Twisted waveguide with a Neumann window

Philippe Briet, Hiba Hammedi *

Aix-Marseille Université, CNRS, CPT, UMR 7332, Case 907, 13288 Marseille and Université de Toulon, CNRS, CPT, UMR 7332 83957, La Garde, France;

January 2016

Dedicated to Pavel Exner on the occasion of his 70th birthday

Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of \mathbb{R}^3 which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

classification Primary 81Q10; Secondary 47F05.

keywords Waveguide, mixed boundary conditions, twisting.

1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_0 := \mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial \Omega_0$.

The cross section ω is supposed to be an open bounded connected subset of \mathbb{R}^2 of diameter d > 0 which is not rotationally invariant. Moreover ω is supposed to have smooth boundary $\partial \omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by \mathcal{N} the Neumann window. It is an open bounded subset of the boundary $\partial \Omega_0$. Let \mathcal{D} be its complement set in $\partial \Omega_0$. When \mathcal{N} is an annulus of size l > 0 we will denote it by,

 $\mathcal{A}_a(l) := I_a(l) \times \partial \omega, I_a(l) := (a, l+a), a \in \mathbb{R}.$

^{*}briet@univ-tln.fr, hammedi@univ-tln.fr.

Consider first the self-adjoint operator $H_0^{\mathcal{N}}$ associated to the following quadratic form. Let $D(Q^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_0) \mid \psi_{\lceil \mathcal{D}} = 0\}$ and for $\psi \in D(Q^{\mathcal{N}})$,

$$Q^{\mathcal{N}}(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

i.e. the Laplace operator defined on Ω_0 with Neumann boundary conditions (NBC) on \mathcal{N} and Dirichlet boundary conditions (DBC) on \mathcal{D} [5, 11].

It is actually shown in the Section 2 of this paper that if \mathcal{N} contains an annulus of size l large enough then $H_0^{\mathcal{N}}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if \mathcal{N} contains an annulus of any size l > 0.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_0^{\mathcal{N}}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function θ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose $\theta \in C_c^1(\mathbb{R})$ and introduce the diffeomorphism

$$\mathcal{L}: \Omega_0 \longrightarrow \mathbb{R}^3$$

$$(s, t_2, t_3) \longmapsto \left(s, t_2 \cos \theta(s) - t_3 \sin \theta(s), t_2 \sin \theta(s) + t_3 \cos \theta(s) \right).$$

$$(1)$$

The twisted tube is given by $\Omega_{\theta} := \mathcal{L}(\Omega_0)$. Let $D(Q_{\theta}^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_{\theta}) \mid \psi_{\lceil \mathcal{L}(\mathcal{D})} = 0\}$ and consider the following quadratic form

$$Q_{\theta}^{\mathcal{N}}(\psi) := \int_{\Omega_{\theta}} |\nabla \psi|^2 dx, \ \psi \in D(Q_{\theta}^{\mathcal{N}}).$$
⁽²⁾

Through unitary equivalence, we then have to consider

$$q_{\theta}^{\mathcal{N}}(\psi) := Q_{\theta}^{\mathcal{N}}(\psi o \mathcal{L}^{-1}) = \| \nabla' \psi \|^2 + \| \partial_s \psi + \dot{\theta} \partial_\tau \psi \|^2,$$
(3)

 $\psi \in D(q_{\theta}^{\mathcal{N}}) := \{ \psi \in \mathcal{H}^1(\Omega_0) \ \mid \psi_{\lceil \mathcal{D}} = 0 \} \text{ and where }$

$$\nabla' := {}^t \left(\partial_{t_2}, \partial_{t_3} \right), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}. \tag{4}$$

Denote by $H_{\theta}^{\mathcal{N}}$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H_{\theta}^{\mathcal{N}}) = \{\psi \in D(q_{\theta}^{\mathcal{N}}), H_{\theta}^{\mathcal{N}}\psi \in L^{2}(\Omega_{0}) \quad \frac{\partial \psi}{\partial n} \lceil_{\mathcal{N}} = 0\}$ with

$$H^{\mathcal{N}}_{\theta}\psi = (-\Delta_{\omega} - (\dot{\theta}\partial_{\tau} + \partial_s)^2)\psi, \qquad (5)$$

where the transverse Laplacian $\Delta_{\omega} := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = \mathcal{A}_a(l), l > 0$, we will denote these forms respectively as $Q_{\theta}^l, q_{\theta}^l$ and the corresponding operator as H_{θ}^l and if $\mathcal{N} = \emptyset$ we denote the associated operator by H_{θ} .

Then the main result of this paper is

Theorem 1.1. *i*) Under conditions stated above on ω and θ , there exists $l_{min} := l_{min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{min}$, $\mathcal{N} \supset \mathcal{A}_a(l)$ then

$$\sigma_d(H^{\mathcal{N}}_{\theta}) \neq \emptyset. \tag{6}$$

ii) Suppose θ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{max} := d_{max}(\theta, \omega) > 0$ such that for all $0 < d \leq d_{max}$ there exists $l_{max} := l_{max}(\omega, d, \theta)$ such as for all $0 < l \leq l_{max}$, if $\mathcal{N} \subset \mathcal{A}_a(l)$ and $supp(\dot{\theta}) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then

$$\sigma_d(H^{\mathcal{N}}_\theta) = \emptyset. \tag{7}$$

Roughly speaking this result implies that for d small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

2 Existence of bound states

First we prove the following. Denote by E_1, E_2, \ldots the eigenvalues (transverse modes) of the Laplacian $-\Delta_{\omega}$ defined on $L^2(\omega)$ with DBC on $\partial \omega$. Let χ_1, χ_2, \ldots be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{ess}(H_{\theta}^{\mathcal{N}}) = [E_1, \infty).$

Proof. We know that $\sigma(H_{\theta}) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $H_{\theta}^{\mathcal{N}} \leq H_{\theta}$, then

$$[E_1, \infty) \subset \sigma_{ess}(H^{\mathcal{N}}_{\theta}).$$
(8)

Let $a' \in \mathbb{R}$ and l' > 0 large enough such that $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial \omega$ and $supp(\dot{\theta}) \subset I_{a'}(l')$. Let $\tilde{H}_{\theta}^{l'}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a'+l'\} \times \omega$. So $H_{\theta}^{\mathcal{N}} \geq \tilde{H}_{\theta}^{l'}$ and then $\sigma_{ess}(H_{\theta}^{\mathcal{N}}) \subset \sigma_{ess}(\tilde{H}_{\theta}^{l'})$ [12]. But $\tilde{H}_{\theta}^{l'} = \tilde{H}_i \oplus \tilde{H}_e$. The interior operator \tilde{H}_i is the corresponding operator

But $H_{\ell}^{l} = H_{i} \oplus H_{e}$. The interior operator H_{i} is the corresponding operator defined on $L^{2}(I_{a'}(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a'+l'\} \times \omega \cup \mathcal{N}$ and DBC elsewhere on $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{ess}(\tilde{H}_{e}^{l'}) = \sigma_{ess}(\tilde{H}_{e})$.

Now the exterior operator \tilde{H}_e is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial \omega \cup (a' + l', \infty) \times \partial \omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for x < a' and x > a' + l', it is easy to see that

$$\tilde{H}_e = \bigoplus_{n \ge 1} (-\partial^2 + E_n)(\chi_n, .)\chi_n.$$

Hence $\sigma(\tilde{H}_e) = \sigma_{ess}(\tilde{H}_e) = [E_1, +\infty).$

The Theorem 1.1 i) follows from

3

Proposition 2.2. Under conditions of the Theorem 1.1 i), there exists $l_{min} := l_{min}(\omega, d) > 0$ such as for all $l > l_{min}$ we have

$$\sigma_d(H^l_\theta) \neq \emptyset. \tag{9}$$

Proof. Let $\varphi_{l,a}$ be the following function

$$\varphi_{l,a}(s) := \begin{cases} \frac{10}{l}(s-a), & \text{on } [a, a + \frac{l}{10});\\ 1, & \text{on } [a + \frac{l}{10}, a + \frac{9l}{10});\\ -\frac{10}{l}(s-l-a), & \text{on } [a + \frac{9l}{10}, a+l);\\ 0, & \text{elsewhere.} \end{cases}$$

It is easy to see that $\varphi_{l,a} \in D(q^l_{\theta})$ and $\|\varphi_{l,a}\|^2 = \frac{13l}{15} |\omega|$. Let us calculate

$$q_{\theta}^{l}(\varphi_{l,a}) - E_{1} \parallel \varphi_{l,a} \parallel^{2} = \parallel \nabla' \varphi_{l,a} \parallel^{2} + \parallel \dot{\theta} \partial_{\tau} \varphi_{l,a} + \partial_{s} \varphi_{l,a} \parallel^{2} - E_{1} \parallel \varphi_{l,a} \parallel^{2} .$$
(10)

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$\| \dot{\theta} \partial_{\tau} \varphi_{l,a} + \partial_{s} \varphi_{l,a} \|^{2} = \| \partial_{s} \varphi_{l,a} \|^{2} = \frac{20}{l} | \omega |.$$

Then

$$q_{\theta}^{l}(\varphi_{l,a}) - E_{1} \parallel \varphi_{l,a} \parallel^{2} = \mid \omega \mid \left(\frac{20}{l} - \frac{13l}{15}E_{1}\right)$$
(11)

and thus if $l \ge l_{min} := \sqrt{\frac{300}{13E_1}}$ we have $q_{\theta}^l(\varphi_{l,a}) - E_1 \parallel \varphi_{l,a} \parallel^2 \le 0$

2.1 Proof of the Theorem 1.1 i)

Using the same notation as in the Theorem 1.1 *i*), then $H_{\theta}^{\mathcal{N}} \leq H_{\theta}^{l}$. Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by $\theta_m = \inf(\operatorname{supp}(\dot{\theta})), \theta_M = \sup(\operatorname{supp}(\dot{\theta}))$ and $L = \theta_M - \theta_m$. Here L > 0. We first consider the case where the Neumann window is an annulus, $\mathcal{A}_a(l) = I_a(l) \times \omega$.

Proposition 3.1. Suppose $\mathcal{A}_a(l)$ is such that $a \ge \theta_M$. Assume that conditions of the Theorem 1.1 ii) hold. Then there exists $d_{max} := d_{max}(\omega, \theta) > 0$, such that for all $0 < d \le d_{max}$ there exists $l_{max}(d, \theta, \omega) > 0$ such as for all $0 < l \le l_{max}$ we have

$$\sigma_d(H^l_\theta) = \emptyset. \tag{12}$$

Remark 3.2. the case where $l+a \leq \theta_m$ follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every $\psi \in D(q_{\theta}^{l})$ it holds,

$$Q(\psi) := q_{\theta}^{l}(\psi) - E_{1} \| \psi \|^{2} \ge 0.$$
(13)

The proof of (13) involves several steps.

3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let g be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_a(l); \\ E_1, & \text{elsewhere.} \end{cases}$$
(14)

Choose $p \in (\theta_m, \theta_M)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$\rho(s) := \begin{cases} \frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases}$$
(15)

Proposition 3.3. Under same conditions of the Proposition 3.1, then there exists a constant C > 0 depending on p and ω and $\dot{\theta}$ such that for any $\psi \in D(q^l_{\theta})$,

$$\|\nabla'\psi\|^2 + \|\dot{\theta}\partial_\tau\psi + \partial_s\psi\|^2 - \int_{\Omega_0} g(s) |\psi|^2 \, dsdt \ge C \int_{\Omega_0} \rho(s) |\psi|^2 \, dsdt.$$

$$\tag{16}$$

We first show the following lemma. Denote by $\Omega_p := (-\infty, p) \times \omega$.

Lemma 3.4. Under same conditions of the Proposition 3.3. Then for any $\psi \in D(q^l_{\theta})$ we have

$$\int_{\Omega_p} |\nabla'\psi|^2 + |\dot{\theta}\partial_\tau\psi + \partial_s\psi|^2 - E_1 |\psi|^2 \, dsdt \ge C \int_{\Omega_p} \rho(s) |\psi|^2 \, dsdt. \tag{17}$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by χ_A the characteristic function of $A \times \omega$. Let $\psi \in D(q_{\theta}^l)$ and define,

$$q_{1}^{A}(\psi) := \| \chi_{A} \nabla' \psi \|^{2} - E_{1} \| \chi_{A} \psi \|^{2}, \quad q_{2}^{A}(\psi) := \| \chi_{A} \partial_{s} \psi \|^{2}, q_{3}^{A}(\psi) := \| \chi_{A} \dot{\theta} \partial_{\tau} \psi \|^{2}, \qquad q_{2,3}^{A}(\psi) := 2\Re(\partial_{s} \psi, \chi_{A} \dot{\theta} \partial_{\tau} \psi).$$
(18)

Denote also by $Q^A(\psi) = q_1^A(\psi) + q_2^A(\psi) + q_3^A(\psi) + q_{2,3}^A(\psi)$. Here and hereafter we often use the fact that for any $\psi \in D(q_{\theta}^A)$

$$q_1^A(\psi) \ge 0,\tag{19}$$

for every $A \subset \mathbb{R}$ such that $A \cap I_a(l) = \emptyset$.

Proof. Choose r > 0 such that $\dot{\theta}(s) \neq 0$ for any $s \in [p - r, p]$. Let f be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty);\\ \frac{p-s}{r}, & \text{on } (p-r, p];\\ 1, & \text{elsewhere.} \end{cases}$$
(20)

For any $\psi \in D(q_{\theta}^{l})$, simple estimates lead to:

$$\int_{\Omega_p} \frac{|\psi(s,t)|^2}{1+(s-p)^2} ds dt = \int_{\Omega_p} \frac{|\psi(s,t)f(s) + (1-f(s))\psi(s,t)|^2}{1+(s-p)^2} ds dt \quad (21)$$

$$\leq 2\Big(\int_{\Omega_p} \frac{|f(s)\psi(s,t)|^2}{(s-p)^2} ds dt + \|\chi_{(p-r,p)}\psi\|^2\Big).$$

Since $f(p)\psi(p,.) = 0$, we can use the usual Hardy inequality (see e.g. [9]), then we get,

$$\int_{\Omega_p} \frac{|\psi(s,t)|^2}{1+(s-p)^2} ds dt \le 8q_2^{(-\infty,p)}(f\psi) + 2\|\chi_{(p-r,p)}\psi\|^2.$$
(22)

Note that with our choice $[p - r, p] \cap [a, a + l] = \emptyset$. Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists $\lambda_0 = \lambda_0(\dot{\theta}, p, r) > 0$ s.t. for any $\psi \in D(q_{\theta}^l)$ we have

$$\|\chi_{(p-r,p)}\psi\|^{2} \leq \frac{1}{\lambda_{0}}Q^{(p-r,p)}(\psi) \leq \frac{1}{\lambda_{0}}Q^{(-\infty,p)}(\psi).$$
(23)

We now want to estimate the first term on the right hand side of (22). We have

$$q_2^{(-\infty,p)}(f\psi) = \int_{\Omega_p} |\partial_s(f\psi)|^2 \, ds dt = q_2^{(-\infty,\theta_m)}(f\psi) + q_2^{(\theta_m,p)}(f\psi).$$
(24)

Evidently since $\dot{\theta} = 0$ and f = 1 in $(-\infty, \theta_m)$, from (19), we have

$$q_2^{(-\infty,\theta_m)}(f\psi) \le Q^{(-\infty,\theta_m)}(\psi).$$
(25)

In the other hand since $f(p)\psi(p,.) = 0$, we can apply the Lemma 4.1 of the Appendix. So for any $0 < \alpha < 1$ there exists $\gamma_{\alpha,1} > 0$ such that

$$|q_{2,3}^{(\theta_m,p)}(f\psi)| \le \gamma_{\alpha,1} q_1^{(\theta_m,p)}(f\psi) + \alpha q_2^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi).$$
(26)

Let $\gamma := \max(1, \gamma_{\alpha, 1})$. Then

$$\gamma^{-1} \mid q_{2,3}^{(\theta_m,p)}(f\psi) \mid \leq q_1^{(\theta_m,p)}(f\psi) + \alpha \gamma^{-1} q_2^{(\theta_m,p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m,p)}(f\psi).$$
(27)

Hence with the decomposition, $q_{2,3}^{(\theta_m,p)} = \gamma^{-1}q_{2,3}^{(\theta_m,p)} + (1-\gamma^{-1})q_{2,3}^{(\theta_m,p)}$ and (27) we have,

$$Q^{(\theta_m,p)}(f\psi) \ge (1-\gamma^{-1}) \Big(q_2^{(\theta_m,p)}(f\psi) + q_{2,3}^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi) \Big)$$
(28)
+ $\gamma^{-1}(1-\alpha) q_2^{(\theta_m,p)}(f\psi)$

and since $q_3^{(\theta_m,p)} + q_{2,3}^{(\theta_m,p)} + q_2^{(\theta_m,p)} \ge 0$, we arrive at,

$$q_2^{(\theta_m,p)}(f\psi) \le \frac{\gamma}{(1-\alpha)} Q^{(\theta_m,p)}(f\psi).$$
⁽²⁹⁾

Now by using that, $q_1^{(\theta_m,p)}(f\psi) \le q_1^{(\theta_m,p)}(\psi)$,

$$\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)(f\psi)\|^2 \le 2(\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)\psi\|^2 + \frac{1}{r^2}\|\chi_{(p-r,p)}\psi\|^2)$$

and (23), we get,

$$q_{2}^{(\theta_{m},p)}(f\psi) \leq \frac{2\gamma}{(1-\alpha)} (Q^{(\theta_{m},p)}(\psi) + \frac{1}{\lambda_{0}r^{2}} Q^{(p-r,p)}(\psi)) \leq c' Q^{(\theta_{m},p)}(\psi)$$
(30)

with $c' = \frac{2\gamma}{(1-\alpha)} (1 + \frac{1}{\lambda_0 r^2})$. Then (25) and (30) imply

$$q_2^{(-\infty,p)}(f\psi) \le (1+c')Q^{(-\infty,p)}(\psi).$$
 (31)

Hence (31) and (23) prove the lemma with

$$C^{-1} = 8(1+c') + \frac{2}{\lambda_0}.$$
(32)

Proof of the proposition 3.3. To prove the proposition we note that for any $\psi \in D(q^l_{\theta})$ and for $p' \in \mathbb{R}$ we have

$$\int_{\omega} \int_{p'}^{\infty} |\nabla'\psi|^2 + |\dot{\theta}\partial_{\tau}\psi + \partial_s\psi|^2 \, dsdt \ge \int_{\omega} \int_{p'}^{\infty} g(s) |\psi|^2 \, dsdt.$$
(33)

Then (33) with p' = p and Lemma 3.4 imply (16).

3.2 Reduction to a one dimensional problem

We now want to prove the following result,

Proposition 3.5. Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by

$$\int_{\mathbb{R}} |\psi'(s)|^2 + 2C\rho(s) |\psi(s)|^2 \, ds - 4E_1 \int_a^{a+l} |\psi(s)|^2 \, ds \ge 0, \tag{34}$$

for any $\psi \in \mathcal{H}^1(\mathbb{R})$ where the constant C is defined in (32).

Remark 3.6. This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on $L^2(\mathbb{R})$,

$$-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 \mathbf{1}_{I_a(l)}.$$
(35)

where $\mathbf{1}_{I_a(l)}$ is the characteristic function of $I_a(l)$.

Proof. Evidently we have

$$Q(\psi) = \frac{1}{2} \left(Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) \mid \psi \mid^2 ds dt + q_{\theta}^l(\psi) - \int_{\Omega_0} g(s) \mid \psi \mid^2 ds dt \right),$$
(36)

where g is defined in (14). By using (16), then

$$Q(\psi) \ge \frac{1}{2} \left(q_{\theta}^{l}(\psi) - E_{1} \parallel \psi \parallel^{2} + C \int_{\Omega_{0}} \rho(s) \mid \psi \mid^{2} ds dt - E_{1} \parallel \chi_{(a,a+l)} \psi \parallel^{2} \right)$$
(37)

Rewrite the expression of q_{θ}^{l} given by (3) as follows:

$$q_{\theta}^{l}(\psi) = \|\nabla'\psi\|^{2} + \|\partial_{s}\psi\|^{2} + \|\dot{\theta}\partial_{\tau}\psi\|^{2} + 2\Re(\partial_{s}\psi,\dot{\theta}\partial_{\tau}\psi).$$
(38)

We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix,

$$|q_{2,3}(\psi)| = |q_{2,3}^{(\theta_m,\theta_M)}(\psi)| \le \gamma_{\frac{1}{2},\frac{1}{2}} q_1^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_2^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_3^{(\theta_m,\theta_M)}(\psi)$$
(39)

where

$$\gamma_{\frac{1}{2},\frac{1}{2}} := \tilde{\gamma}_{\frac{1}{2},\frac{1}{2}} + 4d^2 \parallel \dot{\theta} \parallel_{\infty}^2$$
(40)

 $\text{ with } \widetilde{\gamma}_{\frac{1}{2},\frac{1}{2}} := \max\Big\{\frac{d\|\dot{\theta}\|_{\infty}\|\ddot{\theta}\|_{\infty}\sqrt{f(L)}}{\theta_{0}\sqrt{\lambda}}, \frac{d^{2}\|\ddot{\theta}\|_{\infty}^{2}f(L)}{\lambda\dot{\theta}_{0}^{-2}}, 2d^{2} \parallel \ddot{\theta} \parallel_{\infty}^{2} f(L)\Big\} \text{ for some constant } \lambda > 0 \text{ depending only on the section } \omega \text{ and } f(L) := \max\{2 + \frac{16L^{2}}{r^{2}}, 4L^{2}\}.$

Hence (38) together with (39) give:

$$q_{\theta}^{l}(\psi) \geq \|\nabla'\psi\|^{2} + \frac{1}{2} \|\partial_{s}\psi\|^{2} + \frac{1}{2} \|\dot{\theta}\partial_{\tau}\psi\|^{2} - \gamma_{\frac{1}{2},\frac{1}{2}}q_{1}^{(\theta_{m},\theta_{M})}(\psi).$$
(41)

In view of (19) we have

$$\|\nabla'\psi\|^{2} - E_{1} \|\psi\|^{2} \ge q_{1}^{(\theta_{m},\theta_{M})}(\psi) + q_{1}^{I_{a}(l)}(\psi) \ge q_{1}^{(\theta_{m},\theta_{M})}(\psi) - E_{1}\|\chi_{(a,a+l)}\psi\|^{2}.$$

Thus this last inequality together with (41) in (37) give

$$\begin{aligned} Q(\psi) &\geq \frac{1}{2} \Big(\frac{1}{2} \parallel \partial_s \psi \parallel^2 + \frac{1}{2} \parallel \dot{\theta} \partial_\tau \psi \parallel^2 + C \int_{\Omega_0} \rho(s) \mid \psi \mid^2 ds dt - 2E_1 \parallel \chi_{(a,l+a)} \psi \parallel^2 \\ &+ (1 - \gamma_{\frac{1}{2},\frac{1}{2}}) q_1^{(\theta_m \theta_M)}(\psi) \Big). \end{aligned}$$

Now if $0 < d \le d_{max}$ then $\gamma_{\frac{1}{2},\frac{1}{2}} \le 1$ so the Proposition 3.5 follows.

3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

Proposition 3.7. for all $\varphi \in \mathcal{H}^1(\mathbb{R})$, then there exists $l_{max} > 0$ such that for any $0 < l \leq l_{max}$ we have

$$\int_{\mathbb{R}} |\varphi'(s)|^2 + 2C\rho(s) |\varphi(s)|^2 \, ds \ge 4E_1 \int_{I_a(l)} |\varphi(s)|^2 \, ds. \tag{42}$$

Proof. Introduce the following function:

$$\Phi(s) := \begin{cases} \left(\frac{\pi}{2} + \arctan\left(s - p\right)\right), & \text{if } s < p;\\ \frac{\pi}{2}, & \text{if } s \ge p. \end{cases}$$
(43)

where p is the same real number as in (15). So clearly $\Phi' = \rho$. For any $t \in I_a(l)$ and $\varphi \in \mathcal{H}^1(\mathbb{R})$, we have:

$$\frac{\pi}{2}\varphi(t) = \Phi(t)\varphi(t) = \int_{-\infty}^{t} (\Phi(s)\varphi(s))'ds$$
$$= \int_{-\infty}^{t} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds \qquad (44)$$

and since $\rho(s) = 0$ for any $s \in (p, \infty)$, we get,

$$\frac{\pi}{2}\varphi(t) = \int_{-\infty}^{p} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds.$$
(45)

Then some straightforward estimates lead to,

$$\frac{\pi^2}{4}\varphi^2(t) \leq 2\left(\left(\int_{-\infty}^p \rho(s)\varphi(s)ds\right)^2 + \left(\int_{-\infty}^t \Phi(s)\varphi'(s)ds\right)^2\right) \qquad (46)$$

$$\leq 2\left(\int_{-\infty}^p \rho(s)ds\int_{-\infty}^p \rho(s)\varphi^2(s)ds + \int_{-\infty}^t \Phi^2(s)ds\int_{-\infty}^t \varphi'^2(s)ds\right).$$

By direct calculation $\int_{-\infty}^{p} \rho(s) ds = \frac{\pi}{2}$ and $\int_{-\infty}^{p} \Phi^{2}(s) ds + \int_{p}^{t} \Phi^{2}(s) ds = \pi \ln 2 + \frac{\pi^{2}}{4}(t-p)$. Hence we get,

$$|\varphi(t)|^{2} \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^{2}(s)ds + \left(\frac{8\ln 2}{\pi} + 2(t-p)\right) \int_{\mathbb{R}} |\varphi'(s)|^{2} ds \qquad (47)$$

We integrate both sides of (47) over $I_a(l)$, then

$$\begin{split} \int_{I_a(l)} |\varphi(t)|^2 dt &\leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s) \varphi^2(s) ds + \left(\left(\frac{8\ln 2}{\pi} + 2(a-p)\right)l + l^2 \right) \int_{\mathbb{R}} |\varphi'(s)|^2 ds \\ &\leq c'' \int_{\mathbb{R}} 2C\rho(s) \varphi^2(s) + |\varphi'(s)|^2 ds \end{split}$$

where $c'' = 2l(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p) + l^2$. Finally we get,

$$4E_1 \int_a^{l+a} |\varphi(t)|^2 dt \le 4E_1 c'' \int_{\mathbb{R}} 2C\rho(s) |\varphi(s)|^2 + |\varphi'(s)|^2 ds.$$
(48)

So choose $0 < l \leq l_{max}$ with

$$l_{max} := -\left(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p\right) + \sqrt{\left(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p\right)^2 + (4E_1)^{-1}}$$

then $4E_1c'' \leq 1$ and the proposition 3.7 follows.

3.4 proof of the Theorem 1.1 *ii*)

Under assumptions of the Theorem 1.1 *ii*) $H_{\theta}^{\mathcal{N}} \geq H_{\theta}^{l}$. These two operators have the same essential spectrum so the Theorem 1.1 *ii*) is proved by applying the Proposition 3.1 and the min-max principle.

4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D(q_{\theta}^{l})$ we have for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:

$$q_{2,3}(\psi) \leq \gamma_{\alpha,\beta} q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi).$$

$$(49)$$

Then we have

Lemma 4.1. Let $p \in (\theta_m, \theta_M)$. For all $\psi \in D(q_{\theta}^l)$ such that $\psi(p, .) = 0$, then for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:

$$|q_{2,3}^{(\theta_m,p)}(\psi)| \leq \gamma_{\alpha,\beta} q_1^{(\theta_m,p)}(\psi) + \alpha q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi).$$
(50)

Proof. Let $\psi \in D(q_{\theta}^{l})$ such that $\psi(p, .) = 0$. Then $\psi \in \mathcal{H}_{0}^{1}(\Omega_{p})$. We know that we may first consider vectors $\psi(s,t) = \chi_{1}(t)\phi(s,t)$, where $\phi \in C_{0}^{\infty}(\Omega_{p})$. For such a vector ψ we have,

$$q_1^{(\theta_m,p)}(\psi) = \| \chi_{(\theta_m,p)}\chi_1 \nabla' \phi \|^2, \quad q_2^{(\theta_m,p)}(\psi) = \| \chi_{(\theta_m,p)}\chi_1 \partial_s \phi \|^2 \quad (51)$$

$$q_3^{(\theta_m,p)}(\psi) = \| \chi_{(\theta_m,p)}\dot{\theta}(\chi_1 \partial_\tau \phi + \phi \partial_\tau \chi_1) \|^2$$

and

$$q_{2,3}^{(\theta_m,p)}(\psi) = 2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi,\chi_1\partial_s\phi) + 2(\dot{\theta}\chi_{(\theta_m,p)}\phi\partial_\tau\chi_1,\chi_1\partial_s\phi)$$
(52)

By using simple estimates the first term on the r.h.s of (52) is estimated as :

$$2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi,\chi_1\partial_s\phi) \leq 2 \parallel \dot{\theta} \parallel_{\infty} \parallel \chi_{(\theta_m,p)}\chi_1\nabla'\phi \parallel \parallel \chi_{(\theta_m,p)}\chi_1\partial_s\phi \parallel$$

then

$$|2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi,\chi_1\partial_s\phi)| \le c_1 q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2}q_2^{(\theta_m,p)}(\psi), \tag{53}$$

where $c_1 := \frac{2}{\alpha} d^2 \parallel \dot{\theta} \parallel_{\infty}^2$ and $\alpha > 0$.

Integrating by parts twice and using the fact that $\dot{\theta}(\theta_m) = \phi(p, .) = 0$, the second term of the r.h.s of (52) is written as

$$2(\dot{\theta}\chi_{(\theta_m,p)}\phi\partial_\tau\chi_1,\chi_1\partial_s\phi) = (\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1,\chi_1\partial_\tau\phi).$$
(54)

Then the Cauchy Schwartz inequality implies,

$$|(\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1,\chi_1\partial_\tau\phi)|^2 \le d^2 \parallel \ddot{\theta} \parallel_{\infty}^2 q_1^{(\theta_m,p)} \parallel \chi_{(\theta_m,p)}\chi_1\phi \parallel^2.$$
(55)

Let $p' \in \mathbb{R}$ and r' > 0 such that $(p' - r, p') \subset (\theta_m, p)$ and for $s \in (p' - r, p')$, $|\dot{\theta}(s)| \ge \dot{\theta}_0$ for some $\dot{\theta}_0 > 0$. As in the proof of the Lemma 3 of [6] we have,

$$\|\chi_{(\theta_m,p)}\chi_1\phi\|^2 \le c_2 \Big(q_2^{(\theta_m,p)}(\psi) + \dot{\theta}_0^{-2} \|\chi_{(p'-r,p')}\dot{\theta}\chi_1\phi\|^2\Big)$$
(56)

where $c_2 := \max\left\{2 + 16\frac{(p-\theta_m)^2}{r^2}, 4(p-\theta_m)^2\right\}.$

Moreover, for any $s \in \mathbb{R}$, $\dot{\theta}(s)\chi_1\phi(s,.) \in \mathcal{H}_0^1(\Omega_p)$, then by using the Lemma 1 of [6] there exists $\lambda > 0$ depending on ω such that :

$$\| \chi_{(p'-r,p')} \dot{\theta} \chi_1 \phi \|^2 \le \| \chi_{(\theta_m,p)} \dot{\theta} \chi_1 \phi \|^2 \le \lambda^{-1} \Big(q_3^{(\theta_m,p)}(\psi) + \| \dot{\theta} \|_{\infty}^2 q_1^{(\theta_m,p)}(\psi) \Big).$$
(57)

Hence (56), (57) and (54) give

$$\left|\left(\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1,\chi_1\partial_\tau\phi\right)\right|^2 \le \left(c_3q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2}q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi)\right)^2 \tag{58}$$

where $c_3 := \max\left\{\frac{d\|\ddot{\theta}\|\|\dot{\theta}\|_{\infty}\sqrt{c_2}}{\dot{\theta}_0\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|_{\infty}^2 c_2}{\alpha}, \frac{d^2\|\ddot{\theta}\|_{\infty}^2 c_2}{2\beta\dot{\theta}_0^2\lambda}\right\}$. Then (53) and (58) imply (50) with $\gamma_{\alpha,\beta} := c_1 + c_3$.

Note that we can choose $\chi_1 > 0$ on ω . So that (50) holds for every $\psi \in C_0^{\infty}(\Omega_p)$ and by a density argument this is even true for $\psi \in \mathcal{H}_0^1(\Omega_p)$.

References

- D. Borisov, T. Ekholm, H. Kovařík, Spectrum of the Magnetic Schrödinger Operator in a Waveguide with Combined Boundary Conditions, Ann. Henri Poincar,6 (2005), no. 2, 327–342.
- [2] P. Briet, Spectral analysis for twisted waveguides, *Quatum probabbility and related topics* 27 (2011), 125–130.
- [3] P. Briet, H. Hammedi, D. Krejčiřík, Hardy inequalities in globally twisted waveguides, *lett. Math. Phys.* 105 (2015), no. 7, 939–958.
- [4] P. Briet, H. Kovařík, G. Raikov, E. Soccorsi, Eigenvalue asymptotics in a twisted waveguide, *Commun. in P.D.E.*, **34** (2009), 818–836.
- [5] J. Dittrich, J. Kříž, Straight quantum waveguides with combined boundary conditions. Mathematical results in quantum mechanics (Taxco, 2001), 107– 112, Contemp. Math., 307, Amer. Math. Soc., Providence, RI, 2002.
- [6] T. Ekholm, H. Kovařík, D. Krejčiřík, A Hardy inequality in twisted waveguides, Arch. Ration. Mech. Anal., 188 (2008), 245–264.
- [7] P. Exner, H. Kovařík, Spectrum of the Schrödinger operator in a perturbed periodically twisted tube, *Lett. Math. Phys.* 73 (2005), 183–192.
- [8] H.Hammedi, Analyse spectrale des guides d'ondes "twistés", Thèse de doctorat de Mathématique, Université de Toulon. (2016)
- [9] G. H. Hardy, Note on a theorem of Hilbert, Math, Zeit 6 (1920), 314–317.
- [10] D. Krejčiřík, twisting versus bending in quantum waveguides, Analysis on Graphs and Applications (Cambridge 2007), Proc. Sympos. PureMath., *Amer. Math. Soc.*, Providence, RI, 77 (2008), 617–636.
- [11] J. Kříž, Spectral properties of planar quantum waveguides with combined boundary conditions, P.H.D. Thesis, Charles University Prague, 2003.
- [12] M. Reed, B. Simon, Methods of Modern of Mathematical Physics, IV: Analysis of Operators, Academic Press, New York-San Francisco-London, 1978.