# Twisted waveguide with a Neumann window 

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Dedicated to Pavel Exner on the occasion of his 70th birthday


#### Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of $\mathbb{R}^{3}$ which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.


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## 1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_{0}:=\mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial \Omega_{0}$.
The cross section $\omega$ is supposed to be an open bounded connected subset of $\mathbb{R}^{2}$ of diameter $d>0$ which is not rotationally invariant. Moreover $\omega$ is supposed to have smooth boundary $\partial \omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states 8 .

Let us introduce some notations. Denote by $\mathcal{N}$ the Neumann window. It is an open bounded subset of the boundary $\partial \Omega_{0}$. Let $\mathcal{D}$ be its complement set in $\partial \Omega_{0}$. When $\mathcal{N}$ is an annulus of size $l>0$ we will denote it by,

$$
\mathcal{A}_{a}(l):=I_{a}(l) \times \partial \omega, I_{a}(l):=(a, l+a), a \in \mathbb{R} .
$$

[^0]Consider first the self-adjoint operator $H_{0}^{\mathcal{N}}$ associated to the following quadratic form. Let $D\left(Q^{\mathcal{N}}\right)=\left\{\psi \in \mathcal{H}^{1}\left(\Omega_{0}\right) \mid \psi_{\lceil\mathcal{D}}=0\right\}$ and for $\psi \in D\left(Q^{\mathcal{N}}\right)$,

$$
Q^{\mathcal{N}}(\psi)=\int_{\Omega_{0}}|\nabla \psi|^{2} d x
$$

i.e. the Laplace operator defined on $\Omega_{0}$ with Neumann boundary conditions (NBC) on $\mathcal{N}$ and Dirichlet boundary conditions ( DBC ) on $\mathcal{D}$ [5, 11].

It is actually shown in the Section 2 of this paper that if $\mathcal{N}$ contains an annulus of size $l$ large enough then $H_{0}^{\mathcal{N}}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if $\mathcal{N}$ contains an annulus of any size $l>0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_{0}^{\mathcal{N}}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10 where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function $\theta$ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].
Let us now define the twisting [4, 7. Choose $\theta \in C_{c}^{1}(\mathbb{R})$ and introduce the diffeomorphism

$$
\begin{align*}
\mathcal{L}: \Omega_{0} & \longrightarrow \mathbb{R}^{3}  \tag{1}\\
\left(s, t_{2}, t_{3}\right) & \longmapsto\left(s, t_{2} \cos \theta(s)-t_{3} \sin \theta(s), t_{2} \sin \theta(s)+t_{3} \cos \theta(s)\right)
\end{align*}
$$

The twisted tube is given by $\Omega_{\theta}:=\mathcal{L}\left(\Omega_{0}\right)$. Let $D\left(Q_{\theta}^{\mathcal{N}}\right)=\left\{\psi \in \mathcal{H}^{1}\left(\Omega_{\theta}\right) \mid\right.$ $\left.\psi_{\lceil\mathcal{L}(\mathcal{D})}=0\right\}$ and consider the following quadratic form

$$
\begin{equation*}
Q_{\theta}^{\mathcal{N}}(\psi):=\int_{\Omega_{\theta}}|\nabla \psi|^{2} d x, \psi \in D\left(Q_{\theta}^{\mathcal{N}}\right) \tag{2}
\end{equation*}
$$

Through unitary equivalence, we then have to consider

$$
\begin{equation*}
q_{\theta}^{\mathcal{N}}(\psi):=Q_{\theta}^{\mathcal{N}}\left(\psi o \mathcal{L}^{-1}\right)=\left\|\nabla^{\prime} \psi\right\|^{2}+\left\|\partial_{s} \psi+\dot{\theta} \partial_{\tau} \psi\right\|^{2} \tag{3}
\end{equation*}
$$

$\psi \in D\left(q_{\theta}^{\mathcal{N}}\right):=\left\{\psi \in \mathcal{H}^{1}\left(\Omega_{0}\right) \mid \psi_{\Gamma \mathcal{D}}=0\right\}$ and where

$$
\begin{equation*}
\nabla^{\prime}:={ }^{t}\left(\partial_{t_{2}}, \partial_{t_{3}}\right), \quad \partial_{\tau}:=t_{2} \partial_{t_{3}}-t_{3} \partial_{t_{2}} \tag{4}
\end{equation*}
$$

Denote by $H_{\theta}^{\mathcal{N}}$ the associated self-adjoint operator. It is defined as follows (see [5], 11]. Let $D\left(H_{\theta}^{\mathcal{N}}\right)=\left\{\psi \in D\left(q_{\theta}^{\mathcal{N}}\right), \quad H_{\theta}^{\mathcal{N}} \psi \in L^{2}\left(\Omega_{0}\right) \quad \frac{\partial \psi}{\partial n}\left\lceil_{\mathcal{N}}=0\right\}\right.$ with

$$
\begin{equation*}
H_{\theta}^{\mathcal{N}} \psi=\left(-\Delta_{\omega}-\left(\dot{\theta} \partial_{\tau}+\partial_{s}\right)^{2}\right) \psi \tag{5}
\end{equation*}
$$

where the transverse Laplacian $\Delta_{\omega}:=\partial_{t_{2}}^{2}+\partial_{t_{3}}^{2}$. If $\mathcal{N}=\mathcal{A}_{a}(l), l>0$, we will denote these forms respectively as $Q_{\theta}^{l}, q_{\theta}^{l}$ and the corresponding operator as $H_{\theta}^{l}$ and if $\mathcal{N}=\emptyset$ we denote the associated operator by $H_{\theta}$.

Then the main result of this paper is

Theorem 1.1. i) Under conditions stated above on $\omega$ and $\theta$, there exists $l_{\text {min }}:=$ $l_{\text {min }}(\omega, d)>0$ such as if for some $a \in \mathbb{R}$ and $l>l_{\text {min }}, \mathcal{N} \supset \mathcal{A}_{a}(l)$ then

$$
\begin{equation*}
\sigma_{d}\left(H_{\theta}^{\mathcal{N}}\right) \neq \emptyset . \tag{6}
\end{equation*}
$$

ii) Suppose $\theta$ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{\max }:=d_{\max }(\theta, \omega)>0$ such that for all $0<d \leq d_{\max }$ there exists $l_{\max }:=l_{\max }(\omega, d, \theta)$ such as for all $0<l \leq l_{\text {max }}$, if $\mathcal{N} \subset \mathcal{A}_{a}(l)$ and $\operatorname{supp}(\dot{\theta}) \cap I_{a}(l)=\emptyset$ for some $a \in \mathbb{R}$ then

$$
\begin{equation*}
\sigma_{d}\left(H_{\theta}^{\mathcal{N}}\right)=\emptyset . \tag{7}
\end{equation*}
$$

Roughly speaking this result implies that for $d$ small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.11i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

## 2 Existence of bound states

First we prove the following. Denote by $E_{1}, E_{2}, \ldots$ the eigenvalues (transverse modes) of the Laplacian $-\Delta_{\omega}$ defined on $\mathrm{L}^{2}(\omega)$ with DBC on $\partial \omega$. Let $\chi_{1}, \chi_{2}, \ldots$ be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{\text {ess }}\left(H_{\theta}^{\mathcal{N}}\right)=\left[E_{1}, \infty\right)$.
Proof. We know that $\sigma\left(H_{\theta}\right)=\left[E_{1}, \infty\right)$ see e.g. 2]. But by usual arguments [12], $H_{\theta}^{\mathcal{N}} \leq H_{\theta}$, then

$$
\begin{equation*}
\left[E_{1}, \infty\right) \subset \sigma_{\text {ess }}\left(H_{\theta}^{\mathcal{N}}\right) \tag{8}
\end{equation*}
$$

Let $a^{\prime} \in \mathbb{R}$ and $l^{\prime}>0$ large enough such that $\mathcal{N} \subset \mathcal{A}_{a^{\prime}}\left(l^{\prime}\right)=I_{a^{\prime}}\left(l^{\prime}\right) \times \partial \omega$ and $\operatorname{supp}(\dot{\theta}) \subset I_{a^{\prime}}\left(l^{\prime}\right)$. Let $\tilde{H}_{\theta}^{l^{\prime}}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\left\{a^{\prime}\right\} \times \omega \cup\left\{a^{\prime}+l^{\prime}\right\} \times \omega$. So $H_{\theta}^{\mathcal{N}} \geq \tilde{H}_{\theta}^{l^{\prime}}$ and then $\sigma_{\text {ess }}\left(H_{\theta}^{\mathcal{N}}\right) \subset \sigma_{\text {ess }}\left(\tilde{H}_{\theta}^{l^{\prime}}\right)$ [12].

But $\tilde{H}_{\theta}^{l^{\prime}}=\tilde{H}_{i} \oplus \tilde{H}_{e}$. The interior operator $\tilde{H}_{i}$ is the corresponding operator defined on $L^{2}\left(I_{a^{\prime}}\left(l^{\prime}\right) \times \omega\right)$ with NBC on $\left\{a^{\prime}\right\} \times \omega \cup\left\{a^{\prime}+l^{\prime}\right\} \times \omega \cup \mathcal{N}$ and DBC elsewhere on $\mathcal{A}_{a^{\prime}}\left(l^{\prime}\right)$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{\text {ess }}\left(\tilde{H}_{\theta}^{l^{\prime}}\right)=\sigma_{\text {ess }}\left(\tilde{H}_{e}\right)$.
Now the exterior operator $\tilde{H}_{e}$ is defined on $L^{2}\left(\left(-\infty, a^{\prime}\right) \times \omega \cup\left(a^{\prime}+l^{\prime}, \infty\right) \times \omega\right)$ with DBC on $\left(-\infty, a^{\prime}\right) \times \partial \omega \cup\left(a^{\prime}+l^{\prime}, \infty\right) \times \partial \omega$ and NBC on $\left\{a^{\prime}\right\} \times \omega \cup\left\{a^{\prime}+l^{\prime}\right\} \times \omega$. Since $\theta=0$ for $x<a^{\prime}$ and $x>a^{\prime}+l^{\prime}$, it is easy to see that

$$
\tilde{H}_{e}=\underset{n \geq 1}{\oplus}\left(-\partial^{2}+E_{n}\right)\left(\chi_{n}, .\right) \chi_{n}
$$

Hence $\sigma\left(\tilde{H}_{e}\right)=\sigma_{\text {ess }}\left(\tilde{H}_{e}\right)=\left[E_{1},+\infty\right)$.
The Theorem $1.1 i$ ) follows from

Proposition 2.2. Under conditions of the Theorem 1.1 $i$ ), there exists $l_{\text {min }}:=$ $l_{\min }(\omega, d)>0$ such as for all $l>l_{\text {min }}$ we have

$$
\begin{equation*}
\sigma_{d}\left(H_{\theta}^{l}\right) \neq \emptyset \tag{9}
\end{equation*}
$$

Proof. Let $\varphi_{l, a}$ be the following function

$$
\varphi_{l, a}(s):= \begin{cases}\frac{10}{l}(s-a), & \text { on }\left[a, a+\frac{l}{10}\right) \\ 1, & \text { on }\left[a+\frac{l}{10}, a+\frac{9 l}{10}\right) \\ -\frac{10}{l}(s-l-a), & \text { on }\left[a+\frac{9 l}{10}, a+l\right) \\ 0, & \text { elsewhere. }\end{cases}
$$

It is easy to see that $\varphi_{l, a} \in D\left(q_{\theta}^{l}\right)$ and $\left\|\varphi_{l, a}\right\|^{2}=\frac{13 l}{15}|\omega|$. Let us calculate
$q_{\theta}^{l}\left(\varphi_{l, a}\right)-E_{1}\left\|\varphi_{l, a}\right\|^{2}=\left\|\nabla^{\prime} \varphi_{l, a}\right\|^{2}+\left\|\dot{\theta} \partial_{\tau} \varphi_{l, a}+\partial_{s} \varphi_{l, a}\right\|^{2}-E_{1}\left\|\varphi_{l, a}\right\|^{2}$.
Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$
\left\|\dot{\theta} \partial_{\tau} \varphi_{l, a}+\partial_{s} \varphi_{l, a}\right\|^{2}=\left\|\partial_{s} \varphi_{l, a}\right\|^{2}=\frac{20}{l}|\omega| .
$$

Then

$$
\begin{equation*}
q_{\theta}^{l}\left(\varphi_{l, a}\right)-E_{1}\left\|\varphi_{l, a}\right\|^{2}=|\omega|\left(\frac{20}{l}-\frac{13 l}{15} E_{1}\right) \tag{11}
\end{equation*}
$$

and thus if $l \geq l_{\text {min }}:=\sqrt{\frac{300}{13 E_{1}}}$ we have $q_{\theta}^{l}\left(\varphi_{l, a}\right)-E_{1}\left\|\varphi_{l, a}\right\|^{2} \leq 0$

### 2.1 Proof of the Theorem $1.1 i$ )

Using the same notation as in the Theorem $1.1 i$ ), then $H_{\theta}^{\mathcal{N}} \leq H_{\theta}^{l}$. Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

## 3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1 Denote by $\theta_{m}=\inf (\operatorname{supp}(\dot{\theta})), \theta_{M}=\sup (\operatorname{supp}(\dot{\theta}))$ and $L=\theta_{M}-\theta_{m}$. Here $L>0$. We first consider the case where the Neumann window is an annulus, $\mathcal{A}_{a}(l)=I_{a}(l) \times \omega$.

Proposition 3.1. Suppose $\mathcal{A}_{a}(l)$ is such that $a \geq \theta_{M}$. Assume that conditions of the Theorem 1.1 ii) hold. Then there exists $d_{\max }:=d_{\max }(\omega, \theta)>0$, such that for all $0<d \leq d_{\max }$ there exists $l_{\max }(d, \theta, \omega)>0$ such as for all $0<l \leq l_{\max }$ we have

$$
\begin{equation*}
\sigma_{d}\left(H_{\theta}^{l}\right)=\emptyset . \tag{12}
\end{equation*}
$$

Remark 3.2. the case where $l+a \leq \theta_{m}$ follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every $\psi \in D\left(q_{\theta}^{l}\right)$ it holds,

$$
\begin{equation*}
Q(\psi):=q_{\theta}^{l}(\psi)-E_{1}\|\psi\|^{2} \geq 0 \tag{13}
\end{equation*}
$$

The proof of (13) involves several steps.

### 3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let $g$ be the following function

$$
g(s):=\left\{\begin{array}{lc}
0, & \text { on } I_{a}(l)  \tag{14}\\
E_{1}, & \text { elsewhere }
\end{array}\right.
$$

Choose $p \in\left(\theta_{m}, \theta_{M}\right)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$
\rho(s):= \begin{cases}\frac{1}{1+(s-p)^{2}}, & \text { on }(-\infty, p] ;  \tag{15}\\ 0, & \text { elsewhere. }\end{cases}
$$

Proposition 3.3. Under same conditions of the Proposition 3.1, then there exists a constant $C>0$ depending on $p$ and $\omega$ and $\dot{\theta}$ such that for any $\psi \in$ $D\left(q_{\theta}^{l}\right)$,

$$
\begin{equation*}
\left\|\nabla^{\prime} \psi\right\|^{2}+\left\|\dot{\theta} \partial_{\tau} \psi+\partial_{s} \psi\right\|^{2}-\int_{\Omega_{0}} g(s)|\psi|^{2} d s d t \geq C \int_{\Omega_{0}} \rho(s)|\psi|^{2} d s d t \tag{16}
\end{equation*}
$$

We first show the following lemma. Denote by $\Omega_{p}:=(-\infty, p) \times \omega$.
Lemma 3.4. Under same conditions of the Proposition 3.3. Then for any $\psi \in D\left(q_{\theta}^{l}\right)$ we have

$$
\begin{equation*}
\int_{\Omega_{p}}\left|\nabla^{\prime} \psi\right|^{2}+\left|\dot{\theta} \partial_{\tau} \psi+\partial_{s} \psi\right|^{2}-E_{1}|\psi|^{2} d s d t \geq C \int_{\Omega_{p}} \rho(s)|\psi|^{2} d s d t \tag{17}
\end{equation*}
$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by $\chi_{A}$ the characteristic function of $A \times \omega$. Let $\psi \in D\left(q_{\theta}^{l}\right)$ and define,

$$
\begin{array}{lrl}
q_{1}^{A}(\psi) & :=\left\|\chi_{A} \nabla^{\prime} \psi\right\|^{2}-E_{1}\left\|\chi_{A} \psi\right\|^{2}, & q_{2}^{A}(\psi) \\
q_{3}^{A}(\psi) & :=\left\|\chi_{A} \dot{\theta} \partial_{\tau} \psi\right\|^{2}, & q_{2,3}^{A}(\psi) \tag{18}
\end{array}=2 \Re\left(\partial_{s} \psi \|^{2}, \chi_{A} \dot{\theta} \partial_{\tau} \psi\right) .
$$

Denote also by $Q^{A}(\psi)=q_{1}^{A}(\psi)+q_{2}^{A}(\psi)+q_{3}^{A}(\psi)+q_{2,3}^{A}(\psi)$. Here and hereafter we often use the fact that for any $\psi \in D\left(q_{\theta}^{l}\right)$

$$
\begin{equation*}
q_{1}^{A}(\psi) \geq 0 \tag{19}
\end{equation*}
$$

for every $A \subset \mathbb{R}$ such that $A \cap I_{a}(l)=\emptyset$.
Proof. Choose $r>0$ such that $\dot{\theta}(s) \neq 0$ for any $s \in[p-r, p]$. Let $f$ be the following function:

$$
f(s):= \begin{cases}0, & \text { on }(p, \infty)  \tag{20}\\ \frac{p-s}{r}, & \text { on }(p-r, p] \\ 1, & \text { elsewhere }\end{cases}
$$

For any $\psi \in D\left(q_{\theta}^{l}\right)$, simple estimates lead to:

$$
\begin{align*}
\int_{\Omega_{p}} \frac{|\psi(s, t)|^{2}}{1+(s-p)^{2}} d s d t & =\int_{\Omega_{p}} \frac{|\psi(s, t) f(s)+(1-f(s)) \psi(s, t)|^{2}}{1+(s-p)^{2}} d s d t  \tag{21}\\
& \leq 2\left(\int_{\Omega_{p}} \frac{|f(s) \psi(s, t)|^{2}}{(s-p)^{2}} d s d t+\left\|\chi_{(p-r, p)} \psi\right\|^{2}\right)
\end{align*}
$$

Since $f(p) \psi(p,)=$.0 , we can use the usual Hardy inequality (see e.g. 9]), then we get,

$$
\begin{equation*}
\int_{\Omega_{p}} \frac{|\psi(s, t)|^{2}}{1+(s-p)^{2}} d s d t \leq 8 q_{2}^{(-\infty, p)}(f \psi)+2\left\|\chi_{(p-r, p)} \psi\right\|^{2} \tag{22}
\end{equation*}
$$

Note that with our choice $[p-r, p] \cap[a, a+l]=\emptyset$. Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists $\lambda_{0}=\lambda_{0}(\dot{\theta}, p, r)>0$ s.t. for any $\psi \in D\left(q_{\theta}^{l}\right)$ we have

$$
\begin{equation*}
\left\|\chi_{(p-r, p)} \psi\right\|^{2} \leq \frac{1}{\lambda_{0}} Q^{(p-r, p)}(\psi) \leq \frac{1}{\lambda_{0}} Q^{(-\infty, p)}(\psi) \tag{23}
\end{equation*}
$$

We now want to estimate the first term on the right hand side of (22). We have

$$
\begin{equation*}
q_{2}^{(-\infty, p)}(f \psi)=\int_{\Omega_{p}}\left|\partial_{s}(f \psi)\right|^{2} d s d t=q_{2}^{\left(-\infty, \theta_{m}\right)}(f \psi)+q_{2}^{\left(\theta_{m}, p\right)}(f \psi) \tag{24}
\end{equation*}
$$

Evidently since $\dot{\theta}=0$ and $f=1$ in $\left(-\infty, \theta_{m}\right)$, from (19), we have

$$
\begin{equation*}
q_{2}^{\left(-\infty, \theta_{m}\right)}(f \psi) \leq Q^{\left(-\infty, \theta_{m}\right)}(\psi) \tag{25}
\end{equation*}
$$

In the other hand since $f(p) \psi(p,)=$.0 , we can apply the Lemma 4.1 of the Appendix. So for any $0<\alpha<1$ there exists $\gamma_{\alpha, 1}>0$ such that

$$
\begin{equation*}
\left|q_{2,3}^{\left(\theta_{m}, p\right)}(f \psi)\right| \leq \gamma_{\alpha, 1} q_{1}^{\left(\theta_{m}, p\right)}(f \psi)+\alpha q_{2}^{\left(\theta_{m}, p\right)}(f \psi)+q_{3}^{\left(\theta_{m}, p\right)}(f \psi) \tag{26}
\end{equation*}
$$

Let $\gamma:=\max \left(1, \gamma_{\alpha, 1}\right)$. Then

$$
\begin{equation*}
\gamma^{-1}\left|q_{2,3}^{\left(\theta_{m}, p\right)}(f \psi)\right| \leq q_{1}^{\left(\theta_{m}, p\right)}(f \psi)+\alpha \gamma^{-1} q_{2}^{\left(\theta_{m}, p\right)}(f \psi)+\gamma^{-1} q_{3}^{\left(\theta_{m}, p\right)}(f \psi) \tag{27}
\end{equation*}
$$

Hence with the decomposition, $q_{2,3}^{\left(\theta_{m}, p\right)}=\gamma^{-1} q_{2,3}^{\left(\theta_{m}, p\right)}+\left(1-\gamma^{-1}\right) q_{2,3}^{\left(\theta_{m}, p\right)}$ and (27) we have,

$$
\begin{align*}
Q^{\left(\theta_{m}, p\right)}(f \psi) & \geq\left(1-\gamma^{-1}\right)\left(q_{2}^{\left(\theta_{m}, p\right)}(f \psi)+q_{2,3}^{\left(\theta_{m}, p\right)}(f \psi)+q_{3}^{\left(\theta_{m}, p\right)}(f \psi)\right)  \tag{28}\\
& +\gamma^{-1}(1-\alpha) q_{2}^{\left(\theta_{m}, p\right)}(f \psi)
\end{align*}
$$

and since $q_{3}^{\left(\theta_{m}, p\right)}+q_{2,3}^{\left(\theta_{m}, p\right)}+q_{2}^{\left(\theta_{m}, p\right)} \geq 0$, we arrive at,

$$
\begin{equation*}
q_{2}^{\left(\theta_{m}, p\right)}(f \psi) \leq \frac{\gamma}{(1-\alpha)} Q^{\left(\theta_{m}, p\right)}(f \psi) \tag{29}
\end{equation*}
$$

Now by using that, $q_{1}^{\left(\theta_{m}, p\right)}(f \psi) \leq q_{1}^{\left(\theta_{m}, p\right)}(\psi)$,

$$
\left\|\chi_{\left(\theta_{m}, p\right)}\left(\partial_{s}+\dot{\theta} \partial_{\tau}\right)(f \psi)\right\|^{2} \leq 2\left(\left\|\chi_{\left(\theta_{m}, p\right)}\left(\partial_{s}+\dot{\theta} \partial_{\tau}\right) \psi\right\|^{2}+\frac{1}{r^{2}}\left\|\chi_{(p-r, p)} \psi\right\|^{2}\right)
$$

and (23), we get,

$$
\begin{equation*}
q_{2}^{\left(\theta_{m}, p\right)}(f \psi) \leq \frac{2 \gamma}{(1-\alpha)}\left(Q^{\left(\theta_{m}, p\right)}(\psi)+\frac{1}{\lambda_{0} r^{2}} Q^{(p-r, p)}(\psi)\right) \leq c^{\prime} Q^{\left(\theta_{m}, p\right)}(\psi) \tag{30}
\end{equation*}
$$

with $c^{\prime}=\frac{2 \gamma}{(1-\alpha)}\left(1+\frac{1}{\lambda_{0} r^{2}}\right)$. Then (25) and (30) imply

$$
\begin{equation*}
q_{2}^{(-\infty, p)}(f \psi) \leq\left(1+c^{\prime}\right) Q^{(-\infty, p)}(\psi) \tag{31}
\end{equation*}
$$

Hence (31) and (23) prove the lemma with

$$
\begin{equation*}
C^{-1}=8\left(1+c^{\prime}\right)+\frac{2}{\lambda_{0}} \tag{32}
\end{equation*}
$$

Proof of the proposition 3.3. To prove the proposition we note that for any $\psi \in D\left(q_{\theta}^{l}\right)$ and for $p^{\prime} \in \mathbb{R}$ we have

$$
\begin{equation*}
\int_{\omega} \int_{p^{\prime}}^{\infty}\left|\nabla^{\prime} \psi\right|^{2}+\left|\dot{\theta} \partial_{\tau} \psi+\partial_{s} \psi\right|^{2} d s d t \geq \int_{\omega} \int_{p^{\prime}}^{\infty} g(s)|\psi|^{2} d s d t \tag{33}
\end{equation*}
$$

Then (33) with $p^{\prime}=p$ and Lemma 3.4 imply (16).

### 3.2 Reduction to a one dimensional problem

We now want to prove the following result,
Proposition 3.5. Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by

$$
\begin{equation*}
\int_{\mathbb{R}}\left|\psi^{\prime}(s)\right|^{2}+2 C \rho(s)|\psi(s)|^{2} d s-4 E_{1} \int_{a}^{a+l}|\psi(s)|^{2} d s \geq 0 \tag{34}
\end{equation*}
$$

for any $\psi \in \mathcal{H}^{1}(\mathbb{R})$ where the constant $C$ is defined in (32).
Remark 3.6. This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on $L^{2}(\mathbb{R})$,

$$
\begin{equation*}
-\frac{d^{2}}{d s^{2}}+2 C \rho(s)-4 E_{1} \mathbf{1}_{I_{a}(l)} \tag{35}
\end{equation*}
$$

where $\mathbf{1}_{I_{a}(l)}$ is the characteristic function of $I_{a}(l)$.
Proof. Evidently we have

$$
\begin{equation*}
Q(\psi)=\frac{1}{2}\left(Q(\psi)-\int_{\Omega_{0}}\left(E_{1}-g(s)\right)|\psi|^{2} d s d t+q_{\theta}^{l}(\psi)-\int_{\Omega_{0}} g(s)|\psi|^{2} d s d t\right) \tag{36}
\end{equation*}
$$

where $g$ is defined in (14). By using (16), then

$$
\begin{equation*}
Q(\psi) \geq \frac{1}{2}\left(q_{\theta}^{l}(\psi)-E_{1}\|\psi\|^{2}+C \int_{\Omega_{0}} \rho(s)|\psi|^{2} d s d t-E_{1}\left\|\chi_{(a, a+l)} \psi\right\|^{2}\right) \tag{37}
\end{equation*}
$$

Rewrite the expression of $q_{\theta}^{l}$ given by (3) as follows:

$$
\begin{equation*}
q_{\theta}^{l}(\psi)=\left\|\nabla^{\prime} \psi\right\|^{2}+\left\|\partial_{s} \psi\right\|^{2}+\left\|\dot{\theta} \partial_{\tau} \psi\right\|^{2}+2 \Re\left(\partial_{s} \psi, \dot{\theta} \partial_{\tau} \psi\right) . \tag{38}
\end{equation*}
$$

We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix,

$$
\begin{equation*}
\left|q_{2,3}(\psi)\right|=\left|q_{2,3}^{\left(\theta_{m}, \theta_{M}\right)}(\psi)\right| \leq \gamma_{\frac{1}{2}, \frac{1}{2}} q_{1}^{\left(\theta_{m}, \theta_{M}\right)}(\psi)+\frac{1}{2} q_{2}^{\left(\theta_{m}, \theta_{M}\right)}(\psi)+\frac{1}{2} q_{3}^{\left(\theta_{m}, \theta_{M}\right)}(\psi) \tag{39}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{\frac{1}{2}, \frac{1}{2}}:=\widetilde{\gamma}_{\frac{1}{2}, \frac{1}{2}}+4 d^{2}\|\dot{\theta}\|_{\infty}^{2} \tag{40}
\end{equation*}
$$

with $\widetilde{\gamma}_{\frac{1}{2}, \frac{1}{2}}:=\max \left\{\frac{d\|\dot{\theta}\|_{\infty}\|\ddot{\theta}\|_{\infty} \sqrt{f(L)}}{\dot{\theta}_{0} \sqrt{\lambda}}, \frac{d^{2}\|\ddot{\theta}\|_{\infty}^{2} f(L)}{\lambda \dot{\theta}_{0}^{2}}, 2 d^{2}\|\ddot{\theta}\|_{\infty}^{2} f(L)\right\}$ for some constant $\lambda>0$ depending only on the section $\omega$ and $f(L):=\max \left\{2+\frac{16 L^{2}}{r^{2}}, 4 L^{2}\right\}$.

Hence (38) together with (39) give:

$$
\begin{equation*}
q_{\theta}^{l}(\psi) \geq\left\|\nabla^{\prime} \psi\right\|^{2}+\frac{1}{2}\left\|\partial_{s} \psi\right\|^{2}+\frac{1}{2}\left\|\dot{\theta} \partial_{\tau} \psi\right\|^{2}-\gamma_{\frac{1}{2}, \frac{1}{2}} q_{1}^{\left(\theta_{m}, \theta_{M}\right)}(\psi) \tag{41}
\end{equation*}
$$

In view of (19) we have

$$
\left\|\nabla^{\prime} \psi\right\|^{2}-E_{1}\|\psi\|^{2} \geq q_{1}^{\left(\theta_{m}, \theta_{M}\right)}(\psi)+q_{1}^{I_{a}(l)}(\psi) \geq q_{1}^{\left(\theta_{m}, \theta_{M}\right)}(\psi)-E_{1}\left\|\chi_{(a, a+l)} \psi\right\|^{2} .
$$

Thus this last inequality together with (41) in (37) give

$$
\begin{aligned}
Q(\psi) & \geq \frac{1}{2}\left(\frac{1}{2}\left\|\partial_{s} \psi\right\|^{2}+\frac{1}{2}\left\|\dot{\theta} \partial_{\tau} \psi\right\|^{2}+C \int_{\Omega_{0}} \rho(s)|\psi|^{2} d s d t-2 E_{1}\left\|\chi_{(a, l+a)} \psi\right\|^{2}\right. \\
& \left.+\left(1-\gamma_{\frac{1}{2}, \frac{1}{2}}\right) q_{1}^{\left(\theta_{m} \theta_{M}\right)}(\psi)\right)
\end{aligned}
$$

Now if $0<d \leq d_{\max }$ then $\gamma_{\frac{1}{2}, \frac{1}{2}} \leq 1$ so the Proposition 3.5 follows.

### 3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1] Here we follow a similar strategy as in [1].
Proposition 3.7. for all $\varphi \in \mathcal{H}^{1}(\mathbb{R})$, then there exists $l_{\text {max }}>0$ such that for any $0<l \leq l_{\text {max }}$ we have

$$
\begin{equation*}
\int_{\mathbb{R}}\left|\varphi^{\prime}(s)\right|^{2}+2 C \rho(s)|\varphi(s)|^{2} d s \geq 4 E_{1} \int_{I_{a}(l)}|\varphi(s)|^{2} d s \tag{42}
\end{equation*}
$$

Proof. Introduce the following function:

$$
\Phi(s):= \begin{cases}\left(\frac{\pi}{2}+\arctan (s-p)\right), & \text { if } s<p  \tag{43}\\ \frac{\pi}{2}, & \text { if } s \geq p\end{cases}
$$

where $p$ is the same real number as in (15). So clearly $\Phi^{\prime}=\rho$. For any $t \in I_{a}(l)$ and $\varphi \in \mathcal{H}^{1}(\mathbb{R})$, we have:

$$
\begin{align*}
\frac{\pi}{2} \varphi(t)=\Phi(t) \varphi(t) & =\int_{-\infty}^{t}(\Phi(s) \varphi(s))^{\prime} d s \\
& =\int_{-\infty}^{t} \rho(s) \varphi(s) d s+\int_{-\infty}^{t} \Phi(s) \varphi^{\prime}(s) d s \tag{44}
\end{align*}
$$

and since $\rho(s)=0$ for any $s \in(p, \infty)$, we get,

$$
\begin{equation*}
\frac{\pi}{2} \varphi(t)=\int_{-\infty}^{p} \rho(s) \varphi(s) d s+\int_{-\infty}^{t} \Phi(s) \varphi^{\prime}(s) d s \tag{45}
\end{equation*}
$$

Then some straightforward estimates lead to,

$$
\begin{align*}
\frac{\pi^{2}}{4} \varphi^{2}(t) & \leq 2\left(\left(\int_{-\infty}^{p} \rho(s) \varphi(s) d s\right)^{2}+\left(\int_{-\infty}^{t} \Phi(s) \varphi^{\prime}(s) d s\right)^{2}\right)  \tag{46}\\
& \leq 2\left(\int_{-\infty}^{p} \rho(s) d s \int_{-\infty}^{p} \rho(s) \varphi^{2}(s) d s+\int_{-\infty}^{t} \Phi^{2}(s) d s \int_{-\infty}^{t} \varphi^{\prime 2}(s) d s\right)
\end{align*}
$$

By direct calculation $\int_{-\infty}^{p} \rho(s) d s=\frac{\pi}{2}$ and $\int_{-\infty}^{p} \Phi^{2}(s) d s+\int_{p}^{t} \Phi^{2}(s) d s=$ $\pi \ln 2+\frac{\pi^{2}}{4}(t-p)$. Hence we get,

$$
\begin{equation*}
|\varphi(t)|^{2} \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s) \varphi^{2}(s) d s+\left(\frac{8 \ln 2}{\pi}+2(t-p)\right) \int_{\mathbb{R}}\left|\varphi^{\prime}(s)\right|^{2} d s \tag{47}
\end{equation*}
$$

We integrate both sides of (47) over $I_{a}(l)$, then

$$
\begin{aligned}
\int_{I_{a}(l)}|\varphi(t)|^{2} d t & \leq \frac{4 l}{\pi} \int_{\mathbb{R}} \rho(s) \varphi^{2}(s) d s+\left(\left(\frac{8 \ln 2}{\pi}+2(a-p)\right) l+l^{2}\right) \int_{\mathbb{R}}\left|\varphi^{\prime}(s)\right|^{2} d s \\
& \leq c^{\prime \prime} \int_{\mathbb{R}} 2 C \rho(s) \varphi^{2}(s)+\left|\varphi^{\prime}(s)\right|^{2} d s
\end{aligned}
$$

where $c^{\prime \prime}=2 l\left(\frac{1}{\pi C}+\frac{4 \ln 2}{\pi}+a-p\right)+l^{2}$. Finally we get,

$$
\begin{equation*}
4 E_{1} \int_{a}^{l+a}|\varphi(t)|^{2} d t \leq 4 E_{1} c^{\prime \prime} \int_{\mathbb{R}} 2 C \rho(s)|\varphi(s)|^{2}+\left|\varphi^{\prime}(s)\right|^{2} d s \tag{48}
\end{equation*}
$$

So choose $0<l \leq l_{\max }$ with

$$
l_{\max }:=-\left(\frac{1}{\pi C}+\frac{4 \ln 2}{\pi}+a-p\right)+\sqrt{\left(\frac{1}{\pi C}+\frac{4 \ln 2}{\pi}+a-p\right)^{2}+\left(4 E_{1}\right)^{-1}}
$$

then $4 E_{1} c^{\prime \prime} \leq 1$ and the proposition 3.7 follows.

## 3.4 proof of the Theorem 1.1 ii)

Under assumptions of the Theorem 1.1 ii) $H_{\theta}^{\mathcal{N}} \geq H_{\theta}^{l}$. These two operators have the same essential spectrum so the Theorem 1.1 ii ) is proved by applying the Proposition 3.1 and the min-max principle.

## 4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D\left(q_{\theta}^{l}\right)$ we have for any $\alpha, \beta>0$ there exists $\gamma_{\alpha, \beta}>0$ such that:

$$
\begin{equation*}
\left|q_{2,3}(\psi)\right| \leq \gamma_{\alpha, \beta} q_{1}(\psi)+\alpha q_{2}(\psi)+\beta q_{3}(\psi) \tag{49}
\end{equation*}
$$

Then we have

Lemma 4.1. Let $p \in\left(\theta_{m}, \theta_{M}\right)$. For all $\psi \in D\left(q_{\theta}^{l}\right)$ such that $\psi(p,)=$.0 , then for any $\alpha, \beta>0$ there exists $\gamma_{\alpha, \beta}>0$ such that:

$$
\begin{equation*}
\left|q_{2,3}^{\left(\theta_{m}, p\right)}(\psi)\right| \leq \gamma_{\alpha, \beta} q_{1}^{\left(\theta_{m}, p\right)}(\psi)+\alpha q_{2}^{\left(\theta_{m}, p\right)}(\psi)+\beta q_{3}^{\left(\theta_{m}, p\right)}(\psi) . \tag{50}
\end{equation*}
$$

Proof. Let $\psi \in D\left(q_{\theta}^{l}\right)$ such that $\psi(p,)=$.0 . Then $\psi \in \mathcal{H}_{0}^{1}\left(\Omega_{p}\right)$. We know that we may first consider vectors $\psi(s, t)=\chi_{1}(t) \phi(s, t)$, where $\phi \in C_{0}^{\infty}\left(\Omega_{p}\right)$. For such a vector $\psi$ we have,

$$
\begin{align*}
& q_{1}^{\left(\theta_{m}, p\right)}(\psi)=\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \nabla^{\prime} \phi\right\|^{2}, \quad q_{2}^{\left(\theta_{m}, p\right)}(\psi)=\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \partial_{s} \phi\right\|^{2}  \tag{51}\\
& q_{3}^{\left(\theta_{m}, p\right)}(\psi)=\left\|\chi_{\left(\theta_{m}, p\right)} \dot{\theta}\left(\chi_{1} \partial_{\tau} \phi+\phi \partial_{\tau} \chi_{1}\right)\right\|^{2}
\end{align*}
$$

and

$$
\begin{equation*}
q_{2,3}^{\left(\theta_{m}, p\right)}(\psi)=2\left(\dot{\theta} \chi_{\left(\theta_{m}, p\right)} \chi_{1} \partial_{\tau} \phi, \chi_{1} \partial_{s} \phi\right)+2\left(\dot{\theta} \chi_{\left(\theta_{m}, p\right)} \phi \partial_{\tau} \chi_{1}, \chi_{1} \partial_{s} \phi\right) \tag{52}
\end{equation*}
$$

By using simple estimates the first term on the r.h.s of (52) is estimated as:

$$
\left|2\left(\dot{\theta} \chi_{\left(\theta_{m}, p\right)} \chi_{1} \partial_{\tau} \phi, \chi_{1} \partial_{s} \phi\right)\right| \leq 2\|\dot{\theta}\|_{\infty}\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \nabla^{\prime} \phi\right\|\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \partial_{s} \phi\right\|
$$

then

$$
\begin{equation*}
\left|2\left(\dot{\theta} \chi_{\left(\theta_{m}, p\right)} \chi_{1} \partial_{\tau} \phi, \chi_{1} \partial_{s} \phi\right)\right| \leq c_{1} q_{1}^{\left(\theta_{m}, p\right)}(\psi)+\frac{\alpha}{2} q_{2}^{\left(\theta_{m}, p\right)}(\psi) \tag{53}
\end{equation*}
$$

where $c_{1}:=\frac{2}{\alpha} d^{2}\|\dot{\theta}\|_{\infty}^{2}$ and $\alpha>0$.
Integrating by parts twice and using the fact that $\dot{\theta}\left(\theta_{m}\right)=\phi(p,)=$.0 , the second term of the r.h.s of (52) is written as

$$
\begin{equation*}
2\left(\dot{\theta} \chi_{\left(\theta_{m}, p\right)} \phi \partial_{\tau} \chi_{1}, \chi_{1} \partial_{s} \phi\right)=\left(\chi_{\left(\theta_{m}, p\right)} \ddot{\theta} \phi \chi_{1}, \chi_{1} \partial_{\tau} \phi\right) . \tag{54}
\end{equation*}
$$

Then the Cauchy Schwartz inequality implies,

$$
\begin{equation*}
\left|\left(\chi_{\left(\theta_{m}, p\right)} \ddot{\theta} \phi \chi_{1}, \chi_{1} \partial_{\tau} \phi\right)\right|^{2} \leq d^{2}\|\ddot{\theta}\|_{\infty}^{2} q_{1}^{\left(\theta_{m}, p\right)}\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \phi\right\|^{2} \tag{55}
\end{equation*}
$$

Let $p^{\prime} \in \mathbb{R}$ and $r^{\prime}>0$ such that $\left(p^{\prime}-r, p^{\prime}\right) \subset\left(\theta_{m}, p\right)$ and for $s \in\left(p^{\prime}-r, p^{\prime}\right)$, $|\dot{\theta}(s)| \geq \dot{\theta}_{0}$ for some $\dot{\theta}_{0}>0$. As in the proof of the Lemma 3 of [6] we have,

$$
\begin{equation*}
\left\|\chi_{\left(\theta_{m}, p\right)} \chi_{1} \phi\right\|^{2} \leq c_{2}\left(q_{2}^{\left(\theta_{m}, p\right)}(\psi)+\dot{\theta}_{0}^{-2}\left\|\chi_{\left(p^{\prime}-r, p^{\prime}\right)} \dot{\theta} \chi_{1} \phi\right\|^{2}\right) \tag{56}
\end{equation*}
$$

where $c_{2}:=\max \left\{2+16 \frac{\left(p-\theta_{m}\right)^{2}}{r^{2}}, 4\left(p-\theta_{m}\right)^{2}\right\}$.
Moreover, for any $s \in \mathbb{R}, \dot{\theta}(s) \chi_{1} \phi(s,.) \in \mathcal{H}_{0}^{1}\left(\Omega_{p}\right)$, then by using the Lemma 1 of [6] there exists $\lambda>0$ depending on $\omega$ such that :

$$
\begin{equation*}
\left\|\chi_{\left(p^{\prime}-r, p^{\prime}\right)} \dot{\theta} \chi_{1} \phi\right\|^{2} \leq\left\|\chi_{\left(\theta_{m}, p\right)} \dot{\theta} \chi_{1} \phi\right\|^{2} \leq \lambda^{-1}\left(q_{3}^{\left(\theta_{m}, p\right)}(\psi)+\|\dot{\theta}\|_{\infty}^{2} q_{1}^{\left(\theta_{m}, p\right)}(\psi)\right) \tag{57}
\end{equation*}
$$

Hence (56), (57) and (54) give

$$
\begin{equation*}
\left|\left(\chi_{\left(\theta_{m}, p\right)} \ddot{\theta} \phi \chi_{1}, \chi_{1} \partial_{\tau} \phi\right)\right|^{2} \leq\left(c_{3} q_{1}^{\left(\theta_{m}, p\right)}(\psi)+\frac{\alpha}{2} q_{2}^{\left(\theta_{m}, p\right)}(\psi)+\beta q_{3}^{\left(\theta_{m}, p\right)}(\psi)\right)^{2} \tag{58}
\end{equation*}
$$

where $c_{3}:=\max \left\{\frac{d\|\ddot{\theta}\|\|\dot{\theta}\|_{\infty} \sqrt{c_{2}}}{\dot{\theta}_{0} \sqrt{\lambda}}, \frac{d^{2}\|\ddot{\theta}\|_{\infty}^{2} c_{2}}{\alpha}, \frac{d^{2}\|\ddot{\theta}\|_{\infty}^{2} c_{2}}{2 \beta \dot{\theta}_{0}^{2} \lambda}\right\}$. Then (531) and (58) imply (50) with $\gamma_{\alpha, \beta}:=c_{1}+c_{3}$.

Note that we can choose $\chi_{1}>0$ on $\omega$. So that (50) holds for every $\psi \in$ $C_{0}^{\infty}\left(\Omega_{p}\right)$ and by a density argument this is even true for $\psi \in \mathcal{H}_{0}^{1}\left(\Omega_{p}\right)$.

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