq d$ and any $s \leq-1$, the natural map $R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \rightarrow R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right)$ is zero.

Consider the spectral sequence

$$
E_{2, \mathcal{F}_{l}}^{r, s}:=R^{r} \Gamma\left(R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right) \Rightarrow R^{r+s} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right) .
$$

By the condition (a) and Proposition 3.4 (2), we see that

$$
E_{\infty, \mathcal{F}_{l}}^{r, s} \cong E_{2, \mathcal{F}_{l}}^{r, s}=0 \text { unless } r=0 \text { and }-d \leq s \leq 0
$$

thus for $-d \leq s \leq 0$, the following natural homomorphism is an isomorphism

$$
\gamma_{\mathcal{F}_{l}}^{s}: R^{s} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right) \rightarrow E_{\infty, \mathcal{F}_{l}}^{0, s} \hookrightarrow E_{2, \mathcal{F}_{l}}^{0, s}=H^{0}\left(R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right)
$$

Then consider the spectral sequence

$$
E_{2, \mathcal{F}_{m}}^{r, s}:=R^{r} \Gamma\left(R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right) \Rightarrow R^{r+s} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right)
$$

Since $E_{2, \mathcal{F}_{m}}^{r, s}=0$ whenever $r<0$, we get a natural homomorphism

$$
\left.\gamma_{\mathcal{F}_{k}}^{s}: R^{s} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right)\right) \rightarrow E_{\infty, \mathcal{F}_{m}}^{0, s} \hookrightarrow E_{2, \mathcal{F}_{m}}^{0, s} \cong H^{0}\left(R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right) .
$$

For $i>0$, by the condition (b) and the isomorphism $R^{-i} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right) \cong$ $H^{i}\left(\mathcal{F}_{m} \otimes \hat{H}_{l}{ }^{*}\right)^{*}=0$ in Proposition $3.4(1)$, we see that the following natural map is zero

$$
\beta^{-i}: R^{-i} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \otimes H_{l}\right) \rightarrow R^{-i} \Gamma\left(R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right) \otimes H_{l}\right)=0
$$

Chasing in the following commutative diagram

we can show that for $s \leq-1$, the map $\bar{\beta}^{s}$ is zero, consequently the map $R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{l}\right) \rightarrow$ $R^{s} \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right)$ is zero by the condition (a).

To ease the notation, we denote $\mathcal{E}_{m}=\sigma_{\leq-1} R \Phi_{\mathcal{P}} D_{A}\left(\mathcal{F}_{m}\right)$ for $m=0,1, \cdots, d$. Then $\mathcal{E}_{m} \in D^{[-d,-1]}(\hat{A})$ by Proposition 3.4 (2). And since for every $i, \mathcal{H}^{i}\left(\mathcal{E}_{m}\right) \rightarrow \mathcal{H}^{i}\left(\mathcal{E}_{m+1}\right)$ is a zero map, we can show that for every $p$, the homomorphism $R^{p} \Psi_{\mathcal{P}}\left(\mathcal{H}^{i}\left(\mathcal{E}_{m}\right)\right) \rightarrow$ $R^{p} \Psi_{\mathcal{P}}\left(\mathcal{H}^{i}\left(\mathcal{E}_{m+1}\right)\right)$ is zero.

Step 2: We prove that for $0<m \leq d$, the natural homomorphism

$$
\alpha_{k}: R^{0} \Psi_{\mathcal{P}}\left(\sigma_{>-m-1} \mathcal{E}_{0}\right) \rightarrow R^{0} \Psi_{\mathcal{P}}\left(\sigma_{>-m-1} \mathcal{E}_{m}\right)
$$

is zero, which completes the proof of Lemma 3.8 if setting $m=d$.
We prove this assertion by induction. First note that for every $m$ and $0 \leq i \leq d$, the object $\sigma_{\leq-m}\left(\sigma_{>-m-1} \mathcal{E}_{i}\right) \in D^{b}(\hat{A})$ is quasi-isomorphic to $\mathcal{H}^{-m}\left(\mathcal{E}_{i}\right)[k]$. When $m=1$, the map $\alpha_{1}: R^{0} \Psi_{\mathcal{P}}\left(\sigma_{>-2} \mathcal{E}_{0}\right) \rightarrow R^{0} \Psi_{\mathcal{P}}\left(\sigma_{>-2} \mathcal{E}_{1}\right)$ coincides with the map $R^{1} \Psi_{\mathcal{P}}\left(\mathcal{H}^{-1}\left(\mathcal{E}_{0}\right)\right) \rightarrow R^{1} \Psi_{\mathcal{P}}\left(\mathcal{H}^{-1}\left(\mathcal{E}_{1}\right)\right)$, hence it is zero. Now assume that $\alpha_{m}$ is a zero map for some $m<d$. To prove $\alpha_{m+1}$ is zero, we consider the following commutative diagram

where the horizontal sequences are triangles. Applying the right derived functor $R \Psi_{\mathcal{P}}$ to the above diagram induces

where the horizontal sequences are exact. Since $\alpha_{m}$ is assumed to be a zero map, we have $\operatorname{Im}\left(\beta_{m}\right) \subseteq \operatorname{Im}\left(\mu_{m}\right)$. And since $\delta_{m}$ is zero, from the above commutative diagram we can conclude $\operatorname{Im}\left(\alpha_{m+1}=\beta_{m}^{\prime} \circ \beta_{m}\right) \subseteq \operatorname{Im}\left(\beta_{m}^{\prime} \circ \mu_{m}\right)=0$. Therefore, $\alpha_{m+1}$ is a zero map, and the proof is completed.

## 4. Subadditivity of Kodaira dimensions

In this section, we work over an algebraically closed field $k$ with char $k=p>5$. We will prove the following result on subadditivity of Kodaira dimensions.

Theorem 4.1. Let $f: X \rightarrow Y$ be a fibration from $a \mathbb{Q}$-factorial projective 3-fold to a smooth projective variety of dimension 1 or 2 . Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is klt. Assume that $Y$ is of maximal Albanese dimension, and and assume moreover that
© if $\kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)=\operatorname{dim} X / Y-1$, then $B$ does not intersect the generic fiber $X_{\xi}$ of the relative Iitaka fibration $I: X \rightarrow Z$ induced by $K_{X}+B$ on $X$ over $Y$.

Then

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)+\kappa(Y) .
$$

To prove the theorem above, we will first treat three subcases in the following theorems, which can be seen as complements of the results of [46].
Theorem 4.2. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair of dimension 3. Let $f: X \rightarrow Y=A$ be a fibration to an elliptic curve or a simple abelian surface. Assume that $K_{X}+B$ is $f$-big. Then

$$
\kappa\left(K_{X}+B\right) \geq \kappa\left(X_{\eta}, K_{X_{\eta}}+B_{\eta}\right)
$$

Theorem 4.3. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair of dimension 3. Let $f: X \rightarrow Y$ be a fibration to a normal curve $Y$ of genus $g(Y) \geq 1$. Assume $\kappa\left(X_{\eta},\left.\left(K_{X}+B\right)\right|_{X_{\eta}}\right)=1$ and the condition holds. Then

$$
\kappa\left(X, K_{X}+B\right) \geq 1+\kappa(Y)
$$

Theorem 4.4. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair of dimension 3. Let $f: X \rightarrow Y$ be a fibration to a normal curve $Y$ of genus $g(Y) \geq 1$. Assume $\kappa\left(X_{\eta},\left.\left(K_{X}+B\right)\right|_{X_{\eta}}\right)=0$. Then

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa(Y)
$$

If moreover $K_{X}+B$ is nef then it is semi-ample.
We remark that Theorem 4.3 is a generalization of [12, Theorem 1.2, the subcase $\operatorname{dim} Y=1$ and $\kappa\left(X_{\eta}\right)=1$ ] where the cases without boundary were treated. Since the condition $\boldsymbol{\oplus}$ is assumed, we can adapt the arguments of [12, Sec. 4] to our situation. For the convenience of the reader, we will provide a detailed proof.
4.1. Preparations. First let us recall an invariant introduced by Ejiri [11, Sec.4] to measure the positivity of a sheaf.

Definition 4.5. Let $Y$ be a projective variety, $\mathcal{F}$ a torsion free coherent sheaf and $H$ an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Y$. Let
$t(Y, \mathcal{F}, H)=\sup \left\{a \in \mathbb{Q} \mid\right.$ the sheaf $\left(F_{Y}^{e *} \mathcal{F}\right) \otimes \mathcal{O}_{Y}\left(\left\llcorner-p^{e} a H\right\lrcorner\right)$
is generically globally generated for some $e>0\}$.
We will denote $t(Y, \mathcal{F}) \geq 0$ if $t(Y, \mathcal{F}, H) \geq 0$. This property is independent of the choices of $H$ ([46, Remark 2.13]) and is stronger than weak positivity ([11, Prop. 4.7]).

From the main results of [46] we deduce the following theorem.
Theorem 4.6. Let $f: X \rightarrow Y$ be a separable fibration between smooth projective varieties, and let $D$ be a nef and $f$-big Cartier divisor on $X$.
(1) If $D$ is $f$-semi-ample, then
(1.a) for sufficiently divisible positive integers $n$ and $g$, the sheaf $F_{Y}^{g *} f_{*} \mathcal{O}_{X}(n D+$ $\left.K_{X / Y}\right)$ contains a nonzero subsheaf $V_{n}$ with $t\left(Y, V_{n}\right) \geq 0$, and $\mathrm{rk} V_{n} \geq c n^{\operatorname{dim} X / Y}$ for some $c>0$ independent of $n$; and
(1.b) for any sufficiently divisible $n>0$ and big $\mathbb{Q}$-divisor $H$ on $Y, n D+K_{X / Y}+$ $f^{*} H$ is big.
(2) If $K_{Y}$ is big, then for any sufficiently divisible $n>0, n D+K_{X}$ is big.

Proof. We can assume $D=A+E$ where $A$ is $f$-ample and $E$ is effective. For an integer $n>0$ such that $n E_{\bar{\eta}}$ is Cartier, let $s_{n}$ be a global section of $\mathcal{O}_{X_{\bar{\eta}}}\left(n E_{\bar{\eta}}\right)$ with $\left(s_{n}\right)_{0}=n E_{\bar{\eta}}$. We get an inclusion $S^{0}\left(X_{\bar{\eta}},\left(n A+K_{X}\right)_{\bar{\eta}}\right) \hookrightarrow S^{0}\left(X_{\bar{\eta}},\left(n D+K_{X}\right)_{\bar{\eta}}\right)$ by tensoring with $s_{n}$. Then by applying Lemma 2.14 on $X_{\bar{\eta}}$ for the divisor $\left(n A+K_{X}\right)_{\bar{\eta}}$, we can show that there exists some $c>0$ such that for any sufficiently divisible $n$,

$$
\operatorname{dim}_{k(\bar{\eta})} S^{0}\left(X_{\bar{\eta}},\left(n D+K_{X}\right)_{\bar{\eta}}\right) \geq \operatorname{dim}_{k(\bar{\eta})} S^{0}\left(X_{\bar{\eta}},\left(n A+K_{X}\right)_{\bar{\eta}}\right) \geq c n^{\operatorname{dim} X / Y} .
$$

The assertion (1.a) follows from applying [46, Theorem 1.11]. For (1.b), fix a sufficiently divisible integer $n>0$ such that $F_{Y}^{g *} f_{*} \mathcal{O}_{X}\left(n D+K_{X / Y}\right)$ contains a nonzero subsheaf $V_{n}$ with $t\left(Y, V_{n}\right) \geq 0$. Applying [46, Theorem 4.1] shows that

$$
\kappa\left(X, n D+K_{X / Y}+f^{*} H\right) \geq \kappa\left(X_{\eta},\left.\left(n D+K_{X / Y}\right)\right|_{X_{\eta}}\right)+\operatorname{dim} Y=\operatorname{dim} X
$$

hence $n D+K_{X / Y}+f^{*} H$ is big.
For (2), take an ample divisor $H^{\prime}$ on $Y$ such that $K_{Y}-H^{\prime}$ is big. We can assume $D+f^{*} H^{\prime} \sim_{\mathbb{Q}} A^{\prime}+\Delta$ where $A^{\prime}$ is an ample divisor and $\Delta$ is an effective divisor with index not divisible by $p$. Since $n D+K_{X}-K_{X / Y}-\Delta-f^{*}\left(K_{Y}-H^{\prime}\right) \sim_{\mathbb{Q}}(n-1) D+A^{\prime}$ is nef and $f$-ample, and $\operatorname{dim}_{k(\bar{\eta})} S_{\Delta_{\bar{\eta}}}^{0}\left(X_{\bar{\eta}},\left(n D+K_{X}\right)_{\bar{\eta}}\right)>0$ for sufficiently divisible $n$, we can prove (2) by applying [46, Theorem 1.5].

Corollary 4.7. Let $(X, B)$ be a projective $\mathbb{Q}$-factorial klt pair of dimension 3. Let $f: X \rightarrow Y$ be a fibration to a smooth projective curve or surface $Y$ of maximal Albanese dimension. Assume that $K_{X}+B$ is $f$-big. Then

$$
\kappa_{\sigma}\left(X, K_{X}+B\right) \geq \kappa\left(X_{\eta},\left(K_{X}+B\right)_{\eta}\right)+\kappa(Y)
$$

In particular, if $K_{Y}$ is big, then $K_{X}+B$ is big.
Proof. Let $(\bar{X}, \bar{B})$ be a log minimal model of $(X, B)$ over $\operatorname{Alb}(Y)$. And let $\rho: \tilde{X} \rightarrow X$ be a $\log$ resolution such that the natural map $\mu: \tilde{X} \rightarrow \bar{X}$ is a morphism. Let $D=\mu^{*}\left(K_{\bar{X}}+\bar{B}\right)$. By Theorem 2.5 (3.1) and (3.4), $(\bar{X}, \bar{B})$ is in fact minimal, and $D$ is a nef divisor relatively big and semi-ample over $\operatorname{Alb}(Y)$ (hence over $Y$ ). By Proposition 2.8, we only need to prove $\nu(\tilde{X}, D) \geq \operatorname{dim} X_{\eta}+\kappa(Y)$.

First we reduce to separable fibrations by the following commutative diagram

where $\sigma: \tilde{X} \rightarrow X^{\prime}$ is a purely inseparable morphism constructed in Proposition 2.10 such that $f^{\prime}: X^{\prime} \rightarrow Y$ is separable, $\nu: X^{\prime \prime} \rightarrow X^{\prime}$ is a smooth resolution of singularities, and $\tilde{f}, f^{\prime}, f^{\prime \prime}$ denote natural induced morphisms.

Since $\sigma$ is purely inseparable, there exists $D^{\prime}$ on $X^{\prime}$ such that $\sigma^{*} D^{\prime}=D$. Set $D^{\prime \prime}=\nu^{*} D^{\prime}$, which is a nef divisor relatively big and semi-ample over $Y$. Take a big divisor $H$ on $Y$. By Theorem 4.6, for sufficiently divisible $n>0$, the divisor $n D^{\prime \prime}+K_{X^{\prime \prime}}-f^{\prime \prime *} K_{Y}+f^{\prime \prime *} H$ is big. By Proposition 2.10, there exist a rational number $t>0$ and an effective divisor $\Delta^{\prime}$ such that $K_{\tilde{X}} \sim_{\mathbb{Q}} \sigma^{*} t\left(K_{X^{\prime}}+\Delta^{\prime}\right)$. And we
can write that $K_{X^{\prime \prime}}=\nu^{*}\left(K_{X^{\prime}}+\Delta^{\prime}\right)+E^{\prime \prime}-F^{\prime \prime}$ where $E^{\prime \prime}, F^{\prime \prime}$ are effective divisors on $X^{\prime \prime}$ and $E^{\prime \prime}$ is $\nu$-exceptional. Applying Theorem 2.3, we conclude that

$$
\begin{align*}
& \kappa\left(\tilde{X}, t n D+K_{\tilde{X}}-t \tilde{f}^{*} K_{Y}+t \tilde{f}^{*} H\right) \\
& \geq \kappa\left(X^{\prime}, t\left(n D^{\prime}-f^{\prime *} K_{Y}+f^{\prime *} H\right)+t\left(K_{X^{\prime}}+\Delta^{\prime}\right)\right) \\
& =\kappa\left(X^{\prime \prime}, \nu^{*} t\left(n D^{\prime}-f^{\prime *} K_{Y}+f^{\prime *} H\right)+\nu^{*} t\left(K_{X^{\prime}}+\Delta^{\prime}\right)+t E^{\prime \prime}\right)  \tag{4}\\
& \geq \kappa\left(X^{\prime \prime}, t\left(n D^{\prime \prime}+K_{X^{\prime \prime}}-f^{\prime \prime *} K_{Y}+f^{\prime \prime *} H\right)\right) \geq 3
\end{align*}
$$

Since this is true for any big $\mathbb{Q}$-divisor $H$, and $D$ is nef, we conclude that for $q \gg 0$, $q D+K_{\tilde{X}}-t \tilde{f}^{*} K_{Y}$ is pseudo-effective.

When $\kappa(Y)=0$, if $\operatorname{dim} X_{\eta}=1$ then $\nu(\tilde{X}, D) \geq \operatorname{dim} X_{\eta}$ since $D$ is nonzero and nef; if $\operatorname{dim} X_{\eta}=2$, then for a general fiber $\tilde{F}$ of $\tilde{f}, D^{2} \cdot \tilde{F}>0$ since $D$ is nef and $\tilde{f}$-big, thus $\nu(\tilde{X}, D) \geq 2$.

When $K_{Y}$ is big, by setting $H=K_{Y}$ in Eq. (4), we obtain that $n D+K_{\tilde{X}}$ is big. As we can write that $K_{\tilde{X}}=\mu^{*}\left(K_{\bar{X}}+\bar{B}\right)+E-F$ where $E, F$ are effective divisors on $\tilde{X}$ and $E$ is $\mu$-exceptional, applying Theorem [2.3, it follows that $(n+1) D$ is big.

It remains to consider the case $\kappa(Y)=1$ and $\operatorname{dim} Y=2$. Take an ample divisor $\bar{A}$ on $\bar{X}$. We only need to prove that $D^{2} \cdot \mu^{*} \bar{A}>0$. Fix a $q \gg 0$ such that $q D+K_{\tilde{X}}-t \tilde{f}^{*} K_{Y}$ is pseudo-effective. Then $D^{2} \cdot\left(q D+K_{\tilde{X}}-t \tilde{f}^{*} K_{Y}\right) \geq 0$. And since $E$ is $\mu$-exceptional, by projection formula we have $\mu^{*}\left(K_{\bar{X}}+\bar{B}\right) \cdot \mu^{*} \bar{A} \cdot E=0$. Take a general divisor $H^{\prime} \in\left|N K_{Y}\right|$ for some sufficiently divisible $N$ and set $\tilde{H}^{\prime}=\tilde{f}^{*} H^{\prime}$. Then $\tilde{H}^{\prime}$ contains a component $\tilde{H}$, such that $\left.\mu^{*} \bar{A}\right|_{\tilde{H}}$ is semi-ample and big and that $\left.D\right|_{\tilde{H}}$ is nef and $\left.\tilde{f}\right|_{\tilde{H}}$-big. Therefore,

$$
D \cdot \mu^{*} \bar{A} \cdot \tilde{f}^{*} K_{Y}=\frac{1}{N} D \cdot \mu^{*} \bar{A} \cdot \tilde{H}^{\prime} \geq \frac{1}{N} D \cdot \mu^{*} \bar{A} \cdot \tilde{H}=\frac{1}{N}\left(\left.D\right|_{\tilde{H}}\right) \cdot\left(\left.\mu^{*} \bar{A}\right|_{\tilde{H}}\right)>0
$$

where the last strict inequality is obtained by applying Hodge Index Theorem. Finally the proof is completed by

$$
\begin{aligned}
(q+1) D^{2} \cdot \mu^{*} \bar{A} & =D \cdot\left(q D+\mu^{*}\left(K_{\bar{X}}+\bar{B}\right)\right) \cdot \mu^{*} \bar{A}=D \cdot\left(q D+K_{\tilde{X}}-E+F\right) \cdot \mu^{*} \bar{A} \\
& \geq D \cdot\left(q D+K_{\tilde{X}}-t \tilde{f}^{*} K_{Y}+t \tilde{f}^{*} K_{Y}\right) \cdot \mu^{*} \bar{A}>0 .
\end{aligned}
$$

Recall a positivity result on surfaces.
Lemma 4.8. ([12, Lemma 2.11]) Let $g: Z \rightarrow Y$ be a generically smooth fibration from a smooth projective surface to a smooth projective curve. Let $H$ be a nef and $g$-big divisor on $Z$. Then $g_{*} \mathcal{O}_{Z}\left(K_{Z / Y}+l H\right)$ is a nef vector bundle for every $l \gg 0$.

The following result was proved by Waldron [44] when $\kappa=2$, and the case $\kappa=1$ follows easily from applying Theorem [2.5 (3.2).

Theorem 4.9. ([48, Theorem 3.1]) Let $(X, B)$ be a $\mathbb{Q}$-factorial klt projective 3-fold. Assume that $K_{X}+B$ is nef. If $\kappa\left(X, K_{X}+B\right) \geq 1$, then $K_{X}+B$ is semi-ample.

We extract the following lemma from the strategy of [12, Sec. 4], which will be used in the proof of Theorem 4.2 and 4.3.

Lemma 4.10. Let $(\hat{X}, \hat{B})$ be a minimal projective $\mathbb{Q}$-factorial dlt pair of dimension 3, and let $\hat{f}: \hat{X} \rightarrow Y$ be a fibration to a normal variety. Assume that
(a) $\hat{B}=G_{1}+G_{2}+\cdots+G_{n}$ is a sum of prime Weil divisors.

Then for every $j=1,2, \cdots, n,\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{j}}$ is semi-ample, and a general fiber $F_{j}$ of the Iitaka fibration induced by $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{j}}$ is integral.

Assume moreover that
(b) there exist $N>0$ and two different effective Cartier divisors $\hat{D}_{i}, i=1,2$ such that $\hat{D}_{i} \sim N\left(K_{\hat{X}}+\hat{B}\right)+\hat{f}^{*} L_{i}$ for some $L_{i} \in \operatorname{Pic}^{0}(Y)$ and that

$$
\operatorname{Supp} \hat{D}_{i} \subseteq \operatorname{Supp} \hat{B}
$$

(c) there exist effective divisors $\hat{G}_{1}, \hat{G}_{2}, \hat{G}_{1}^{\prime}, \hat{G}_{2}^{\prime}$ such that $\hat{D}_{1}=a_{11} \hat{G}_{1}+a_{12} \hat{G}_{2}+$ $\hat{G}_{1}^{\prime}$ and $\hat{D}_{2}=a_{21} \hat{G}_{1}+a_{22} \hat{G}_{2}+\hat{G}_{2}^{\prime}$ where $a_{11}>a_{21} \geq 0$ and $a_{22}>a_{12} \geq 0$; and
(d) there exist two irreducible components, say, $G_{1}, G_{2}$ of $\hat{G}_{1}, \hat{G}_{2}$ respectively, such that for $i, j \in\{1,2\}$ and $i \neq j, F_{j}$ is dominant over $Y$ and

$$
F_{j} \cap \operatorname{Supp}\left(\hat{G}_{j}^{\prime \prime}:=\hat{G}_{i}+\hat{G}_{1}^{\prime}+\hat{G}_{2}^{\prime}\right)=\emptyset .
$$

Then both $L_{1}$ and $L_{2}$ are torsion.
Furthermore, condition (d) holds, if for $j=1,2, G_{j}$ is not a component of $\hat{G}_{j}^{\prime \prime}$ and $\kappa\left(F_{j}\right) \geq 0$.

Proof. Note that since each $G_{i}$ is a dlt center of $(\hat{X}, \hat{B}), G_{i}$ is a normal surface by [2, Lemma 4.2]. Moreover we have $\left.\left(\hat{B}-G_{i}\right)\right|_{G_{i}} \geq 0$, and $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}=K_{G_{i}}+\left.\left(\hat{B}-G_{i}\right)\right|_{G_{i}}$ is $\log$ canonical. Then since $K_{\hat{X}}+\hat{B}$ is assumed nef, by [41, Theorem 1.2], $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}$ is semi-ample. And as $\operatorname{dim} G_{i}=2, F_{j}$ is always integral by Proposition [2.1.

Assume (b), (c) and (d). Let's prove that $L_{1}, L_{2}$ are torsion. First considering the restrictions on $F_{1}$, by $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{F_{1}} \sim_{\mathbb{Q}} 0$, applying these assumptions we can show

$$
\begin{aligned}
\left.a_{21} \hat{f}^{*} L_{1}\right|_{F_{1}} & \left.\sim_{\mathbb{Q}} a_{21}\left(N\left(K_{\hat{X}}+\hat{B}\right)+\hat{f}^{*} L_{1}\right)\right|_{F_{1}} \\
& \left.\left.\left.\sim a_{21} \hat{D}_{1}\right|_{F_{1}} \sim a_{11} a_{21} \hat{G}_{1}\right|_{F_{1}} \sim a_{11} \hat{D}_{2}\right|_{F_{1}} \\
& \left.\left.\sim a_{11}\left(N\left(K_{\hat{X}}+\hat{B}\right)+\hat{f}^{*} L_{2}\right)\right|_{F_{1}} \sim_{\mathbb{Q}} a_{11} \hat{f}^{*} L_{2}\right|_{F_{1}}
\end{aligned}
$$

thus $a_{21} L_{1} \sim_{\mathbb{Q}} a_{11} L_{2}$ by Lemma [2.4. And restricting on $F_{2}$, in the same way we can show $a_{22} L_{1} \sim_{\mathbb{Q}} a_{12} L_{2}$. Then granted these two relations, we can conclude $L_{i} \sim_{\mathbb{Q}} 0, i=1,2$ from the fact that $\left(\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right)$ is invertible over $\mathbb{Q}$, which is because $a_{11}>a_{21} \geq 0$ and $a_{22}>a_{12} \geq 0$.

It remains to prove the third assertion. As $\kappa\left(F_{j}\right) \geq 0$, we can assume the canonical divisor $K_{F_{j}^{\prime}} \geq 0$ where $F_{j}^{\prime}$ is the normalization of $F_{j}$. Applying the adjunction formula, we get

$$
\begin{aligned}
\left.0 \sim_{\mathbb{Q}}\left(K_{\hat{X}}+\hat{B}\right)\right|_{F_{j}^{\prime}} & \left.\sim_{\mathbb{Q}}\left(\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{j}}\right)\right|_{F_{j}^{\prime}} \\
& \left.\sim_{\mathbb{Q}}\left(K_{G_{j}}+\left(\hat{B}-G_{j}\right)\right)\right|_{F_{j}^{\prime}} \sim_{\mathbb{Q}} K_{F_{j}^{\prime}}+C_{j}^{\prime}+\left.\left(\hat{B}-G_{j}\right)\right|_{F_{j}^{\prime}}
\end{aligned}
$$

where $C_{j}^{\prime} \geq 0$ on $F_{j}^{\prime}$. As $F_{j}$ is general, we may assume $F_{j}$ is not contained in $\hat{B}-G_{j}$. In turn we conclude that $\left.\left(\hat{B}-G_{j}\right)\right|_{F_{j}^{\prime}}=0$. This, combing with the assumption that $G_{j}$ is not a component of $\hat{G}_{j}^{\prime \prime}$, indicates that $F_{j} \cap \operatorname{Supp} \hat{G}_{j}^{\prime \prime}=\emptyset$.
4.2. Proof of Theorem 4.2. For a sufficiently large integer $e$, the Weil index of $B^{\prime}=\frac{p^{e}}{p^{e}+1} B$ is not divisible by $p$, and $K_{X}+B^{\prime}$ is still $f$-big. Replacing $B$ with $B^{\prime}$, we can assume the Weil index of $B$ is not divisible by $p$. By Theorem [2.5 (3.1, 3.4), we can replace $X$ with the relative $\log$ canonical model over $A$ with the loss of $X$ being $\mathbb{Q}$-factorial, thus $K_{X}+B$ is a nef and $f$-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

We claim that we only need to prove $\kappa\left(K_{X}+B\right) \geq 1$. Indeed, if this is true then $K_{X}+B$ is semi-ample by Theorem 4.9, thus for a sufficiently divisible $M>0$, the linear system $\left|M\left(K_{X}+B\right)\right|$ has no base point. Since $\left(K_{X}+B\right)_{\eta}$ is big, the restriction $\mid M\left(K_{X}+B\right) \|_{X_{\eta}}$ on the generic fiber defines a generically finite morphism, which indicates that $\kappa\left(X, K_{X}+B\right) \geq \kappa\left(X_{\eta},\left(K_{X}+B\right)_{\eta}\right)$.

Let $l, g>0$ be two integers such that $l\left(K_{X}+B\right)$ is Cartier and $\left(p^{g}-1\right) B$ is integral. For an integer $e>0$ divisible by $g$, we have the trace map
$\operatorname{Tr}_{X, B}^{e, l}: \mathcal{F}_{e, l}:=f_{*}\left(F_{X *}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}\right)\left(K_{X}+B\right)\right) \otimes \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \rightarrow f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)$, and denote its image by $\mathcal{F}_{e}^{l}$. The restriction of $\mathcal{F}_{e}^{l}$ on the generic point $\eta$ defines a linear system $\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$ contained in $\left|l\left(K_{X}+B\right)_{\eta}\right|$.

We claim the following assertions with the proof postponed.
C1 For every $l>0$ such that $l\left(K_{X}+B\right)$ is Cartier, there exists some $e(l)>0$ such that for any $e \geq e(l)$ divisible by $g$, the rank of $\mathcal{F}_{e}^{l}$ is a stable number $r_{l}$; and there exists an integer $K>0$ such that, for any ldivisible by $K$ and any e divisible by $g$, the linear system $\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$ defines a generically finite map.
C2 If $L$ is an ample line bundle on $\hat{A}$ and $l \geq 2$ is an integer such that $l\left(K_{X}+B\right)$ is Cartier, then for any $i>0$ and sufficiently divisible integer $e>0$,

$$
H^{i}\left(A, \mathcal{F}_{e, l} \otimes \hat{L}^{*}\right)=0
$$

Fix a positive integer $l$ divisible by $K$ and an integer $e_{0}$ such that rk $\mathcal{F}_{e_{0}}^{l}=r_{l}$. Let $\mathcal{F}=\mathcal{F}_{e_{0}}^{l}$. Then by (C1), the linear system $\left|\mathcal{F}_{\eta}\right|$ defines a generically finite map of $X_{\eta}$. Recall that for positive integers $e^{\prime}>e$ divisible by $g$, there exists a natural trace map $\mathcal{F}_{e^{\prime}, l} \rightarrow \mathcal{F}_{e, l}$ (Sec. 2.7). Applying (C2), by induction we can find two integers $e_{1}<e_{2}$ divisible by $g$ and bigger than $e_{0}$, two ample line bundles $H_{0}, H_{1}$ on $\hat{A}$ and three sheaves $\mathcal{F}_{0}:=\mathcal{F}, \mathcal{F}_{1}:=\mathcal{F}_{e_{1}, l}, \mathcal{F}_{2}:=\mathcal{F}_{e_{2}, l}$ which satisfy the conditions of Theorem 3.7. Note that if $\operatorname{dim} A=1$ we only need two sheaves $\mathcal{F}_{0}, \mathcal{F}_{1}$. Applying Theorem 3.7 (i) and Proposition 3.4 (2), the cohomological locus $V^{0}(\mathcal{F})=(-1)_{\hat{A}}^{*} \operatorname{Supp} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F}) \neq \emptyset$.

The statement is trivial when $\nu\left(K_{X}+B\right)=3$, so from now on we assume $\nu\left(K_{X}+\right.$ $B) \leq 2$. We break the proof into three steps.

Step 1: We prove that if $\operatorname{dim} V^{0}(\mathcal{F})>0$, then $\kappa\left(K_{X}+B\right) \geq 1$.
In this case $V^{0}(\mathcal{F})$ generates $\hat{A}$ since $\hat{A}$ is simple. Hence so does $V^{0}\left(f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+\right.\right.\right.$ $B)$ )) too, as $V^{0}(\mathcal{F}) \subseteq V^{0}\left(f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right.$. So for $m \geq \operatorname{dim} A$, the map

$$
\times^{m} V^{0}\left(f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \rightarrow \hat{A} \text { via }\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right) \mapsto \alpha_{1}+\alpha_{2}+\cdots+\alpha_{m}
$$

is surjective. In particular if $m>2 \operatorname{dim} A$, there exist infinitely many $m$-tuples $\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ mapped to $\hat{0}$. Consider the natural map

$$
\begin{aligned}
H^{0}\left(X, l\left(K_{X}+B\right)\right. & \left.+f^{*} \alpha_{1}\right) \times H^{0}\left(X, l\left(K_{X}+B\right)+f^{*} \alpha_{2}\right) \times \cdots \times H^{0}\left(X, l\left(K_{X}+B\right)+f^{*} \alpha_{m}\right) \\
& \rightarrow H^{0}\left(X, m l\left(K_{X}+B\right)+f^{*}\left(\alpha_{1}+\alpha_{2} \cdots+\alpha_{m}\right)\right) \cong H^{0}\left(X, m l\left(K_{X}+B\right)\right)
\end{aligned}
$$

We can show $h^{0}\left(X, m l\left(K_{X}+B\right)\right) \geq 2$, thus $\kappa\left(K_{X}+B\right) \geq 1$.
From now on we assume that $V^{0}(\mathcal{F})=(-1)_{\hat{A}}^{*} \operatorname{Supp} R^{0} \Phi_{\mathcal{P}} D_{A}(\mathcal{F})$ consists of finitely many closed points.

Step 2: We will find an integer $m_{1}$ and some divisors $D_{i} \in\left|m_{1}\left(K_{X}+B\right)+f^{*} L_{i}\right|, i=$ $1,2, \cdots, r$ for some $L_{i} \in \operatorname{Pic}^{0}(A)$, such that the sub-linear system of $\left|m_{1}\left(K_{X}+B\right)_{\eta}\right|$ generated by $\left(D_{i}\right)_{\eta}, i=1,2, \cdots, r$ defines a generically finite map of $X_{\eta}$.

By Theorem 3.7(ii), there exist an isogeny $\pi: A_{1} \rightarrow A, P_{1}, P_{2}, \cdots, P_{s} \in \operatorname{Pic}^{0}\left(A_{1}\right)$ and a generically surjective homomorphism $\bigoplus_{j} P_{j} \rightarrow \pi^{*} \mathcal{F}$. Consider the following commutative diagram

where $X_{1}^{\prime}$ is the normalization of the reduced scheme structure of $X_{1}$. There exists a natural composition homomorphism by [17, Chapter III. Prop. 9.3]

$$
\begin{aligned}
\alpha: \bigoplus_{j} P_{j} \rightarrow \pi^{*} \mathcal{F} \hookrightarrow & \pi^{*} f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right) \cong f_{1 *} \pi_{1}^{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right) \\
& \rightarrow f_{1 *}^{\prime}\left(\nu^{*} \pi_{1}^{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \cong f_{1 *}^{\prime}\left(\pi_{1}^{\prime *} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) .
\end{aligned}
$$

The linear system $\left|(\operatorname{Im} \alpha)_{\eta}\right| \subseteq\left|\pi_{1}^{\prime *}\left(l\left(K_{X}+B\right)\right)\right|_{\left(X_{1}^{\prime}\right)_{\eta}} \mid$ defines a generically finite map of $\left(X_{1}^{\prime}\right)_{\eta}$. If the direct summand $P_{j}$ is not mapped to zero via $\alpha$, then $h^{0}\left(X_{1}^{\prime}, \pi_{1}^{\prime *}\left(l\left(K_{X}+\right.\right.\right.$ $\left.B)-f_{1}^{\prime *} P_{j}\right) \geq 1$. Since $\pi^{*}: \operatorname{Pic}^{0}(A) \rightarrow \operatorname{Pic}^{0}\left(A_{1}\right)$ is an isogeny, there exist $Q_{j} \in$ $\operatorname{Pic}^{0}(A)$ such that $P_{j}=\pi^{*} Q_{j}$. Applying Theorem [2.3, if $\alpha\left(P_{j}\right) \neq 0$ then

$$
\kappa\left(X, l\left(K_{X}+B\right)-f^{*} Q_{j}\right)=\kappa\left(X^{\prime}, \pi_{1}^{\prime *}\left(l\left(K_{X}+B\right)-f_{1}^{\prime *} P_{j}\right) \geq 0\right.
$$

We can find a sufficiently divisible integer $l_{1}>0$ such that for every $j$, if $\alpha\left(P_{j}\right) \neq 0$, the pull-back linear system $\left.\left(\pi_{1}^{\prime *}\left|l_{1}\left(l\left(K_{X}+B\right)-f_{1}^{\prime *} P_{j}\right)\right|\right)\right|_{\left(X_{1}^{\prime}\right)_{\eta}}$ defines a rational map $\phi_{j}:\left(X_{1}^{\prime}\right)_{\eta} \rightarrow \mathbb{P}^{r_{j}}$ where $r_{j}=\operatorname{dim} \mid l_{1}\left(\left(l\left(K_{X}+B\right)-f_{1}^{\prime \prime} P_{j}\right) \mid\right.$, whose image has dimension $\kappa\left(X, l\left(K_{X}+B\right)-f^{*} Q_{j}\right)$. By the construction, the sub-linear system of $\left|\pi_{1}^{\prime *}\left(l_{1} l\left(K_{X}+B\right)\right)_{\eta}\right|$ generated by the divisors of $\left.\left(\pi_{1}^{\prime *}\left|l_{1}\left(l\left(K_{X}+B\right)-f_{1}^{\prime *} P_{j}\right)\right|\right)\right|_{\left(X_{1}^{\prime}\right)_{\eta}}, j=$ $1,2, \cdots, s$ defines a generically finite map of $\left(X_{1}^{\prime}\right)_{\eta}$. Therefore, there exist $L_{i}=$ $-l_{1} Q_{j_{i}} \in \operatorname{Pic}^{0}(A)$ for some $j_{1}, j_{2}, \cdots, j_{r}$ and effective divisors $D_{i} \in \mid m_{1}\left(K_{X}+B\right)+$ $f^{*} L_{i} \mid$ where $m_{1}=l_{1} l$, such that the linear system generated by $\left(D_{i}\right)_{\eta}, i=1,2, \cdots, r$ defines a generically finite map of $X_{\eta}$.

We aim to prove that there exist at least two different divisors among $D_{i}$, say, $D_{1} \neq D_{2}$, such that $L_{1}, L_{2}$ are torsion in $\operatorname{Pic}^{0}(A)$. Then for some sufficiently divisible $N>0$ such that $N L_{1} \sim N L_{2} \sim 0$, we have $N D_{1}, N D_{2} \in\left|N m_{1}\left(K_{X}+B\right)\right|$, which concludes $\kappa\left(X, K_{X}+B\right) \geq 1$.

Step 3: We will construct a minimal dlt pair $(\hat{X}, \hat{B})$ and divisors $\hat{D}_{1}, \hat{D}_{2}$ satisfying the conditions of Lemma 4.10.
(3.1) Take a $\log$ resolution $\mu: \tilde{X} \rightarrow X$ of $B+\sum_{i} D_{i}$. Let $\tilde{B}$ be the reduced divisor supported on the union of $\mu^{-1}\left(B+\sum_{i} D_{i}\right)$ and the $\mu$-exceptional divisors. Then $(\tilde{X}, \tilde{B})$ is dlt, and $K_{\tilde{X}}+\tilde{B}$ has a weak Zariski decomposition. By Theorem 2.5, running a relative $\log$ MMP for $(\tilde{X}, \tilde{B})$ over $A$, we can get a dlt log minimal model $(\hat{X}, \hat{B})$ and a fibration $\hat{f}: \hat{X} \rightarrow A$. The divisor $\tilde{E}=K_{\tilde{X}}+\tilde{B}-\mu^{*}\left(K_{X}+B\right)$ is effective. Take a sufficiently divisible integer $l_{2}>0$ such that $l_{2} \tilde{E}$ is Cartier. Let $m_{2}=m_{1} l_{2}$. We get effective divisors

$$
\tilde{D}_{i}=l_{2} \mu^{*} D_{i}+l_{2} \tilde{E} \sim m_{2}\left(K_{\tilde{X}}+\tilde{B}\right)+l_{2} \mu^{*} f^{*} L_{i}
$$

and the push-forward divisors via the natural map $\tilde{X} \rightarrow \hat{X}$

$$
\hat{D}_{i} \sim m_{2}\left(K_{\hat{X}}+\hat{B}\right)+l_{2} \hat{f}^{*} L_{i} .
$$

(3.2) We prove $\nu\left(K_{\hat{X}}+\hat{B}\right)=\nu\left(K_{X}+B\right)$ as follows. Applying Proposition 2.8, on one hand since $K_{\tilde{X}}+\tilde{B} \geq \mu^{*}\left(K_{X}+B\right)$ we have $\nu\left(K_{\hat{X}}+\hat{B}\right)=\kappa_{\sigma}\left(K_{\tilde{X}}+\tilde{B}\right) \geq \nu\left(K_{X}+\right.$ $B)$, on the other hand since there exists an effective $\mu$-exceptional divisor $\tilde{E}^{\prime}$ such that $K_{\tilde{X}}+\tilde{B} \leq \mu^{*}\left(K_{X}+B\right)+\sum_{i} \mu^{*} D_{i}+\tilde{E}^{\prime} \equiv\left(r m_{1}+1\right) \mu^{*}\left(K_{X}+B\right)+\tilde{E}^{\prime}$, we conclude $\kappa_{\sigma}\left(K_{\tilde{X}}+\tilde{B}\right) \leq \nu\left(K_{X}+B\right)$. In summary, we get the equality $\nu\left(K_{\hat{X}}+\hat{B}\right)=\nu\left(K_{X}+B\right)$.
(3.3) The restrictions of $\hat{D}_{i}$ on $\hat{X}_{\eta}$ generate a linear system $|\hat{V}| \subseteq\left|m_{2}\left(K_{\hat{X}}+\hat{B}\right)_{\eta}\right|$, which defines a generically finite map $\hat{X}_{\eta} \rightarrow \mathbb{P}_{k(\eta)}^{r-1}$ by the construction in Step 2. Let $\hat{C}_{\eta}$ be the fixed part of $|\hat{V}|$, and set $\hat{A}_{i, \eta}=\left(\hat{D}_{i}\right)_{\eta}-\hat{C}_{\eta}$. We may assume $\hat{A}_{1, \eta} \neq \hat{A}_{2, \eta}$. Since $\hat{A}_{1, \eta} \sim \hat{A}_{2, \eta}$, we can choose two irreducible components $G_{1, \eta}, G_{2, \eta}$ of $\hat{A}_{1, \eta}+\hat{A}_{2, \eta}$ such that,

- if $b_{i j}, i, j=1,2$ are the coefficients of $G_{j, \eta}$ in $\hat{A}_{i, \eta}$ respectively, i.e.,

$$
\hat{A}_{1, \eta}=b_{11} G_{1, \eta}+b_{12} G_{2, \eta}+G_{1, \eta}^{\prime} \text { and } \hat{A}_{2, \eta}=b_{21} G_{1, \eta}+b_{22} G_{2, \eta}+G_{2, \eta}^{\prime}
$$

where neither of $G_{1, \eta}, G_{2, \eta}$ are contained in $G_{1, \eta}^{\prime}+G_{2, \eta}^{\prime}$, then $b_{11}>b_{21} \geq 0$ and $b_{22}>b_{12} \geq 0$.
(3.4) If $\operatorname{dim} \hat{X}_{\eta}=2$, since $K_{\hat{X}}+\hat{B}$ is relatively big over $A$, we can choose $\hat{A}_{2, \eta}$ and $G_{i, \eta}, i=1,2$ such that $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i, \eta}}$ is big as follows. First we can choose a component $G_{1, \eta}$ of $\hat{A}_{1, \eta}$ such that $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{1, \eta}}$ is big. Second since $\hat{A}_{i, \eta}, i=$ $1,2, \cdots, r$ have no common component, we can choose $\hat{A}_{2, \eta}$ not containing $G_{1, \eta}$, i.e., $b_{21}=0$. Finally since $\hat{A}_{1, \eta} \sim \hat{A}_{2, \eta}$ and the intersection number $\left(K_{\hat{X}}+\hat{B}\right) \cdot\left(\hat{A}_{1, \eta}-\right.$ $\left.b_{11} G_{1, \eta}\right)<\left(K_{\hat{X}}+\hat{B}\right) \cdot \hat{A}_{2, \eta}, \hat{A}_{2, \eta}$ must have an irreducible component $G_{2, \eta}$ such that $\left(K_{\hat{X}}+\hat{B}\right) \cdot G_{2, \eta}>0$ and the coefficients $b_{22}>b_{12}$.
(3.5) Let $G_{i}, i=1,2$ be the reduced irreducible divisors on $\hat{X}$ such that $\left(G_{i}\right)_{\eta}=$ $G_{i, \eta}$. By (3.3) we can write that

$$
\hat{D}_{1}=a_{11} G_{1}+a_{12} G_{2}+G_{1}^{\prime} \text { and } \hat{D}_{2}=a_{21} G_{1}+a_{22} G_{2}+G_{2}^{\prime}
$$

where $a_{11}>a_{21} \geq 0$ and $a_{22}>a_{12} \geq 0$, and neither of $G_{1}, G_{2}$ are contained in $G_{1}^{\prime}+G_{2}^{\prime}$.
(3.6) By Lemma4.10, we know that $G_{i}$ is normal, and $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}$ is semi-ample. Since $\hat{D}_{i} \equiv m_{2}\left(K_{\hat{X}}+\hat{B}\right)$ and $G_{i}$ is a component of $\hat{D}_{i}$, by Proposition 2.8 we have

$$
\kappa\left(G_{i},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}\right)=\nu\left(G_{i},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}\right) \leq \nu\left(\hat{X}, K_{\hat{X}}+\hat{B}\right)-1
$$

For the case $\operatorname{dim} A=2$, we have $\nu\left(G_{i},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}\right)=0$ or 1 . The case $\nu\left(G_{i},\left(K_{\hat{X}}+\right.\right.$ $\left.\hat{B})\left.\right|_{G_{i}}\right)=1$ does not happen, because otherwise, the divisor $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}} \sim_{\mathbb{Q}} K_{G_{i}}+$ $\left.\left(\hat{B}-G_{i}\right)\right|_{G_{i}}$ will induce a fibration fibred by curves of arithmetic genus $\leq 1$ on $G_{i}$, which is impossible since $G_{i}$ is dominant over $A$ and $A$ is simple. Thus the Iitaka fibration induced by $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}$ is trivial. By (3.5), the surface $G_{i}$ is dominant over $A$, hence $\kappa\left(G_{i}, K_{G_{i}}\right) \geq 0$ (see for example [1, Prop. 13.1]).

For the case $\operatorname{dim} A=1, \hat{X}_{\eta}$ is a surface over $k(\eta)$, and by (3.4)

$$
\kappa\left(G_{i},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}\right)=\nu\left(G_{i},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}\right)=1
$$

Take a general fiber $F_{i}$ of the Iitaka fibration induced by $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i}}$ on $G_{i}$. Since $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{i, \eta}}$ is big, $F_{i}$ is a curve dominant over $A$, thus $\kappa\left(F_{i}, K_{F_{i}}\right) \geq 0$.

Let $\hat{G}_{i}=G_{i}$ and $\hat{G}_{i}^{\prime}=G_{i}^{\prime}$ for $i=1,2$. By (3.1), (3.5) and (3.6), the conditions of Lemma 4.10 are satisfied, hence $l_{2} L_{1}, l_{2} L_{2}$ are torsion, which finishes the proof.

It remains to prove Claim (C1) and (C2).
(C1) For every $l>0$, if $e^{\prime}>e$ then $\mathcal{F}_{e^{\prime}}^{l} \subseteq \mathcal{F}_{e}^{l}$, hence there exists some $e(l)$ such that for every $e \geq e(l)$, we have $\operatorname{rk} \mathcal{F}_{e}^{l}=\operatorname{rk} \mathcal{F}_{e(l)}^{l}$, which is denoted by $r_{l}$.

To study the linear system $\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$, we may replace $B$ with $B+f^{*} H$ for some ample divisor $H$ on $A$, so $K_{X}+B$ is ample. And we may replace $B$ with a bigger divisor $B+t D$ for some effective divisor $D \sim_{\mathbb{Q}} K_{X}+B$ and a rational number $t>0$, to make the Cartier index of $K_{X}+B$ not divisible by $p$. ${ }^{2}$ Moreover we remark that for some fixed $l$, to show that the linear system $\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$ defines a generically finite map of $X_{\eta}$, we only need to prove this assertion for any sufficiently divisible $e$.

Fix a sufficiently divisible integer $g^{\prime}>0$ such that $\left(1-p^{g^{\prime}}\right)\left(K_{X}+B\right)$ is Cartier, and that for the non- $F$-pure ideal $\sigma(B)$ (Sec. 2.7)

$$
\operatorname{Tr}_{X, B}^{g^{\prime}}\left(F_{X *}^{g^{\prime}}\left(\sigma(B) \cdot \mathcal{O}_{X}\left(\left(1-p^{g^{\prime}}\right)\left(K_{X}+B\right)\right)\right)\right)=\sigma(B)
$$

Since $K_{X}+B$ is ample, applying Lemma 2.14, we can find a sufficiently divisible integer $K$ such that for every positive integer $l$ divisible by $K$ the trace map

$$
\begin{aligned}
\Phi_{e, l}: H^{0}\left(X, F_{X *}^{e g^{\prime}}\left(\sigma ( B ) \cdot \mathcal { O } _ { X } \left(\left(1-p^{e g^{\prime}}\right)\right.\right.\right. & \left.\left.\left.\left(K_{X}+B\right)\right)\right) \otimes \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \\
& \rightarrow H^{0}\left(X, \sigma(B) \cdot \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) .
\end{aligned}
$$

[^1]is surjective. It follows that
$$
\left|\sigma(B) \cdot \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right|\left\|_{X_{\eta}}=\mid \operatorname{Im}\left(\Phi_{e, l}\right)\right\|_{X_{\eta}}
$$

If necessary, we can enlarge $K$ to assume that the linear system $\mid \sigma(B) \cdot \mathcal{O}_{X}\left(K\left(K_{X}+\right.\right.$ $B)) \mid$ defines a generically finite map of $X$. Then by $\left|\operatorname{Im}\left(\Phi_{e, l}\right)\right|_{X_{\eta}} \subseteq\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$, we show that $\left|\left(\mathcal{F}_{e}^{l}\right)_{\eta}\right|$ defines a generically finite map of $X_{\eta}$.
(C2) Before the proof we remind that the Weil index $q_{0}$ of $K_{X}+B$ is not divisible by $p$, but the Cartier index $q_{1}$ is possibly divisible by $p$. From now on to the end of the proof of (C2), we assume the integer $e$ always satisfies $q_{0} \mid p^{e}-1$. Let $d=\frac{q_{1}}{q_{0}}$. Then $H=q_{1}\left(K_{X}+B\right)$ is a nef and $f$-ample divisor, and we have a set consisting of finitely many coherent sheaves

$$
\left\{\mathcal{G}_{r}=\mathcal{O}_{X}\left(r q_{0}\left(K_{X}+B\right)\right) \mid r=0,1, \cdots d-1\right\} .
$$

Define $\phi_{L}: \hat{A} \rightarrow A$ via $\hat{t} \mapsto T_{\hat{t}}^{*} L \otimes L^{-1} \in \operatorname{Pic}^{0}(\hat{A})=A$. Then by results of [31, Sec. 13, 16], the morphism $\phi_{L}$ is an isogeny since $L$ is ample, and

$$
\phi^{*} \hat{L}^{*} \cong \bigoplus^{r} L \text { where } r=h^{0}(\hat{A}, L)
$$

The isogeny $\phi_{L}: \hat{A} \rightarrow A$ is not necessarily separable. Denote by $\hat{K}$ the kernel of $\phi_{L}$, let $\hat{K}_{0}$ be the maximal sub-group of $\hat{K}$ supported at $\hat{0}$ and let $A^{\prime}=\hat{A} / \hat{K}_{0}$. Then we have a factorization

$$
\phi_{L}: \hat{A} \xrightarrow{\nu} A^{\prime} \xrightarrow{\mu} A
$$

where $\nu: \hat{A} \rightarrow A^{\prime}$ is the natural quotient map which is purely inseparable, and $\mu: A^{\prime} \rightarrow A$ is étale. Fix a sufficiently large integer $g_{1}$ such that $F_{A^{\prime}}^{g_{1}}: A^{\prime g_{1}} \rightarrow A^{\prime}$ factors through $\nu$. Let

$$
\phi=\mu \circ F_{A^{\prime}}^{g_{1}}: A^{\prime g_{1}} \rightarrow A^{\prime} \rightarrow A .
$$

Then $\phi^{*} \hat{L}^{*} \cong \bigoplus^{r} L^{\prime}$ where $L^{\prime}$ is an ample line bundle on $A^{\prime g_{1}}$. For a larger integer $e>g_{1}$, we get the following commutative diagram

where $W_{X^{\prime}}^{e}, h_{1}^{e}, f^{\prime}, h$ denote the natural projections of the corresponding fiber products, and $\sigma: X^{\prime e} \rightarrow X_{1}^{\prime e}$ arises from the universal property of the fiber product. Since the base change $h: X^{\prime} \rightarrow X$ is étale, $\sigma$ is an isomorphism.

Assume $l \geq 2, q_{1} \mid l$. Let $n_{e}=\left\llcorner\frac{1-p^{e}+l p^{e}}{q_{1}}\right\lrcorner$. Then we can write that $1-p^{e}+l p^{e}=$ $n_{e} q_{1}+r_{e} q_{0}$ where $0 \leq r_{e}<d$. It follows that

$$
\mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right) \cong \mathcal{O}_{X}\left(n_{e} H\right) \otimes \mathcal{G}_{r_{e}}
$$

By Theorem 3.5 (3), to get that for every $i>0$,

$$
H^{i}\left(A, f_{*}\left(F_{X *}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}\right)\left(K_{X}+B\right)\right) \otimes \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \otimes \hat{L}^{*}\right)=0
$$

we only need to verify that for every $i>0$,

$$
\left.H^{i}\left(A^{\prime}, \mu^{*}\left(f_{*} F_{X *}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \hat{L}^{*}\right)\right)=0
$$

This is true when $e \gg 0$, which is proved as follows

$$
\begin{aligned}
& \left.H^{i}\left(A^{\prime}, \mu^{*}\left(f_{*} F_{X^{*}}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \hat{L}^{*}\right)\right) \\
& \left.\cong H^{i}\left(A^{\prime}, \mu^{*}\left(f_{*} F_{X *}^{e} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \mu^{*} \hat{L}^{*}\right)\right) \\
& \cong H^{i}\left(A^{\prime}, f_{*}^{\prime} W_{X^{\prime} *}^{e}\left(h_{1}^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \mu^{*} \hat{L}^{*}\right) \\
& \cong H^{i}\left(A^{\prime}, f_{*}^{\prime} W_{X^{\prime} *}^{e} \sigma_{*}\left(\sigma^{*} h_{1}^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \mu^{*} \hat{L}^{*}\right) \\
& \cong H^{i}\left(A^{\prime}, f_{*}^{\prime} F_{X^{\prime} *}^{e}\left(h^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \mu^{*} \hat{L}^{*}\right) \\
& \cong H^{i}\left(A^{\prime}, F_{A^{\prime} *}^{e} f_{*}^{\prime e}\left(h^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes \mu^{*} \hat{L}^{*}\right) \\
& \cong H^{i}\left(A^{\prime e}, f_{*}^{\prime e}\left(h^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes F_{A^{\prime}}^{e *} \mu^{*} \hat{L}^{*}\right) \\
& \cong H^{i}\left(A^{\prime e}, f_{*}^{\prime e}\left(\left(h^{e *} \mathcal{O}_{X}\left(\left(1-p^{e}+l p^{e}\right)\left(K_{X}+B\right)\right)\right) \otimes\left(f^{\prime e *} F_{A^{\prime}}^{\left(e-g_{1}\right) *} F_{A^{\prime}}^{g_{1} *} \mu^{*} \hat{L}^{*}\right)\right)\right. \\
& \cong H^{i}(A^{\prime}, f_{*}^{\prime}(\mathcal{O}_{X^{\prime}}\left(h^{*} n_{e} H\right) \otimes h^{*} \mathcal{G}_{r_{e}} \otimes \bigoplus \overbrace{}^{\prime *}\left(L^{\prime}\right)^{p^{e-g_{1}}})) \\
& \cong \bigoplus^{r} H^{i}\left(X^{\prime}, \mathcal{O}_{X^{\prime}}\left(h^{*} n_{e} H+p^{e-g_{1}} f^{\prime *} L^{\prime}\right) \otimes h^{*} \mathcal{G}_{r_{e}}\right)=0
\end{aligned}
$$

where

- the $2^{\text {nd }} \cong$ is due to $\mu^{*} f_{*} F_{X *}^{e} \cong f_{*}^{\prime} W_{X^{\prime} *}^{e} h_{1}^{e *}$ since $\mu$ is a flat base change;
- the $3^{\text {rd }} \cong$ is due to the fact that $\sigma$ is an isomorphism;
- the $6^{\text {th }} \cong$ is from applying projection formula and $R F_{A^{\prime} *}^{e} \cong F_{A^{\prime} *}^{e}$;
- the $9^{\text {th }} \cong$ is from applying Lerray spectral sequence and relative Fujita vanishing (Lemma 2.2) $R^{j} f_{*}^{\prime}\left(\mathcal{O}_{X^{\prime}}\left(h^{*} n_{e} H\right) \otimes h^{*} \mathcal{G}_{r_{e}}\right)=0$ for $j>0$ since $n_{e} H$ is sufficiently $f^{\prime}$-ample if $e \gg 0$, and the last vanishing follows from applying Fujita vanishing since $h^{*} n_{e} H+p^{e-g_{1}} f^{\prime *} L^{\prime}$ is sufficiently ample if $e \gg 0$.
4.3. Proof of Theorem 4.3. We may assume that $K_{X}+B$ is nef by working on a log minimal model of $(X, B)$ over $Y$ (Theorem [2.5](3.4)). As $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}+B_{\bar{\eta}}\right)=1$, there exist a $\log$ resolution $\sigma: W \rightarrow X$ and a fibration $h: W \rightarrow Z$ to a smooth projective surface $Z$, which is birational to the relative Iitaka fibration induced by $\sigma^{*}\left(K_{X}+B\right)$ on $W$ over $Y$. These varieties fit into the following commutative diagram


By the assumption $\boldsymbol{\uparrow}$ and Proposition 2.11, the geometric generic fiber of $h$ is a smooth elliptic curve over $\overline{K(Z)}$. Applying flattening trick ([4, Lemma 5.6]), we can assume $\sigma^{*}\left(K_{X}+B\right) \sim_{\mathbb{Q}} h^{*} C$ where $C$ is a nef and $g$-big divisor on $Z$. By [8, Claim 3.1 and 3.2], there exists an effective divisor $E$ on $W$ such that $K_{W} \sim_{\mathbb{Q}} h^{*} K_{Z}+E$.

For the case $g(Y)>1$, applying Theorem 4.6 for $g: Z \rightarrow Y$, we show that for $n \gg 0, n C+K_{Z}$ is big. By $K_{W} \sim_{\mathbb{Q}} h^{*} K_{Z}+E$, applying Theorem 2.3, we have

$$
\begin{aligned}
\kappa\left(X, K_{X}+B\right) & =\kappa\left(X,(n+1)\left(K_{X}+B\right)\right) \geq \kappa\left(X, n\left(K_{X}+B\right)+K_{X}\right) \\
& =\kappa\left(W, \sigma^{*} n\left(K_{X}+B\right)+K_{W}\right) \geq \kappa\left(Z, n C+K_{Z}\right)=2 .
\end{aligned}
$$

Let's restrict on the case $g(Y)=1$. We aim to prove that $\kappa\left(X, K_{X}+B\right) \geq 1$. First applying [8, Theorem 1.2 and 1.3], we have

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa(X)=\kappa(W) \geq \kappa(Z) \geq \kappa\left(Z, K_{Z_{\bar{\eta}}}\right)
$$

So we may assume that $\kappa\left(Z, K_{Z_{\bar{\eta}}}\right) \leq 0$, i.e., $p_{a}\left(Z_{\bar{\eta}}\right) \leq 1$, hence the geometric generic fiber $Z_{\bar{\eta}}$ is either a smooth elliptic curve (Proposition 2.11) or a rational curve over $k(\bar{\eta})$. And if $\nu(Z, C)=2$ then $C$ is big, applying Theorem 2.3 we are done by

$$
\kappa\left(X, K_{X}+B\right)=\kappa\left(W, \sigma^{*}\left(K_{X}+B\right)\right)=\kappa(Z, C)=2 .
$$

So we may assume additionally that $\nu(Z, C)=1$.
We will mimic the proof of Theorem 4.2 and break the arguments into three steps for similar purposes.

Step 1: By the assumptions above, $g: Z \rightarrow Y$ is generically smooth. And since $K_{Y} \sim 0$ and $C$ is nef and $g$-big, we can apply Lemma 4.8 and obtain that, for sufficiently divisible $n>0, V^{\prime}:=g_{*} \mathcal{O}_{Z}\left(n C+K_{Z}\right)$ is a nef vector bundle of rank $\geq 2$. Fix a sufficiently divisible $N>0$ such that $N K_{W} \sim h^{*} N K_{Z}+N E$ where $N E$ is integral, and that $N\left(K_{X}+B\right)$ is Cartier. Applying the projection formula, we can get a natural inclusion $\mathcal{O}_{Z}\left(N K_{Z}\right) \hookrightarrow h_{*} \mathcal{O}_{W}\left(N K_{W}\right)$. In turn we have

$$
\begin{aligned}
& \operatorname{Sym}^{N} V^{\prime}=\operatorname{Sym}^{N} g_{*} \mathcal{O}_{Z}\left(n C+K_{Z}\right) \rightarrow g_{*} \mathcal{O}_{Z}\left(n N C+N K_{Z}\right) \\
& \hookrightarrow g_{*} h_{*} \mathcal{O}_{W}\left(n N \sigma^{*}\left(K_{X}+B\right)+N K_{W}\right) \cong f_{*} \sigma_{*} \mathcal{O}_{W}\left(n N \sigma^{*}\left(K_{X}+B\right)+N K_{W}\right) \\
& \subseteq f_{*} \mathcal{O}_{X}\left(N n\left(K_{X}+B\right)+N K_{X}\right) \subseteq f_{*} \mathcal{O}_{X}\left(N n\left(K_{X}+B\right)+N\left(K_{X}+B\right)\right) \\
& =f_{*} \mathcal{O}_{X}\left(N(n+1)\left(K_{X}+B\right)\right)
\end{aligned}
$$

Denote by $V$ the image of $S y m^{N} V^{\prime}$ via the above composition map. Let $l=N(n+1)$. Then $V$ is a nef sub-vector bundle of $f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)$, and rk $V=r^{\prime} \geq 2$.

By [12, Proposition 2.7], there exists a flat base change $\pi: Y_{1} \rightarrow Y$ between elliptic curves such that $\pi^{*} V \cong \bigoplus_{i=1}^{i=r^{\prime}} L_{i}^{\prime}$ where each $L_{i}^{\prime}$ is a line bundle with $\operatorname{deg} L_{i}^{\prime} \geq 0$. Consider the following commutative diagram

where $X_{1}^{\prime}$ is the normalization of $X_{1}$. We have a natural inclusion map

$$
\begin{aligned}
\alpha: \bigoplus_{i=1}^{i=r^{\prime}} L_{i}^{\prime} \cong & \cong \pi^{*} V \subseteq \pi^{*}\left(f_{*} \mathcal{O}_{X}\left(l\left(K_{X}+B\right)\right)\right) \\
& \cong f_{1 *} \mathcal{O}_{X_{1}}\left(\pi_{1}^{*}\left(l\left(K_{X}+B\right)\right)\right) \subseteq f_{1 *}^{\prime} \mathcal{O}_{X_{1}^{\prime}}\left(\pi_{1}^{\prime *}\left(l\left(K_{X}+B\right)\right)\right)
\end{aligned}
$$

If some $L_{i_{0}}^{\prime}$ satisfies that $\operatorname{deg} L_{i_{0}}^{\prime}>0$, by doing a further étale base change, we may assume $\operatorname{deg} L_{i_{0}}^{\prime} \geq 2$, thus

$$
h^{0}\left(X_{1}^{\prime}, \pi_{1}^{\prime *}\left(m\left(K_{X}+B\right)\right)\right)=h^{0}\left(Y_{1}, f_{1 *}^{\prime} \mathcal{O}_{X_{1}^{\prime}}\left(\pi_{1}^{\prime *} m\left(K_{X}+B\right)\right)\right) \geq h^{0}\left(Y_{1}, L_{i_{0}}^{\prime}\right) \geq 2
$$

which implies $\kappa\left(X, K_{X}+B\right)=\kappa\left(X^{\prime}, \pi_{1}^{\prime *}\left(K_{X}+B\right)\right) \geq 1$ by Theorem 2.3,
From now on, we assume every $L_{i}^{\prime} \in \operatorname{Pic}^{0}\left(Y_{1}\right)$.
Step 2: Granted the inclusion map $\alpha$ and the commutative diagram (5) in Step 1, as in Step 2 of the proof of Theorem 4.2, we can find an integer $m_{1}=l_{1} l$ and some divisors $D_{i} \in\left|m_{1}\left(K_{X}+B\right)+f^{*} L_{i}\right|, i=1,2, \cdots, r$ for some $L_{i} \in \operatorname{Pic}^{0}(Y)$, such that the sub-linear system of $\left|m_{1}\left(K_{X}+B\right)_{\eta}\right|$ generated by $\left(D_{i}\right)_{\eta}, i=1,2, \cdots, r$ defines a nontrivial map of $X_{\eta}$.

We only need to prove that there exist at least two different divisors among $D_{i}$, say, $D_{1} \neq D_{2}$, such that $L_{1}, L_{2}$ are torsion in $\operatorname{Pic}^{0}(Y)$ (Step 2 of Theorem 4.2).

Step 3: We will construct a minimal dlt pair $(\hat{X}, \hat{B})$ and divisors $\hat{D}_{1}, \hat{D}_{2}$ satisfying the conditions of Lemma 4.10.
(3.1) Take a $\log$ resolution $\mu: \tilde{X} \rightarrow X$ of the pair $\left(X, B+\sum_{i} D_{i}\right)$. Denote by $\tilde{f}: \tilde{X} \rightarrow Y$ the natural morphism. Let $\tilde{B}$ be the reduced divisor supported on the union of $\sum_{i} \mu^{*} D_{i}$ and the exceptional divisors. By running a log MMP, we get a minimal dlt model $(\hat{X}, \hat{B})$, and a natural morphism $\hat{f}: \hat{X} \rightarrow Y$. The divisor $\tilde{E}=K_{\tilde{X}}+\tilde{B}-\mu^{*}\left(K_{X}+B\right)$ is effective. Take a sufficiently divisible integer $l_{2}>0$ such that $l_{2} \tilde{E}$ is Cartier. Let $m_{2}=m_{1} l_{2}$. We get effective divisors

$$
\tilde{D}_{i}=l_{2} \mu^{*} D_{i}+l_{2} \tilde{E} \sim m_{2}\left(K_{\tilde{X}}+\tilde{B}\right)+l_{2} \mu^{*} f^{*} L_{i}
$$

and the push-forward divisors via the natural map $\tilde{X} \rightarrow \hat{X}$

$$
\hat{D}_{i} \sim m_{2}\left(K_{\hat{X}}+\hat{B}\right)+l_{2} \hat{f}^{*} L_{i} .
$$

(3.2) We can prove $\nu\left(K_{\hat{X}}+\hat{B}\right)=\nu\left(K_{X}+B\right)=1$ as in Step 3 (3.2) of the proof of Theorem 4.2. Note that $\left(K_{\hat{X}}+\hat{B}\right)_{\eta}$ is semi-ample by Theorem 2.5 (3.2), thus $\kappa\left(\hat{X}_{\eta},\left(K_{\hat{X}}+\hat{B}\right)_{\eta}\right)=1$. Considering the relative Iitaka fibration $\hat{h}^{\prime}: \hat{X} \rightarrow \hat{Z}^{\prime}$ of $\hat{X}$ over $Y$ induced by $K_{\hat{X}}+\hat{B}$ and applying flattening trick ([4, Lemma 5.6]), we get a commutative diagram

such that $\hat{\sigma}^{*}\left(K_{\hat{X}}+\hat{B}\right) \sim_{\mathbb{Q}} \hat{h}^{*} \hat{C}$ for some nef and $\hat{g}$-big divisor $\hat{C}$ on $\hat{Z}$, where $\hat{W}$ and $\hat{Z}$ are smooth, birational to $\hat{X}$ and $\hat{Z}^{\prime}$ respectively, and $\hat{h}$ is a fibration birational to $\hat{h}^{\prime}$. By the construction, $\hat{h}: \hat{W} \rightarrow \hat{Z}$ is an elliptic fibration, and $\nu(\hat{Z}, \hat{C})=1$.
(3.3) We can take effective divisors $\hat{C}_{i} \sim m_{2} \hat{C}+l_{2} \hat{g}^{*} L_{i}$ on $\hat{Z}$ such that $\hat{h}^{*} \hat{C}_{i}=\hat{\sigma}^{*} \hat{D}_{i}$. Note that $\hat{C}_{i}$ is nef and $\hat{C}_{i}^{2}=0$. Considering the connected components of the union of $\hat{C}_{i}, 0 \leq i \leq r$, we can show that there exist effective Cartier divisors $\hat{C}_{1}^{\prime}, \ldots, \hat{C}_{s}^{\prime}$ satisfying the following conditions:

- $\operatorname{Supp}\left(\hat{C}_{j}^{\prime}\right)$ is connected for each $j$, and $\operatorname{Supp}\left(\hat{C}_{j_{1}}^{\prime}\right) \cap \operatorname{Supp}\left(\hat{C}_{j_{2}}^{\prime}\right)=\emptyset$ if $j_{1} \neq j_{2}$;
- every $\hat{C}_{j}^{\prime}$ is nef and $\left(\hat{C}_{j}^{\prime}\right)^{2}=0$;
- the greatest common divisor of the coefficients of every $\hat{C}_{j}^{\prime}$ is equal to one;
- for each $i$, there exist $a_{i 1}, \ldots, a_{i s} \in \mathbb{Z}^{\geq 0}$ such that $\hat{C}_{i}=a_{i 1} \hat{C}_{1}^{\prime}+\cdots+a_{i s} \hat{C}_{s}^{\prime}$. As $\hat{\sigma}^{*} \hat{D}_{i}=\hat{h}^{*} \hat{C}_{i}$, by the construction, $\hat{h}^{*} \hat{C}_{1}^{\prime}, \ldots, \hat{h}^{*} \hat{C}_{s}^{\prime}$ are disjoint connected components of $\hat{\sigma}^{*}\left(\sum \hat{D}_{i}\right)$. Let $\hat{G}_{j}:=\hat{\sigma}_{*} \hat{h}^{*} \hat{C}_{j}^{\prime}$. Then $\operatorname{Supp}\left(\hat{G}_{j_{1}}\right) \cap \operatorname{Supp}\left(\hat{G}_{j_{2}}\right)=\emptyset$ if $j_{1} \neq j_{2}$, and $\hat{D}_{i}=a_{i 1} \hat{G}_{1}+\cdots+a_{i s} \hat{G}_{s}$.
(3.4) We can take two divisors among $\hat{D}_{i}$, say, $\hat{D}_{1}, \hat{D}_{2}$ such that $\left(\hat{D}_{1}\right)_{\eta} \neq\left(\hat{D}_{2}\right)_{\eta}$. And since $\left(\hat{D}_{1}\right)_{\eta} \sim\left(\hat{D}_{2}\right)_{\eta}$, we can find two connected components among $\hat{G}_{j}$, say, $\hat{G}_{1}, \hat{G}_{2}$, such that $\left(\hat{G}_{1}\right)_{\eta},\left(\hat{G}_{2}\right)_{\eta}>0$ and $a_{11}>a_{21} \geq 0$ and $a_{22}>a_{12} \geq 0$. And if we write that

$$
\hat{D}_{1}=a_{11} \hat{G}_{1}+a_{12} \hat{G}_{2}+\hat{G}_{1}^{\prime} \text { and } \hat{D}_{2}=a_{21} \hat{G}_{1}+a_{22} \hat{G}_{2}+\hat{G}_{2}^{\prime}
$$

then for $i, j \in\{1,2\}$ and $i \neq j, \hat{G}_{j}$ does not intersect $\hat{G}_{j}^{\prime \prime}:=\hat{G}_{i}+\hat{G}_{1}^{\prime}+\hat{G}_{2}^{\prime}$.
(3.5) Take two reduced, irreducible and dominant over $Y$ components $G_{1}, G_{2}$ of $\hat{G}_{1}, \hat{G}_{2}$ respectively. Since $K_{\hat{X}}+\hat{B}$ is nef and $\nu\left(\hat{X}, \hat{D}_{j}\right)=\nu\left(\hat{X}, K_{\hat{X}}+\hat{B}\right)=1$, applying Proposition 2.8 we obtain that for $j=1,2$,

$$
\nu\left(G_{j},\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{j}}\right)=\nu\left(G_{j},\left.\hat{D}_{j}\right|_{G_{j}}\right) \leq \nu\left(\hat{X}, \hat{D}_{j}\right)-1=0
$$

hence $\left.\left(K_{\hat{X}}+\hat{B}\right)\right|_{G_{j}} \sim_{\mathbb{Q}} 0$ (Lemma 4.10), so the Iitaka fiber $F_{j}=G_{j}$.
Finally, as the conditions of Lemma 4.10 are verified for the pair $(\hat{X}, \hat{B})$ and the divisors $\hat{D}_{1}, \hat{D}_{2}$, we conclude that $L_{1}$ and $L_{2}$ are torsion. The proof is completed.
4.4. Proof of Theorem 4.4. We can assume $(X, B)$ is relatively minimal over $Y$. Then by Theorem $2.5(3.3,3.4), K_{X}+B$ is nef and $f$ - $\mathbb{Q}$-trivial, so we can assume $l\left(K_{X}+B\right) \sim f^{*} L$ for some integer $l>0$ and a nef line bundle $L$ on $Y$. If $\operatorname{deg} L>0$ then $\kappa\left(X, K_{X}+B\right) \geq 1 \geq \kappa(Y)$. So we may assume $\operatorname{deg} L=0$. And we only need to prove that $g(Y)=1$ and $L$ is torsion in $\operatorname{Pic}^{0}(Y)$.

Lemma 4.11. Let $D$ be a pseudo-effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. If $D$ is $f$-nef, then $D$ is nef.

Proof. Assume the contrary. We can take an ample $\mathbb{Q}$-divisor $H$ on $X$ such that $D+H$ is not nef. Since $D$ is pseudo-effective, there exists an effective $\mathbb{Q}$-divisor $B^{\prime} \sim_{\mathbb{Q}} D+H$. Take a small rational number $t>0$ such that $\left(X, B+t B^{\prime}\right)$ is klt. Since $K_{X}+B+t B^{\prime}$ is $f$-nef, applying Theorem 2.5 shows that $t B^{\prime} \equiv K_{X}+B+t B^{\prime}$ is nef, which contradicts the assumption.

Let $H$ be an ample Cartier divisor on $X$ and $F$ a general fiber of $f$. We can find two positive integers $a, b$ such that $D^{3}=0$ where $D=a H-b F$.

We claim that $D$ is nef, thus $\nu(D)=2$. By Lemma 4.11 it suffices to show that $D$ is pseudo-effective. Otherwise, denote by $t_{0}$ the maximal real number such that $H-t_{0} F$ is pseudo-effective. Then $t_{0}<\frac{b}{a}$ and $\left(H-t_{0} F\right)^{3}>0$. Since for every rational number $t<t_{0}$ the divisor $H-t F$ is $f$-ample and pseudo-effective, so $H-t F$ is nef by Lemma 4.11. Taking the limit shows that $H-t_{0} F$ is nef, thus $H-t_{0} F$ is big
since $\left(H-t_{0} F\right)^{3}>0\left(\left[33\right.\right.$, Chap. V Lemma 2.1]) $3^{3}$ It follows that for a sufficiently small $\epsilon>0$ such that $t_{0}+\epsilon \in \mathbb{Q}$, the divisor $H-t_{0} F-\epsilon F$ is $\mathbb{Q}$-linearly equivalent to an effective divisor. However, this contradicts the definition of $t_{0}$.

Take a smooth resolution of singularities $\sigma: W \rightarrow X$ and let $h=f \circ \sigma: W \rightarrow$ $X \rightarrow Y$. Since $n \sigma^{*} D+K_{W / Y}-K_{W / Y}=n \sigma^{*} D$ is nef, $h$-big and $h$-semi-ample, applying Theorem 4.6 shows that for sufficiently divisible integers $n$ and $g$, the sheaf $F_{Y}^{g *} h_{*} \mathcal{O}_{W}\left(n \sigma^{*} D+K_{W / Y}\right)$ contains a nonzero nef subbundle $V$.

We exclude the case $g(Y)>1$ as follows. Otherwise, for sufficiently large $n$, the divisor $n \sigma^{*} D+K_{W}$ is big by Theorem 4.7. We can find an effective $\sigma$-exceptional divisor $E$ on $W$ such that $K_{W} \leq \sigma^{*} K_{X}+E$. Then $n \sigma^{*} D+\sigma^{*}\left(K_{X}+B\right)+E \geq$ $n \sigma^{*} D+K_{W}+\sigma^{*} B$ is big. Applying Theorem 2.3,

$$
\kappa\left(X, n D+\left(K_{X}+B\right)\right)=\kappa\left(W, n \sigma^{*} D+\sigma^{*}\left(K_{X}+B\right)+E\right)=3
$$

As $K_{X}+B$ is numerically trivial, we conclude that $D$ is big, but this contradicts that $\nu(D)=2$.

We may assume $Y$ is an elliptic curve. Next we will prove that there exists a semi-ample divisor $D^{\prime} \equiv t D$ for some rational number $t>0$. Since the subbundle $V \subseteq F_{Y}^{g *} h_{*} \mathcal{O}_{W}\left(n \sigma^{*} D+K_{W / Y}\right)$ is nef, for every $L^{\prime \prime} \in \operatorname{Pic}^{0}(Y)$, applying RiemannRoch formula gives

$$
h^{0}\left(Y, V \otimes L^{\prime \prime}\right)-h^{1}\left(Y, V \otimes L^{\prime \prime}\right)=\chi\left(Y, V \otimes L^{\prime \prime}\right)=\operatorname{deg} V \geq 0
$$

We claim that there exists some $L^{\prime \prime} \in \operatorname{Pic}^{0}(Y)$ such that $h^{0}\left(Y, V \otimes L^{\prime \prime}\right)>0$. Otherwise, for every $L^{\prime \prime} \in \operatorname{Pic}^{0}(Y)$ and $i=0,1, h^{i}\left(Y, V \otimes L^{\prime \prime}\right)=0$, thus $R^{i} \Phi_{\mathcal{P}} V=$ $0, i=0,1$, i.e., the Fourier-Mukai transform $R \Phi_{\mathcal{P}} V$ is acyclic ([16, p.53]). Therefore $V=0$ by Theorem 3.2, and the claim follows from this contradiction. Note that since $\operatorname{dim} Y=1$, by Proposition 2.1 the fibration $h: W \rightarrow Y$ is separable, hence $W_{Y^{g}}=W \times_{Y} Y^{g}$ is integral. Consider the following commutative diagram

where $W^{\prime}$ denotes the normalization of $W_{Y^{g}}$ and $\rho, \rho^{\prime}, h^{\prime}$ denote the natural morphisms. From the commutative diagram above, by [17, Chapter III, Prop. 9.3] we obtain

$$
\begin{aligned}
F_{Y}^{g *} h_{*} \mathcal{O}_{W}\left(n \sigma^{*} D+K_{W}\right) & \cong h_{g *}\left(\pi^{g *} \mathcal{O}_{W}\left(n \sigma^{*} D+K_{W}\right)\right) \cong h_{g *} \mathcal{O}_{W_{Y g}}\left(\pi^{g *}\left(n \sigma^{*} D+K_{W}\right)\right) \\
& \hookrightarrow h_{*}^{\prime} \mathcal{O}_{W^{\prime}}\left(\rho^{\prime *} \pi^{g *}\left(n \sigma^{*} D+K_{W}\right)\right)=h_{*}^{\prime} \mathcal{O}_{W^{\prime}}\left(\rho^{*}\left(n \sigma^{*} D+K_{W}\right)\right)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
h^{0}\left(\mathcal{O}_{W^{\prime}}\left(\rho^{*}\left(n \sigma^{*} D+K_{W}\right)\right) \otimes h^{*} L^{\prime \prime}\right) & =h^{0}\left(Y, h_{*}^{\prime} \mathcal{O}_{W^{\prime}}\left(\rho^{*}\left(n \sigma^{*} D+K_{W}\right)\right) \otimes L^{\prime \prime}\right) \\
& \geq h^{0}\left(Y, V \otimes L^{\prime \prime}\right)>0
\end{aligned}
$$

[^2]Take $L^{\prime} \in \operatorname{Pic}^{0}(Y)$ such that $F_{Y}^{g *} L^{\prime} \sim L^{\prime \prime}$. Then applying Theorem 2.3, we can show

$$
\begin{aligned}
\kappa\left(X, n D+\left(K_{X}+B\right)+f^{*} L^{\prime}\right) & \geq \kappa\left(W, n \sigma^{*} D+K_{W}+h^{*} L^{\prime}\right) \\
& =\kappa\left(W^{\prime}, \rho^{*}\left(n \sigma^{*} D+K_{W}\right)+h^{\prime *} L^{\prime \prime}\right) \geq 0
\end{aligned}
$$

There exists an effective divisor $D_{1} \sim_{\mathbb{Q}} n D+\left(K_{X}+B\right)+f^{*} L^{\prime}$. The pair $\left(X, B+t D_{1}\right)$ is klt for a sufficiently small rational number $t$, so the divisor $K_{X}+B+t D_{1}$ is semiample by Theorem 4.2. Since $K_{X}+B \equiv 0$, we can set $D^{\prime}=K_{X}+B+t D_{1}$.

Since $D^{\prime}$ is $f$-ample and $\nu\left(D^{\prime}\right)=2$, the associated morphism to $D^{\prime}$ is a fibration $g: X \rightarrow Z$ to a surface. Let $X_{\xi}$ be the generic fiber of $g$. Then $X_{\xi}$ is a normal curve defined over $k(\xi)=K(Z)$ and dominant over $Y \otimes_{k} k(\xi)$. It follows that $p_{a}\left(X_{\xi}\right) \geq 1$, thus $K_{X_{\xi}}$ is semi-ample. Since $\left.l\left(K_{X}+B\right)\right|_{X_{\xi}} \sim_{\mathbb{Q}} l\left(K_{X_{\xi}}+\left.B\right|_{X_{\xi}}\right)$ is numerically trivial, we conclude that $\left.K_{X_{\xi}} \sim_{\mathbb{Q}} K_{X}\right|_{X_{\xi}} \sim_{\mathbb{Q}} 0$ and $\left.B\right|_{X_{\xi}}=0$, thus $\left.\left.l\left(K_{X}+B\right)\right|_{X_{\xi}} \sim_{\mathbb{Q}} f^{*} L\right|_{X_{\xi}} \sim_{\mathbb{Q}} 0$. Though the geometric generic fiber $X_{\bar{\xi}}$ is not necessarily reduced, we can apply Lemma 2.4 to the morphism $\left(X_{\bar{\xi}}\right)_{\text {red }} \rightarrow Y \otimes_{k} k(\bar{\xi})$ and show that $L$ is torsion.
4.5. Proof of Theorem 4.1. In the following we assume that $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}+B_{\bar{\eta}}\right) \geq 0$.

Case (i) $\operatorname{dim} Y=1$. If $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}+B_{\bar{\eta}}\right)=0$ then we can use Theorem 4.4. If $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\pi}}}+B_{\bar{\eta}}\right)=1$ then we can use Theorem 4.3. If $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}+B_{\bar{\eta}}\right)=2$, then we can use Theorem 4.2 when $g(Y)=1$ and use Corollary 4.7 when $g(Y)>1$.

Case (ii) $\operatorname{dim} Y=2$. If $\kappa\left(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}+B_{\bar{\eta}}\right)=0$, then by the assumption $\boldsymbol{\phi}, f$ is an elliptic fibration (Proposition [2.11), applying [8, Theorem 1.2] shows that

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa(X) \geq \kappa(Y)
$$

So we may assume $K_{X}+B$ is $f$-big. We fall into three cases according to $\kappa(Y)$.
When $\kappa(Y)=2$, we can use Corollary 4.7.
When $\kappa(Y)=1$, we need to prove that $\kappa\left(X, K_{X}+B\right) \geq 2$. Denote by $g: Y \rightarrow C$ the Iitaka fibration of $Y$ and by $\operatorname{alb}_{Y}: Y \rightarrow \operatorname{Alb}(Y)$ the Albanese map which is generically finite. Take a general fiber $G$ of $g$, which is an elliptic curve. Let $E=\operatorname{alb}_{Y}(G)$. We may assume $\operatorname{Alb}(Y)$ is an abelian variety and $E$ passes through the origin. Let $A(E)$ be the sub-abelian variety generated by $E$. It follows that $1 \leq$ $\operatorname{dim} A(E) \leq g(E) \leq g(G)=1$, hence the equalities are attained, and $E=A(E)$ is a sub-abelian variety. Since the composition morphism $\phi: Y \rightarrow \operatorname{Alb}(Y) \rightarrow \operatorname{Alb}(Y) / E$ contracts the fiber $G$, applying Rigidity Lemma ([32, Proposition 6.1]), every fiber of $g$ is contracted by $\phi$, and $\phi: Y \rightarrow \operatorname{Alb}(Y) / E$ factors through $g: Y \rightarrow C$. By the universal property of the Albanese map we have the following commutative diagram


Wee see that the morphism $\operatorname{Alb}(C) \rightarrow \operatorname{Alb}(Y) / E$ is in fact an isomorphism, hence $h: \operatorname{Alb}(Y) \rightarrow \operatorname{Alb}(C)$ is of relative dimension one, in particular $g(C) \geq 1$. Next let $Y \xrightarrow{\sigma} \bar{Y} \rightarrow \mathrm{Alb}(Y)$ be the Stein factorization. Then $\sigma: Y \rightarrow \bar{Y}$ is a birational morphism, and the $\sigma$-exceptional locus does not intersect the generic fiber of $g$ :
$Y \rightarrow C$. Therefore, the natural morphism $\bar{Y} \rightarrow \operatorname{Alb}(C)$ factors through a morphism $\bar{g}: \bar{Y} \rightarrow C$, which is induced by the Stein factorization. Let $(\bar{X}, \bar{B})$ be a $\log$ minimal model of $(X, B)$ over $\bar{Y}$. By Theorem 2.5 (3.4), $(\bar{X}, \bar{B})$ is in fact minimal. Then we have the following commutative diagram

where $\bar{f}^{\prime}: \bar{X} \rightarrow C^{\prime}$ is the fibration induced by the Stein factorization of $\bar{X} \rightarrow C$. Note that $g\left(C^{\prime}\right) \geq g(C) \geq 1$. Denote by $\zeta^{\prime}$ the generic point of $C^{\prime}$. Consider the fibration $\bar{f}^{\prime}: \bar{X} \rightarrow C^{\prime}$. By Theorem $2.5(3.2),\left(K_{\bar{X}}+\bar{B}\right)_{\zeta^{\prime}}$ is semi-ample. And since $K_{\bar{X}}+\bar{B}$ is big over $\operatorname{Alb}(Y)$, it is not numerically trivial over the generic point of $C^{\prime}$, which implies that $\kappa\left(\bar{X}_{\zeta^{\prime}},\left(K_{\bar{X}}+\bar{B}\right)_{\zeta^{\prime}}\right) \geq 1$.
Claim 4.12. If $\kappa\left(\bar{X}_{\zeta^{\prime}},\left(K_{\bar{X}}+\bar{B}\right)_{\zeta^{\prime}}\right)=1$ then $\bar{f}^{\prime}$ satisfies the assumption
Proof. Denote by $\varphi: \bar{X} \rightarrow Z$ the relative Iitaka fibration induced by $K_{\bar{X}}+\bar{B}$ on $\bar{X}$ over $C^{\prime}$ and by $\bar{X}_{\xi}$ the generic fiber of $\varphi$. Then $\bar{X}_{\xi}$ is a normal curve over $K(Z)$. As $K_{\bar{X}}+\bar{B}$ is big over $\operatorname{Alb}(Y)$ while $\left.\left(K_{\bar{X}}+\bar{B}\right)\right|_{\bar{X}_{\xi}} \sim_{\mathbb{Q}} 0$, we see that $\bar{X}_{\xi}$ is not contracted by $\bar{X} \rightarrow \operatorname{Alb}(Y)$, hence is generically finite over $\operatorname{Alb}(Y) \otimes_{k} K(Z)$. Therefore, $p_{a}\left(\bar{X}_{\xi}\right) \geq 1$, and $K_{\bar{X}_{\xi}}$ is semi-ample. Since $\left.\left(K_{\bar{X}}+\bar{B}\right)\right|_{\bar{X}_{\xi}} \sim_{\mathbb{Q}} K_{\bar{X}_{\xi}}+\left.\bar{B}\right|_{\bar{X}_{\xi}}$ is numerically trivial, we conclude that $\left.\bar{B}\right|_{\bar{X}_{\xi}}=0$, thus $\bar{f}^{\prime}$ satisfies the assumption

So we can apply the results of Case (i) to the fibration $\overline{f^{\prime}}: \bar{X} \rightarrow C^{\prime}$ and show that

$$
\kappa\left(\bar{X}, K_{\bar{X}}+\bar{B}\right) \geq \kappa\left(\bar{X}_{\zeta^{\prime}},\left(K_{\bar{X}}+\bar{B}\right)_{\zeta^{\prime}}\right)+\kappa\left(C^{\prime}\right) \geq 1
$$

Hence $K_{\bar{X}}+\bar{B}$ is semi-ample by Theorem 4.9. And by Corollary 4.7, we have $\nu\left(\bar{X}, K_{\bar{X}}+\bar{B}\right) \geq 2$, hence $\kappa\left(\bar{X}, K_{\bar{X}}+\bar{B}\right) \geq 2$.

When $\kappa(Y)=0$, we need to prove that $\kappa\left(X, K_{X}+B\right) \geq 1$. In this case $Y$ is birationally equivalent to an abelian surface ([1, Sec. 10]). We may assume $Y$ is an abelian surface and assume $(X, B)$ is minimal by working on a log minimal model over $Y$. If $Y$ is simple then we can use Theorem 4.2, Otherwise, $Y$ admits a fibration to an elliptic curve, then we get a fibration $f^{\prime}: X \rightarrow C^{\prime}$ with $g\left(C^{\prime}\right)=1$. Let $\zeta^{\prime}$ denote the generic point of $C^{\prime}$. Again since $K_{X}+B$ is big over $Y$, we have $\kappa\left(X_{\zeta^{\prime}}, K_{X_{\zeta^{\prime}}}+B_{\zeta^{\prime}}\right) \geq 1$, and if $\kappa\left(X_{\zeta^{\prime}}, K_{X_{\zeta^{\prime}}}+B_{\zeta^{\prime}}\right)=1$ we can verify that $f^{\prime}$ satisfies the assumption $\boldsymbol{\uparrow}$ by the same proof of Claim 4.12. Applying the results of Case (i) to the fibration $f^{\prime}: X \rightarrow C^{\prime}$, we complete the proof by

$$
\kappa\left(X, K_{X}+B\right) \geq \kappa\left(X_{\zeta^{\prime}}, K_{X_{\zeta^{\prime}}}+B_{\zeta^{\prime}}\right)+\kappa\left(C^{\prime}\right) \geq 1
$$

## 5. Abundance

In this section, we will prove Theorem 1.2. By Theorem 4.9, we only need to show that either $\kappa\left(X, K_{X}+B\right) \geq 1$ or $K_{X}+B \sim_{\mathbb{Q}} 0$. Hence we may assume $\kappa\left(X, K_{X}+B\right) \leq 0$. If $X$ is of maximal Albanese dimension, then we are done by [48, Theorem 1.1] or [20, Theorem 0.3]. Let $f: X \rightarrow Y$ be the fibration arising from the Stein factorization of $a_{X}$. Then $\kappa\left(X_{\eta},\left(K_{X}+B\right)_{\eta}\right) \geq 0$ by Theorem 2.5 (3.2).

Therefore, by Theorem 4.1 it is only possible that $\kappa(Y)=\kappa\left(X_{\eta},\left(K_{X}+B\right)_{\eta}\right)=0$. If $\operatorname{dim} Y=1$ then we can use Theorem 4.4. If $\operatorname{dim} Y=2$ then $B=0$ by the assumption (1), and $f$ is an elliptic fibration by Proposition 2.11, so $X$ is non-uniruled, and this case has been treated in [48, Theorem 1.1].

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[^0]:    ${ }^{1}$ Existence of smooth resolution of singularities has been proved in dimension $\leq 3$ by 6 .

[^1]:    ${ }^{2}$ The divisor $D$ can be obtained as follows. Take an effective Weil divisor $D_{1} \sim K_{X}+A$ for some ample divisor $A$ on $X$. For sufficiently large $v$ the divisor $K_{X}+B-\frac{1}{p^{v}-1}\left(D_{1}+B\right)$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and hence is $\mathbb{Q}$-linearly equivalent to an effective divisor $D_{2}$, which can be assumed to have Cartier index not divisible by $p$. Let $D=D_{2}+\frac{1}{p^{v}-1}\left(D_{1}+B\right)$. Identifying $K_{X}$ with $D_{1}-A$ similarly as in [37, Lemma 3.15], one can verify that $p$ does not divide the Cartier index of

    $$
    K_{X}+B+\frac{1}{p^{v}+1} D=-A+\frac{1}{p^{v}+1} D_{2}+\frac{p^{2 v}}{p^{2 v}-1}\left(D_{1}+B\right)
    $$

[^2]:    ${ }^{3}$ this is well known for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors ([24, Prop. 2.61]), but the proof also applies for $\mathbb{R}$-Cartier $\mathbb{R}$-divisors as mentioned by [33, Chap. V Lemma 2.1]

