# ABUNDANCE FOR 3-FOLDS WITH NON-TRIVIAL ALBANESE MAPS IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper, we prove abundance for 3-folds with non-trivial Albanese maps, over an algebraically closed field of characteristic p > 5.

## 1. INTRODUCTION

Over an algebraically closed field of characteristic p > 5, existence of log minimal models of 3-folds has been proved by Birkar, Hacon and Xu ([2] [21]); and log abundance has been proved for minimal klt pairs (X, B) when  $K_X + B$  is big or Bis big ([2] [7] [45]), and when X is non-uniruled and has non-trivial Albanese map ([48]).

This paper aims to prove abundance for 3-folds with non-trivial Albanese maps.

**Theorem 1.1.** Let X be a klt,  $\mathbb{Q}$ -factorial, projective minimal 3-fold, over an algebraically closed field k with char k = p > 5. Assume that the Albanese map  $a_X : X \to A_X$  is non-trivial. Then  $K_X$  is semi-ample.

Precisely we prove log abundance in some cases.

**Theorem 1.2.** Let (X, B) be a klt,  $\mathbb{Q}$ -factorial, projective minimal pair of dimension 3, over an algebraically closed field k with char k = p > 5. Assume that the Albanese map  $a_X : X \to A_X$  is non-trivial. Denote by  $f : X \to Y$  the fibration arising from the Stein factorization of  $a_X$  and by  $X_\eta$  the generic fiber of f. Assume moreover that B = 0 if

(1) dim Y = 2 and  $\kappa(X_{\eta}, (K_X + B)|_{X_{\eta}}) = 0$ , or

(2) dim Y = 1 and  $\kappa(X_{\eta}, (K_X + B)|_{X_{\eta}}) = 1$ .

Then  $K_X + B$  is semi-ample.

Throughout of this paper, as  $f : X \to Y$  frequently appears as a projective morphism of varieties, we denote by  $\eta$  (resp.  $\bar{\eta}$ ) the generic (resp. geometric generic) point of Y and by  $X_{\eta}$  (resp.  $X_{\bar{\eta}}$ ) the generic (resp. geometric generic) fiber of f. Moreover we say  $f : X \to Y$  is a fibration if  $f_* \mathcal{O}_X = \mathcal{O}_Y$ .

To study abundance for varieties with non-trivial Albanese maps, it is necessary to study the following conjecture on subadditivity of Kodaira dimensions.

**Conjecture 1.3** (litaka conjecture). Let  $f : X \to Y$  be a fibration between two smooth projective varieties over an algebraically closed field k, with dim X = n and dim Y = m. Then

$$C_{n,m}:\kappa(X) \ge \kappa(Y) + \kappa(X_{\eta}, K_{X_{\eta}}).$$

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The divisor  $K_{X_{\eta}} = K_X|_{X_{\eta}}$  is a Cartier divisor corresponding to the dualizing sheaf of  $X_{\eta}$ , which is invertible since  $X_{\eta}$  is regular. In characteristic zero, since the geometric generic fiber  $X_{\bar{\eta}}$  is smooth over  $k(\bar{\eta})$ , so  $\kappa(X_{\bar{\eta}}) = \kappa(X_{\eta}, K_{X_{\eta}})$ . In positive characteristic,  $X_{\bar{\eta}}$  is not necessarily smooth over  $k(\bar{\eta})$ , if  $X_{\bar{\eta}}$  has a smooth projective birational model  $\tilde{X}_{\bar{\eta}}$ ,<sup>1</sup> then by [8, Corollary 2.5]

$$\kappa(X_{\eta}, K_{X_{\eta}}) = \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}) \ge \kappa(\tilde{X}_{\bar{\eta}}).$$

We refer to [46] for more discussions on this conjecture. In this paper we prove the following theorem, which implies Theorem 1.2 after combined with the results of [48].

**Theorem 1.4** (= Theorem 4.1). Let  $f : X \to Y$  be a fibration from a Q-factorial projective 3-fold to a smooth projective variety of dimension 1 or 2, over an algebraically closed field k with char k = p > 5. Let B be an effective Q-divisor on X such that (X, B) is klt. Assume that Y is of maximal Albanese dimension, and assume moreover that

• if  $\kappa(X_{\eta}, K_{X_{\eta}} + B_{\eta}) = \dim X/Y - 1$ , then B does not intersect the generic fiber  $X_{\xi}$  of the relative Iitaka fibration  $I : X \dashrightarrow Z$  induced by  $K_X + B$  on X over Y.

Then

$$\kappa(X, K_X + B) \ge \kappa(X_\eta, K_{X_\eta} + B_\eta) + \kappa(Y).$$

Remark 1.5. As char k = p > 5, for a fibration  $h: X \to Z$ , if the generic fiber is a curve with arithmetic genus one, then the geometric generic fiber must be a smooth elliptic curve (Proposition 2.11). So in case  $\kappa(X_{\eta}, K_{X_{\eta}} + B_{\eta}) = \dim X/Y - 1$ , the assumption  $\blacklozenge$  guarantees that the relative Iitaka fibration  $I: X \dashrightarrow Z$  is fibred by elliptic curves. The advantage of an elliptic fibration is the canonical bundle formula: if  $h: W \to Z$  is a flat relative minimal elliptic fibration then there exists an effective divisor  $B_Z$  on Z such that  $K_W \sim_{\mathbb{Q}} h^*(K_Z + B_Z)$  ([8, 3.2]). Canonical bundle formula is the key technique to study log abundance ([25]). Here we mention that in positive characteristic, only under some very strong conditions,  $B_Z$  has been proved to be effective ([7, Lemma 6.7], [11, Theorem 3.18], [10, Theorem B]).

The result above has been proved when the geometric generic fiber  $X_{\bar{\eta}}$  is smooth and B = 0 ([8, 12]), which was used in [48] to prove log abundance for non-uniruled 3-folds with non-trivial Albanese maps. To study the uniruled case, we have to treat fibrations with singular geometric generic fibers. For a separable fibration with possibly singular geometric generic fiber,  $C_{3,m}$  has been proved by the author [46] when  $K_X + B$  is f-big,  $K_Y$  is big and Y is of maximal Albanese dimension. To prove Theorem 1.4, the essentially new results are the following two theorems.

**Theorem 1.6** (= Theorem 4.2). Let (X, B) be a projective Q-factorial klt pair of dimension 3, over an algebraically closed field k with char k = p > 5. Let  $f : X \to Y = A$  be a fibration to an elliptic curve or a simple abelian surface. Assume that  $K_X + B$  is f-big. Then

$$\kappa(K_X + B) \ge \kappa(X_\eta, K_{X_\eta} + B_\eta).$$

<sup>&</sup>lt;sup>1</sup>Existence of smooth resolution of singularities has been proved in dimension  $\leq 3$  by [6].

**Theorem 1.7** (= Theorem 4.4). Let (X, B) be a Q-factorial klt pair of dimension 3, over an algebraically closed field k with char k = p > 5. Let  $f : X \to Y$  be a fibration to a normal curve Y of genus  $g(Y) \ge 1$ . Assume  $\kappa(X_{\eta}, (K_X + B)|_{X_{\eta}}) = 0$ . Then

$$\kappa(X, K_X + B) \ge \kappa(Y).$$

If moreover  $K_X + B$  is nef then it is semi-ample.

We summarize the techniques to study  $C_{3,m}$  which appeared in the papers [37, 11, 46, 12], then explain the ideas of the proof of Theorem 1.6 and 1.7.

(1) Positivity results. Let  $f: X \to Y$  be a separable fibration of normal projective varieties. In positive characteristic, Parakfalvi [37] first proved that for sufficiently divisible n, the sheaf  $f_*\mathcal{O}_X(n(K_{X/Y}+B))$  is weakly positive under the condition that  $(X_{\overline{\eta}}, B_{\overline{\eta}})$  is strongly F-regular and  $K_{X/Y} + B$  is f-ample, then Ejiri [11] reproved it and made some generalizations using different method. This result may fail when  $X_{\overline{\eta}}$  is singular. But in dimension three and over an algebraically closed field kwith char k = p > 5, we can take advantage of minimal model theory. Under the condition that  $K_X + B$  is nef, relatively big and semi-ample over Y, the author [46] (or [38] under stronger conditions) proved that for sufficiently divisible n, g > 0, the sheaf  $F_Y^{g*}(f_*\mathcal{O}_X(n(K_{X/Y}+B)) \otimes \omega_Y^{n-1})$  contains a non-zero weakly positive sub-sheaf  $V_n$ , which plays a similar role as  $f_*\mathcal{O}_X(n(K_{X/Y}+B))$  does in studying subadditivity of Kodaira dimensions.

If  $f: X \to Y$  is a fibration from a 3-fold to a smooth curve, when  $\kappa(X_{\overline{\eta}}) = 1$ the relative Iitaka fibration of X over Y is fibred by elliptic curves since p > 5(Proposition 2.11). By canonical bundle formula ([8, 3.2]) and minimal model theory, one can reduce to a pair  $(Z, B_Z)$  of dimension two with  $K_Z + B_Z$  being relatively big over Y. Then one can show that  $f_* \omega_{X/Y}^n$  contains a non-zero nef sub-sheaf  $V_n$ ([12, Theorem 2.8]).

Using these positivity results above, one can usually treat the case when  $K_Y$  is big or the case when det  $V_n$  is big. Note that this approach only requires  $K_X + B$  to be nef (not necessarily klt). This paper also treats inseparable fibrations, which can be reduced to a separable fibration of a pair (X', B') not necessarily klt by applying foliation theory (Proposition 2.10).

To treat the case when Y is an elliptic curve and deg  $V_n = 0$ , we have the following two approaches.

(2.1) Trace maps of relative Frobenius. Ejiri [11, Theorem 1.7] introduced a clever trick as follows. First there exists an isogeny  $\pi : Y' \to Y$  such that  $\pi^* V_n = \bigoplus L_i$  where  $L_i \in \operatorname{Pic}^0(Y')$  by [27, 1.4. Satz] and [34, Corollary 2.10]. Then by applying trace maps of relative Frobenius, one gets many relations of  $L_i$ , from which one can prove that every  $L_i$  is torsion in  $\operatorname{Pic}^0(Y')$ . This indicates that for sufficiently divisible N, the line bundle  $\mathcal{O}(NK_X)$  has many global sections.

This method heavily relies on the smoothness of the geometric generic fiber  $X_{\overline{\eta}}$ , it was applied to prove  $C_{3,1}$  when  $X_{\overline{\eta}}$  is smooth and either  $K_X$  is f-big ([11, Thm. 1.7]) or is f-Q-trivial ([12, Sec. 3]).

(2.2) Adjunction formula. Granted the isogeny  $\pi : Y' \to Y$  and the splitting  $\pi^* V_n = \bigoplus L_i$ , first by applying covering theorem (Theorem 2.3) one can construct effective divisors  $D_i \sim NK_X + P_i$  for some  $P_i \in f^* \operatorname{Pic}^0(Y)$  and an integer N. Then

applying adjunction formula to different components of  $D_i$  and log abundance for surfaces ([41]), one can get some relations of  $P_i$  which conclude that  $P_i$  are torsion.

This approach was used in [12, Sec. 4] and [47]. In fact it applies once granted positivity results as in (1) without requiring  $X_{\overline{\eta}}$  to be smooth.

- Now we explain the ideas of the proof of Theorem 1.6 and 1.7.
- (3) To prove Theorem 1.6, we consider the cohomological locus

$$V^{0}(f_{*}\mathcal{O}_{X}(n(K_{X}+B))) = \{L \in \operatorname{Pic}^{0}(A) | h^{0}(A, f_{*}\mathcal{O}_{X}(n(K_{X}+B)) \otimes L) > 0\}.$$

If dim $(V^0(f_*\mathcal{O}_X(n(K_X + B)))) > 0$ , then  $V^0(f_*\mathcal{O}_X(n(K_X + B)))$  generates  $\operatorname{Pic}^0(A)$ since  $\operatorname{Pic}^0(A)$  is simple, and it is easy to show that  $\kappa(X, K_X + B) \ge \dim A$ . For the remaining case dim  $V^0(f_*\mathcal{O}_X(n(K_X + B))) \le 0$ , we follow the approach in (2.2). The key point is to find at least two effective divisors  $D_i \sim n(K_X + B) + P_i$  for some  $P_i \in f^*\operatorname{Pic}^0(A)$ . We try to find a subsheaf  $\mathcal{F}$  of  $f_*\mathcal{O}_X(n(K_X + B))$  and an isogeny  $\pi : A' \to A$  such that  $\pi^*\mathcal{F} = \bigoplus L_i$  where  $L_i \in \operatorname{Pic}^0(A')$ . The sheaf  $\mathcal{F}$  is obtained as the image of the trace map

$$Tr_{X,B}^{e,n}: f_*(F_{X*}^e\mathcal{O}_X((1-p^e)(K_X+B)) \otimes \mathcal{O}_X(n(K_X+B))) \to f_*\mathcal{O}_X(n(K_X+B)).$$

We apply Frobenius amplitude to show that  $\mathcal{F}$  satisfies a property slightly weaker than generic vanishing (Theorem 3.7), then apply "killing cohomology" ([40, Prop. 12 and Sec. 9] or [20, Lemma 1.3]) to get an isogeny  $\pi : A' \to A$ , some  $P_i \in \operatorname{Pic}^0(A')$ and a generically surjective homomorphism  $\bigoplus_i P_i \to \pi^* \mathcal{F}$ .

(4) To prove Theorem 1.7, we can assume  $K_X + B$  to be f-nef by replacing (X, B) with a relative minimal model, then for a sufficiently divisible integer k > 0,  $k(K_X + B) = f^*L$  for some nef  $L \in \operatorname{Pic}(Y)$  ([43]). It suffices to show that either deg L > 0 or L is torsion. For the case deg L = 0, we use the strategy of [1, Theorem 8.10]. First we construct a nef divisor D = aH - bF on X with  $\nu(D) = 2$  where H is an ample divisor and F is a general fiber of f, second we prove that there exists a semi-ample divisor  $D' \equiv D$ , which induces a fibration  $g: X \to Z$  to a surface, finally we can show the restriction on the generic fiber  $k(K_X + B)|_G = k(K_G + B|_G) \sim 0$ , which implies that L is torsion.

*Remark* 1.8. The crucial results to be used include minimal model theory of 3-folds [2, 4, 21, 45], abundance for surfaces [41, 43] and results on positivity and subadditivity of Kodaira dimensions in [46]. We try to make the proof self-contained. But for some cases we only sketch the proof and refer to [12] and [48] for details.

This paper is organized as follows. In Section 2, we collect some useful results and study inseparable fibrations as preparations. Section 3 is devoted to studying sheaves on abelian varieties. In Section 4, we study subadditivity of Kodaira dimensions. Finally in Section 5 we finish the proof of abundance in Theorem 1.2.

**Conventions:** Sometimes we do not distinguish between line bundles and Cartier divisors, and abuse the notation additivity and tensor product if no confusion occurs.

Let X be a normal projective variety. Denote by  $\operatorname{Wdiv}(X)$  the set of Weil divisors and by  $\operatorname{Cdiv}(X)$  the set of Cartier divisors on X. Assume  $\mathbb{K} = \mathbb{Q}$  or  $\mathbb{R}$ . The divisors in  $\operatorname{Wdiv}(X) \otimes_{\mathbb{Z}} \mathbb{K}$  are called  $\mathbb{K}$ -divisors, and the ones in  $\operatorname{Cdiv}(X) \otimes_{\mathbb{Z}} \mathbb{K}$  are called  $\mathbb{K}$ -Cartier  $\mathbb{K}$ -divisors. We use  $\equiv$  for the numerical equivalence relation in  $\operatorname{Cdiv}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ . Let D be a  $\mathbb{Q}$ -divisor on a normal variety X. The Weil index of D is the minimal positive integer l such that lD is integral. If D is  $\mathbb{Q}$ -Cartier, the Cartier index is defined similarly. We use  $\sim$  (resp.  $\sim_{\mathbb{Q}}$ ) for linear (resp.  $\mathbb{Q}$ -linear) equivalence between Cartier (resp.  $\mathbb{Q}$ -Cartier) divisors and line bundles. For two ( $\mathbb{Q}$ -)divisors  $D_1, D_2$  on X, if  $D_1|_{X^{sm}} \sim D_2|_{X^{sm}}$  (resp.  $D_1|_{X^{sm}} \sim_{\mathbb{Q}} D_2|_{X^{sm}}$ ), we also denote  $D_1 \sim D_2$  (resp.  $D_1 \sim_{\mathbb{Q}} D_2$ ).

Let X be a normal variety and D a Weil divisor on X. Then  $\mathcal{O}_X(D)$  is a subsheaf of the constant sheaf K(X) of rational functions, with the stalk at a point x being defined by

$$\mathcal{O}_X(D)_x := \{ f \in K(X) | ((f) + D)|_U \ge 0 \text{ for some open set } U \text{ containing } x \}.$$

For notions in minimal model theory such as lc, klt dlt pairs, flip and divisorial contraction and so on, we refer to [2].

For a variety X, we usually use  $F_X^e: X \to X$  to denote the  $e^{\text{th}}$  iteration of absolute Frobenius, we sometimes use  $X^e$  for the origin scheme of  $F_X^e$  to avoid confusions.

Let  $\varphi : X \to T$  be a morphism of schemes and let T' be a T-scheme. Then we denote by  $X_{T'}$  the fiber product  $X \times_T T'$ . For a divisor D on X (resp. an  $\mathcal{O}_X$ -module  $\mathcal{G}$ ), the pullback of D (resp.  $\mathcal{G}$ ) to  $X_{T'}$  is denoted by  $D_{T'}$  or  $D|_{X_{T'}}$  (resp.  $\mathcal{G}_{T'}$  or  $\mathcal{G}|_{X_{T'}}$ ) if it is well-defined.

If X is a projective variety in dimension  $\leq 3$ , it always has a smooth birational model  $\tilde{X}$ , then  $\kappa(X) := \kappa(\tilde{X})$  which is birational invariant.

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## 2. Preliminaries

In this section we collect some technical results which will be used in the paper.

## 2.1. Separability of fibrations.

**Proposition 2.1.** Let  $f : X \to Y$  be a fibration of normal varieties over an algebraically closed field k of characteristic p > 0. Then

(1) f is separable if and only if  $X_{\bar{\eta}}$  is reduced, and if and only if  $X_{\bar{\eta}}$  is integral;

(2) if dim Y = 1 then f is separable.

*Proof.* Since f is a fibration we have  $H^0(\mathcal{O}_{X_\eta}) = K(Y)$ , and since  $X_\eta$  is normal we can show K(Y) is algebraically closed in K(X). Then the assertion (1) follows from applying [28, Sec. 3.2.2 Cor. 2.14 and Prop. 2.15]. The assertion (2) is [1, Lemma 7.2].

2.2. Relative Fujita Vanishing. The following result is [26, Theorem 1.5].

**Lemma 2.2.** Let  $f : X \to Y$  be a projective morphism over a Noetherian scheme, H an f-ample line bundle and  $\mathcal{F}$  a coherent sheaf on X. Then there exists a positive integer N such that, for every n > N and every nef line bundle L,

$$R^i f_*(\mathcal{F} \otimes H^n \otimes L) = 0, \text{ if } i > 0.$$

2.3. Covering Theorem. The result below is [[22], Theorem 10.5] when X and Y are both smooth, and the proof also applies when varieties are normal.

**Theorem 2.3.** ([22, Theorem 10.5]) Let  $f: X \to Y$  be a proper surjective morphism between complete normal varieties. If D is a Cartier divisor on Y and E an effective f-exceptional divisor on X, then

$$\kappa(X, f^*D + E) = \kappa(Y, D).$$

As a corollary we get the following useful result, which also appeared in [12].

**Lemma 2.4.** ([12, Lemma 2.3]) Let  $g: W \to Y$  be a surjective projective morphism between projective varieties. Assume Y is normal and let  $L_1, L_2 \in \text{Pic}(Y)$  be two line bundles on Y. If  $g^*L_1 \sim_{\mathbb{Q}} g^*L_2$  then  $L_1 \sim_{\mathbb{Q}} L_2$ .

*Proof.* Let  $L = L_1 \otimes L_2^{-1}$ . Denote by  $\sigma : W' \to W$  the normalization and let  $g' = g \circ \sigma : W' \to Y$ . Then  $g'^*L \sim_{\mathbb{Q}} 0$ . Applying Theorem 2.3 to  $g' : W' \to Y$  shows that  $L \sim_{\mathbb{Q}} 0$ , which is equivalent to that  $L_1 \sim_{\mathbb{Q}} L_2$ .

2.4. Minimal model theory of 3-folds. The following theorem includes some recent results of minimal model theory for 3-folds in positive characteristic.

**Theorem 2.5.** Assume char k = p > 5. Let (X, B) be a  $\mathbb{Q}$ -factorial projective pair of dimension three and  $f: X \to Y$  a projective surjective morphism.

(1) If either (X, B) is klt and  $K_X + B$  is pseudo-effective over Y, or (X, B) is lc and  $K_X + B$  has a weak Zariski decomposition over Y, then (X, B) has a log minimal model over Y.

(2) If (X, B) is dlt and  $K_X + B$  is not pseudo-effective over Y, then (X, B) has a Mori fibre space over Y.

(3) Assume that (X, B) is klt in (3.1) and is dlt in other cases, and that  $K_X + B$  is nef over Y.

- (3.1) If  $K_X + B$  or B is big over Y, then  $K_X + B$  is semi-ample over Y.
- (3.2) If dim  $Y \ge 1$ , then  $(K_X + B)_\eta$  is semi-ample on  $X_\eta$ .
- (3.3) If Y is a smooth curve,  $X_{\eta}$  is integral and  $\kappa(X_{\eta}, (K_X + B)_{\eta}) = 0$  or 2, then  $K_X + B$  is semi-ample over Y.
- (3.4) If Y contains no rational curves, then  $K_X + B$  is nef.

(4) Assume (X, B) is klt. If Y is a non-uniruled surface and  $K_X + B$  is pseudoeffective over Y, then  $K_X + B$  is pseudo-effective, and there exists a map  $\sigma : X \dashrightarrow \bar{X}$ to a minimal model  $(\bar{X}, \bar{B})$  of (X, B) such that, the restriction map  $\sigma|_{X_{\eta}}$  is an isomorphism from  $X_{\eta}$  to its image.

*Proof.* Assertion (1) is from [2, Theorem 1.2 and Proposition 7.3]; (2) is [4, Theorem 1.7]; (3.1) is [2, Theorem 1.4]; (3.2) is from [43, Theorem 1.1]; and (3.3) is from [3, Theorem 1.5 and 1.6 and the remark below 1.6].

Assertion (3.4) follows from the cone theorem [4, Theorem 1.1]. Indeed, otherwise we can find an extremal ray R generated by a rational curve  $\Gamma$ , so  $\Gamma$  is contained in a fiber of f since Y contains no rational curves, this contradicts that  $K_X + B$  is f-nef.

For (4),  $K_X + B$  is obviously pseudo-effective because otherwise, X will be ruled by horizontal (w.r.t. f) rational curves by (2), which contradicts that Y is nonuniruled. The exceptional locus of a flip contraction is of dimension one, so it does not intersect  $X_{\eta}$ , neither does that of an extremal divisorial contraction because it is uniruled (see the proof of [4, Lemma 3.2]). Running an LMMP for  $K_X + B$ , by induction we get a map  $\sigma : X \dashrightarrow \overline{X}$  as required.  $\Box$ 

# 2.5. Numerical dimension.

**Definition 2.6.** Let D be an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor on a smooth projective variety X of dimension n. The numerical dimension  $\kappa_{\sigma}(D)$  is defined as the biggest natural number k such that

$$\liminf_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0 \text{ for some ample divisor } A \text{ on } X.$$

If such a k does not exist then we define  $\kappa_{\sigma}(D) = -\infty$ .

Remark 2.7. In arbitrary characteristics, since smooth alterations exist due to de Jong [23], this invariant can be defined for  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisors on normal varieties by pulling back to a smooth variety, which does not depend on the choices of smooth alterations by [5, 2.5-2.7].

**Proposition 2.8.** Let X be a normal projective variety and D an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor on X.

(1) When D is nef, then  $\kappa_{\sigma}(D)$  coincides with

 $\nu(D) = \max\{k \in \mathbb{N} | D^k \cdot A^{n-k} > 0 \text{ for an ample divisor } A \text{ on } X\}.$ 

If moreover D is effective and S is a normal component of D, then

$$\nu(D|_S) \le \nu(D) - 1.$$

(2) Let  $\mu : W \to X$  be a generically finite surjective morphism between two normal projective varieties and E an effective  $\mu$ -exceptional  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor on W. Then

$$\kappa_{\sigma}(D) = \kappa_{\sigma}(\mu^*D + E).$$

(3) If  $(X, \Delta)$  is a Q-factorial log canonical 3-fold, and  $(X', \Delta')$  is a minimal model of  $(X, \Delta)$ , then

$$\kappa_{\sigma}(K_X + \Delta) = \nu(K_{X'} + \Delta').$$

*Proof.* For (1), the first assertion is [33, V, 2.7 (6)] in characteristic zero, and is [5, Porposition 4.5] in characteristic p > 0. The second assertion follows from the relation [9, Sec. 1.2]

$$(D|_S)^{k-1} \cdot (A|_S)^{n-k} = D^{k-1} \cdot A^{n-k} \cdot S \le D^k \cdot A^{n-k}.$$

(2) is from applying Theorem 2.3 (cf. [5, 2.7]).

Finally for (3), taking a common log resolution of  $(X, \Delta)$  and  $(X', \Delta')$ , this assertion follows from applying (2) and some standard arguments of minimal model theory.

2.6. Inseparable fibrations. In this section we work over an algebraically closed field k of characteristic p > 0. Let X be a smooth variety. Recall that a (1-)*foliation* is a saturated subsheaf  $\mathcal{F} \subset T_X$  which is involutive (i.e.,  $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$ ) and p-closed (i.e.,  $\xi^p \in \mathcal{F}, \forall \xi \in \mathcal{F}$ ). A foliation  $\mathcal{F}$  is called *smooth* if it is a subbundle of  $T_X$ .

**Proposition 2.9.** Let X be a smooth variety and  $\mathcal{F}$  a foliation on X.

(1) We get a normal variety  $Y = X/\mathcal{F} = \operatorname{Spec}Ann(\mathcal{F})$ , and there exist natural morphisms  $\pi : X \to Y$  and  $\pi' : Y \to X^{(1)}$  fitting into the following commutative diagram



Moreover deg  $\pi = p^r$  where  $r = \operatorname{rk} \mathcal{F}$ .

(2) There is a one-to-one correspondence between foliations and normal varieties between X and  $X^{(1)}$ , by the correspondence  $\mathcal{F} \mapsto X/\mathcal{F}$  and the inverse correspondence  $Y \mapsto Ann(\mathcal{O}_Y) := \{\xi \in T_X | \xi(a) = 0, \forall a \in \mathcal{O}_Y\}.$ 

(3) The variety Y is regular if and only if  $\mathcal{F}$  is smooth.

(4) If  $Y_0$  denotes the regular locus of Y and  $X_0 = \pi^{-1}Y_0$ , then

$$K_{X_0} \sim \pi^* K_{Y_0} + (p-1) \det \mathcal{F}|_{X_0}$$

*Proof.* We refer to [29, p.56-58] or [13].

The following result helps us to reduce an inseparable fibration to a separable one.

**Proposition 2.10.** Let  $f: X \to Y$  be a fibration from a normal projective 3-fold to a normal surface of maximal Albanese dimension, and let B be an effective  $\mathbb{Q}$ divisor on X. Then there exist a purely inseparable morphism  $\sigma: X \to X'$ , a separable fibration  $f': X' \to Y$ , a rational number t > 0 and an effective  $\mathbb{Q}$ -divisor B' on X' such that

$$K_X + B \sim_{\mathbb{O}} t(\sigma^*(K_{X'} + B'))$$

*Proof.* Let  $a_Y : Y \to A$  be the Albanese map and let  $a_X = a_Y \circ f$ . If f is a separable morphism, then we are done.

Assume that f is inseparable. Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  denote the saturation of the image of the natural homomorphisms  $a_X^* \Omega_{A_X}^1 \to \Omega_X^1$  and  $f^* \Omega_Y^1 \to \Omega_X^1$  respectively. Then  $\mathcal{L}_1 \subseteq \mathcal{L}_2$  and rk  $\mathcal{L}_2 \leq 1$  since f is inseparable ([17, Prop. 8.6A]). And by Igusa's result ([39, Theorem 4]), we have  $\mathcal{L}_1$  is generically globally generated, and  $h^0(X, \mathcal{L}_1) \geq h^0(A_X, \Omega_{A_X}^1) \geq 2 > \text{rk } \mathcal{L}_1$ . Therefore  $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$  is of rank one and  $h^0(X, \mathcal{L}^{**}) \geq 2$  ([48, Lemma 4.2]).

We get a natural foliation  $\mathcal{F} = \mathcal{L}^{\perp} \subset T_X$  of rank 2, and a quotient map  $\rho : X \to X_1 = X/\mathcal{F}$ , which is a factor of f by the construction above. Denote by  $X^0$ 

the maximal smooth open subset of X such that  $\mathcal{F}|_{X_0}$  is smooth, and let  $X_1^0 = X^0/(\mathcal{F}|_{X^0})$ . Then

(1) 
$$K_{X^0} \sim \rho^* K_{X_1^0} + (p-1) \det \mathcal{F}|_{X^0}.$$

On the other hand, we have the following exact sequence

$$0 \to \mathcal{L}|_{X^0} \to \Omega^1_{X^0} \to \mathcal{F}^*|_{X^0} \to 0,$$

which gives

$$\det \mathcal{F}|_{X^0} \sim \mathcal{L}|_{X^0} - K_{X^0}.$$

Combining with Equation (1), we get

(2) 
$$K_{X^0} \sim_{\mathbb{Q}} \frac{1}{p} (\rho^* K_{X_1^0} + (p-1)\mathcal{L}|_{X^0}).$$

Since  $\mathcal{L}$  is generically globally generated, there exists an effective Weil divisor B'on X such that  $B' \sim \mathcal{L}$ . And since  $\rho$  is purely inseparable, there exist  $\mathbb{Q}$ -divisors  $B_1, B'_1$  on  $X_1$  such that  $\rho^* B_1 = B$  and  $\rho^* B'_1 = B'$ . Let  $B_1 = pB_1 + (p-1)B'_1$ . Then since  $X \setminus X^0$  is of codimension  $\geq 2$  in X, by Equation (2) we have that

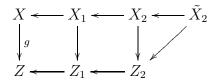
$$K_X + B \sim_{\mathbb{Q}} \frac{1}{p} (\rho^* (K_{X_1} + B_1)).$$

If the natural fibration  $f_1 : X_1 \to Y$  is separable then we are done. If not, we consider the pair  $(X_1, B_1)$  instead and repeat the process above. We can prove that  $\operatorname{mult}((X_1)_{\overline{\eta}}) < \operatorname{mult}(X_{\overline{\eta}})$  by the argument of [48, the latter case of Thm. 4.3]. So this process will terminate, and we can show the assertion by induction.  $\Box$ 

For a fibration fibred by curves of arithmetic genus one, we have the following result if char  $k = p \ge 5$ .

**Proposition 2.11.** Assume char  $k = p \ge 5$ . Let  $g: X \to Z$  be a fibration of normal varieties of relative dimension one. Assume that the generic fiber  $X_{\xi}$  of g is a curve with arithmetic genus  $p_a(X_{\xi}) = 1$ . Then the geometric generic fiber  $X_{\overline{\xi}}$  of g is a smooth elliptic curve over  $\overline{K(Z)}$ .

*Proof.* Assume by contrary that  $X_{\overline{\xi}}$  is singular. Applying [42, Lemma 2.3 and 2.4], we get the following commutative diagram



where  $Z_1 \to Z$  is a purely inseparable base change (or identity),  $Z_2 \to Z_1$  is a degree p purely inseparable extension,  $X_1 = X \times_Z Z_1$  and  $X_2 = X \times_Z Z_2$ , such that  $X_{\xi_1}$  is regular and that  $X_{\xi_2}$  is reduced but not normal, here  $\xi_1, \xi_2$  denote the generic point of  $Z_1, Z_2$  respectively. The normalization  $\tilde{X}_{\xi_2}$  of  $X_{\xi_2}$  has smaller arithmetic genus hence must be a smooth curve of genus zero. If necessary by shrinking  $Z_i$ , we can assume both  $\tilde{X}_2$  and  $X_1$  are smooth, so the natural morphism  $\pi: \tilde{X}_2 \to X_1$  is the

quotient induced by a smooth foliation  $\mathcal{F}$  on  $\tilde{X}_2$ , which is a subbundle of  $T_{\tilde{X}_2}$  of rank one. Since

$$K_{\tilde{X}_2} = \pi^* K_{X_1} + (p-1)\mathcal{F}$$

we get a contradiction by  $(p-1) \deg \mathcal{F}_{\xi} = \deg K_{\tilde{X}_{\xi_2}} = -2$  and  $p \ge 5$ .

2.7. Trace maps of absolute Frobenius morphisms. Let X be a projective variety over an algebraically closed field k of characteristic p > 0. We will consider the trace maps in the following two settings.

Notation 2.12. Assume X is normal. Denote by  $X_0$  the smooth open subset of X. Let B be an effective Q-Weil divisor with Weil index not divisible by p. There exists a positive integer g such that  $(p^g - 1)B$  is integral, thus  $(p^{eg} - 1)B$  is integral for every integer e > 0. The composition map of the natural inclusion

$$F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+B))|_{X_0} \hookrightarrow F_{X_0*}^{eg}\mathcal{O}_{X_0}((1-p^{eg})K_{X_0})$$

and the trace map  $Tr_{F_{X_0}^{eg}}: F_{X_0*}^{eg}\mathcal{O}_{X_0}((1-p^{eg})K_{X_0}) \to \mathcal{O}_{X_0}$  extends to a map on X:

$$Tr_{X,B}^{eg}: F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+B)) \to \mathcal{O}_X.$$

Let D be a Cartier divisor on X. Twisting  $Tr_{X,B}^{eg}$  with  $\mathcal{O}_X(D)$  induces

$$Tr_{X,B}^{eg}(D): F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+B)) \otimes \mathcal{O}_X(D)$$
$$\cong F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+B)+p^{eg}D) \to \mathcal{O}_X(D).$$

Then taking global sections induces a trace map

$$\Phi_{eg}: H^0(X, F^{eg}_{X_*}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)+p^{eg}D)) \to H^0(X, D).$$

Denote  $S_B^{eg}(X, D) = \operatorname{Im} \Phi_{eg}$  and  $S_B^0(X, D) = \bigcap_{e \ge 0} S_B^{eg}(X, D)$ . If B = 0, we usually use the notation  $S^0(X, D)$  instead of  $S_0^0(X, D)$ . Note that for e' > e, we have  $S_B^{e'g}(X, D) \subseteq S_B^{eg}(X, D)$  by the factorization

$$Tr_{X,B}^{e'g}(D) : F_{X*}^{eg} F_{X*}^{(e'-e)g} \mathcal{O}_X((1-p^{e'g})(K_X+B) + p^{e'g}D) \xrightarrow{F_{X*}^{eg} Tr_{X,\Delta}^{(e'-e)g}((1-p^{eg})(K_X+B) + p^{eg}D)} \xrightarrow{F_{X*}^{eg} \mathcal{O}_X((1-p^{eg})(K_X+B) + p^{eg}D)} \mathcal{O}_X(D).$$

Notation 2.13. Assume X is Gorenstein in codimension one and satisfies Serre condition  $S_2$ . Let B be an effective Q-AC divisor such that  $K_X + B$  is Q-Cartier ([46, Sec. 2.1 and 2.3], namely,  $B = \frac{M-nK_X}{n}$  for some n > 0, where M is a Cartier divisor and  $M - nK_X$  is effective in codimension one). Assume moreover that the Cartier index of  $K_X + B$  is not divisible by p. Let g > 0 be an integer such that  $(1 - p^g)(K_X + B)$  is Cartier. Then we can define trace maps  $Tr_{X,B}^{eg}, Tr_{X,B}^{eg}(D)$  as in 2.12 (see [46, Sec. 2.3] for details). By [14, Lemma 13.1], there exists an ideal  $\sigma(B)$ , namely, the non-F-pure ideal of (X, B), such that for some sufficiently divisible g' > 0 and any e > 0,

$$\mathrm{Im}Tr_{X,B}^{eg'} = \sigma(B) = Tr_{X,B}^{eg'}(F_{X*}^{eg'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{eg'})(K_X+B)))).$$

Borrowing the idea of the proof of [37, Lemma 2.20], we prove the following lemma.

**Lemma 2.14.** Using Notation 2.13, let A, D be two Cartier divisors on X. If A is ample, then there exists M > 0 such that for any m > M and e > 0, the trace map

$$\Phi_{e,m}: H^0(X, F_{X*}^{eg'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{eg'})(K_X+B))) \otimes \mathcal{O}_X(mA+D)) \rightarrow H^0(X, \sigma(B) \cdot \mathcal{O}_X(mA+D))$$

is surjective. In particular, there exists some c > 0 such that for any m > 0,  $\dim S^0_B(X, mA + D) \ge cm^{\dim X}$ .

*Proof.* We only need to prove the first assertion, which implies the second one. Let  $K_{eg'} = \ker(Tr_{X,B}^{eg'}: F_{X*}^{eg'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{eg'})(K_X+B))) \to \sigma(B))$ . Then we have the following commutative diagram of short exact sequences

where  $\gamma_2$  is obtained by applying  $F_{X_*}^{(e-1)g'}$  to the trace map

$$F_{X*}^{g'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{g'})(K_X+B))) \otimes \mathcal{O}_X((1-p^{(e-1)g'})(K_X+B)) \\ \cong F_{X*}^{g'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{eg'})(K_X+B))) \to \sigma(B) \cdot \mathcal{O}_X((1-p^{(e-1)g'})(K_X+B)),$$

and  $\gamma_1$  arises naturally. Since  $F_X^{(e-1)g'}$  is an affine morphism,  $\gamma_2$  is surjective and

$$\ker(\gamma_2) = F_{X*}^{(e-1)g'}(K_{g'} \otimes \mathcal{O}_X((1-p^{(e-1)g'})(K_X+B))).$$

Let  $K' = \ker(\gamma_1)$ . Since  $\gamma_2$  is surjective, applying Snake Lemma we obtain that  $\gamma_1$  is surjective and  $K' \cong F_{X*}^{(e-1)g'}(K_{g'} \otimes \mathcal{O}_X((1-p^{(e-1)g'})(K_X+B)))$ . It follows an exact sequence

(3) 
$$0 \to F_{X*}^{(e-1)g'}(K_{g'} \otimes \mathcal{O}_X((1-p^{(e-1)g'})(K_X+B))) \to K_{eg'} \to K_{(e-1)g'} \to 0$$

And since A is ample, we have

- (a) there exists  $M_0 > 0$  such that for any  $t \in [0, 1]$ , the divisor  $M_0A + D t(K_X + B)$  is nef; and
- (b) applying Fujita vanishing (Lemma 2.2), there exists an integer  $M_1$  such that for any  $l > M_1$  and any nef Cartier divisor P,  $H^1(X, K_{g'} \otimes \mathcal{O}_X(lA + P)) = 0$ .

Let  $M = M_0 + M_1$ . Fix an integer m > M. We aim to show that the trace map  $\Phi_{e,m}$  is surjective. Tensoring the following exact sequence with  $\mathcal{O}_X(mA + D)$ 

$$0 \to K_{eg'} \to F_{X*}^{eg'}(\sigma(B) \cdot \mathcal{O}_X((1 - p^{eg'})(K_X + B))) \to \sigma(B) \to 0,$$

and taking cohomology, it is sufficient to show that for every e > 0,

$$H^1(X, K_{eg'} \otimes \mathcal{O}_X(mA + D)) = 0.$$

By  $mA + D = (m - M_0)A + (M_0A + D)$ , applying (a) and (b) we can show the case e = 1. Assume by induction that  $H^1(X, K_{(e-1)q'} \otimes \mathcal{O}_X(mA + D)) = 0$  for some

 $e \geq 2$ . The condition (a) implies that  $p^{(e-1)g'}(M_0A + D) + (1 - p^{(e-1)g'})(K_X + B)$  is nef. Then applying the condition (b), it follows that

$$H^{1}(X, F_{X*}^{(e-1)g'}(K_{g'} \otimes \mathcal{O}_{X}((1-p^{(e-1)g'})(K_{X}+B))) \otimes \mathcal{O}_{X}(mA+D))$$
  

$$\cong H^{1}(X, K_{g'} \otimes \mathcal{O}_{X}(p^{(e-1)g'}(m-M_{0})A+p^{(e-1)g'}(M_{0}A+D)+(1-p^{(e-1)g'})(K_{X}+B))) = 0$$

Tensoring the exact sequence (3) with  $\mathcal{O}_X(mA + D)$  and taking cohomology, we deduce that  $H^1(X, K_{eg'} \otimes \mathcal{O}_X(mA + D)) = 0$ .

2.8. Derived categories. Let X be a projective variety, we denote by  $D^b(X)$  the bounded derived category of coherent sheaves on X.

For an object  $\mathcal{F} \in D^b(X)$  represented by the complex

$$\mathcal{K}^{\bullet}: \cdots \to K^{n-1} \to K^n \to K^{n+1} \to \cdots$$

we have truncations ([16, p. 69])

 $\sigma_{\leq n}(\mathcal{K}^{\bullet}): \dots \to K^{n-1} \to \ker d^n \to 0 \to \dots \text{ with } \mathcal{H}^i(\sigma_{\leq n}(\mathcal{K}^{\bullet})) \cong \mathcal{H}^i(\mathcal{K}^{\bullet}) \text{ for } i \leq n$ and

 $\sigma_{>n}(\mathcal{K}^{\bullet}): \dots \to 0 \to \text{im } d^n \to K^{n+1} \to \dots \text{ with } \mathcal{H}^i(\sigma_{>n}(\mathcal{K}^{\bullet})) \cong \mathcal{H}^i(\mathcal{K}^{\bullet}) \text{ for } i > n.$ 

The exact sequence below

$$0 \to \sigma_{\leq n}(\mathcal{K}^{\bullet}) \to \mathcal{K}^{\bullet} \to \sigma_{>n}(\mathcal{K}^{\bullet}) \to 0,$$

descends to a triangle in  $D^b(X)$  ([16, p. 63 Remark after Prop. 6.1])

$$\sigma_{\leq n}(\mathcal{F}) \to \mathcal{F} \to \sigma_{>n}(\mathcal{F}) \to \sigma_{\leq n}(\mathcal{F})[1].$$

Let  $f: X \to Y$  be a projective morphism of projective varieties. Assume Y is smooth. We have derived functors  $Rf_*: D^b(X) \to D^b(Y)$  and  $Lf^*: D^b(Y) \to D^b(X)$  of  $f_*$  and  $f^*$  respectively ([16, Chap. II Sec. 2,4]. By Grothendieck duality ([16, Chap. III Sec. 11]) there exists a functor  $f^!$  such that

$$R\mathcal{H}om_Y(Rf_*\mathcal{E},\mathcal{F})\cong Rf_*R\mathcal{H}om_X(\mathcal{E},f^!\mathcal{F})$$

where  $\mathcal{E} \in D^b(X), \mathcal{F} \in D^b(Y)$ . In particular if both X and Y are smooth, then  $f^! \mathcal{F} \cong Lf^* \mathcal{F} \otimes \mathcal{O}_X(K_{X/Y})[\dim X/Y]$  ([16, Chap. VI Sec. 4]).

2.9. Cohomology of flat complexes under base changes. Let's recall the following result, which is an adaption of [17, Chap. III Cor. 12.11] to flat bounded complexes. Though this is known to experts ([36, Remark of 3.6]), we explain the modifications of the proof for the convenience of the reader.

**Theorem 2.15.** Let  $f : X \to Y$  be a projective morphism of varieties. Let  $\mathcal{K}^{\bullet}$  be a bounded complex of coherent sheaves on X such that every  $\mathcal{K}^i$  is flat over Y. For a closed point  $y \in Y$ ,

(1) if the natural map  $\varphi_y^i : R^i f_* \mathcal{K}^{\bullet} \otimes k(y) \to R^i \Gamma(X_y, \mathcal{K}_y^{\bullet})$  is surjective, then  $\varphi_y^i$  is an isomorphism, and there exists a neighborhood U of y such that for any  $y' \in U$ , the map  $\varphi_{y'}^i$  is surjective;

(2) if  $\varphi_y^i$  is surjective then the following two conditions are equivalent to each other

(2.1)  $\varphi_y^{i-1}: R^{i-1}f_*\mathcal{K}^{\bullet} \otimes k(y) \to R^{i-1}\Gamma(X_y, \mathcal{K}_y^{\bullet})$  is surjective;

(2.2)  $R^i f_* \mathcal{K}^{\bullet}$  is locally free at y.

*Proof.* We assume Y = Spec A is affine. For an A-module M, define the functor

$$T^{i}(M) = R^{i}\Gamma(X, \mathcal{K}^{\bullet} \otimes_{A} M).$$

To adapt the arguments of [17, Chap. III, Sec. 12] to flat complexes, we only need to verify the analogue of [17, Chap. III, Prop. 12.1 and 12.2].

(a) Tensoring  $\mathcal{K}^{\bullet}$  with a short exact sequence  $0 \to M' \to M \to M'' \to 0$  of A-modules, since  $\mathcal{K}^{\bullet}$  is flat, we get a short exact sequence of complexes of sheaves on X

$$0 \to \mathcal{K}^{\bullet} \otimes_A M' \to \mathcal{K}^{\bullet} \otimes_A M \to \mathcal{K}^{\bullet} \otimes_A M'' \to 0.$$

Taking cohomology shows that  $T^i$  is exact in the middle. So the analogue of [17, Chap. III, Prop. 12.1] holds.

(b) Fix an open affine cover  $\mathcal{U} = \{U_i\}_{i \in I}$  of X. Consider the double complex

$$C^{\bullet}(\mathcal{U}, \mathcal{K}^{\bullet}) = (C^q(\mathcal{U}, \mathcal{K}^p), d'^{pq}, d''^{pq})$$

where  $d'^{pq}$  come from Čech complexes  $C^{\bullet}(\mathcal{U}, \mathcal{K}^p)$  and  $d''^{pq}$  are induced by the differentials of  $\mathcal{K}^{\bullet}$ . Consider the total complex  $C^{\bullet}$ 

$$C^{i} = \bigoplus_{p+q=i} C^{q}(\mathcal{U}, \mathcal{K}^{p}), d^{i} = \sum_{p+q=i} (-1)^{p} d'^{pq} + d''^{pq} : C^{i} \to C^{i+1}.$$

Note that  $C^i$  is a flat A-module, the total complex of  $C^{\bullet}(\mathcal{U}, \mathcal{K}^{\bullet} \otimes_A M)$  coincides with  $C^{\bullet} \otimes_A M$ , thus  $H^i(C^{\bullet} \otimes_A M) \cong R^i \Gamma(X, \mathcal{K}^{\bullet} \otimes_A M)$ . Then we can show the analogue of [17, Chap. III, Prop. 12.2] by similar arguments.  $\Box$ 

2.10. Abelian subvarieties generated by subschemes. Let A be an abelian variety of dimension d and V a closed subscheme of A. Let  $W_n$  be the reduced subscheme supported on the *n*-fold sum  $V + V + \cdots + V$ . Let N be an integer (say,  $N = \dim A$ ) such that  $d' = \dim W_N$  attains the maximum of  $\dim W_n$ . Let Z be an irreducible component of  $W_N$  with  $\dim Z = d'$ .

**Lemma 2.16.** With the above notation, take a closed point  $z \in Z$  and let  $Z_0 = Z - z$ . Then

(1)  $Z_0$  is an abelian subvariety of A; and

(2) every component of  $W_n$  is contained in a translate of  $Z_0$ .

Proof. (1) Note that  $Z_0 + Z_0$  is irreducible because it is the image of  $m : Z_0 \times Z_0 \to A$ via  $m(z_1, z_2) = z_1 + z_1$ . Since  $0 \in Z_0$ , we have  $\operatorname{Supp} Z_0 \subseteq \operatorname{Supp} (Z_0 + Z_0)$ , then dim  $Z_0 = \dim(Z_0 + Z_0)$  implies that  $\operatorname{Supp} Z_0 = \operatorname{Supp} (Z_0 + Z_0)$ . Applying [31, p. 44, Theorem of Appendix to Sec. 4], we see that  $Z_0$  is an abelian subvariety of A.

(2) For a component Z', take  $z' \in Z'$  and let  $Z'_0 = Z' - z'$ . Since  $0 \in Z'_0, Z'_0 + Z_0$  is irreducible and contains both  $Z'_0$  and  $Z_0$ . And since dim  $Z_0$  attains maximum,  $Z'_0 + Z_0 = Z_0$ , which shows  $Z'_0 \subseteq Z_0$ . Then (2) follows easily.

## 3. Sheaves on Abelian Varieties

In this section, we will study sheaves on abelian varieties. The main result is Theorem 3.7, which, as a generalization of [12, Proposition 2.7], is used in the present paper to study fibrations over abelian varieties. The idea of the proof is stimulated by [19, Theorem 3.1.1].

Notation 3.1. We work over an algebraically closed field k. Let A be an abelian variety of dimension d,  $\hat{A} = \operatorname{Pic}^{0}(A)$  and  $\mathcal{P}$  the Poincaré line bundle on  $A \times \hat{A}$ . Let p, q denote the projections from  $A \times \hat{A}$  to  $A, \hat{A}$  respectively. The Fourier-Mukai transform  $R\Phi_{\mathcal{P}}: D^b(A) \to D^b(\hat{A})$  w.r.t.  $\mathcal{P}$  is defined as

$$R\Phi_{\mathcal{P}}(-) := Rq_*(Lp^*(-) \otimes \mathcal{P})$$

which is a right derived functor. Similarly  $R\Psi_{\mathcal{P}}: D^b(\hat{A}) \to D^b(A)$  is defined as

$$R\Psi_{\mathcal{P}}(-) := Rp_*(Lq^*(-) \otimes \mathcal{P}).$$

If L is an ample line bundle on  $\hat{A}$ , then  $H^i(A, L \otimes \mathcal{P}_t) = 0$  for i > 0 and every  $t \in A$  ([31, Sec. 13]), thus  $\hat{L} := R^0 \Psi_{\mathcal{P}} L \cong R \Psi_{\mathcal{P}} L$  is a locally free sheaf of rank  $h^0(\hat{A}, L)$  by Theorem 2.15.

Note that since p, q are smooth morphisms, we have  $Lp^* \cong p^*, Lq^* \cong q^*$ . In what follows, for an isogeny  $\pi: A_1 \to A$  of abelian varieties, by abuse of the notation of the pull-back map of sheaves, we use  $\pi^* : \hat{A} \to \hat{A}_1$  for the dual map of  $\pi$ .

For a coherent sheaf  $\mathcal{F}$  on A, we define

$$D_A(\mathcal{F}) := R\mathcal{H}om_X(\mathcal{F}, \omega_A)[d],$$

then applying Grothendieck duality we have

$$D_k(R\Gamma(\mathcal{F})) \cong R\Gamma(D_A(\mathcal{F})).$$

For a closed point  $t_0 \in A$ , the translating morphism  $T_{t_0} : A \to A$  is defined via  $t \mapsto t + t_0$ . For  $\hat{t}_0 \in \hat{A}$ ,  $T_{\hat{t}_0}$  is similarly defined.

**Theorem 3.2** ([30]). Using Notation 3.1, we have

- (1)  $R\Psi_{\mathcal{P}} \circ R\Phi_{\mathcal{P}} \cong (-1)^*_{A}[-d], \ R\Phi_{\mathcal{P}} \circ R\Psi_{\mathcal{P}} \cong (-1)^*_{\hat{A}}[-d];$ (2)  $R\Phi_{\mathcal{P}} \circ (-1)^*_{A} \cong (-1)^*_{\hat{A}} \circ R\Phi_{\mathcal{P}}, R\Psi_{\mathcal{P}} \circ (-1)^*_{\hat{A}} \cong (-1)^*_{A} \circ R\Psi_{\mathcal{P}}; and$ (3) for  $\hat{t}_0 \in \hat{A}, \ R\Psi_{\mathcal{P}} \circ T^*_{\hat{t}_0} \cong \mathcal{P}_{-\hat{t}_0} \otimes R\Psi_{\mathcal{P}}.$

**Definition 3.3.** ([36, Def. 3.1]) Given a coherent sheaf  $\mathcal{F}$  on an abelian variety A, its  $i^{\text{th}}$  cohomological support locus is defined as

$$V^{i}(\mathcal{F}) := \{ \alpha \in \operatorname{Pic}^{0}(A) | h^{i}(\mathcal{F} \otimes \alpha) > 0 \}$$

which are Zariski closed by semi-continuity. If

$$gv(\mathcal{F}) := \min_{i>0} \{ \operatorname{codim}_{\operatorname{Pic}^0(A)} V^i(\mathcal{F}) - i \} \ge 0$$

we say  $\mathcal{F}$  is a *GV*-sheaf.

**Proposition 3.4.** Using Notation 3.1, let  $\mathcal{F}$  be a coherent sheaf on A.

(1) For an ample line bundle H on  $\hat{A}$ , there is a natural isomorphism

$$D_k(R\Gamma(\mathcal{F}\otimes \hat{H}^*))\cong R\Gamma(R\Phi_{\mathcal{P}}D_A(\mathcal{F})\otimes H)$$

in particular,  $H^i(A, \mathcal{F} \otimes \hat{H}^*)^* \cong R^{-i}\Gamma(R\Phi_{\mathcal{P}}D_A(\mathcal{F}) \otimes H)$  for every integer *i*. (2) The Fourier-Mukai transform

 $R\Phi_{\mathcal{P}}D_A(\mathcal{F}) \in D^{[-d,0]}(A)$  and  $\operatorname{Supp}(-1)^*_{\hat{A}}R^0\Phi_{\mathcal{P}}D_A(\mathcal{F}) = V^0(\mathcal{F}).$ 

- (3) The following three conditions are equivalent to each other
- (i) the sheaf  $\mathcal{F}$  is a GV-sheaf;
- (ii)  $R\Phi_{\mathcal{P}}D_A(\mathcal{F}) \cong R^0\Phi_{\mathcal{P}}D_A(\mathcal{F});$
- (iii) for any sufficiently ample line bundle L on  $\hat{A}$ ,

$$H^i(A, \mathcal{F} \otimes \hat{L}^*) = 0 \text{ for } i > 0.$$

*Proof.* (1) is contained in the proof of [18, Theorem 1.2], which follows from applying Grothendieck duality and projection formula.

- (2) Take an ample line bundle H on  $\hat{A}$ . By (1) we have that
- (a)  $R^{-i}\Gamma(R\Phi_{\mathcal{P}}D_A(\mathcal{F})\otimes H) = 0$  unless  $0 \le i \le d$ .

Since  $R\Phi_{\mathcal{P}}D_A(\mathcal{F}) \in D^b(\hat{A})$ , we can assume *H* is sufficiently ample such that, for every q,

- (b)  $R^q \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H$  is globally generated, and
- (c)  $R^p \Gamma(R^q \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H) = 0$  whenever  $p \neq 0$ .

Since H is a line bundle, we have the spectral sequence

 $E_2^{p,q} := R^p \Gamma(R^q \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H) \cong R^p \Gamma(\mathcal{H}^q(R \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes^L H)) \Rightarrow R^{p+q} \Gamma(R \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H).$ By (c) we can show

$$E^{0,q}_{\infty} = E^{0,q}_2 = \Gamma(R^q \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H) \cong R^q \Gamma(R \Phi_{\mathcal{P}} D_A(\mathcal{F}) \otimes H).$$

Then by (a) and (b), we conclude that  $R^i \Phi_{\mathcal{P}} D_A(\mathcal{F}) = 0$  unless  $-d \leq i \leq 0$ , that is,  $R \Phi_{\mathcal{P}} D_A(\mathcal{G}) \in D^{[-d,0]}(A)$ .

For  $\alpha \in \hat{A}$ , applying Grothendieck duality we have

$$H^{j}(\mathcal{F}\otimes\mathcal{P}_{\alpha})^{*}\cong R^{-j}\Gamma(D_{A}(\mathcal{F})\otimes\mathcal{P}_{-\alpha}).$$

In particular,  $R^i \Gamma(D_A(\mathcal{F}) \otimes \mathcal{P}_{-\alpha}) = 0$  for i > 0. Applying  $Rq_*$  to  $p^* D_A(\mathcal{F}) \otimes \mathcal{P}$  on  $A \times \hat{A}$ , since  $R^1 \Phi_{\mathcal{P}} D_A(\mathcal{F}) = 0$ , by Theorem 2.15, we conclude

$$H^{0}(\mathcal{F} \otimes \mathcal{P}_{\alpha})^{*} \cong R^{0}\Gamma(D_{A}(\mathcal{F}) \otimes \mathcal{P}_{-\alpha}) \cong R^{0}\Phi_{\mathcal{P}}D_{A}(\mathcal{F}) \otimes k(-\alpha),$$

thus  $\operatorname{Supp}(-1)_{\hat{A}}^* R^0 \Phi_{\mathcal{P}} D_A(\mathcal{F}) = V^0(\mathcal{F}).$ 

(3) follows from applying [18, Theorem 1.2] and [36, Lemma 3.6].

Part of the following theorem is known to experts, in particular assertion (4) also appeared in [20, Sec. 1.2], which states a special reature of positive characteristic.

## Theorem 3.5. Using Notation 3.1, then

(1) If  $\tau$  is a coherent sheaf supported at finitely many closed points on  $\hat{A}$  of length r, then  $U = R\Psi_{\mathcal{P}}\tau = R^{0}\Psi_{\mathcal{P}}\tau$  is a vector bundle of rank r (homogenous vector bundle); moreover if  $\operatorname{Supp}\tau = \{\hat{0}\}$  then U is a unipotent vector bundle, that is, U admits a filtration of vector bundles

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_r = U$$

such that  $U_i/U_{i-1} \cong \mathcal{O}_X$ .

(2) If  $\pi: A_1 \to A$  is an isogeny of abelian varieties, then  $\pi_* \mathcal{O}_{A_1} \cong \bigoplus_i U_{\hat{t}_i}$  where

- $\hat{t}_i$  are finitely many closed points on  $\hat{A}$  such that  $\pi^* \mathcal{P}_{\hat{t}_i} \cong \mathcal{O}_{A_1}$ , and
- $U_{\hat{t}_i} = R \Psi_{\mathcal{P}} \tau_i$  where  $\tau_i$  is a skyscraper sheaf supported at  $\hat{t}_i$ .

(3) Let  $\mathcal{F}$  be a coherent sheaf on A. Set  $V^{\geq m}(\mathcal{F}) = \bigcup_{j\geq m} V^j(\mathcal{F})$ . If  $\pi : A_1 \to A$  is an isogeny of abelian varieties, then

$$V^m(\pi^*\mathcal{F}) \subseteq \pi^*V^m(\mathcal{F}) \text{ and } V^{\geq m}(\pi^*\mathcal{F}) = \pi^*V^{\geq m}(\mathcal{F}).$$

In particular, if for every j > 0,  $h^j(A_1, \pi^* \mathcal{F}) = 0$ , then  $h^j(A, \mathcal{F}) = 0$  for j > 0. Moreover, we have  $V^0(\pi^* \mathcal{F}) = \pi^* V^0(\mathcal{F})$ .

(4) Assume char k = p > 0. Let  $\tau$  be a coherent sheaf supported on the union of finitely many closed points on  $\hat{A}$  and let  $U = R\Psi_{\mathcal{P}}\tau$ . Then there exists an isogeny  $\mu: B \to A$  of abelian varieties such that  $\mu^*U \cong \bigoplus_i \mathcal{P}'_{\hat{s}_i}$  for some closed points  $\hat{s}_i$  on  $\hat{B} = \operatorname{Pic}^0(B)$ , where  $\mathcal{P}'$  denotes the Poincaré line bundle on  $B \times \hat{B}$ .

*Proof.* (1) Since dim Supp $\tau = 0$ , for any closed point  $t \in A$  we have

$$h^i(\hat{A}, \tau \otimes \mathcal{P}_t) = 0$$
 if  $i \neq 0$  and  $h^0(\hat{A}, \tau \otimes \mathcal{P}_t) = r$ .

The first assertion of (1) follows from applying Theorem 2.15.

If moreover  $\tau$  is a skyscraper sheaf supported at 0, then we have a filtration

$$0 = \tau_0 \subset \tau_1 \subset \tau_2 \subset \cdots \subset \tau_r = \tau$$

such that  $\tau_i/\tau_{i-1} \cong k(\hat{0})$ , which, by applying Fourier-Mukai transform  $R\Psi_{\mathcal{P}}$ , induces a filtration of U as wanted.

(2) For  $\hat{t} \in \hat{A}$ , we have  $H^i(A_1, \pi^* \mathcal{P}_{\hat{t}}) \cong H^i(A, \pi_* \mathcal{O}_{A_1} \otimes \mathcal{P}_{\hat{t}})$ . It follows that

$$V^{0}(\pi_{*}\mathcal{O}_{A_{1}}) = V^{1}(\pi_{*}\mathcal{O}_{A_{1}}) = \dots = V^{d}(\pi_{*}\mathcal{O}_{A_{1}}) = S_{\pi} := \{\hat{t}' \in \hat{A} | \pi^{*}\mathcal{P}_{\hat{t}'} \cong \mathcal{O}_{A_{1}} \}.$$

We have  $S_{\pi} = \text{Supp}(\ker \pi^*)$ , thus dim  $V^k(\pi_*\mathcal{O}_{A_1}) = 0$  for  $0 \leq k \leq d$ , and  $\pi_*\mathcal{O}_{A_1}$  is a GV-sheaf. By Grothendieck duality

$$D_A(\pi_*\mathcal{O}_{A_1}) \cong \pi_*D_{A_1}(\mathcal{O}_{A_1}) \cong \pi_*\mathcal{O}_{A_1}[d]$$

applying Proposition 3.4(3) we have

$$R\Phi_{\mathcal{P}}\pi_*\mathcal{O}_{A_1}[d] \cong R^d\Phi_{\mathcal{P}}\pi_*\mathcal{O}_{A_1} \text{ and } \operatorname{Supp} R^d\Phi_{\mathcal{P}}\pi_*\mathcal{O}_{A_1} = S_{\pi}.$$

We can assume that  $R^d \Phi_{\mathcal{P}} \pi_* \mathcal{O}_{A_1} = \bigoplus_{i=1}^{i=n} \tau'_i$  where every  $\tau'_i$  is a skyscraper sheaf supported at some  $\hat{t}'_i \in S_{\pi}$ . Applying Theorem 3.2, we have that

$$\pi_*\mathcal{O}_{A_1} \cong (-1)^*_A R^0 \Psi_{\mathcal{P}} \bigoplus_{i=1}^{i=n} \tau'_i \cong \bigoplus_{i=1}^{i=n} R^0 \Psi_{\mathcal{P}}(-1)^*_{\hat{A}} \tau'_i.$$

So we only need to set  $\tau_i = (-1)^*_{\hat{A}} \tau'_i, \hat{t}_i = (-1)^*_{\hat{A}} \hat{t}'_i$  and  $U_{\hat{t}_i} = R^0 \Psi_{\mathcal{P}} \tau_i$ .

(3) We use the notation of (2). For  $\hat{t} \in \hat{A}$ , we have

$$H^{j}(A_{1}, \pi^{*}(\mathcal{F} \otimes \mathcal{P}_{\hat{t}})) \cong H^{j}(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes \pi_{*}\mathcal{O}_{A_{1}}) \cong H^{j}(A, \bigoplus_{i} \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes U_{\hat{t}_{i}}).$$

Let  $\tau_i^0 = T_{\hat{t}_i}^* \tau_i$  and  $U_i^0 = R^0 \Psi_{\mathcal{P}} \tau_i^0$ . Then  $\tau_i^0$  is supported at  $\hat{0}, U_{\hat{t}_i} \cong \mathcal{P}_{\hat{t}_i} \otimes U_i^0$  and

$$H^{j}(A_{1}, \pi^{*}(\mathcal{F} \otimes \mathcal{P}_{\hat{t}})) \cong \bigoplus_{i} H^{j}(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_{i}} \otimes U_{i}^{0}).$$

**Claim**: For a coherent sheaf  $\mathcal{G}$  and a unipotent vector bundle U on A, (3.1') if  $H^{l}(A, \mathcal{G}) = 0$  then  $H^{l}(A, \mathcal{G} \otimes U) = 0$ ; and

(3.2) if  $H^j(A, \mathcal{G} \otimes U) = 0$  for every  $j \ge m$ , then  $H^j(A, \mathcal{G}) = 0$  for  $j \ge m$ .

Granted the claim above, we prove assertion (3) by the following arguments.

(3.1) For  $\hat{t} \in \hat{A}$ , if  $\pi^* \hat{t} \in V^m(\pi^* \mathcal{F})$ , then there exists some  $\hat{t}_i \in S_{\pi}$  such that  $H^m(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_i} \otimes U_i^0) \neq 0$ . Applying (3.1') shows that  $H^m(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}+\hat{t}_i}) \neq 0$ , i.e.,  $\hat{t} + \hat{t}_i \in V^m(\mathcal{F})$ . Since  $\pi^* \hat{t}_i = \hat{0}$  and  $\pi^* : \hat{A} \to \hat{A}_1$  is an epimorphism, we see that

$$V^m(\pi^*\mathcal{F}) \subseteq \pi^*V^m(\mathcal{F}).$$

(3.2) For  $\hat{t} \in \hat{A}$ , if  $\hat{t} \in V^{\geq m}(\mathcal{F})$  then  $H^{j}(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}}) \neq 0$  for some  $j \geq m$ . Since  $\hat{0} \in S_{\pi}, \pi_{*}\mathcal{O}_{A_{1}}$  has a unipotent direct summand U. Applying (3.2') shows that there exists some  $j' \geq m$  such that  $H^{j'}(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}} \otimes U) \neq 0$ , thus  $H^{j'}(A_{1}, \pi^{*}(\mathcal{F} \otimes \mathcal{P}_{\hat{t}})) \neq 0$ . Therefore,  $\pi^{*}V^{\geq m}(\mathcal{F}) \subseteq V^{\geq m}(\pi^{*}\mathcal{F})$ , and the equality holds by combining with (3.1). (3.3) To show  $V^{0}(\pi^{*}\mathcal{F}) = \pi^{*}V^{0}(\mathcal{F})$ , by (3.1) it suffices to show that  $\pi^{*}V^{0}(\mathcal{F}) \subseteq V^{\geq m}(\mathcal{F})$ .

(3.3) To show  $V^{\circ}(\pi^{*}\mathcal{F}) = \pi^{*}V^{\circ}(\mathcal{F})$ , by (3.1) It suffices to show that  $\pi^{*}V^{\circ}(\mathcal{F}) \subseteq V^{0}(\pi^{*}\mathcal{F})$ , which follows from the fact that the natural map  $\pi^{*}: H^{0}(A, \mathcal{F} \otimes \mathcal{P}_{\hat{t}}) \to H^{0}(A_{1}, \pi^{*}(\mathcal{F} \otimes \mathcal{P}_{\hat{t}}))$  is injective since  $\pi$  is flat.

*Proof of Claim.* Let  $r = \operatorname{rk} U$  and  $U_r = U$ . By (1) we have a filtration of vector bundles

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_r = U,$$

in turn we get the following short exact sequences

$$\begin{array}{ccc} 0 \to \mathcal{G} \otimes U_{r-1} \to \mathcal{G} \otimes U_r \to \mathcal{G} \to 0 & (r) \\ 0 \to \mathcal{G} \otimes U_{r-2} \to \mathcal{G} \otimes U_{r-1} \to \mathcal{G} \to 0 & (r-1) \\ & & \cdots & \cdots \\ 0 \to \mathcal{G} \to \mathcal{G} \otimes U_2 \to \mathcal{G} \to 0 & (2). \end{array}$$

(3.1') Taking cohomology of those short exact sequences  $(2, 3, \dots, r)$ , since  $H^{l}(A, \mathcal{G}) = 0$  we can show that

$$H^{l}(A, \mathcal{G} \otimes U_{2}) = H^{l}(A, \mathcal{G} \otimes U_{3}) = \cdots = H^{l}(A, \mathcal{G} \otimes U_{r}) = 0.$$

(3.2') We will prove  $H^j(A, \mathcal{G}) = 0$  for  $j \ge m$  by induction on j. This is trivial if j > d. Assume that we have proved for some fixed l > m,

$$H^i(A, \mathcal{G}) = 0$$
 whenever  $i \ge l$ ,

which, by (3.1'), implies that for  $i \ge l$ 

$$H^{i}(A, \mathcal{G} \otimes U_{2}) = H^{i}(A, \mathcal{G} \otimes U_{3}) = \cdots = H^{i}(A, \mathcal{G} \otimes U_{r}) = 0.$$

Taking cohomology of the short exact sequence (r), by the vanishing  $H^{l-1}(A, \mathcal{G} \otimes U_r) = 0$  and  $H^l(A, \mathcal{G} \otimes U_{r-1}) = 0$ , we show that

$$H^{l-1}(A,\mathcal{G}) = 0.$$

By induction, we finish the proof of this claim.

(4) We can write that  $\tau = \bigoplus_i \tau_i$  where every  $\tau_i$  is a sheaf supported at one certain closed point  $\hat{t}_i$ . Let  $U_i = R^0 \Psi_{\mathcal{P}} \tau_i$ . Then  $U = \bigoplus_i U_i$ . We only need to show assertion (4) for a single  $U_i$ .

We can assume  $\tau$  is supported at exactly one closed point  $\hat{t} \in \hat{A}$ . Let

$$\bar{\tau} = T_{\hat{t}}^* \tau$$
 and  $\bar{U} = R \Psi_{\mathcal{P}} \bar{\tau}$ .

Then  $\operatorname{Supp} \bar{\tau} = \{\hat{0}\}, U \cong \bar{U} \otimes \mathcal{P}_{\hat{t}}$ , and  $\bar{U}$  is a unipotent vector bundle. We can consider  $\bar{U}$  instead and do induction on the rank. By (1) there is a filtration of vector bundles

$$0 = \bar{U}_0 \subset \bar{U}_1 \subset \bar{U}_2 \subset \cdots \bar{U}_{i-1} \subset \bar{U}_i \subset \cdots \subset \bar{U}_r = \bar{U}.$$

Assume by induction that for some  $i \leq r$  there exists an isogeny  $\mu_{i-1} : B_{i-1} \to A$ such that  $\mu_{i-1}^* \overline{U}_{i-1} \cong \bigoplus^{i-1} \mathcal{O}_{B_{i-1}}$ . Then we have the extension

$$0 \to \bigoplus^{i-1} \mathcal{O}_{B_{i-1}} \to \mu_{i-1}^* \bar{U}_i \to \mathcal{O}_{B_{i-1}} \to 0$$

which corresponds to

$$(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}) \in \bigoplus^{i-1} H^1(\mathcal{O}_{B_{i-1}}) \cong Ext^1(\mathcal{O}_{B_{i-1}}, \bigoplus^{i-1} \mathcal{O}_{B_{i-1}}).$$

Recall "killing cohomology", which says that for a projective variety X and  $\alpha \in H^1(\mathcal{O}_X)$ , there exists a morphism  $\pi : Y \to X$  composed with some Frobenius iterations and étale  $\mathbb{Z}/(p)$ -covers such that  $\pi^*\alpha = 0$  in  $H^1(\mathcal{O}_Y)$  ([40, Prop. 12 and Sec. 9]). So applying "killing cohomology", we get a base change  $\nu_i : B_i \to B_{i-1}$ , which is an isogeny of abelian varieties such that

$$\nu_i^* \alpha_1 = \nu_i^* \alpha_2 = \cdots \nu_i^* \alpha_{i-1} = 0 \in H^1(\mathcal{O}_{B_i}).$$

Let  $\mu_i = \mu_{i-1} \circ \nu_i : B_i \to A$ . Then

$$\mu_i^* \bar{U}_i \cong \nu_i^* \mu_{i-1}^* \bar{U}_i \cong \bigoplus^{i} \mathcal{O}_{B_i}.$$

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We finish the proof.

**Lemma 3.6.** Using Notation 3.1, for a coherent sheaf  $\mathcal{G}$  on A, there exists a natural homomorphism

$$\alpha_{\mathcal{G}}: \mathcal{G}^* = \mathcal{E}xt^0(\mathcal{G}, \mathcal{O}_A) \to (-1)^*_A R^0 \Psi_{\mathcal{P}} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{G})$$

with the kernel  $\mathcal{K}_{\mathcal{G}} \cong (-1)^*_A R^0 \Psi_{\mathcal{P}}(\sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G})).$ 

*Proof.* Apply  $(-1)^*_A R \Psi_{\mathcal{P}}$  to the following triangle

$$\sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G}) \to R \Phi_{\mathcal{P}} D_A(\mathcal{G}) \to \sigma_{> -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G}) \to \sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G})[1]$$

and take cohomology. Since  $\sigma_{>-1}R\Phi_{\mathcal{P}}D_A(\mathcal{G}) \cong R^0\Phi_{\mathcal{P}}D_A(\mathcal{G})$  (Proposition 3.4 (2)), we get the following exact sequence

$$(-1)^*_A R^{-1} \Psi_{\mathcal{P}} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{G}) \to (-1)^*_A R^0 \Psi_{\mathcal{P}} (\sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G})) \to (-1)^*_A R^0 \Psi_{\mathcal{P}} R \Phi_{\mathcal{P}} D_A(\mathcal{G}) \xrightarrow{\alpha'_{\mathcal{G}}} (-1)^*_A R^0 \Psi_{\mathcal{P}} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{G}).$$

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Applying Theorem 3.2, we have an isomorphism

$$\mathcal{E}xt^0(\mathcal{G},\mathcal{O}_A) \cong (-1)^*_A \mathcal{H}^0(R\Psi_{\mathcal{P}}R\Phi_{\mathcal{P}}D_A(\mathcal{G})),$$

then we get the homomorphism  $\alpha_{\mathcal{G}}$  by composing  $\alpha'_{\mathcal{G}}$  with this isomorphism. Since  $R^{-1}\Psi_{\mathcal{P}}R^0\Phi_{\mathcal{P}}D_A(\mathcal{G})=0$ , it follows that the kernel of  $\alpha_{\mathcal{G}}$ 

$$\mathcal{K}_{\mathcal{G}} \cong (-1)_A^* R^0 \Psi_{\mathcal{P}}(\sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{G})).$$

Let us prove the main theorem of this section.

**Theorem 3.7.** Using Notation 3.1, let  $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_d$  be torsion free coherent sheaves on A equipped with homomorphisms  $\phi_e : \mathcal{F}_e \to \mathcal{F}_{e-1}, e = 1, 2, \cdots, d$ . Let  $\varphi_e = \phi_e \circ \phi_{e-1} \circ \cdots \circ \phi_1 : \mathcal{F}_e \to \mathcal{F}$ . And let  $H_l, l = 0, 1, \cdots, d-1$  be ample line bundles on  $\hat{A}$ . Assume that

- (a) for  $0 \leq l \leq d-1$  and every *i*, the sheaf  $R^i \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l$  is globally generated, and if j > 0 then  $H^j(\hat{A}, R^i \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l) = 0$ ;
- (b) for  $0 \leq l < m \leq d$ , if j > 0 then  $H^{j}(A, \mathcal{F}_{m} \otimes \hat{H}_{l}^{*}) = 0$ ; and (c) the dual homomorphism  $\mathcal{F}^{*} \to \mathcal{F}_{d}^{*}$  of  $\varphi_{d}$  is injective.

Then

- (i) the homomorphism  $\alpha_{\mathcal{F}}: \mathcal{F}^* \to (-1)^*_A R^0 \Psi_{\mathcal{P}} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{F})$  (introduced in Lemma 3.6) is injective; and
- (ii) if moreover char k = p > 0 and  $R^0 \Phi_{\mathcal{P}} D_A(\mathcal{F}) = \tau$  is supported at finitely many closed points, then there exist an isogeny  $\pi: A_1 \to A$  of abelian varieties, some  $P_i \in \operatorname{Pic}^0(A_1)$  and a generically surjective homomorphism

$$\beta_{\mathcal{F}}: \bigoplus_i P_i \to \pi^* \mathcal{F}.$$

*Proof.* The maps  $\varphi_e : \mathcal{F}_e \to \mathcal{F}$  induce  $\varphi_e^* : D_A(\mathcal{F}) \to D_A(\mathcal{F}_e)$  by taking dual, then applying  $R\Phi_{\mathcal{P}}$  induces natural homomorphisms

$$R\Phi_{\mathcal{P}}D_A(\mathcal{F}) \to R\Phi_{\mathcal{P}}D_A(\mathcal{F}_e).$$

We have the following lemma with the proof postponed.

Lemma 3.8. The natural homomorphism

$$R^{0}\Psi_{\mathcal{P}}(\sigma_{\leq -1}R\Phi_{\mathcal{P}}D_{A}(\mathcal{F})) \to R^{0}\Psi_{\mathcal{P}}(\sigma_{\leq -1}R\Phi_{\mathcal{P}}D_{A}(\mathcal{F}_{d}))$$

is zero.

(i) Applying Lemma 3.6, we have the following commutative diagram

and the vertical map  $\mathcal{K}_{\mathcal{F}} \to \mathcal{K}_{\mathcal{F}_d}$  is zero by Lemma 3.8. Then from the condition (c), we conclude  $\mathcal{K}_{\mathcal{F}} = 0$ , thus  $\alpha_{\mathcal{F}}$  is injective.

(ii) Let  $U = (-1)_A^* R^0 \Psi_P \tau$ . Then by Theorem 3.5 (1) and (4), the sheaf U is locally free, and there exists an isogeny  $\pi : A_1 \to A$  of abelian varieties such that

$$\pi^* U \cong \bigoplus_i Q_i \text{ for some } Q_i \in \operatorname{Pic}^0(A_1).$$

Applying  $(-1)^*_A R \Psi_P$  to the natural homomorphism  $R \Phi_P D_A(\mathcal{F}) \to \tau$  induces a homomorphism

$$\gamma: R\mathcal{H}om(\mathcal{F}, \mathcal{O}_A) \cong D_A(\mathcal{F})[-d] \to U,$$

then applying  $R\mathcal{H}om(\cdot, \mathcal{O}_A)$  induces

$$\gamma^*: U^* \to \mathcal{F}.$$

Since by (i) the homomorphism  $\alpha_{\mathcal{F}} : \mathcal{F}^* \to U$  is injective,  $\gamma^*$  is generically surjective. It follows that the pull-back homomorphism via  $\pi$ 

$$\beta_{\mathcal{F}}: \pi^* U^* \to \pi^* \mathcal{F}$$

is surjective over the generic point of  $A_1$ . So we are done by setting  $P_i = Q_i^*$ .

Proof of Lemma 3.8. We divide the proof into two steps.

Step 1: We prove that for fixed  $0 \leq l < m \leq d$  and any  $s \leq -1$ , the natural map  $R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \to R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_m)$  is zero.

Consider the spectral sequence

$$E_{2,\mathcal{F}_l}^{r,s} := R^r \Gamma(R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l) \Rightarrow R^{r+s} \Gamma(R \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l).$$

By the condition (a) and Proposition 3.4 (2), we see that

$$E_{\infty,\mathcal{F}_l}^{r,s} \cong E_{2,\mathcal{F}_l}^{r,s} = 0$$
 unless  $r = 0$  and  $-d \le s \le 0$ ,

thus for  $-d \leq s \leq 0$ , the following natural homomorphism is an isomorphism

$$\gamma_{\mathcal{F}_l}^s : R^s \Gamma(R\Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l) \to E_{\infty, \mathcal{F}_l}^{0, s} \hookrightarrow E_{2, \mathcal{F}_l}^{0, s} = H^0(R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \otimes H_l).$$

Then consider the spectral sequence

$$E_{2,\mathcal{F}_m}^{r,s} := R^r \Gamma(R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_m) \otimes H_l) \Rightarrow R^{r+s} \Gamma(R \Phi_{\mathcal{P}} D_A(\mathcal{F}_m) \otimes H_l).$$

Since  $E_{2,\mathcal{F}_m}^{r,s} = 0$  whenever r < 0, we get a natural homomorphism

 $\gamma^s_{\mathcal{F}_k} : R^s \Gamma(R\Phi_{\mathcal{P}} D_A(\mathcal{F}_m) \otimes H_l)) \to E^{0,s}_{\infty,\mathcal{F}_m} \hookrightarrow E^{0,s}_{2,\mathcal{F}_m} \cong H^0(R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_m) \otimes H_l).$ For i > 0, by the condition (b) and the isomorphism  $R^{-i} \Gamma(R\Phi_{\mathcal{P}} D_A(\mathcal{F}_m) \otimes H_l) \cong$ 

 $H^{i}(\mathcal{F}_{m} \otimes \hat{H}_{l}^{*})^{*} = 0$  in Proposition 3.4 (1), we see that the following natural map is zero

$$\beta^{-i}: R^{-i}\Gamma(R\Phi_{\mathcal{P}}D_A(\mathcal{F}_l)\otimes H_l) \to R^{-i}\Gamma(R\Phi_{\mathcal{P}}D_A(\mathcal{F}_m)\otimes H_l) = 0.$$

Chasing in the following commutative diagram

$$\begin{array}{ccc} R^{s}\Gamma(R\Phi_{\mathcal{P}}D_{A}(\mathcal{F}_{l})\otimes H_{l}) &\xrightarrow{\gamma_{\mathcal{F}_{l}}^{s}} H^{0}(R^{s}\Phi_{\mathcal{P}}D_{A}(\mathcal{F}_{l})\otimes H_{l}) \\ & \downarrow^{\beta^{s}} & \downarrow^{\bar{\beta}^{s}} \\ R^{s}\Gamma(R\Phi_{\mathcal{P}}D_{A}(\mathcal{F}_{m})\otimes H_{l})) &\xrightarrow{\gamma_{\mathcal{F}_{m}}^{s}} H^{0}(R^{s}\Phi_{\mathcal{P}}D_{A}(\mathcal{F}_{m})\otimes H_{l}) \end{array}$$

we can show that for  $s \leq -1$ , the map  $\bar{\beta}^s$  is zero, consequently the map  $R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_l) \to R^s \Phi_{\mathcal{P}} D_A(\mathcal{F}_m)$  is zero by the condition (a).

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To ease the notation, we denote  $\mathcal{E}_m = \sigma_{\leq -1} R \Phi_{\mathcal{P}} D_A(\mathcal{F}_m)$  for  $m = 0, 1, \dots, d$ . Then  $\mathcal{E}_m \in D^{[-d,-1]}(\hat{A})$  by Proposition 3.4 (2). And since for every  $i, \mathcal{H}^i(\mathcal{E}_m) \to \mathcal{H}^i(\mathcal{E}_{m+1})$  is a zero map, we can show that for every p, the homomorphism  $R^p \Psi_{\mathcal{P}}(\mathcal{H}^i(\mathcal{E}_m)) \to R^p \Psi_{\mathcal{P}}(\mathcal{H}^i(\mathcal{E}_{m+1}))$  is zero.

Step 2: We prove that for  $0 < m \leq d$ , the natural homomorphism

$$\alpha_k : R^0 \Psi_{\mathcal{P}}(\sigma_{>-m-1}\mathcal{E}_0) \to R^0 \Psi_{\mathcal{P}}(\sigma_{>-m-1}\mathcal{E}_m)$$

is zero, which completes the proof of Lemma 3.8 if setting m = d.

We prove this assertion by induction. First note that for every m and  $0 \leq i \leq d$ , the object  $\sigma_{\leq -m}(\sigma_{>-m-1}\mathcal{E}_i) \in D^b(\hat{A})$  is quasi-isomorphic to  $\mathcal{H}^{-m}(\mathcal{E}_i)[k]$ . When m = 1, the map  $\alpha_1 : R^0 \Psi_{\mathcal{P}}(\sigma_{>-2}\mathcal{E}_0) \to R^0 \Psi_{\mathcal{P}}(\sigma_{>-2}\mathcal{E}_1)$  coincides with the map  $R^1 \Psi_{\mathcal{P}}(\mathcal{H}^{-1}(\mathcal{E}_0)) \to R^1 \Psi_{\mathcal{P}}(\mathcal{H}^{-1}(\mathcal{E}_1))$ , hence it is zero. Now assume that  $\alpha_m$  is a zero map for some m < d. To prove  $\alpha_{m+1}$  is zero, we consider the following commutative diagram

where the horizontal sequences are triangles. Applying the right derived functor  $R\Psi_{\mathcal{P}}$  to the above diagram induces

where the horizontal sequences are exact. Since  $\alpha_m$  is assumed to be a zero map, we have  $\operatorname{Im}(\beta_m) \subseteq \operatorname{Im}(\mu_m)$ . And since  $\delta_m$  is zero, from the above commutative diagram we can conclude  $\operatorname{Im}(\alpha_{m+1} = \beta'_m \circ \beta_m) \subseteq \operatorname{Im}(\beta'_m \circ \mu_m) = 0$ . Therefore,  $\alpha_{m+1}$  is a zero map, and the proof is completed.

### 4. Subadditivity of Kodaira dimensions

In this section, we work over an algebraically closed field k with char k = p > 5. We will prove the following result on subadditivity of Kodaira dimensions.

**Theorem 4.1.** Let  $f: X \to Y$  be a fibration from a  $\mathbb{Q}$ -factorial projective 3-fold to a smooth projective variety of dimension 1 or 2. Let B be an effective  $\mathbb{Q}$ -divisor on X such that (X, B) is klt. Assume that Y is of maximal Albanese dimension, and and assume moreover that

• if  $\kappa(X_{\eta}, K_{X_{\eta}} + B_{\eta}) = \dim X/Y - 1$ , then B does not intersect the generic fiber  $X_{\xi}$  of the relative Iitaka fibration  $I : X \dashrightarrow Z$  induced by  $K_X + B$  on X over Y.

Then

$$\kappa(X, K_X + B) \ge \kappa(X_\eta, K_{X_\eta} + B_\eta) + \kappa(Y).$$

To prove the theorem above, we will first treat three subcases in the following theorems, which can be seen as complements of the results of [46].

**Theorem 4.2.** Let (X, B) be a projective Q-factorial klt pair of dimension 3. Let  $f : X \to Y = A$  be a fibration to an elliptic curve or a simple abelian surface. Assume that  $K_X + B$  is f-big. Then

$$\kappa(K_X + B) \ge \kappa(X_\eta, K_{X_\eta} + B_\eta).$$

**Theorem 4.3.** Let (X, B) be a projective  $\mathbb{Q}$ -factorial klt pair of dimension 3. Let  $f : X \to Y$  be a fibration to a normal curve Y of genus  $g(Y) \ge 1$ . Assume  $\kappa(X_{\eta}, (K_X + B)|_{X_{\eta}}) = 1$  and the condition  $\blacklozenge$  holds. Then

$$\kappa(X, K_X + B) \ge 1 + \kappa(Y).$$

**Theorem 4.4.** Let (X, B) be a projective Q-factorial klt pair of dimension 3. Let  $f : X \to Y$  be a fibration to a normal curve Y of genus  $g(Y) \ge 1$ . Assume  $\kappa(X_{\eta}, (K_X + B)|_{X_{\eta}}) = 0$ . Then

$$\kappa(X, K_X + B) \ge \kappa(Y).$$

If moreover  $K_X + B$  is nef then it is semi-ample.

We remark that Theorem 4.3 is a generalization of [12, Theorem 1.2, the subcase dim Y = 1 and  $\kappa(X_{\eta}) = 1$ ] where the cases without boundary were treated. Since the condition  $\blacklozenge$  is assumed, we can adapt the arguments of [12, Sec. 4] to our situation. For the convenience of the reader, we will provide a detailed proof.

4.1. **Preparations.** First let us recall an invariant introduced by Ejiri [11, Sec.4] to measure the positivity of a sheaf.

**Definition 4.5.** Let Y be a projective variety,  $\mathcal{F}$  a torsion free coherent sheaf and H an ample Q-Cartier Q-divisor on Y. Let

 $t(Y, \mathcal{F}, H) = \sup\{a \in \mathbb{Q} | \text{the sheaf } (F_Y^{e*}\mathcal{F}) \otimes \mathcal{O}_Y(\llcorner -p^e a H \lrcorner)$ 

is generically globally generated for some e > 0.

We will denote  $t(Y, \mathcal{F}) \ge 0$  if  $t(Y, \mathcal{F}, H) \ge 0$ . This property is independent of the choices of H ([46, Remark 2.13]) and is stronger than weak positivity ([11, Prop. 4.7]).

From the main results of [46] we deduce the following theorem.

**Theorem 4.6.** Let  $f : X \to Y$  be a separable fibration between smooth projective varieties, and let D be a nef and f-big Cartier divisor on X.

(1) If D is f-semi-ample, then

(1.a) for sufficiently divisible positive integers n and g, the sheaf  $F_Y^{g*} f_* \mathcal{O}_X(nD + K_{X/Y})$  contains a nonzero subsheaf  $V_n$  with  $t(Y, V_n) \ge 0$ , and  $\operatorname{rk} V_n \ge cn^{\dim X/Y}$  for some c > 0 independent of n; and

(1.b) for any sufficiently divisible n > 0 and big  $\mathbb{Q}$ -divisor H on Y,  $nD + K_{X/Y} + f^*H$  is big.

(2) If  $K_Y$  is big, then for any sufficiently divisible n > 0,  $nD + K_X$  is big.

Proof. We can assume D = A + E where A is f-ample and E is effective. For an integer n > 0 such that  $nE_{\overline{\eta}}$  is Cartier, let  $s_n$  be a global section of  $\mathcal{O}_{X_{\overline{\eta}}}(nE_{\overline{\eta}})$  with  $(s_n)_0 = nE_{\overline{\eta}}$ . We get an inclusion  $S^0(X_{\overline{\eta}}, (nA + K_X)_{\overline{\eta}}) \hookrightarrow S^0(X_{\overline{\eta}}, (nD + K_X)_{\overline{\eta}})$  by tensoring with  $s_n$ . Then by applying Lemma 2.14 on  $X_{\overline{\eta}}$  for the divisor  $(nA + K_X)_{\overline{\eta}}$ , we can show that there exists some c > 0 such that for any sufficiently divisible n,

$$\dim_{k(\bar{\eta})} S^0(X_{\overline{\eta}}, (nD + K_X)_{\overline{\eta}}) \ge \dim_{k(\bar{\eta})} S^0(X_{\overline{\eta}}, (nA + K_X)_{\overline{\eta}}) \ge cn^{\dim X/Y}$$

The assertion (1.a) follows from applying [46, Theorem 1.11]. For (1.b), fix a sufficiently divisible integer n > 0 such that  $F_Y^{g*} f_* \mathcal{O}_X(nD + K_{X/Y})$  contains a nonzero subsheaf  $V_n$  with  $t(Y, V_n) \ge 0$ . Applying [46, Theorem 4.1] shows that

$$\kappa(X, nD + K_{X/Y} + f^*H) \ge \kappa(X_{\eta}, (nD + K_{X/Y})|_{X_{\eta}}) + \dim Y = \dim X,$$

hence  $nD + K_{X/Y} + f^*H$  is big.

For (2), take an ample divisor H' on Y such that  $K_Y - H'$  is big. We can assume  $D + f^*H' \sim_{\mathbb{Q}} A' + \Delta$  where A' is an ample divisor and  $\Delta$  is an effective divisor with index not divisible by p. Since  $nD + K_X - K_{X/Y} - \Delta - f^*(K_Y - H') \sim_{\mathbb{Q}} (n-1)D + A'$  is nef and f-ample, and  $\dim_{k(\bar{\eta})} S^0_{\Delta_{\bar{\eta}}}(X_{\bar{\eta}}, (nD + K_X)_{\bar{\eta}}) > 0$  for sufficiently divisible n, we can prove (2) by applying [46, Theorem 1.5].

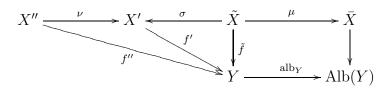
**Corollary 4.7.** Let (X, B) be a projective  $\mathbb{Q}$ -factorial klt pair of dimension 3. Let  $f : X \to Y$  be a fibration to a smooth projective curve or surface Y of maximal Albanese dimension. Assume that  $K_X + B$  is f-big. Then

$$\kappa_{\sigma}(X, K_X + B) \ge \kappa(X_{\eta}, (K_X + B)_{\eta}) + \kappa(Y).$$

In particular, if  $K_Y$  is big, then  $K_X + B$  is big.

Proof. Let  $(\bar{X}, \bar{B})$  be a log minimal model of (X, B) over Alb(Y). And let  $\rho : \bar{X} \to X$ be a log resolution such that the natural map  $\mu : \tilde{X} \to \bar{X}$  is a morphism. Let  $D = \mu^*(K_{\bar{X}} + \bar{B})$ . By Theorem 2.5 (3.1) and (3.4),  $(\bar{X}, \bar{B})$  is in fact minimal, and D is a nef divisor relatively big and semi-ample over Alb(Y) (hence over Y). By Proposition 2.8, we only need to prove  $\nu(\tilde{X}, D) \ge \dim X_{\eta} + \kappa(Y)$ .

First we reduce to separable fibrations by the following commutative diagram



where  $\sigma : \tilde{X} \to X'$  is a purely inseparable morphism constructed in Proposition 2.10 such that  $f' : X' \to Y$  is separable,  $\nu : X'' \to X'$  is a smooth resolution of singularities, and  $\tilde{f}, f', f''$  denote natural induced morphisms.

Since  $\sigma$  is purely inseparable, there exists D' on X' such that  $\sigma^*D' = D$ . Set  $D'' = \nu^*D'$ , which is a nef divisor relatively big and semi-ample over Y. Take a big divisor H on Y. By Theorem 4.6, for sufficiently divisible n > 0, the divisor  $nD'' + K_{X''} - f''^*K_Y + f''^*H$  is big. By Proposition 2.10, there exist a rational number t > 0 and an effective divisor  $\Delta'$  such that  $K_{\tilde{X}} \sim_{\mathbb{Q}} \sigma^* t(K_{X'} + \Delta')$ . And we

can write that  $K_{X''} = \nu^*(K_{X'} + \Delta') + E'' - F''$  where E'', F'' are effective divisors on X'' and E'' is  $\nu$ -exceptional. Applying Theorem 2.3, we conclude that

(4)  

$$\kappa(\tilde{X}, tnD + K_{\tilde{X}} - tf^*K_Y + tf^*H) \\ \geq \kappa(X', t(nD' - f'^*K_Y + f'^*H) + t(K_{X'} + \Delta')) \\ = \kappa(X'', \nu^*t(nD' - f'^*K_Y + f'^*H) + \nu^*t(K_{X'} + \Delta') + tE'') \\ \geq \kappa(X'', t(nD'' + K_{X''} - f''^*K_Y + f''^*H)) \geq 3.$$

Since this is true for any big  $\mathbb{Q}$ -divisor H, and D is nef, we conclude that for  $q \gg 0$ ,  $qD + K_{\tilde{X}} - t\tilde{f}^*K_Y$  is pseudo-effective.

When  $\kappa(Y) = 0$ , if dim  $X_{\eta} = 1$  then  $\nu(\tilde{X}, D) \ge \dim X_{\eta}$  since D is nonzero and nef; if dim  $X_{\eta} = 2$ , then for a general fiber  $\tilde{F}$  of  $\tilde{f}$ ,  $D^2 \cdot \tilde{F} > 0$  since D is nef and  $\tilde{f}$ -big, thus  $\nu(\tilde{X}, D) \ge 2$ .

When  $K_Y$  is big, by setting  $H = K_Y$  in Eq. (4), we obtain that  $nD + K_{\tilde{X}}$  is big. As we can write that  $K_{\tilde{X}} = \mu^*(K_{\bar{X}} + \bar{B}) + E - F$  where E, F are effective divisors on  $\tilde{X}$  and E is  $\mu$ -exceptional, applying Theorem 2.3, it follows that (n+1)D is big.

It remains to consider the case  $\kappa(Y) = 1$  and dim Y = 2. Take an ample divisor  $\overline{A}$  on  $\overline{X}$ . We only need to prove that  $D^2 \cdot \mu^* \overline{A} > 0$ . Fix a  $q \gg 0$  such that  $qD + K_{\tilde{X}} - t\tilde{f}^*K_Y$  is pseudo-effective. Then  $D^2 \cdot (qD + K_{\tilde{X}} - t\tilde{f}^*K_Y) \ge 0$ . And since E is  $\mu$ -exceptional, by projection formula we have  $\mu^*(K_{\bar{X}} + \overline{B}) \cdot \mu^* \overline{A} \cdot E = 0$ . Take a general divisor  $H' \in |NK_Y|$  for some sufficiently divisible N and set  $\tilde{H}' = \tilde{f}^*H'$ . Then  $\tilde{H}'$  contains a component  $\tilde{H}$ , such that  $\mu^* \overline{A}|_{\tilde{H}}$  is semi-ample and big and that  $D|_{\tilde{H}}$  is nef and  $\tilde{f}|_{\tilde{H}}$ -big. Therefore,

$$D \cdot \mu^* \bar{A} \cdot \tilde{f}^* K_Y = \frac{1}{N} D \cdot \mu^* \bar{A} \cdot \tilde{H}' \ge \frac{1}{N} D \cdot \mu^* \bar{A} \cdot \tilde{H} = \frac{1}{N} (D|_{\tilde{H}}) \cdot (\mu^* \bar{A}|_{\tilde{H}}) > 0$$

where the last strict inequality is obtained by applying Hodge Index Theorem. Finally the proof is completed by

$$(q+1)D^{2} \cdot \mu^{*}\bar{A} = D \cdot (qD + \mu^{*}(K_{\bar{X}} + \bar{B})) \cdot \mu^{*}\bar{A} = D \cdot (qD + K_{\tilde{X}} - E + F) \cdot \mu^{*}\bar{A}$$
  

$$\geq D \cdot (qD + K_{\tilde{X}} - t\tilde{f}^{*}K_{Y} + t\tilde{f}^{*}K_{Y}) \cdot \mu^{*}\bar{A} > 0.$$

Recall a positivity result on surfaces.

**Lemma 4.8.** ([12, Lemma 2.11]) Let  $g: Z \to Y$  be a generically smooth fibration from a smooth projective surface to a smooth projective curve. Let H be a nef and g-big divisor on Z. Then  $g_*\mathcal{O}_Z(K_{Z/Y} + lH)$  is a nef vector bundle for every  $l \gg 0$ .

The following result was proved by Waldron [44] when  $\kappa = 2$ , and the case  $\kappa = 1$  follows easily from applying Theorem 2.5 (3.2).

**Theorem 4.9.** ([48, Theorem 3.1]) Let (X, B) be a  $\mathbb{Q}$ -factorial klt projective 3-fold. Assume that  $K_X + B$  is nef. If  $\kappa(X, K_X + B) \ge 1$ , then  $K_X + B$  is semi-ample.

We extract the following lemma from the strategy of [12, Sec. 4], which will be used in the proof of Theorem 4.2 and 4.3.

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**Lemma 4.10.** Let  $(\hat{X}, \hat{B})$  be a minimal projective  $\mathbb{Q}$ -factorial dlt pair of dimension 3, and let  $\hat{f} : \hat{X} \to Y$  be a fibration to a normal variety. Assume that

(a)  $\hat{B} = G_1 + G_2 + \dots + G_n$  is a sum of prime Weil divisors.

Then for every  $j = 1, 2, \dots, n$ ,  $(K_{\hat{X}} + \hat{B})|_{G_j}$  is semi-ample, and a general fiber  $F_j$  of the Iitaka fibration induced by  $(K_{\hat{X}} + \hat{B})|_{G_j}$  is integral.

Assume moreover that

(b) there exist N > 0 and two different effective Cartier divisors  $\hat{D}_i, i = 1, 2$  such that  $\hat{D}_i \sim N(K_{\hat{X}} + \hat{B}) + \hat{f}^*L_i$  for some  $L_i \in \text{Pic}^0(Y)$  and that

$$\operatorname{Supp}\hat{D}_i \subseteq \operatorname{Supp}\hat{B};$$

(c) there exist effective divisors  $\hat{G}_1, \hat{G}_2, \hat{G}'_1, \hat{G}'_2$  such that  $\hat{D}_1 = a_{11}\hat{G}_1 + a_{12}\hat{G}_2 + \hat{G}'_1$  and  $\hat{D}_2 = a_{21}\hat{G}_1 + a_{22}\hat{G}_2 + \hat{G}'_2$  where  $a_{11} > a_{21} \ge 0$  and  $a_{22} > a_{12} \ge 0$ ; and

(d) there exist two irreducible components, say,  $G_1, G_2$  of  $\hat{G}_1, \hat{G}_2$  respectively, such that for  $i, j \in \{1, 2\}$  and  $i \neq j$ ,  $F_j$  is dominant over Y and

$$F_j \cap \text{Supp } (\hat{G}_j'' := \hat{G}_i + \hat{G}_1' + \hat{G}_2') = \emptyset.$$

Then both  $L_1$  and  $L_2$  are torsion.

Furthermore, condition (d) holds, if for  $j = 1, 2, G_j$  is not a component of  $\hat{G}''_j$ and  $\kappa(F_j) \geq 0$ .

Proof. Note that since each  $G_i$  is a dlt center of  $(\hat{X}, \hat{B})$ ,  $G_i$  is a normal surface by [2, Lemma 4.2]. Moreover we have  $(\hat{B}-G_i)|_{G_i} \ge 0$ , and  $(K_{\hat{X}}+\hat{B})|_{G_i} = K_{G_i}+(\hat{B}-G_i)|_{G_i}$  is log canonical. Then since  $K_{\hat{X}}+\hat{B}$  is assumed nef, by [41, Theorem 1.2],  $(K_{\hat{X}}+\hat{B})|_{G_i}$  is semi-ample. And as dim  $G_i = 2$ ,  $F_j$  is always integral by Proposition 2.1.

Assume (b), (c) and (d). Let's prove that  $L_1, L_2$  are torsion. First considering the restrictions on  $F_1$ , by  $(K_{\hat{\chi}} + \hat{B})|_{F_1} \sim_{\mathbb{Q}} 0$ , applying these assumptions we can show

$$\begin{aligned} a_{21}\hat{f}^*L_1|_{F_1} \sim_{\mathbb{Q}} a_{21}(N(K_{\hat{X}}+\hat{B})+\hat{f}^*L_1)|_{F_1} \\ \sim a_{21}\hat{D}_1|_{F_1} \sim a_{11}a_{21}\hat{G}_1|_{F_1} \sim a_{11}\hat{D}_2|_{F_1} \\ \sim a_{11}(N(K_{\hat{X}}+\hat{B})+\hat{f}^*L_2)|_{F_1} \sim_{\mathbb{Q}} a_{11}\hat{f}^*L_2|_{F_1}, \end{aligned}$$

thus  $a_{21}L_1 \sim_{\mathbb{Q}} a_{11}L_2$  by Lemma 2.4. And restricting on  $F_2$ , in the same way we can show  $a_{22}L_1 \sim_{\mathbb{Q}} a_{12}L_2$ . Then granted these two relations, we can conclude  $L_i \sim_{\mathbb{Q}} 0, i = 1, 2$  from the fact that  $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  is invertible over  $\mathbb{Q}$ , which is because  $a_{11} > a_{21} \ge 0$  and  $a_{22} > a_{12} \ge 0$ .

It remains to prove the third assertion. As  $\kappa(F_j) \ge 0$ , we can assume the canonical divisor  $K_{F'_j} \ge 0$  where  $F'_j$  is the normalization of  $F_j$ . Applying the adjunction formula, we get

$$0 \sim_{\mathbb{Q}} (K_{\hat{X}} + B)|_{F'_j} \sim_{\mathbb{Q}} ((K_{\hat{X}} + B)|_{G_j})|_{F'_j} \\ \sim_{\mathbb{Q}} (K_{G_j} + (\hat{B} - G_j))|_{F'_j} \sim_{\mathbb{Q}} K_{F'_j} + C'_j + (\hat{B} - G_j)|_{F'_j}$$

where  $C'_j \ge 0$  on  $F'_j$ . As  $F_j$  is general, we may assume  $F_j$  is not contained in  $\hat{B} - G_j$ . In turn we conclude that  $(\hat{B} - G_j)|_{F'_j} = 0$ . This, combing with the assumption that  $G_j$  is not a component of  $\hat{G}''_j$ , indicates that  $F_j \cap \text{Supp } \hat{G}''_j = \emptyset$ .

4.2. **Proof of Theorem 4.2.** For a sufficiently large integer e, the Weil index of  $B' = \frac{p^e}{p^e+1}B$  is not divisible by p, and  $K_X + B'$  is still f-big. Replacing B with B', we can assume the Weil index of B is not divisible by p. By Theorem 2.5 (3.1, 3.4), we can replace X with the relative log canonical model over A with the loss of X being  $\mathbb{Q}$ -factorial, thus  $K_X + B$  is a nef and f-ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor.

We claim that we only need to prove  $\kappa(K_X + B) \geq 1$ . Indeed, if this is true then  $K_X + B$  is semi-ample by Theorem 4.9, thus for a sufficiently divisible M > 0, the linear system  $|M(K_X + B)|$  has no base point. Since  $(K_X + B)_\eta$  is big, the restriction  $|M(K_X + B)||_{X_\eta}$  on the generic fiber defines a generically finite morphism, which indicates that  $\kappa(X, K_X + B) \geq \kappa(X_\eta, (K_X + B)_\eta)$ .

Let l, g > 0 be two integers such that  $l(K_X + B)$  is Cartier and  $(p^g - 1)B$  is integral. For an integer e > 0 divisible by g, we have the trace map

$$Tr_{X,B}^{e,l}: \mathcal{F}_{e,l}:=f_*(F_{X*}^e\mathcal{O}_X((1-p^e)(K_X+B))\otimes\mathcal{O}_X(l(K_X+B))) \to f_*\mathcal{O}_X(l(K_X+B)),$$

and denote its image by  $\mathcal{F}_{e}^{l}$ . The restriction of  $\mathcal{F}_{e}^{l}$  on the generic point  $\eta$  defines a linear system  $|(\mathcal{F}_{e}^{l})_{\eta}|$  contained in  $|l(K_{X} + B)_{\eta}|$ .

We claim the following assertions with the proof postponed.

- C1 For every l > 0 such that  $l(K_X + B)$  is Cartier, there exists some e(l) > 0such that for any  $e \ge e(l)$  divisible by g, the rank of  $\mathcal{F}_e^l$  is a stable number  $r_l$ ; and there exists an integer K > 0 such that, for any l divisible by K and any e divisible by g, the linear system  $|(\mathcal{F}_e^l)_{\eta}|$  defines a generically finite map.
- C2 If L is an ample line bundle on A and  $l \ge 2$  is an integer such that  $l(K_X + B)$  is Cartier, then for any i > 0 and sufficiently divisible integer e > 0,

$$H^i(A, \mathcal{F}_{e,l} \otimes \tilde{L}^*) = 0.$$

Fix a positive integer l divisible by K and an integer  $e_0$  such that rk  $\mathcal{F}_{e_0}^l = r_l$ . Let  $\mathcal{F} = \mathcal{F}_{e_0}^l$ . Then by (C1), the linear system  $|\mathcal{F}_{\eta}|$  defines a generically finite map of  $X_{\eta}$ . Recall that for positive integers e' > e divisible by g, there exists a natural trace map  $\mathcal{F}_{e',l} \to \mathcal{F}_{e,l}$  (Sec. 2.7). Applying (C2), by induction we can find two integers  $e_1 < e_2$  divisible by g and bigger than  $e_0$ , two ample line bundles  $H_0, H_1$  on  $\hat{A}$  and three sheaves  $\mathcal{F}_0 := \mathcal{F}, \mathcal{F}_1 := \mathcal{F}_{e_1,l}, \mathcal{F}_2 := \mathcal{F}_{e_2,l}$  which satisfy the conditions of Theorem 3.7. Note that if dim A = 1 we only need two sheaves  $\mathcal{F}_0, \mathcal{F}_1$ . Applying Theorem 3.7 (i) and Proposition 3.4 (2), the cohomological locus  $V^0(\mathcal{F}) = (-1)^*_{\hat{A}} \operatorname{Supp} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{F}) \neq \emptyset$ .

The statement is trivial when  $\nu(K_X + B) = 3$ , so from now on we assume  $\nu(K_X + B) \leq 2$ . We break the proof into three steps.

Step 1: We prove that if dim  $V^0(\mathcal{F}) > 0$ , then  $\kappa(K_X + B) \ge 1$ .

In this case  $V^0(\mathcal{F})$  generates  $\hat{A}$  since  $\hat{A}$  is simple. Hence so does  $V^0(f_*\mathcal{O}_X(l(K_X + B)))$  too, as  $V^0(\mathcal{F}) \subseteq V^0(f_*\mathcal{O}_X(l(K_X + B)))$ . So for  $m \ge \dim A$ , the map

$$\times^{m} V^{0}(f_{*}\mathcal{O}_{X}(l(K_{X}+B))) \to \hat{A} \text{ via } (\alpha_{1},\alpha_{2},\cdots,\alpha_{m}) \mapsto \alpha_{1}+\alpha_{2}+\cdots+\alpha_{m}$$

$$H^{0}(X, l(K_{X} + B) + f^{*}\alpha_{1}) \times H^{0}(X, l(K_{X} + B) + f^{*}\alpha_{2}) \times \dots \times H^{0}(X, l(K_{X} + B) + f^{*}\alpha_{m}) \\ \to H^{0}(X, ml(K_{X} + B) + f^{*}(\alpha_{1} + \alpha_{2} \dots + \alpha_{m})) \cong H^{0}(X, ml(K_{X} + B)).$$

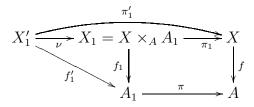
We can show  $h^0(X, ml(K_X + B)) \ge 2$ , thus  $\kappa(K_X + B) \ge 1$ .

 $(\alpha_1, \alpha_2, \cdots, \alpha_m)$  mapped to 0. Consider the natural map

From now on we assume that  $V^0(\mathcal{F}) = (-1)^*_{\hat{A}} \operatorname{Supp} R^0 \Phi_{\mathcal{P}} D_A(\mathcal{F})$  consists of finitely many closed points.

Step 2: We will find an integer  $m_1$  and some divisors  $D_i \in |m_1(K_X+B)+f^*L_i|, i = 1, 2, \cdots, r$  for some  $L_i \in \text{Pic}^0(A)$ , such that the sub-linear system of  $|m_1(K_X+B)_{\eta}|$  generated by  $(D_i)_{\eta}, i = 1, 2, \cdots, r$  defines a generically finite map of  $X_{\eta}$ .

By Theorem 3.7 (ii), there exist an isogeny  $\pi : A_1 \to A, P_1, P_2, \cdots, P_s \in \operatorname{Pic}^0(A_1)$ and a generically surjective homomorphism  $\bigoplus_j P_j \to \pi^* \mathcal{F}$ . Consider the following commutative diagram



where  $X'_1$  is the normalization of the reduced scheme structure of  $X_1$ . There exists a natural composition homomorphism by [17, Chapter III. Prop. 9.3]

$$\alpha : \bigoplus_{j} P_{j} \to \pi^{*} \mathcal{F} \hookrightarrow \pi^{*} f_{*} \mathcal{O}_{X}(l(K_{X} + B)) \cong f_{1*} \pi_{1}^{*} \mathcal{O}_{X}(l(K_{X} + B))$$
$$\to f_{1*}'(\nu^{*} \pi_{1}^{*} \mathcal{O}_{X}(l(K_{X} + B))) \cong f_{1*}'(\pi_{1}'^{*} \mathcal{O}_{X}(l(K_{X} + B))).$$

The linear system  $|(\operatorname{Im}\alpha)_{\eta}| \subseteq |\pi_1'^*(l(K_X+B))|_{(X_1')_{\eta}}|$  defines a generically finite map of  $(X_1')_{\eta}$ . If the direct summand  $P_j$  is not mapped to zero via  $\alpha$ , then  $h^0(X_1', \pi_1'^*(l(K_X+B)) - f_1'^*P_j) \geq 1$ . Since  $\pi^* : \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(A_1)$  is an isogeny, there exist  $Q_j \in \operatorname{Pic}^0(A)$  such that  $P_j = \pi^*Q_j$ . Applying Theorem 2.3, if  $\alpha(P_j) \neq 0$  then

$$\kappa(X, l(K_X + B) - f^*Q_j) = \kappa(X', \pi_1'^*(l(K_X + B) - f_1'^*P_j) \ge 0$$

We can find a sufficiently divisible integer  $l_1 > 0$  such that for every j, if  $\alpha(P_j) \neq 0$ , the pull-back linear system  $(\pi_1'^*|l_1(l(K_X + B) - f_1'^*P_j)|)|_{(X'_1)_\eta}$  defines a rational map  $\phi_j : (X'_1)_\eta \longrightarrow \mathbb{P}^{r_j}$  where  $r_j = \dim |l_1((l(K_X + B) - f_1'^*P_j))|$ , whose image has dimension  $\kappa(X, l(K_X + B) - f^*Q_j)$ . By the construction, the sub-linear system of  $|\pi_1'^*(l_1l(K_X + B))_\eta|$  generated by the divisors of  $(\pi_1'^*|l_1(l(K_X + B) - f_1'^*P_j)|)|_{(X'_1)_\eta}, j =$  $1, 2, \cdots, s$  defines a generically finite map of  $(X'_1)_\eta$ . Therefore, there exist  $L_i =$  $-l_1Q_{j_i} \in \operatorname{Pic}^0(A)$  for some  $j_1, j_2, \cdots, j_r$  and effective divisors  $D_i \in |m_1(K_X + B) + f^*L_i|$  where  $m_1 = l_1l$ , such that the linear system generated by  $(D_i)_\eta, i = 1, 2, \cdots, r$ defines a generically finite map of  $X_\eta$ .

We aim to prove that there exist at least two different divisors among  $D_i$ , say,  $D_1 \neq D_2$ , such that  $L_1, L_2$  are torsion in  $\operatorname{Pic}^0(A)$ . Then for some sufficiently divisible N > 0 such that  $NL_1 \sim NL_2 \sim 0$ , we have  $ND_1, ND_2 \in |Nm_1(K_X + B)|$ , which concludes  $\kappa(X, K_X + B) \geq 1$ .

Step 3: We will construct a minimal dlt pair  $(\hat{X}, \hat{B})$  and divisors  $\hat{D}_1, \hat{D}_2$  satisfying the conditions of Lemma 4.10.

(3.1) Take a log resolution  $\mu : \tilde{X} \to X$  of  $B + \sum_i D_i$ . Let  $\tilde{B}$  be the reduced divisor supported on the union of  $\mu^{-1}(B + \sum_i D_i)$  and the  $\mu$ -exceptional divisors. Then  $(\tilde{X}, \tilde{B})$  is dlt, and  $K_{\tilde{X}} + \tilde{B}$  has a weak Zariski decomposition. By Theorem 2.5, running a relative log MMP for  $(\tilde{X}, \tilde{B})$  over A, we can get a dlt log minimal model  $(\hat{X}, \hat{B})$  and a fibration  $\hat{f} : \hat{X} \to A$ . The divisor  $\tilde{E} = K_{\tilde{X}} + \tilde{B} - \mu^*(K_X + B)$  is effective. Take a sufficiently divisible integer  $l_2 > 0$  such that  $l_2\tilde{E}$  is Cartier. Let  $m_2 = m_1 l_2$ . We get effective divisors

$$\tilde{D}_i = l_2 \mu^* D_i + l_2 \tilde{E} \sim m_2 (K_{\tilde{X}} + \tilde{B}) + l_2 \mu^* f^* L_i$$

and the push-forward divisors via the natural map  $\tilde{X} \dashrightarrow \hat{X}$ 

$$\hat{D}_i \sim m_2(K_{\hat{X}} + \hat{B}) + l_2 \hat{f}^* L_i.$$

(3.2) We prove  $\nu(K_{\hat{X}} + \hat{B}) = \nu(K_X + B)$  as follows. Applying Proposition 2.8, on one hand since  $K_{\tilde{X}} + \tilde{B} \ge \mu^*(K_X + B)$  we have  $\nu(K_{\hat{X}} + \hat{B}) = \kappa_{\sigma}(K_{\tilde{X}} + \tilde{B}) \ge \nu(K_X + B)$ , on the other hand since there exists an effective  $\mu$ -exceptional divisor  $\tilde{E}'$  such that  $K_{\tilde{X}} + \tilde{B} \le \mu^*(K_X + B) + \sum_i \mu^* D_i + \tilde{E}' \equiv (rm_1 + 1)\mu^*(K_X + B) + \tilde{E}'$ , we conclude  $\kappa_{\sigma}(K_{\tilde{X}} + \tilde{B}) \le \nu(K_X + B)$ . In summary, we get the equality  $\nu(K_{\hat{X}} + \hat{B}) = \nu(K_X + B)$ .

(3.3) The restrictions of  $\hat{D}_i$  on  $\hat{X}_\eta$  generate a linear system  $|\hat{V}| \subseteq |m_2(K_{\hat{X}} + \hat{B})_\eta|$ , which defines a generically finite map  $\hat{X}_\eta \dashrightarrow \mathbb{P}_{k(\eta)}^{r-1}$  by the construction in Step 2. Let  $\hat{C}_\eta$  be the fixed part of  $|\hat{V}|$ , and set  $\hat{A}_{i,\eta} = (\hat{D}_i)_\eta - \hat{C}_\eta$ . We may assume  $\hat{A}_{1,\eta} \neq \hat{A}_{2,\eta}$ . Since  $\hat{A}_{1,\eta} \sim \hat{A}_{2,\eta}$ , we can choose two irreducible components  $G_{1,\eta}, G_{2,\eta}$  of  $\hat{A}_{1,\eta} + \hat{A}_{2,\eta}$ such that,

• if  $b_{ij}, i, j = 1, 2$  are the coefficients of  $G_{j,\eta}$  in  $\hat{A}_{i,\eta}$  respectively, i.e.,

 $\hat{A}_{1,\eta} = b_{11}G_{1,\eta} + b_{12}G_{2,\eta} + G'_{1,\eta}$  and  $\hat{A}_{2,\eta} = b_{21}G_{1,\eta} + b_{22}G_{2,\eta} + G'_{2,\eta}$ 

where neither of  $G_{1,\eta}, G_{2,\eta}$  are contained in  $G'_{1,\eta} + G'_{2,\eta}$ , then  $b_{11} > b_{21} \ge 0$ and  $b_{22} > b_{12} \ge 0$ .

(3.4) If dim  $\hat{X}_{\eta} = 2$ , since  $K_{\hat{X}} + \hat{B}$  is relatively big over A, we can choose  $\hat{A}_{2,\eta}$ and  $G_{i,\eta}, i = 1, 2$  such that  $(K_{\hat{X}} + \hat{B})|_{G_{i,\eta}}$  is big as follows. First we can choose a component  $G_{1,\eta}$  of  $\hat{A}_{1,\eta}$  such that  $(K_{\hat{X}} + \hat{B})|_{G_{1,\eta}}$  is big. Second since  $\hat{A}_{i,\eta}, i =$  $1, 2, \dots, r$  have no common component, we can choose  $\hat{A}_{2,\eta}$  not containing  $G_{1,\eta}$ , i.e.,  $b_{21} = 0$ . Finally since  $\hat{A}_{1,\eta} \sim \hat{A}_{2,\eta}$  and the intersection number  $(K_{\hat{X}} + \hat{B}) \cdot (\hat{A}_{1,\eta} - b_{11}G_{1,\eta}) < (K_{\hat{X}} + \hat{B}) \cdot \hat{A}_{2,\eta}, \hat{A}_{2,\eta}$  must have an irreducible component  $G_{2,\eta}$  such that  $(K_{\hat{X}} + \hat{B}) \cdot G_{2,\eta} > 0$  and the coefficients  $b_{22} > b_{12}$ .

(3.5) Let  $G_i, i = 1, 2$  be the reduced irreducible divisors on  $\hat{X}$  such that  $(G_i)_{\eta} = G_{i,\eta}$ . By (3.3) we can write that

$$\hat{D}_1 = a_{11}G_1 + a_{12}G_2 + G'_1$$
 and  $\hat{D}_2 = a_{21}G_1 + a_{22}G_2 + G'_2$ 

where  $a_{11} > a_{21} \ge 0$  and  $a_{22} > a_{12} \ge 0$ , and neither of  $G_1, G_2$  are contained in  $G'_1 + G'_2$ .

(3.6) By Lemma 4.10, we know that  $G_i$  is normal, and  $(K_{\hat{X}} + \hat{B})|_{G_i}$  is semi-ample. Since  $\hat{D}_i \equiv m_2(K_{\hat{X}} + \hat{B})$  and  $G_i$  is a component of  $\hat{D}_i$ , by Proposition 2.8 we have

$$\kappa(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) = \nu(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) \le \nu(\hat{X}, K_{\hat{X}} + \hat{B}) - 1.$$

For the case dim A = 2, we have  $\nu(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) = 0$  or 1. The case  $\nu(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) = 1$  does not happen, because otherwise, the divisor  $(K_{\hat{X}} + \hat{B})|_{G_i} \sim_{\mathbb{Q}} K_{G_i} + (\hat{B} - G_i)|_{G_i}$  will induce a fibration fibred by curves of arithmetic genus  $\leq 1$  on  $G_i$ , which is impossible since  $G_i$  is dominant over A and A is simple. Thus the Iitaka fibration induced by  $(K_{\hat{X}} + \hat{B})|_{G_i}$  is trivial. By (3.5), the surface  $G_i$  is dominant over A, hence  $\kappa(G_i, K_{G_i}) \geq 0$  (see for example [1, Prop. 13.1]).

For the case dim A = 1,  $\hat{X}_{\eta}$  is a surface over  $k(\eta)$ , and by (3.4)

$$\kappa(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) = \nu(G_i, (K_{\hat{X}} + \hat{B})|_{G_i}) = 1.$$

Take a general fiber  $F_i$  of the Iitaka fibration induced by  $(K_{\hat{X}} + \hat{B})|_{G_i}$  on  $G_i$ . Since  $(K_{\hat{X}} + \hat{B})|_{G_{i,\eta}}$  is big,  $F_i$  is a curve dominant over A, thus  $\kappa(F_i, K_{F_i}) \ge 0$ .

Let  $\hat{G}_i = G_i$  and  $\hat{G}'_i = G'_i$  for i = 1, 2. By (3.1), (3.5) and (3.6), the conditions of Lemma 4.10 are satisfied, hence  $l_2L_1, l_2L_2$  are torsion, which finishes the proof.

It remains to prove Claim (C1) and (C2).

(C1) For every l > 0, if e' > e then  $\mathcal{F}_{e'}^l \subseteq \mathcal{F}_e^l$ , hence there exists some e(l) such that for every  $e \ge e(l)$ , we have rk  $\mathcal{F}_e^l = \operatorname{rk} \mathcal{F}_{e(l)}^l$ , which is denoted by  $r_l$ .

To study the linear system  $|(\mathcal{F}_e^l)_{\eta}|$ , we may replace B with  $B + f^*H$  for some ample divisor H on A, so  $K_X + B$  is ample. And we may replace B with a bigger divisor B + tD for some effective divisor  $D \sim_{\mathbb{Q}} K_X + B$  and a rational number t > 0, to make the Cartier index of  $K_X + B$  not divisible by  $p^2$ . Moreover we remark that for some fixed l, to show that the linear system  $|(\mathcal{F}_e^l)_{\eta}|$  defines a generically finite map of  $X_{\eta}$ , we only need to prove this assertion for any sufficiently divisible e.

Fix a sufficiently divisible integer g' > 0 such that  $(1 - p^{g'})(K_X + B)$  is Cartier, and that for the non-*F*-pure ideal  $\sigma(B)$  (Sec. 2.7)

$$Tr_{X,B}^{g'}(F_{X*}^{g'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{g'})(K_X+B)))) = \sigma(B).$$

Since  $K_X + B$  is ample, applying Lemma 2.14, we can find a sufficiently divisible integer K such that for every positive integer l divisible by K the trace map

$$\Phi_{e,l}: H^0(X, F_{X*}^{eg'}(\sigma(B) \cdot \mathcal{O}_X((1-p^{eg'})(K_X+B))) \otimes \mathcal{O}_X(l(K_X+B))) \rightarrow H^0(X, \sigma(B) \cdot \mathcal{O}_X(l(K_X+B))).$$

$$K_X + B + \frac{1}{p^v + 1}D = -A + \frac{1}{p^v + 1}D_2 + \frac{p^{2v}}{p^{2v} - 1}(D_1 + B)$$

<sup>&</sup>lt;sup>2</sup>The divisor D can be obtained as follows. Take an effective Weil divisor  $D_1 \sim K_X + A$  for some ample divisor A on X. For sufficiently large v the divisor  $K_X + B - \frac{1}{p^v - 1}(D_1 + B)$  is an ample Q-Cartier Q-divisor and hence is Q-linearly equivalent to an effective divisor  $D_2$ , which can be assumed to have Cartier index not divisible by p. Let  $D = D_2 + \frac{1}{p^v - 1}(D_1 + B)$ . Identifying  $K_X$ with  $D_1 - A$  similarly as in [37, Lemma 3.15], one can verify that p does not divide the Cartier index of

is surjective. It follows that

$$\sigma(B) \cdot \mathcal{O}_X(l(K_X + B))||_{X_\eta} = |\mathrm{Im}(\Phi_{e,l})||_{X_\eta}.$$

If necessary, we can enlarge K to assume that the linear system  $|\sigma(B) \cdot \mathcal{O}_X(K(K_X + B))|$  defines a generically finite map of X. Then by  $|\text{Im}(\Phi_{e,l})||_{X_\eta} \subseteq |(\mathcal{F}_e^l)_{\eta}|$ , we show that  $|(\mathcal{F}_e^l)_{\eta}|$  defines a generically finite map of  $X_{\eta}$ .

(C2) Before the proof we remind that the Weil index  $q_0$  of  $K_X + B$  is not divisible by p, but the Cartier index  $q_1$  is possibly divisible by p. From now on to the end of the proof of (C2), we assume the integer e always satisfies  $q_0|p^e - 1$ . Let  $d = \frac{q_1}{q_0}$ . Then  $H = q_1(K_X + B)$  is a nef and f-ample divisor, and we have a set consisting of finitely many coherent sheaves

$$\{\mathcal{G}_r = \mathcal{O}_X(rq_0(K_X + B)) | r = 0, 1, \cdots d - 1\}.$$

Define  $\phi_L : \hat{A} \to A$  via  $\hat{t} \mapsto T^*_{\hat{t}}L \otimes L^{-1} \in \operatorname{Pic}^0(\hat{A}) = A$ . Then by results of [31, Sec. 13, 16], the morphism  $\phi_L$  is an isogeny since L is ample, and

$$\phi^* \hat{L}^* \cong \bigoplus^i L$$
 where  $r = h^0(\hat{A}, L)$ .

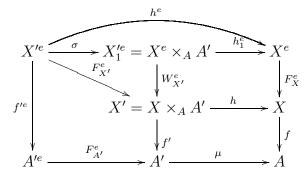
The isogeny  $\phi_L : \hat{A} \to A$  is not necessarily separable. Denote by  $\hat{K}$  the kernel of  $\phi_L$ , let  $\hat{K}_0$  be the maximal sub-group of  $\hat{K}$  supported at  $\hat{0}$  and let  $A' = \hat{A}/\hat{K}_0$ . Then we have a factorization

$$\phi_L: \hat{A} \xrightarrow{\nu} A' \xrightarrow{\mu} A$$

where  $\nu : \hat{A} \to A'$  is the natural quotient map which is purely inseparable, and  $\mu : A' \to A$  is étale. Fix a sufficiently large integer  $g_1$  such that  $F_{A'}^{g_1} : A'^{g_1} \to A'$  factors through  $\nu$ . Let

$$\phi = \mu \circ F_{A'}^{g_1} : A'^{g_1} \to A' \to A$$

Then  $\phi^* \hat{L}^* \cong \bigoplus^r L'$  where L' is an ample line bundle on  $A'^{g_1}$ . For a larger integer  $e > g_1$ , we get the following commutative diagram



where  $W_{X'}^e, h_1^e, f', h$  denote the natural projections of the corresponding fiber products, and  $\sigma : X'^e \to X_1'^e$  arises from the universal property of the fiber product. Since the base change  $h : X' \to X$  is étale,  $\sigma$  is an isomorphism.

Assume  $l \ge 2$ ,  $q_1|l$ . Let  $n_e = \lfloor \frac{1-p^e+lp^e}{q_1} \rfloor$ . Then we can write that  $1-p^e+lp^e = n_e q_1 + r_e q_0$  where  $0 \le r_e < d$ . It follows that

$$\mathcal{O}_X((1-p^e+lp^e)(K_X+B))\cong \mathcal{O}_X(n_eH)\otimes \mathcal{G}_{r_e}.$$

By Theorem 3.5 (3), to get that for every i > 0,

$$H^{i}(A, f_{*}(F_{X*}^{e}\mathcal{O}_{X}((1-p^{e})(K_{X}+B))\otimes\mathcal{O}_{X}(l(K_{X}+B)))\otimes\hat{L}^{*})=0,$$

we only need to verify that for every i > 0,

$$H^{i}(A', \mu^{*}(f_{*}F^{e}_{X_{*}}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \hat{L}^{*}))=0.$$

This is true when  $e \gg 0$ , which is proved as follows

$$\begin{aligned} H^{i}(A', \mu^{*}(f_{*}F_{X*}^{e}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes L^{*})) \\ &\cong H^{i}(A', \mu^{*}(f_{*}F_{X*}^{e}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \mu^{*}\hat{L}^{*})) \\ &\cong H^{i}(A', f'_{*}W_{X'*}^{e}(h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \mu^{*}\hat{L}^{*}) \\ &\cong H^{i}(A', f'_{*}F_{X'*}^{e}(h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \mu^{*}\hat{L}^{*}) \\ &\cong H^{i}(A', f'_{*}F_{X'*}^{e}(h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \mu^{*}\hat{L}^{*}) \\ &\cong H^{i}(A', F_{A'*}^{e}f_{*}^{fe}(h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes \mu^{*}\hat{L}^{*}) \\ &\cong H^{i}(A'^{e}, f'^{e}(h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes F_{A'}^{e*}\mu^{*}\hat{L}^{*}) \\ &\cong H^{i}(A'^{e}, f'^{e}((h^{e*}\mathcal{O}_{X}((1-p^{e}+lp^{e})(K_{X}+B)))\otimes (f'^{e*}F_{A'}^{(e-g_{1})*}F_{A'}^{g_{1}*}\mu^{*}\hat{L}^{*})) \\ &\cong H^{i}(A', f'_{*}(\mathcal{O}_{X'}(h^{*}n_{e}H)\otimes h^{*}\mathcal{G}_{r_{e}}\otimes \bigoplus^{r} f'^{*}(L')^{p^{e-g_{1}}})) \\ &\cong \bigoplus^{r} H^{i}(X', \mathcal{O}_{X'}(h^{*}n_{e}H+p^{e-g_{1}}f'^{*}L')\otimes h^{*}\mathcal{G}_{r_{e}}) = 0 \end{aligned}$$

where

- the 2<sup>nd</sup> ≃ is due to μ\*f\*F\*e ≃ f'\*W\*\*h1e\* since μ is a flat base change;
  the 3<sup>rd</sup> ≃ is due to the fact that σ is an isomorphism;
- the 6<sup>th</sup>  $\cong$  is from applying projection formula and  $RF_{A'*}^e \cong F_{A'*}^e$ ;
- the  $9^{\text{th}} \cong$  is from applying Lerray spectral sequence and relative Fujita vanishing (Lemma 2.2)  $R^j f'_*(\mathcal{O}_{X'}(h^*n_eH) \otimes h^*\mathcal{G}_{r_e}) = 0$  for j > 0 since  $n_eH$  is sufficiently f'-ample if  $e \gg 0$ , and the last vanishing follows from applying Fujita vanishing since  $h^* n_e H + p^{e-g_1} f'^* L'$  is sufficiently ample if  $e \gg 0$ .

4.3. **Proof of Theorem 4.3.** We may assume that  $K_X + B$  is nef by working on a log minimal model of (X, B) over Y (Theorem 2.5 (3.4)). As  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 1$ , there exist a log resolution  $\sigma: W \to X$  and a fibration  $h: W \to Z$  to a smooth projective surface Z, which is birational to the relative Iitaka fibration induced by  $\sigma^*(K_X+B)$  on W over Y. These varieties fit into the following commutative diagram

$$W \xrightarrow{b} X$$

$$h \bigvee f$$

$$Z \xrightarrow{g} Y$$

By the assumption  $\blacklozenge$  and Proposition 2.11, the geometric generic fiber of h is a smooth elliptic curve over K(Z). Applying flattening trick ([4, Lemma 5.6]), we can assume  $\sigma^*(K_X + B) \sim_{\mathbb{Q}} h^*C$  where C is a nef and g-big divisor on Z. By [8, Claim 3.1 and 3.2], there exists an effective divisor E on W such that  $K_W \sim_{\mathbb{Q}} h^* K_Z + E$ .

For the case g(Y) > 1, applying Theorem 4.6 for  $g : Z \to Y$ , we show that for  $n \gg 0$ ,  $nC + K_Z$  is big. By  $K_W \sim_{\mathbb{Q}} h^* K_Z + E$ , applying Theorem 2.3, we have

$$\kappa(X, K_X + B) = \kappa(X, (n+1)(K_X + B)) \ge \kappa(X, n(K_X + B) + K_X) = \kappa(W, \sigma^* n(K_X + B) + K_W) \ge \kappa(Z, nC + K_Z) = 2.$$

Let's restrict on the case g(Y) = 1. We aim to prove that  $\kappa(X, K_X + B) \ge 1$ . First applying [8, Theorem 1.2 and 1.3], we have

$$\kappa(X, K_X + B) \ge \kappa(X) = \kappa(W) \ge \kappa(Z) \ge \kappa(Z, K_{Z_{\bar{\eta}}}).$$

So we may assume that  $\kappa(Z, K_{Z_{\bar{\eta}}}) \leq 0$ , i.e.,  $p_a(Z_{\bar{\eta}}) \leq 1$ , hence the geometric generic fiber  $Z_{\bar{\eta}}$  is either a smooth elliptic curve (Proposition 2.11) or a rational curve over  $k(\bar{\eta})$ . And if  $\nu(Z, C) = 2$  then C is big, applying Theorem 2.3 we are done by

$$\kappa(X, K_X + B) = \kappa(W, \sigma^*(K_X + B)) = \kappa(Z, C) = 2.$$

So we may assume additionally that  $\nu(Z, C) = 1$ .

We will mimic the proof of Theorem 4.2 and break the arguments into three steps for similar purposes.

Step 1: By the assumptions above,  $g: Z \to Y$  is generically smooth. And since  $K_Y \sim 0$  and C is nef and g-big, we can apply Lemma 4.8 and obtain that, for sufficiently divisible n > 0,  $V' := g_* \mathcal{O}_Z(nC + K_Z)$  is a nef vector bundle of rank  $\geq 2$ . Fix a sufficiently divisible N > 0 such that  $NK_W \sim h^*NK_Z + NE$  where NE is integral, and that  $N(K_X + B)$  is Cartier. Applying the projection formula, we can get a natural inclusion  $\mathcal{O}_Z(NK_Z) \hookrightarrow h_*\mathcal{O}_W(NK_W)$ . In turn we have

$$Sym^{N}V' = Sym^{N}g_{*}\mathcal{O}_{Z}(nC + K_{Z}) \rightarrow g_{*}\mathcal{O}_{Z}(nNC + NK_{Z})$$
  

$$\hookrightarrow g_{*}h_{*}\mathcal{O}_{W}(nN\sigma^{*}(K_{X} + B) + NK_{W}) \cong f_{*}\sigma_{*}\mathcal{O}_{W}(nN\sigma^{*}(K_{X} + B) + NK_{W})$$
  

$$\subseteq f_{*}\mathcal{O}_{X}(Nn(K_{X} + B) + NK_{X}) \subseteq f_{*}\mathcal{O}_{X}(Nn(K_{X} + B) + N(K_{X} + B))$$
  

$$= f_{*}\mathcal{O}_{X}(N(n + 1)(K_{X} + B)).$$

Denote by V the image of  $Sym^N V'$  via the above composition map. Let l = N(n+1). Then V is a nef sub-vector bundle of  $f_*\mathcal{O}_X(l(K_X + B))$ , and  $\operatorname{rk} V = r' \geq 2$ .

By [12, Proposition 2.7], there exists a flat base change  $\pi : Y_1 \to Y$  between elliptic curves such that  $\pi^* V \cong \bigoplus_{i=1}^{i=r'} L'_i$  where each  $L'_i$  is a line bundle with deg  $L'_i \ge 0$ . Consider the following commutative diagram

(5) 
$$X_{1}^{\prime} \xrightarrow{\nu} X_{1} = X \times_{Y} Y_{1} \xrightarrow{\pi_{1}} X$$
$$\downarrow f_{1} \qquad \qquad \downarrow f_{1} \qquad \qquad \downarrow f \qquad \qquad \downarrow f$$
$$Y_{1} \xrightarrow{\pi} Y$$

where  $X'_1$  is the normalization of  $X_1$ . We have a natural inclusion map

$$\alpha : \bigoplus_{i=1}^{i=r'} L'_i \cong \pi^* V \subseteq \pi^* (f_* \mathcal{O}_X (l(K_X + B)))$$
$$\cong f_{1*} \mathcal{O}_{X_1} (\pi_1^* (l(K_X + B))) \subseteq f'_{1*} \mathcal{O}_{X'_1} (\pi_1'^* (l(K_X + B))).$$

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If some  $L'_{i_0}$  satisfies that deg  $L'_{i_0} > 0$ , by doing a further étale base change, we may assume deg  $L'_{i_0} \ge 2$ , thus

$$h^{0}(X'_{1}, \pi'^{*}(m(K_{X} + B))) = h^{0}(Y_{1}, f'_{1*}\mathcal{O}_{X'_{1}}(\pi'^{*}m(K_{X} + B))) \ge h^{0}(Y_{1}, L'_{i_{0}}) \ge 2,$$

which implies  $\kappa(X, K_X + B) = \kappa(X', \pi_1'^*(K_X + B)) \ge 1$  by Theorem 2.3. From now on, we assume every  $L'_i \in \text{Pic}^0(Y_1)$ .

Step 2: Granted the inclusion map  $\alpha$  and the commutative diagram (5) in Step 1, as in Step 2 of the proof of Theorem 4.2, we can find an integer  $m_1 = l_1 l$  and some divisors  $D_i \in |m_1(K_X + B) + f^*L_i|, i = 1, 2, \cdots, r$  for some  $L_i \in \text{Pic}^0(Y)$ , such that the sub-linear system of  $|m_1(K_X + B)_{\eta}|$  generated by  $(D_i)_{\eta}, i = 1, 2, \cdots, r$  defines a nontrivial map of  $X_{\eta}$ .

We only need to prove that there exist at least two different divisors among  $D_i$ , say,  $D_1 \neq D_2$ , such that  $L_1, L_2$  are torsion in  $\text{Pic}^0(Y)$  (Step 2 of Theorem 4.2).

Step 3: We will construct a minimal dlt pair  $(\hat{X}, \hat{B})$  and divisors  $\hat{D}_1, \hat{D}_2$  satisfying the conditions of Lemma 4.10.

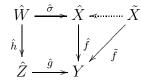
(3.1) Take a log resolution  $\mu : \tilde{X} \to X$  of the pair  $(X, B + \sum_i D_i)$ . Denote by  $\tilde{f} : \tilde{X} \to Y$  the natural morphism. Let  $\tilde{B}$  be the reduced divisor supported on the union of  $\sum_i \mu^* D_i$  and the exceptional divisors. By running a log MMP, we get a minimal dlt model  $(\hat{X}, \hat{B})$ , and a natural morphism  $\hat{f} : \hat{X} \to Y$ . The divisor  $\tilde{E} = K_{\tilde{X}} + \tilde{B} - \mu^*(K_X + B)$  is effective. Take a sufficiently divisible integer  $l_2 > 0$  such that  $l_2\tilde{E}$  is Cartier. Let  $m_2 = m_1 l_2$ . We get effective divisors

$$\tilde{D}_i = l_2 \mu^* D_i + l_2 \tilde{E} \sim m_2 (K_{\tilde{X}} + \tilde{B}) + l_2 \mu^* f^* L_i$$

and the push-forward divisors via the natural map  $\tilde{X} \dashrightarrow \hat{X}$ 

$$\hat{D}_i \sim m_2(K_{\hat{X}} + \hat{B}) + l_2 \hat{f}^* L_i.$$

(3.2) We can prove  $\nu(K_{\hat{X}} + \hat{B}) = \nu(K_X + B) = 1$  as in Step 3 (3.2) of the proof of Theorem 4.2. Note that  $(K_{\hat{X}} + \hat{B})_{\eta}$  is semi-ample by Theorem 2.5 (3.2), thus  $\kappa(\hat{X}_{\eta}, (K_{\hat{X}} + \hat{B})_{\eta}) = 1$ . Considering the relative Iitaka fibration  $\hat{h}' : \hat{X} \longrightarrow \hat{Z}'$  of  $\hat{X}$ over Y induced by  $K_{\hat{X}} + \hat{B}$  and applying flattening trick ([4, Lemma 5.6]), we get a commutative diagram



such that  $\hat{\sigma}^*(K_{\hat{X}} + \hat{B}) \sim_{\mathbb{Q}} \hat{h}^* \hat{C}$  for some nef and  $\hat{g}$ -big divisor  $\hat{C}$  on  $\hat{Z}$ , where  $\hat{W}$  and  $\hat{Z}$  are smooth, birational to  $\hat{X}$  and  $\hat{Z}'$  respectively, and  $\hat{h}$  is a fibration birational to  $\hat{h}'$ . By the construction,  $\hat{h}: \hat{W} \to \hat{Z}$  is an elliptic fibration, and  $\nu(\hat{Z}, \hat{C}) = 1$ .

(3.3) We can take effective divisors  $\hat{C}_i \sim m_2 \hat{C} + l_2 \hat{g}^* L_i$  on  $\hat{Z}$  such that  $\hat{h}^* \hat{C}_i = \hat{\sigma}^* \hat{D}_i$ . Note that  $\hat{C}_i$  is nef and  $\hat{C}_i^2 = 0$ . Considering the connected components of the union of  $\hat{C}_i, 0 \leq i \leq r$ , we can show that there exist effective Cartier divisors  $\hat{C}'_1, \ldots, \hat{C}'_s$  satisfying the following conditions:

• Supp $(\hat{C}'_j)$  is connected for each j, and Supp $(\hat{C}'_{j_1}) \cap$  Supp $(\hat{C}'_{j_2}) = \emptyset$  if  $j_1 \neq j_2$ ;

- every  $\hat{C}'_j$  is nef and  $(\hat{C}'_j)^2 = 0;$
- the greatest common divisor of the coefficients of every  $\hat{C}'_i$  is equal to one;
- for each *i*, there exist  $a_{i1}, \ldots, a_{is} \in \mathbb{Z}^{\geq 0}$  such that  $\hat{C}_i = a_{i1}\hat{C}'_1 + \cdots + a_{is}\hat{C}'_s$ .

As  $\hat{\sigma}^* \hat{D}_i = \hat{h}^* \hat{C}_i$ , by the construction,  $\hat{h}^* \hat{C}'_1, \dots, \hat{h}^* \hat{C}'_s$  are disjoint connected components of  $\hat{\sigma}^*(\sum \hat{D}_i)$ . Let  $\hat{G}_j := \hat{\sigma}_* \hat{h}^* \hat{C}'_j$ . Then  $\operatorname{Supp}(\hat{G}_{j_1}) \cap \operatorname{Supp}(\hat{G}_{j_2}) = \emptyset$  if  $j_1 \neq j_2$ , and  $\hat{D}_i = a_{i1}\hat{G}_1 + \dots + a_{is}\hat{G}_s$ .

(3.4) We can take two divisors among  $\hat{D}_i$ , say,  $\hat{D}_1, \hat{D}_2$  such that  $(\hat{D}_1)_n \neq (\hat{D}_2)_n$ . And since  $(\hat{D}_1)_{\eta} \sim (\hat{D}_2)_{\eta}$ , we can find two connected components among  $\hat{G}_j$ , say,  $\hat{G}_1, \hat{G}_2$ , such that  $(\hat{G}_1)_{\eta}, (\hat{G}_2)_{\eta} > 0$  and  $a_{11} > a_{21} \ge 0$  and  $a_{22} > a_{12} \ge 0$ . And if we write that

$$\hat{D}_1 = a_{11}\hat{G}_1 + a_{12}\hat{G}_2 + \hat{G}'_1$$
 and  $\hat{D}_2 = a_{21}\hat{G}_1 + a_{22}\hat{G}_2 + \hat{G}'_2$ 

then for  $i, j \in \{1, 2\}$  and  $i \neq j$ ,  $\hat{G}_j$  does not intersect  $\hat{G}''_j := \hat{G}_i + \hat{G}'_1 + \hat{G}'_2$ . (3.5) Take two reduced, irreducible and dominant over Y components  $G_1, G_2$  of  $\hat{G}_1, \hat{G}_2$  respectively. Since  $K_{\hat{X}} + \hat{B}$  is nef and  $\nu(\hat{X}, \hat{D}_j) = \nu(\hat{X}, K_{\hat{X}} + \hat{B}) = 1$ , applying Proposition 2.8 we obtain that for j = 1, 2,

$$\nu(G_j, (K_{\hat{X}} + \hat{B})|_{G_j}) = \nu(G_j, \hat{D}_j|_{G_j}) \le \nu(\hat{X}, \hat{D}_j) - 1 = 0,$$

hence  $(K_{\hat{X}} + \hat{B})|_{G_j} \sim_{\mathbb{Q}} 0$  (Lemma 4.10), so the Iitaka fiber  $F_j = G_j$ .

Finally, as the conditions of Lemma 4.10 are verified for the pair (X, B) and the divisors  $\hat{D}_1, \hat{D}_2$ , we conclude that  $L_1$  and  $L_2$  are torsion. The proof is completed.

4.4. Proof of Theorem 4.4. We can assume (X, B) is relatively minimal over Y. Then by Theorem 2.5 (3.3, 3.4),  $K_X + B$  is nef and f-Q-trivial, so we can assume  $l(K_X + B) \sim f^*L$  for some integer l > 0 and a nef line bundle L on Y. If deg L > 0then  $\kappa(X, K_X + B) \ge 1 \ge \kappa(Y)$ . So we may assume deg L = 0. And we only need to prove that q(Y) = 1 and L is torsion in  $\operatorname{Pic}^{0}(Y)$ .

**Lemma 4.11.** Let D be a pseudo-effective  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on X. If D is f-nef, then D is nef.

*Proof.* Assume the contrary. We can take an ample  $\mathbb{Q}$ -divisor H on X such that D + H is not nef. Since D is pseudo-effective, there exists an effective Q-divisor  $B' \sim_{\mathbb{O}} D + H$ . Take a small rational number t > 0 such that (X, B + tB') is klt. Since  $K_X + B + tB'$  is f-nef, applying Theorem 2.5 shows that  $tB' \equiv K_X + B + tB'$ is nef, which contradicts the assumption. 

Let H be an ample Cartier divisor on X and F a general fiber of f. We can find two positive integers a, b such that  $D^3 = 0$  where D = aH - bF.

We claim that D is nef, thus  $\nu(D) = 2$ . By Lemma 4.11 it suffices to show that D is pseudo-effective. Otherwise, denote by  $t_0$  the maximal real number such that  $H-t_0F$  is pseudo-effective. Then  $t_0 < \frac{b}{a}$  and  $(H-t_0F)^3 > 0$ . Since for every rational number  $t < t_0$  the divisor H - tF is f-ample and pseudo-effective, so H - tF is nef by Lemma 4.11. Taking the limit shows that  $H - t_0 F$  is nef, thus  $H - t_0 F$  is big since  $(H - t_0 F)^3 > 0$  ([33, Chap. V Lemma 2.1]).<sup>3</sup> It follows that for a sufficiently small  $\epsilon > 0$  such that  $t_0 + \epsilon \in \mathbb{Q}$ , the divisor  $H - t_0 F - \epsilon F$  is  $\mathbb{Q}$ -linearly equivalent to an effective divisor. However, this contradicts the definition of  $t_0$ .

Take a smooth resolution of singularities  $\sigma : W \to X$  and let  $h = f \circ \sigma : W \to X \to Y$ . Since  $n\sigma^*D + K_{W/Y} - K_{W/Y} = n\sigma^*D$  is nef, h-big and h-semi-ample, applying Theorem 4.6 shows that for sufficiently divisible integers n and g, the sheaf  $F_Y^{g*}h_*\mathcal{O}_W(n\sigma^*D + K_{W/Y})$  contains a nonzero nef subbundle V.

We exclude the case g(Y) > 1 as follows. Otherwise, for sufficiently large n, the divisor  $n\sigma^*D + K_W$  is big by Theorem 4.7. We can find an effective  $\sigma$ -exceptional divisor E on W such that  $K_W \leq \sigma^*K_X + E$ . Then  $n\sigma^*D + \sigma^*(K_X + B) + E \geq n\sigma^*D + K_W + \sigma^*B$  is big. Applying Theorem 2.3,

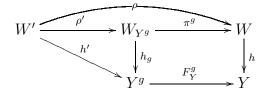
$$\kappa(X, nD + (K_X + B)) = \kappa(W, n\sigma^*D + \sigma^*(K_X + B) + E) = 3.$$

As  $K_X + B$  is numerically trivial, we conclude that D is big, but this contradicts that  $\nu(D) = 2$ .

We may assume Y is an elliptic curve. Next we will prove that there exists a semi-ample divisor  $D' \equiv tD$  for some rational number t > 0. Since the subbundle  $V \subseteq F_Y^{g*}h_*\mathcal{O}_W(n\sigma^*D + K_{W/Y})$  is nef, for every  $L'' \in \operatorname{Pic}^0(Y)$ , applying Riemann-Roch formula gives

$$h^0(Y, V \otimes L'') - h^1(Y, V \otimes L'') = \chi(Y, V \otimes L'') = \deg V \ge 0.$$

We claim that there exists some  $L'' \in \operatorname{Pic}^{0}(Y)$  such that  $h^{0}(Y, V \otimes L'') > 0$ . Otherwise, for every  $L'' \in \operatorname{Pic}^{0}(Y)$  and  $i = 0, 1, h^{i}(Y, V \otimes L'') = 0$ , thus  $R^{i}\Phi_{\mathcal{P}}V = 0, i = 0, 1$ , i.e., the Fourier-Mukai transform  $R\Phi_{\mathcal{P}}V$  is acyclic ([16, p.53]). Therefore V = 0 by Theorem 3.2, and the claim follows from this contradiction. Note that since dim Y = 1, by Proposition 2.1 the fibration  $h : W \to Y$  is separable, hence  $W_{Y^{g}} = W \times_{Y} Y^{g}$  is integral. Consider the following commutative diagram



where W' denotes the normalization of  $W_{Y^g}$  and  $\rho, \rho', h'$  denote the natural morphisms. From the commutative diagram above, by [17, Chapter III, Prop. 9.3] we obtain

$$F_Y^{g*}h_*\mathcal{O}_W(n\sigma^*D + K_W) \cong h_{g*}(\pi^{g*}\mathcal{O}_W(n\sigma^*D + K_W)) \cong h_{g*}\mathcal{O}_{W_Yg}(\pi^{g*}(n\sigma^*D + K_W))$$
$$\hookrightarrow h'_*\mathcal{O}_{W'}(\rho'^*\pi^{g*}(n\sigma^*D + K_W)) = h'_*\mathcal{O}_{W'}(\rho^*(n\sigma^*D + K_W)).$$

It follows that

$$h^{0}(\mathcal{O}_{W'}(\rho^{*}(n\sigma^{*}D+K_{W}))\otimes h^{*}L'') = h^{0}(Y, h'_{*}\mathcal{O}_{W'}(\rho^{*}(n\sigma^{*}D+K_{W}))\otimes L'')$$
$$\geq h^{0}(Y, V\otimes L'') > 0.$$

<sup>&</sup>lt;sup>3</sup>this is well known for  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisors ([24, Prop. 2.61]), but the proof also applies for  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisors as mentioned by [33, Chap. V Lemma 2.1]

Take  $L' \in \operatorname{Pic}^{0}(Y)$  such that  $F_{Y}^{g*}L' \sim L''$ . Then applying Theorem 2.3, we can show

$$\kappa(X, nD + (K_X + B) + f^*L') \ge \kappa(W, n\sigma^*D + K_W + h^*L') = \kappa(W', \rho^*(n\sigma^*D + K_W) + h'^*L'') \ge 0.$$

There exists an effective divisor  $D_1 \sim_{\mathbb{Q}} nD + (K_X + B) + f^*L'$ . The pair  $(X, B + tD_1)$  is klt for a sufficiently small rational number t, so the divisor  $K_X + B + tD_1$  is semiample by Theorem 4.2. Since  $K_X + B \equiv 0$ , we can set  $D' = K_X + B + tD_1$ .

Since D' is f-ample and  $\nu(D') = 2$ , the associated morphism to D' is a fibration  $g: X \to Z$  to a surface. Let  $X_{\xi}$  be the generic fiber of g. Then  $X_{\xi}$  is a normal curve defined over  $k(\xi) = K(Z)$  and dominant over  $Y \otimes_k k(\xi)$ . It follows that  $p_a(X_{\xi}) \geq 1$ , thus  $K_{X_{\xi}}$  is semi-ample. Since  $l(K_X + B)|_{X_{\xi}} \sim_{\mathbb{Q}} l(K_{X_{\xi}} + B|_{X_{\xi}})$  is numerically trivial, we conclude that  $K_{X_{\xi}} \sim_{\mathbb{Q}} K_X|_{X_{\xi}} \sim_{\mathbb{Q}} 0$  and  $B|_{X_{\xi}} = 0$ , thus  $l(K_X + B)|_{X_{\xi}} \sim_{\mathbb{Q}} f^*L|_{X_{\xi}} \sim_{\mathbb{Q}} 0$ . Though the geometric generic fiber  $X_{\bar{\xi}}$  is not necessarily reduced, we can apply Lemma 2.4 to the morphism  $(X_{\bar{\xi}})_{\mathrm{red}} \to Y \otimes_k k(\bar{\xi})$  and show that L is torsion.

4.5. **Proof of Theorem 4.1.** In the following we assume that  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) \ge 0$ .

**Case (i)** dim Y = 1. If  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 0$  then we can use Theorem 4.4. If  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 1$  then we can use Theorem 4.3. If  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 2$ , then we can use Theorem 4.2 when g(Y) = 1 and use Corollary 4.7 when g(Y) > 1.

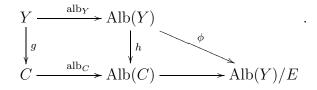
**Case (ii)** dim Y = 2. If  $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 0$ , then by the assumption  $\blacklozenge$ , f is an elliptic fibration (Proposition 2.11), applying [8, Theorem 1.2] shows that

$$\kappa(X, K_X + B) \ge \kappa(X) \ge \kappa(Y).$$

So we may assume  $K_X + B$  is f-big. We fall into three cases according to  $\kappa(Y)$ .

When  $\kappa(Y) = 2$ , we can use Corollary 4.7.

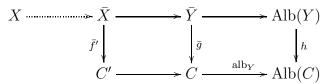
When  $\kappa(Y) = 1$ , we need to prove that  $\kappa(X, K_X + B) \geq 2$ . Denote by  $g: Y \to C$ the Iitaka fibration of Y and by  $\operatorname{alb}_Y : Y \to \operatorname{Alb}(Y)$  the Albanese map which is generically finite. Take a general fiber G of g, which is an elliptic curve. Let  $E = \operatorname{alb}_Y(G)$ . We may assume  $\operatorname{Alb}(Y)$  is an abelian variety and E passes through the origin. Let A(E) be the sub-abelian variety generated by E. It follows that  $1 \leq \dim A(E) \leq g(E) \leq g(G) = 1$ , hence the equalities are attained, and E = A(E) is a sub-abelian variety. Since the composition morphism  $\phi: Y \to \operatorname{Alb}(Y) \to \operatorname{Alb}(Y)/E$ contracts the fiber G, applying Rigidity Lemma ([32, Proposition 6.1]), every fiber of g is contracted by  $\phi$ , and  $\phi: Y \to \operatorname{Alb}(Y)/E$  factors through  $g: Y \to C$ . By the universal property of the Albanese map we have the following commutative diagram



We see that the morphism  $Alb(C) \to Alb(Y)/E$  is in fact an isomorphism, hence  $h : Alb(Y) \to Alb(C)$  is of relative dimension one, in particular  $g(C) \ge 1$ . Next let  $Y \xrightarrow{\sigma} \overline{Y} \to Alb(Y)$  be the Stein factorization. Then  $\sigma : Y \to \overline{Y}$  is a birational morphism, and the  $\sigma$ -exceptional locus does not intersect the generic fiber of g:

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 $Y \to C$ . Therefore, the natural morphism  $\overline{Y} \to \text{Alb}(C)$  factors through a morphism  $\overline{g}: \overline{Y} \to C$ , which is induced by the Stein factorization. Let  $(\overline{X}, \overline{B})$  be a log minimal model of (X, B) over  $\overline{Y}$ . By Theorem 2.5 (3.4),  $(\overline{X}, \overline{B})$  is in fact minimal. Then we have the following commutative diagram



where  $\bar{f}': \bar{X} \to C'$  is the fibration induced by the Stein factorization of  $\bar{X} \to C$ . Note that  $g(C') \ge g(C) \ge 1$ . Denote by  $\zeta'$  the generic point of C'. Consider the fibration  $\bar{f}': \bar{X} \to C'$ . By Theorem 2.5 (3.2),  $(K_{\bar{X}} + \bar{B})_{\zeta'}$  is semi-ample. And since  $K_{\bar{X}} + \bar{B}$  is big over Alb(Y), it is not numerically trivial over the generic point of C', which implies that  $\kappa(\bar{X}_{\zeta'}, (K_{\bar{X}} + \bar{B})_{\zeta'}) \ge 1$ .

Claim 4.12. If  $\kappa(\bar{X}_{\zeta'}, (K_{\bar{X}} + \bar{B})_{\zeta'}) = 1$  then  $\bar{f}'$  satisfies the assumption  $\blacklozenge$ .

Proof. Denote by  $\varphi : \bar{X} \dashrightarrow Z$  the relative Iitaka fibration induced by  $K_{\bar{X}} + \bar{B}$ on  $\bar{X}$  over C' and by  $\bar{X}_{\xi}$  the generic fiber of  $\varphi$ . Then  $\bar{X}_{\xi}$  is a normal curve over K(Z). As  $K_{\bar{X}} + \bar{B}$  is big over Alb(Y) while  $(K_{\bar{X}} + \bar{B})|_{\bar{X}_{\xi}} \sim_{\mathbb{Q}} 0$ , we see that  $\bar{X}_{\xi}$  is not contracted by  $\bar{X} \to \text{Alb}(Y)$ , hence is generically finite over Alb(Y)  $\otimes_k K(Z)$ . Therefore,  $p_a(\bar{X}_{\xi}) \ge 1$ , and  $K_{\bar{X}_{\xi}}$  is semi-ample. Since  $(K_{\bar{X}} + \bar{B})|_{\bar{X}_{\xi}} \sim_{\mathbb{Q}} K_{\bar{X}_{\xi}} + \bar{B}|_{\bar{X}_{\xi}}$ is numerically trivial, we conclude that  $\bar{B}|_{\bar{X}_{\xi}} = 0$ , thus  $\bar{f}'$  satisfies the assumption  $\blacklozenge$ .

So we can apply the results of Case (i) to the fibration  $\bar{f}': \bar{X} \to C'$  and show that

$$\kappa(\bar{X}, K_{\bar{X}} + \bar{B}) \ge \kappa(\bar{X}_{\zeta'}, (K_{\bar{X}} + \bar{B})_{\zeta'}) + \kappa(C') \ge 1.$$

Hence  $K_{\bar{X}} + \bar{B}$  is semi-ample by Theorem 4.9. And by Corollary 4.7, we have  $\nu(\bar{X}, K_{\bar{X}} + \bar{B}) \geq 2$ , hence  $\kappa(\bar{X}, K_{\bar{X}} + \bar{B}) \geq 2$ .

When  $\kappa(Y) = 0$ , we need to prove that  $\kappa(X, K_X + B) \geq 1$ . In this case Y is birationally equivalent to an abelian surface ([1, Sec. 10]). We may assume Y is an abelian surface and assume (X, B) is minimal by working on a log minimal model over Y. If Y is simple then we can use Theorem 4.2. Otherwise, Y admits a fibration to an elliptic curve, then we get a fibration  $f' : X \to C'$  with g(C') = 1. Let  $\zeta'$  denote the generic point of C'. Again since  $K_X + B$  is big over Y, we have  $\kappa(X_{\zeta'}, K_{X_{\zeta'}} + B_{\zeta'}) \geq 1$ , and if  $\kappa(X_{\zeta'}, K_{X_{\zeta'}} + B_{\zeta'}) = 1$  we can verify that f' satisfies the assumption  $\blacklozenge$  by the same proof of Claim 4.12. Applying the results of Case (i) to the fibration  $f' : X \to C'$ , we complete the proof by

$$\kappa(X, K_X + B) \ge \kappa(X_{\zeta'}, K_{X_{\zeta'}} + B_{\zeta'}) + \kappa(C') \ge 1.$$

# 5. Abundance

In this section, we will prove Theorem 1.2. By Theorem 4.9, we only need to show that either  $\kappa(X, K_X + B) \geq 1$  or  $K_X + B \sim_{\mathbb{Q}} 0$ . Hence we may assume  $\kappa(X, K_X + B) \leq 0$ . If X is of maximal Albanese dimension, then we are done by [48, Theorem 1.1] or [20, Theorem 0.3]. Let  $f: X \to Y$  be the fibration arising from the Stein factorization of  $a_X$ . Then  $\kappa(X_{\eta}, (K_X + B)_{\eta}) \geq 0$  by Theorem 2.5 (3.2).

Therefore, by Theorem 4.1 it is only possible that  $\kappa(Y) = \kappa(X_{\eta}, (K_X + B)_{\eta}) = 0$ . If dim Y = 1 then we can use Theorem 4.4. If dim Y = 2 then B = 0 by the assumption (1), and f is an elliptic fibration by Proposition 2.11, so X is non-uniruled, and this case has been treated in [48, Theorem 1.1].

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