

Spin networks and $\mathrm{SL}(2, \mathbb{C})$ -character varieties

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1 Introduction

The purpose of this chapter is to demonstrate the utility of a graphical calculus in the algebraic study of $\mathrm{SL}(2, \mathbb{C})$ -representations of the fundamental group of an oriented surface of Euler characteristic -1 .

Let F_2 be a rank 2 free group, the fundamental group of both the three-holed sphere and the one-holed torus. The set $\mathcal{R} = \mathrm{Hom}(F_2, \mathrm{SL}(2, \mathbb{C}))$ of representations inherits the structure of an algebraic set from $\mathrm{SL}(2, \mathbb{C})$. The subset of representations that are *completely reducible*, denoted by \mathcal{R}^{ss} , have closed orbits under conjugation. Consequently, the orbit space $\mathcal{R}^{ss}/\mathrm{SL}(2, \mathbb{C}) = \mathcal{R}/\mathrm{SL}(2, \mathbb{C})$ is an algebraic set referred to as the *character variety*. The character variety encodes both Teichmüller Space and moduli of geometric structures [17].

Graphs known as *spin networks* permit a concise description of a natural additive basis for the coordinate ring of the character variety

$$\mathbb{C}[\mathcal{R}/\mathrm{SL}(2, \mathbb{C})] = \mathbb{C}[\mathcal{R}]^{\mathrm{SL}(2, \mathbb{C})}.$$

We will refer to the basis elements as *central functions*. The central functions are indexed by Clebsch-Gordan injections

$$V_c \hookrightarrow V_a \otimes V_b,$$

where $V_c = \mathrm{Sym}^c(\mathbb{C}^2)$ denotes an irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. Our main results use the spin network calculus to describe a strong symmetry within the central function basis, a graphical means of computing the product of two central functions, and an algorithm for computing central functions. This provides a concrete description of the regular functions on the $\mathrm{SL}(2, \mathbb{C})$ -character variety of F_2 and a new proof of a classical result of Fricke, Klein, and Vogt.

We are motivated by a greater understanding of the invariant ring, and the subsequent knowledge of various geometric objects of interest encoded within the character variety. Consequently, the main results in this chapter concern the structure of the central function basis. The results and methods of this chapter may also provide new insight into gauge theoretic questions. However, we are most interested in a methodology and point of view that allows for generalizations to other Lie groups and other surface groups.

History of Central Functions and Spin Networks.

The first reference to the central function basis in the literature appears in [2], where Baez used spin networks to describe a basis of quantum mechanical “state vectors.” He considered the basis abstractly, showing that the space of square integrable functions on a related space of connections modulo gauge transformations is spanned by a set of labelled graphs. He also demonstrated that the basis is orthonormal with respect to the L^2 inner product. His basis, when restricted to $SU(2)$, is precisely the one under consideration here.

More recently, Florentino, Mourão, and Nunes use a like basis to produce distributions related to geometric quantization of moduli spaces of flat connections on a surface [13]. Adam Sikora has also used spin networks to study the character variety for $SL(n, \mathbb{C})$, although without using the central function basis [30]. The construction of arbitrary rank $SL(2, \mathbb{C})$ central functions is described in [25], while much of the diagrammatic theory required for the $SL(n, \mathbb{C})$ case is covered in [8, 9, 25, 30].

The history of the diagrammatic calculus in this chapter is hard to trace, due to the historical difficulty in publishing papers making extensive use of figures. While it is likely that many works on diagrammatic notation have been lost over the years, the specific notation used in this chapter is due to Roger Penrose. In a 1981 letter to Predrag Cvitanović, a physicist who also used diagrams extensively, Penrose recalls developing the notation in the early 1950s while “trying to cope with Hodge’s lectures on differential geometry” [24].

Diagrammatic notations have also played an important role in modern physics. Feynman diagrams are probably the most famous example, but spin networks have also been used for many years, as a graphical description of quantum angular momentum [23]. The use of diagrams in physics is probably best summarized in [31]. Cvitanović also has a thorough description of such notations, which he calls *birdtracks* in [8, 9]. In his work, birdtracks play a starring role in a new classification of semi-simple Lie algebras. Using primitive invariants, which have unique diagrammatic depictions, the exceptional Lie algebras arise in a single series in a construction that he calls the “Magic Triangle.”

The remainder of this chapter is organized as follows.

Section 2 gives some basic definitions and results from invariant theory, as well as a short history of $\mathrm{SL}(2, \mathbb{C})$ invariant theory. It also covers necessary material from representation theory.

In Section 3, we introduce spin networks, which are special types of graphs that may be identified with functions between tensor powers of \mathbb{C}^2 . We give a full treatment of the *spin network calculus*, a powerful means for working with regular functions on $\mathcal{R}/\mathrm{SL}(2, \mathbb{C})$.

Section 4 begins by constructing an additive basis for $\mathbb{C}[\mathcal{R}/\mathrm{SL}(2, \mathbb{C})]$. This basis, denoted by $\{\chi^{a,b,c}\}$, is indexed by triples of nonnegative integers (a, b, c) satisfying the *admissibility condition*:

$$\frac{1}{2}(-a + b + c), \frac{1}{2}(a - b + c), \frac{1}{2}(a + b - c) \in \mathbb{N}.$$

The functions $\chi^{a,b,c} \in \mathbb{C}[\mathcal{R}/\mathrm{SL}(2, \mathbb{C})]$ are central in

$$\mathrm{End}(V_c) \hookrightarrow \mathrm{End}(V_a) \otimes \mathrm{End}(V_b),$$

and are referred to as *central functions*. The construction of the central function basis uses the decomposition

$$\mathbb{C}[\mathrm{SL}(2, \mathbb{C})] \cong \sum_{n \geq 0} V_n^* \otimes V_n.$$

We include a constructive proof of this decomposition, since it is hard to find in the literature. The section concludes by examining the $\mathrm{SL}(2, \mathbb{C})$ -central functions of a rank one free group.

Section 5 contains the main results of this chapter, which concern the case of a rank two free group. In this case, central functions may be written as polynomials in three trace variables, a consequence of a theorem due to Fricke, Klein, and Vogt [14, 32]. The results we prove are summarized below.

- Theorem 5.2 describes a symmetry property of the central function basis: permuting the indices of a central function is equivalent to permuting the variables of its polynomial representation.
- Corollary 5.7 states that, with an appropriate definition of rank, any central function may be written in terms of at most four central functions of lower rank:

$$\begin{aligned} \chi^{a,b,c} = x \cdot \chi^{a-1,b,c-1} &- \frac{(a+b-c)^2}{4a(a-1)} \chi^{a-2,b,c} - \frac{(-a+b+c)^2}{4c(c-1)} \chi^{a,b,c-2} \\ &- \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)} \chi^{a-2,b,c-2}. \end{aligned}$$

Together with Theorem 5.2, this result gives an algorithm for computing central functions explicitly.

- Proposition 5.8 states that central functions are monic, and gives the leading term of the central function $\chi^{a,b,c}$.
- Proposition 5.9 describes a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading on the central function basis.

- Theorem 5.11 gives the coefficients in the expression of the product of two central functions as a sum of central functions, and therefore a precise description of the ring structure of $\mathbb{C}[\mathcal{R}]^{\mathrm{SL}(2, \mathbb{C})}$ in terms of central functions.

Finally, as another consequence of the recurrence relation and Theorem 5.2, we provide a new constructive proof of the following classical theorem [14, 32]:

Theorem 5.12 (Fricke-Klein-Vogt Theorem). *Let $G = \mathrm{SL}(2, \mathbb{C})$ act on $G \times G$ by simultaneous conjugation. Then*

$$\mathbb{C}[G \times G]^G \cong \mathbb{C}[t_x, t_y, t_z],$$

the complex polynomial ring in three indeterminates. In particular, every regular function $f : \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathbb{C}$ satisfying

$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1g^{-1}, g\mathbf{x}_2g^{-1}) \quad \text{for all } g \in \mathrm{SL}(2, \mathbb{C}),$$

can be written uniquely as a polynomial in the three trace variables $x = \mathrm{tr}(\mathbf{x}_1)$, $y = \mathrm{tr}(\mathbf{x}_2)$, and $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$.

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2 Preliminaries

2.1 Algebraic Structure of the Character Variety $\mathcal{R} // \mathrm{SL}(2, \mathbb{C})$

The group $G = \mathrm{SL}(2, \mathbb{C})$ has the structure of an irreducible algebraic set, since it is the zero set of the irreducible polynomial $\det(\mathbf{x}) - 1$. Since the product of two varieties is again a variety, the *representation variety* $\mathcal{R} = \mathrm{Hom}(\mathrm{F}_2, G) \cong G \times G$ of a rank 2 free group F_2 is an irreducible algebraic set as well. The *coordinate ring* of \mathcal{R} is

$$\mathbb{C}[\mathcal{R}] = \frac{\mathbb{C}[x_{ij}^k : 1 \leq i, j, k \leq 2]}{(\det(\mathbf{x}_1) - 1, \det(\mathbf{x}_2) - 1)}.$$

Stated otherwise, it is the free commutative polynomial ring in 8 indeterminates over \mathbb{C} subject to the ideal generated by the two polynomials $\det(\mathbf{x}_k) - 1$, where $\mathbf{x}_k = (x_{ij}^k)$ are called *generic matrices*.

There is an action of G on \mathcal{R} by simultaneous conjugation. Given $(\mathbf{x}_1, \mathbf{x}_2) \in G \times G$, then

$$g \cdot (\mathbf{x}_1, \mathbf{x}_2) = (g\mathbf{x}_1g^{-1}, g\mathbf{x}_2g^{-1}).$$

This is a *polynomial action*, since $\mathcal{R} \times G \rightarrow \mathcal{R}$ is a regular mapping.

Definition 2.1. The *ring of invariants* $\mathbb{C}[\mathcal{R}]^G$ consists of elements of the coordinate ring $\mathbb{C}[\mathcal{R}]$ which are invariant under the action of simultaneous conjugation:

$$\mathbb{C}[\mathcal{R}]^G = \{f \in \mathbb{C}[\mathcal{R}] : g \cdot f = f\}.$$

Recall that an algebraic group is *linearly reductive* if its finite dimensional rational representations are decomposable as direct sums of irreducible representations. Since $G = \mathrm{SL}(2, \mathbb{C})$ is linearly reductive, the ring of invariants $\mathbb{C}[\mathcal{R}]^G = \{f \in \mathbb{C}[\mathcal{R}] : g \cdot f = f\}$ is finitely generated [10]. This implies that the space of maximal ideals of $\mathbb{C}[\mathcal{R}]^G$ is also an irreducible algebraic set, permitting the following definition:

Definition 2.2. The *G-character variety* of F_2 is the space of maximal ideals

$$\mathfrak{X} = \mathrm{Spec}_{\max}(\mathbb{C}[\mathcal{R}]^G) = \mathcal{R} // G.$$

The character variety \mathfrak{X} is identified with conjugacy classes of *completely reducible* representations in \mathcal{R} [1, 27]. Procesi [26] has shown that $\mathbb{C}[\mathcal{R}]^G$ is generated by traces of products of matrix variables of word length less than or equal to three [26]. Hence $\mathbb{C}[\mathfrak{X}]$ is generated, although not minimally, by

$$\{\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^2), \mathrm{tr}(\mathbf{x}_2\mathbf{x}_1^2)\}.$$

2.2 History of $\mathrm{SL}(2, \mathbb{C})$ Invariant Theory

The invariant theory of $\mathrm{SL}(2, \mathbb{C})$ has a long history. Two pioneering papers on the subject were authored by Vogt in 1889 [32], and by Fricke and Klein in 1896 [14]. Both investigated the invariants of pairs of unimodular 2×2 matrices with respect to simultaneous conjugation. They showed this ring of invariants to be the free commutative polynomial ring in three indeterminants, given by the trace of each generic matrix and the trace of their product. This chapter concludes with a reproof of this classical result using the spin network calculus.

In 1972, Horowitz investigated the algebraic structure of this ring, saying that Fricke’s approach was principally analytic, and partially incomplete [20]. In 1980, Magnus made clear the priority of Vogt’s approach [32] and worked out the defining polynomial relations for an arbitrary number of matrices under simultaneous conjugation [22]. In 1983, Culler and Shalen defined the character variety and showed that it is in fact an algebraic set [6]; the set is the image under a “trace” map. González-Acuña and Montesinos-Amilibia showed in 1993 that the relations of Magnus in fact determine the algebraic set that Culler and Shalen had defined [19]. In 2001, Sikora, using results of Procesi [26], showed that the character variety of $\mathrm{SL}(n, \mathbb{C})$ can be realized as spaces of graphs subject to topologically motivated relations [30]. These graphs correspond to the spin networks discussed in this chapter when $n = 2$.

Closely related is the ring of invariants of *arbitrary* generic 2×2 matrices under simultaneous conjugation. The works of Procesi (1976) and Razmyslov (1974) generalized the work above to the case of $n \times n$ matrices [26, 28], and showed that the invariant ring is generated by traces of words in generic matrices. Methods from geometric invariant theory (see Dolgachev [10]) show that the character variety is the variety whose coordinate ring is the ring of invariants. Restricting to unimodular matrices gives like results for the unimodular ring of invariants. From this point of view, the character variety begins as an algebraic set and so is obviously closed. However, the defining relations and minimal generators are not at all obvious.

A central question in invariant theory is a description of the generators and relations of an invariant ring. Indeed, a theorem that characterizes the generators of an invariant ring is called a *first fundamental theorem*, and a theorem giving the relations is called a *second fundamental theorem*. In [26, 28] both Procesi and Razmyslov gave the two fundamental theorems, although they offered only sufficient generators and an implicit description of the relations.

It is much more difficult to determine *minimal* generators and *explicit* relations. In this more general context, which bears strongly on the unimodular case, minimal generators and defining relations for the invariants of an arbitrary number of generic 2×2 matrices were found only recently by Drensky in 2003 [11].

2.3 Representation Theory of $\mathrm{SL}(2, \mathbb{C})$

The coordinate ring $\mathbb{C}[G]$ decomposes into a direct sum of tensor products of the finite-dimensional irreducible representations of G . We will use this decomposition, given explicitly by Theorem 4.1, to understand $\mathbb{C}[\mathfrak{X}]$. To this end, we review the representation theory of G (see [3, 10, 15]).

The symmetric powers of the standard representation of G are all irreducible representations and moreover they comprise a complete list. Let $V_0 = \mathbb{C} = V_0^*$ be the trivial representation of G . Denote the standard basis for \mathbb{C}^2 by $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the dual basis by $e_1^* = e_1^T$ and $e_2^* = e_2^T$. Then the standard representation and its dual are

$$V = V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \quad \text{and} \quad V^* = V_1^* = \mathbb{C}e_1^* \oplus \mathbb{C}e_2^*,$$

respectively. Denote the symmetric powers of these representations by

$$V_n = \mathrm{Sym}^n(V) \quad \text{and} \quad V_n^* = \mathrm{Sym}^n(V^*).$$

Since V_n admits an invariant non-degenerate bilinear form, $V_n \cong (V_n)^*$.

Moreover, V_n^* is naturally isomorphic to $(V_n)^*$, so elements in V_n pair with elements in V_n^* . Denote the projection of $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to V_n by $v_1 \circ v_2 \circ \cdots \circ v_n$. There exist bases for V_n and V_n^* , given by the elements

$$\begin{aligned} \mathfrak{n}_{n-k} &= e_1^{n-k} e_2^k = \underbrace{e_1 \circ e_1 \circ \cdots \circ e_1}_{n-k} \circ \underbrace{e_2 \circ e_2 \circ \cdots \circ e_2}_k \quad \text{and} \\ \mathfrak{n}_{n-k}^* &= (e_1^*)^{n-k} (e_2^*)^k = \underbrace{e_1^* \circ e_1^* \circ \cdots \circ e_1^*}_{n-k} \circ \underbrace{e_2^* \circ e_2^* \circ \cdots \circ e_2^*}_k, \end{aligned}$$

respectively, where $0 \leq k \leq n$. In these terms, this pairing is given by

$$\mathfrak{n}_{n-k}^*(v_1 \circ v_2 \circ \cdots \circ v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\mathfrak{n}_{n-k})^*(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}),$$

where Σ_n is the symmetric group on n elements. In particular,

$$\mathfrak{n}_{n-k}^*(\mathfrak{n}_{n-l}) = \frac{(n-k)!k!}{n!} \delta_{kl} = \delta_{kl} / \binom{n}{k}.$$

Let $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in G$. Then the G -action on V_n is given by

$$\begin{aligned} g \cdot \mathfrak{n}_{n-k} &= (g_{11}e_1 + g_{21}e_2)^{n-k} (g_{12}e_1 + g_{22}e_2)^k \\ &= \sum_{\substack{0 \leq j \leq n-k \\ 0 \leq i \leq k}} \binom{n-k}{j} \binom{k}{i} \left(g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^j g_{22}^i \right) \mathfrak{n}_{n-(i+j)}. \end{aligned}$$

For the dual, G acts on V_n^* in the usual way:

$$(g \cdot \mathfrak{n}_{n-k}^*)(v) = \mathfrak{n}_{n-k}^*(g^{-1} \cdot v) \quad \text{for } v \in V_n.$$

The tensor product $V_a \otimes V_b$, where $a, b \in \mathbb{N}$, is also a representation of G and decomposes into irreducible representations as follows:

Proposition 2.3 (Clebsch-Gordan formula).

$$V_a \otimes V_b \cong \bigoplus_{j=0}^{\min(a,b)} V_{a+b-2j}.$$

Finally, we give several versions of Schur's Lemma, which will be used frequently.

Proposition 2.4 (Schur's Lemma). *Let G be a group, V and W representations of G , and $f \in \text{Hom}_G(V, W)$ with $f \neq 0$.*

- (1) *If V is irreducible, then f is injective.*
- (2) *If W is irreducible, then f is surjective.*
- (3) *If $V = W$ is irreducible, then f is a homothety.*
- (4) *Suppose V, W are irreducible:*
 - if $V \cong W$, then $\dim_{\mathbb{C}} \text{Hom}_G(V, W) = 1$;*
 - if $V \not\cong W$, then $\dim_{\mathbb{C}} \text{Hom}_G(V, W) = 0$.*

See [3] or [7] for proof of Propositions 2.3 and 2.4.

3 The Spin Network Calculus

This section provides a self-contained introduction to spin networks and the spin network calculus. Our treatment employs a nonstandard definition of spin networks which is more natural when working with traces. This definition leads to different versions of the usual spin network relations in the literature [5, 8, 9, 21, 23, 31].

3.1 Spin Networks and Representation Theory

At its heart, a spin network is a graph that is identified with a specific function between tensor powers of $V = \mathbb{C}^2$, the standard $\text{SL}(2, \mathbb{C})$ representation.

In order for this function to be well-defined, the edges incident to each vertex of the spin network must have a cyclic ordering. This ordering is often called a *ciliation*, since it is represented on paper by a small mark drawn between two of the edges. The edges adjacent to a ciliated vertex are ordered

by proceeding in a clockwise fashion from this mark. For example, in the degree 2 case, there are two possible ciliations: $\begin{smallmatrix} 1 \\ \downarrow \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 2 \\ \downarrow \\ 1 \end{smallmatrix}$.

Definition 3.1. A *spin network* \S is a graph with vertex set $\S_i \sqcup \S_o \sqcup \S_v$ consisting of degree 1 ‘inputs’ \S_i , degree 1 ‘outputs’ \S_o and degree 2 ‘ciliated vertices’ \S_v . If there are $k_i = |\S_i|$ inputs and $k_o = |\S_o|$ outputs, then \S is identified with a function $f_\S : V^{\otimes k_i} \rightarrow V^{\otimes k_o}$. If the spin network is *closed*, meaning $k_i = 0 = k_o$, it is identified with a complex scalar $f_\S \in \mathbb{C}$.

Spin networks are drawn in *general position* inside an oriented rectangle with inputs at the bottom and outputs at the top. This convention allows us to equate the composition of functions $f_{\S'} \circ f_\S$ with the concatenation of diagrams $\S' \circ \S$ formed by placing \S' on top of \S .

For example, the following spin network has two ciliated vertices and represents a function from $V^{\otimes 5} \rightarrow V^{\otimes 3}$:

$$\begin{array}{c} \text{3 outputs} \\ \hline \text{Diagram} \\ \hline \text{5 inputs} \end{array} = \left(\begin{array}{c} | \curvearrowright | \end{array} \right) \circ \left(\begin{array}{c} \times \curvearrowright | \end{array} \right) \circ \left(\begin{array}{c} | \quad | \quad | \quad \times \end{array} \right).$$

Note that the marks on the local extrema do not indicate vertices of the graph, but are indicators of how to decompose the graph.

Since spin networks are just graphs with ciliations, it does not matter how the graph is represented inside the square. Strands may be moved about freely and ciliations may “slide” along the strands. As long as the endpoints remain fixed, the underlying spin network does not change.

Let $v, w \in V$ and let $\{e_1, e_2\}$ be the standard basis for \mathbb{C}^2 . The function f_\S of a spin network \S is computed by decomposing \S into the four *spin network component maps*:

- the *identity* $\left| : V \rightarrow V, \quad v \mapsto v; \right.$
- the *cap* $\curvearrowright : V \otimes V \rightarrow \mathbb{C}, \quad v \otimes w \mapsto v^T w$ (inner product);
- the *cup* $\curvearrowleft : \mathbb{C} \rightarrow V \otimes V, \quad 1 \mapsto e_1 \otimes e_1 + e_2 \otimes e_2;$
- the *cap vertex* $\curvearrowright : V \otimes V \rightarrow \mathbb{C}, \quad v \otimes w \mapsto \det[v \ w].$

For example, since \curvearrowright and \bigcirc are the same ciliated graph,

$$\curvearrowright(v \otimes w) = \bigcirc(v \otimes w) = \curvearrowright \circ \times(v \otimes w) = \curvearrowright(w \otimes v) = \det[w \ v].$$

The definition given here differs from the literature [5, 21, 23]. In particular, we omit the $i = \sqrt{-1}$ factor in the definition of \curvearrowright to gain an advantage in trace calculations. Also, the maps \curvearrowright and \cup are included in order to simplify the proof that f_{\S} is well-defined.

Theorem 3.2. *The spin network function f_{\S} is well-defined.*

Proof. We need to show that every decomposition of \S into the component maps gives the same function.

If \S has n ciliated vertices, then any decomposition of \S into component maps has n occurrences of \curvearrowright . The remainder of the diagram consists of loops or arcs without any vertices. Two corresponding arcs in different decompositions will differ only by the insertion or deletion of a number of ‘kinks’ of the form \cup . Finally, since

$$\cup(v) = \curvearrowright \Big| \circ \Big| \cup(v) = \Big| (v)$$

for all $v \in V$, these kinks do not change the resulting function. For alternate proofs, see [5, 21]. \square

This theorem allows us to freely interpret a spin network \S as a function. The computation of f_{\S} will be easier once the functions for a few simple spin networks are known.

Proposition 3.3. *As spin network functions,*

- (1) the swap $\times : V \otimes V \rightarrow V \otimes V$ takes $v \otimes w \mapsto w \otimes v$;
- (2) the vertex on a straight line $\vdash : V \rightarrow V$ takes $v \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$;
- (3) the vertex on a cup $\cup : \mathbb{C} \rightarrow V \otimes V$ takes $1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$;
- (4) with opposite ciliations, $\curvearrowright = -\curvearrowleft$, $\vdash = -\dashv$, and $\cup = -\cup$.

Proof. First (1) is the statement that crossings change only the order of the outputs. Statement (2) follows from, for $v = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$:

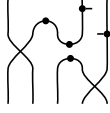
$$\begin{aligned} \vdash(v) &= \left(\curvearrowright \Big| \right) \circ \left(\Big| \cup \right) (v) = \left(\curvearrowright \Big| \right) (v \otimes e_1 \otimes e_1 + v \otimes e_2 \otimes e_2) \\ &= \det[v \ e_1]e_1 + \det[v \ e_2]e_2 = -v^2e_1 + v^1e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v. \end{aligned}$$

Statement (3) is computed similarly, using the decomposition

$$\cup = \left(\Big| \curvearrowright \Big| \right) \circ (\cup \cup).$$

Finally, (4) follows from the observation $\cap = \bigcirc = -\cup$, which has already been demonstrated. \square

Given these facts, the function of the earlier example can be computed. The reader may check that the function given by



takes $e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_2$ to $-e_2 \otimes e_2 \otimes e_2$.

The maps \cap and \cup are unnecessary for trace computations, and so we make the following assumption:

Convention 3.4. For the remainder of this chapter, the set of ciliated vertices will *coincide exactly with the set of local extrema*. The ciliations are usually omitted, with the understanding that

$$\cup = \cup : 1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$$

and $\cap = \cap : v \otimes w \mapsto \det[v \ w].$

Under this assumption, each straightened kink $\cap \leftrightarrow \cup$ introduces a sign, and more generally

$$\cap^n = (-1)^n \cup^n.$$

Thus, any diagram manipulation in which kinks are straightened must be done carefully.

Spin networks exhibit considerable symmetry, which can be exploited for calculations. For example:

Proposition 3.5. Let \S be a spin network with function $f_{\S} : V^{\otimes k_i} \rightarrow V^{\otimes k_o}$. Denote its images under reflection through vertical and horizontal lines by $\overleftarrow{\S}$ and \S^{\uparrow} , respectively. Then

$$f_{\overleftarrow{\S}} = (-1)^{|\S_v|} \overleftarrow{f_{\S}} : V^{\otimes k_i} \rightarrow V^{\otimes k_o},$$

where $|\S_v|$ is the number of local extrema in the diagram and \overleftarrow{f} indicates that the ordering of inputs and outputs is reversed. Also, $f_{\S^{\uparrow}} = (f_{\S})^*$ where

$$(f_{\S})^*(v_1 \otimes \cdots \otimes v_{k_i}) = \sum_{e_b \in \mathcal{B}(V^{\otimes k_i})} (f_{\S}(e_b) \cdot (v_1 \otimes \cdots \otimes v_{k_o})) e_b,$$

where \cdot indicates the dot product with respect to the standard basis for $V^{\otimes k_o}$ and $\mathcal{B}(V^{\otimes k_i})$ is the basis for $V^{\otimes k_i}$. That is, $(f_{\S})^*$ and f_{\S} are dual with respect to the standard inner product on V .

Proof. The first statement is an extension of the fact that reflecting \curvearrowright through a vertical line gives $\curvearrowleft = -\curvearrowright$.

For the second statement, consider $\S = \cup$. If $v_i = \begin{bmatrix} v_i^1 \\ v_i^2 \end{bmatrix}$, then

$$\begin{aligned} (f_{\S})^*(v_1 \otimes v_2) &= \cup(1) \cdot (v_1 \otimes v_2) = (e_1 \otimes e_2 - e_2 \otimes e_1) \cdot (v_1 \otimes v_2) \\ &= v_1^1 v_2^2 - v_1^2 v_2^1 = \det[v_1 \ v_2] = \curvearrowright(v_1 \otimes v_2). \end{aligned}$$

This computation, together with the corresponding one for $\S = \curvearrowright$, are sufficient to prove the second claim (see [25] for details). \square

The next theorem, which follows from Proposition 3.5, describes how to apply these symmetries to relations among spin networks:

Theorem 3.6 (Spin Network Reflection Theorem). *A relation*

$$\sum_m \alpha_m \S^m = 0$$

among some collection of spin networks $\{\S^m\}$ is equivalent to the same relation for the vertically reflected spin networks $\{\S^{\uparrow m}\}$ and (up to sign) for the horizontally reflected spin networks $\{\overleftrightarrow{\S}^m\}$, that is

$$\sum_m \alpha_m \S^{\uparrow m} = 0 \quad \text{and} \quad \sum_m \alpha_m (-1)^{|\S_v^m|} \overleftrightarrow{\S}^m = 0.$$

3.2 Basic Diagram Manipulations

In this section, we describe the *spin network calculus*, which governs diagram manipulations.

Proposition 3.7. *Any spin network can be expressed as a sum of diagrams with no crossings or loops. In particular,*

$$\times = \left| \right| - \cup; \quad \bigcirc \S = \text{tr}(I)\S = 2\S. \quad (1)$$

The proof is given in [25]. The first of these relations is called the *Fundamental Binor Identity*, and represents a fundamental type of structure in mathematics; it is the core concept in defining both the *Kauffman Bracket Skein Module* in knot theory [4] and the *Poisson bracket* on the set of loops

on a surface, which Goldman describes in [16]. It can also be identified with the *characteristic equation* for 2×2 matrices [25, 30].

Since 2×2 matrices act on V , the definition of spin networks may be extended to allow matrices to act on diagrams: $\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array}$ is the action $v \mapsto \mathbf{x} \cdot v$. The corresponding action on the tensor product $V^{\otimes n}$ is represented by

$$\begin{array}{c} \circ \quad \circ \quad \cdots \quad \circ \\ \diagup \diagdown \quad \diagup \diagdown \quad \cdots \quad \diagup \diagdown \\ \circ \quad \circ \quad \cdots \quad \circ \end{array}^n (v_1 \otimes \cdots \otimes v_n) = \mathbf{x}v_1 \otimes \cdots \otimes \mathbf{x}v_n.$$

The matrices $\mathbf{x} \in \mathrm{SL}(2, \mathbb{C})$ of interest in this chapter satisfy the following special property:

Proposition 3.8. *The spin network component maps $\left| \right|, \cup = \cup, \text{ and } \cap = \cap$, and therefore all spin networks, are equivariant under the natural action of $\mathrm{SL}(2, \mathbb{C})$ on V described above.*

Proof. The case for the identity $\left| \right|$ is clear, while

$$\begin{aligned} \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} (v \otimes w) &= \det[\mathbf{x}v \ \mathbf{x}w] = \det(\mathbf{x} \cdot [v \ w]) \\ &= \det(\mathbf{x}) \cdot \det[v \ w] = 1 \cdot \det[v \ w] = \cap(v \otimes w) \end{aligned}$$

shows that $\cap \circ \mathbf{x} = \cap = \mathbf{x} \circ \cap$.

The proof for \cup follows by reflecting this relation. \square

This means that matrices in such a diagram can “slide across” a vertex (local extremum) by simply inverting the matrix, so that

$$\text{if } \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} = \mathbf{x}^{-1} \in \mathrm{SL}(2, \mathbb{C}), \quad \text{then } \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} = \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array}.$$

For a general matrix $\mathbf{x} \in M_{2 \times 2}$, the determinant is introduced in such relations since $\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} = \det \left(\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} \right) \cup$. If \mathbf{x} is invertible, this implies

$$\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} = \det \left(\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array} \right) \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \end{array}.$$

A *closed* spin network with one or more matrices is called a *trace diagram*, and may be identified with a map $G \times \cdots \times G \rightarrow \mathbb{C}$. One of the primary motivations for this chapter is the study of invariance properties of such maps. The simplest cases are given by:

Proposition 3.9. For $\mathbf{x} \in M_{2 \times 2}$ and $\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$\bigcirc = 2 = \text{tr}(\mathbb{I}); \quad \bigcirc \otimes = \text{tr}(\mathbf{x}); \quad \bigcirc \otimes \bigcirc = \det(\mathbf{x}) \cdot \text{tr}(\mathbb{I}). \quad (2)$$

3.3 Symmetrizers and Irreducible Representations

Another important $\text{SL}(2, \mathbb{C})$ -equivariant map is the symmetrizer, defined by:

Definition 3.10. The *symmetrizer* $\begin{array}{c} || \bullet \bullet \\ \boxed{n} \\ || \bullet \bullet \end{array} : V^{\otimes n} \rightarrow V^{\otimes n}$ is the map taking

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}, \quad (3)$$

where $v_i \in V$ and Σ_n is the group of permutations on n elements.

For example,

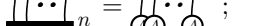
$$\begin{array}{c} || \\ \boxed{2} \\ || \end{array} = \frac{1}{2} \left(\begin{array}{c} | \\ | \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \begin{array}{c} | \\ | \end{array} - \frac{1}{2} \left(\begin{array}{c} \cup \\ \cap \end{array} \right);$$


$$\begin{aligned} \begin{array}{c} || \\ \boxed{3} \\ || \end{array} &= \frac{1}{6} \left(\begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagup \diagup \end{array} + \begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \end{array} \right) \\ &= \begin{array}{c} | \\ | \\ | \end{array} - \frac{2}{3} \left(\begin{array}{c} \cup \\ \cap \end{array} + \begin{array}{c} \cap \\ \cup \end{array} \right) + \frac{1}{3} \left(\begin{array}{c} \cup \\ \cup \end{array} + \begin{array}{c} \cap \\ \cap \end{array} \right) \end{aligned}$$

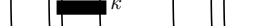
Note that the crossings are removed by applying the Fundamental Binor Identity.

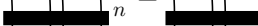
The defining equation (3) of $\begin{array}{c} || \bullet \bullet \\ \boxed{n} \\ || \bullet \bullet \end{array}$ should look familiar: its image is a subspace of $V^{\otimes n}$ isomorphic to the n th symmetric power $\text{Sym}^n V$, and thus it can be thought of as either the projection $\pi : V^{\otimes n} \rightarrow \text{Sym}^n V$ or as the inclusion $i : \text{Sym}^n V \rightarrow V^{\otimes n}$ (see [15], page 473).

What does this mean for us? If a diagram from $V^{\otimes k_i}$ to $V^{\otimes k_o}$ has symmetrizers at its top and bottom, it can be thought of as a map between V_{k_i} and V_{k_o} . We freely interpret such spin networks as maps between tensor powers of these irreducible $\text{SL}(2, \mathbb{C})$ -representations.

Invariance: ; (4)

stacking relation: ; (5)

capping/cupping: ; (6)

symmetrizer sliding: ; (7)

For the *capping* and *cupping relations*, notice that

This implies the general case because, by the stacking relation, one may insert $\begin{array}{|c|} \hline 2 \\ \hline \end{array}$ between \cap and $\begin{array}{|c|} \hline n \\ \hline \end{array}$. The other case is similar.

[25] for more details. □

We now move on to some more involved relations among symmetrizers. Although it is easy to write down an arbitrary $\begin{smallmatrix} \bullet & \bullet \\ n \\ \bullet & \bullet \end{smallmatrix}$ in terms of permutations, it is usually rather difficult to write it down in terms of diagrams without crossings (the Temperley-Lieb algebra). The next two propositions describe how to do exactly this. As such, they are a fundamental step in the proof of Theorem 5.6, which permits a fast computation of rank two central functions.

Proposition 3.12. *The symmetrizer $\begin{bmatrix} & & \vdots \\ & n & \\ & & \vdots \end{bmatrix}$ satisfies:*

$$\begin{aligned} \text{Diagram 1}^n &= \text{Diagram 1}^{n-1} - \left(\frac{n-1}{n}\right) \text{Diagram 2}^{n-1} + \left(\frac{n-2}{n}\right) \text{Diagram 3}^{n-1} + \dots \\ &\quad + (-1)^i \left(\frac{n-i}{n}\right) \text{Diagram 4}^{n-1} + \dots + (-1)^{n-1} \left(\frac{1}{n}\right) \text{Diagram 5}^{n-1}. \end{aligned} \quad (8)$$

Proof. If Σ_n is the group of permutations on the set $N_n = \{1, 2, \dots, n\}$, then

$$|\Sigma_n| = |N_n| |\Sigma_{n-1}|.$$

Interpret $|\Sigma_n|$ as the number of ways to arrange n people in a line. To do this, one may first select someone to be at the front of the line ($|N_n|$ choices), and then rearrange the remaining $n - 1$ people ($|\Sigma_{n-1}|$ choices).

In diagram form, the selection of someone to head the line corresponds to one of the diagrams




$$||| \cdot \cdot |, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \cdot \cdot |, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \cdot \cdot |, \dots, \begin{array}{c} \cdot \cdot \\ \diagup \diagdown \\ \cdot \cdot \end{array} \cdot \cdot | \cdot \cdot |, \dots, \begin{array}{c} \cdot \cdot \\ \diagup \diagdown \\ \cdot \cdot \end{array} \cdot \cdot | \cdot \cdot |.$$

The arrangement of the remaining people corresponds to $\left| \begin{array}{c} \blacksquare \\ \vdots \\ \vdots \end{array} \right|^{n-1}$. Thus, the diagrammatic form of the above interpretation is:

$$\left[\begin{array}{c} \text{---} \\ | \\ | \\ \vdots \\ | \\ \text{---} \end{array} \right]^n = \frac{1}{n} \left[\begin{array}{c} \text{---} \\ | \\ | \\ \vdots \\ | \\ \text{---} \end{array} \right]^{n-1} \circ \left(\left| \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \right| \cdot \left| \begin{array}{c} | \\ | \\ \vdots \\ | \end{array} \right| + \left| \begin{array}{c} \diagup \\ | \\ \vdots \\ | \end{array} \right| \cdot \left| \begin{array}{c} \diagdown \\ | \\ \vdots \\ | \end{array} \right| + \left| \begin{array}{c} \diagup \\ \diagdown \\ \vdots \\ \diagdown \\ \diagup \end{array} \right| \cdot \left| \begin{array}{c} \diagdown \\ | \\ \vdots \\ | \end{array} \right| + \cdots + \left| \begin{array}{c} \diagup \\ \diagdown \\ \vdots \\ \diagdown \\ \diagup \end{array} \right| \cdot \left| \begin{array}{c} \diagdown \\ | \\ \vdots \\ | \end{array} \right| + \cdots + \left| \begin{array}{c} \diagup \\ \diagdown \\ \vdots \\ \diagdown \\ \diagup \end{array} \right| \cdot \left| \begin{array}{c} \diagdown \\ | \\ \vdots \\ | \end{array} \right| \right).$$

Now, use the binor identity to remove crossings. Most of the resulting terms disappear, since any term whose cups are not in the ‘first position’ on top will vanish due to the *capping relation*. In particular:


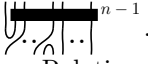
$$\left| \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right|^{n-1} \circ \left| \begin{array}{c} \cdot \\ \vdots \\ \cdot \end{array} \right| = \left| \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right|^{n-1} - \left| \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right|^{n-1} + \left| \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right|^{n-1} + \cdots + (-1)^i \left| \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \right|^{n-1},$$

where i is the number of ‘kinks’  in  or 1 plus the number of kinks in  ^{$n-1$} . Finally, group the number of terms on the righthand side with the same number of kinks together: there will be $n - i - 1$ terms with i kinks. \square

Proposition 3.13. $\left| \begin{smallmatrix} \cdot & \cdot \\ n & \\ \cdot & \cdot \end{smallmatrix} \right|$ also satisfies the recurrence relations:

$$\left| \begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \right| n = \sum_i \left| \begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \right|^{n-1}_i \left| \begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \right|^{n-i}_{n-i} + (-1)^i \binom{n-i}{n} \left| \begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \right|^{n-1}_i \left| \begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{array} \right|^{n-i}_{n-i}; \quad (9)$$

$$\left| \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \right|^n = \left| \begin{array}{c} \vdots \\ \vdots \\ \text{---} \\ \vdots \\ \vdots \end{array} \right|^{n-1} - \left(\frac{n-1}{n} \right) \left| \begin{array}{c} \text{---} \\ \vdots \\ \vdots \\ \text{---} \end{array} \right|^{n-1}. \quad (10)$$

Proof. Compose relation (8) with $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^i \otimes \begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-i}$. This has no effect on the lefthand side, by the *stacking relation*. On the righthand side, all but one of the terms with a cap on the bottom vanish, due to the *capping relation*, since they will cap off either the $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^i$ or the $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-i}$. The one term which remains ‘caps between’ these two symmetrizers. The coefficient is $(-1)^i \binom{n-i}{n}$ since in recurrence (8), i is equal to one more than the number of kinks  in .

Relation (10) is a special case of (9) for $i = 1$. \square

The next relations follow directly from these recurrences:

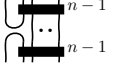
Proposition 3.14 (Looping Relations).

$$\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n = \left(\frac{n+1}{n} \right) \begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-1}. \quad (11)$$

When k strands of $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n$ are closed off:

$$k \left\{ \begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n \right\} = \left(\frac{n+1}{n-k+1} \right) \begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-k}. \quad (12)$$

$$\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n = n+1. \quad (13)$$

Proof. Close off the left strand in (10) above. Then, $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n$, $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-1}$, and  become $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^n$, $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-1} = 2 \begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-1}$ and $\begin{array}{|c|} \hline \cdot\cdot \\ \hline \cdot\cdot \\ \hline \end{array}^{n-1}$, respectively. Now collect terms to get (11), and proceed to (12) by applying the first relation k times. Finally, (13) is a special case of (12) with $k = n$. \square

3.4 Symmetrizers and Trivalent Spin Networks

Recall the Clebsch-Gordan decomposition (Proposition 2.3):

$$V_a \otimes V_b \cong \bigoplus_{c \in [a, b]} V_c, \quad [a, b] = \{a+b, a+b-2, \dots, |a-b|\}.$$

The requirement $c \in [a, b]$ is equivalent to the following symmetric condition:

Definition 3.15. A triple (a, b, c) of nonnegative integers is *admissible*, and we write $c \in [a, b]$, if

$$\frac{1}{2}(-a + b + c), \quad \frac{1}{2}(a - b + c), \quad \frac{1}{2}(a + b - c) \in \mathbb{N}. \quad (14)$$

Two maps arise from the Clebsch-Gordon decomposition: an injection $\iota_c^{a,b} : V_c \rightarrow V_a \otimes V_b$ and a projection $(\iota^*)_{a,b}^c : V_a \otimes V_b \rightarrow V_c$. Both have simple diagrammatic depictions [5]:

$$\iota_c^{a,b} = \begin{array}{c} a \quad b \\ \text{---} \quad \text{---} \\ \vdots \\ \text{---} \quad \text{---} \\ c \end{array} : V_c \rightarrow V_a \otimes V_b; \quad (\iota^*)_{a,b}^c = \begin{array}{c} c \\ \text{---} \\ \vdots \\ \text{---} \\ a \quad b \end{array} : V_a \otimes V_b \rightarrow V_c.$$

The admissibility condition (14) is the requirement that there is a nonnegative number of strands connecting each pair of symmetrizers. These “strand numbers” appear frequently in diagram manipulations, and will be referenced by the Greek letters α, β, γ :

Convention 3.16. Given an admissible triple (a, b, c) , denote by α , β , and γ the total number of strands connecting V_b to V_c , V_a to V_c , and V_a to V_b , respectively. Also, denote by δ the total number of strands in the diagram. Then:

$$\alpha = \frac{1}{2}(-a + b + c), \quad \beta = \frac{1}{2}(a - b + c), \quad \gamma = \frac{1}{2}(a + b - c); \quad \delta = \frac{1}{2}(a + b + c).$$

Note that (a, b, c) is admissible if and only if $\alpha, \beta, \gamma \in \mathbb{N}$.

Convention 3.17. Because the maps $\iota_c^{a,b}$ and $(\iota^*)_{a,b}^c$ will be so important for the remainder of this chapter, we introduce a notation which simplifies their depiction. Let n lines with a symmetrizer be represented by one **thick** line

$$\text{labelled } n, \text{ so that } \int^n \equiv \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \\ n \end{array}.$$

Definition 3.18. A *trivalent spin network* \S is a graph drawn on the plane with vertices of degree ≤ 3 and edges labelled by positive integers such that:

- 2-vertices are ciliated and coincide with local extrema;
- 3-vertices are drawn ‘up’ $\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array}$ or ‘down’ $\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \end{array}$;
- any two edges meeting at a 2-vertex have the same label;
- the three labels adjacent to any vertex form an admissible triple.

If there are m input edges with labels l_i for $i = 1, \dots, m$ and n output edges with labels l'_i for $i = 1, \dots, n$, the network is identified with a map between tensor products of irreducible $\text{SL}(2, \mathbb{C})$ representations,

$$f_{\S} : V_{l_1} \otimes \dots \otimes V_{l_m} \rightarrow V_{l'_1} \otimes \dots \otimes V_{l'_n}.$$

This map is computed by identifying § with a regular spin network using the following identifications:

Note that ciliations are normally chosen to be on the local extrema, and degree-3 vertices, when expanded, also have a number of ciliated vertices. The need to keep track of these ciliations makes diagram manipulation a more delicate operation.

3.5 Trivalent Diagram Manipulations

This section describes in detail the relations which may be used to manipulate trivalent spin networks. For the remainder of this chapter, we assume that all sets of labels incident to a common vertex in a diagram are admissible. Moreover, whenever we sum over a label in a diagram, the sum is taken over all possible values of that label which make the requisite triples in the diagram admissible.

Any closed trivalent spin network may be interpreted as a constant. The simplest such diagrams are given by

Proposition 3.19. *Let $\Theta(a, b, c) = \text{diagram with edges } a, b, c$ and $\Delta(c) = \text{diagram with edge } c$. Then $\Theta(a, b, c)$*

is symmetric in $\{a, b, c\}$ and explicitly (recall the $\alpha, \beta, \gamma, \delta$ given in Convention 3.16):

$$\Delta(c) = c + 1 = \dim(V_c); \quad (15)$$

$$\Theta(a, b, c) = \frac{\left(\frac{-a+b+c}{2}\right)! \left(\frac{a-b+c}{2}\right)! \left(\frac{a+b-c}{2}\right)! \left(\frac{a+b+c+2}{2}\right)!}{a!b!c!} = \frac{\alpha!\beta!\gamma!(\delta+1)!}{a!b!c!}; \quad (16)$$


$$\Theta(1, a, a+1) = \Delta(a+1) = a+2. \quad (17)$$

Proof. The first equation (15) is a consequence of the *Looping Relation* (11). That $\Theta(1, a, a+1) = \Delta(a+1)$ is a consequence of the *stacking relation*, and demonstrates (17). We refer the reader to [5] for the $\Theta(a, b, c)$ formula. \square

Ratios of Δ and Θ show up in the next two propositions, which tell us how to “pop bubbles” and how to “fuse together” two thick edges. The first demonstrates the usefulness of Schur’s Lemma (Proposition 2.4) in diagrammatic techniques.

Proposition 3.20 (Bubble Identity). $\text{a} \begin{array}{c} \text{c} \\ | \\ \text{d} \end{array} \text{b} = \left(\frac{\Theta(a,b,c)}{\Delta(c)} \Big| \text{c} \right) \delta_{cd}$, where δ_{cd} is the Kronecker delta.

Proof. Schur's Lemma requires $\begin{array}{c} \text{---} c \\ | \\ a \text{---} \text{---} b \\ | \\ \text{---} d \end{array} = C \int^c \delta_{cd}$ for some constant C , since


 is a map between irreducible representations. This equation remains true if we “close off” the diagrams, giving:

$$a \text{---} b^c = C \text{---}^c \implies C = \frac{\Theta(a, b, c)}{\Delta(c)}. \quad \square$$

Proposition 3.21 (Fusion Identities).

$$\begin{aligned} a \text{) } b &= \sum_{c \in [a, b]} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) a \text{) }_c b \\ a \text{ X } b &= \sum_{c \in [a, b]} (-1)^{\frac{1}{2}(a-b+c)} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) a \text{) }_c b \text{ X }_c a. \end{aligned}$$

Proof. Maps of the form $\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \\ \diagup \quad \diagdown \\ a \quad b \end{array}$ for $c \in [a, b]$ form a basis for the space of $\mathrm{SL}(2, \mathbb{C})$ -equivariant maps $V_a \otimes V_b \rightarrow V_a \otimes V_b$ [5]. Thus, we may express the first diagram as a linear combination:

$${}^a \cup {}^b = \sum_{c \in [a,b]} C(c) \begin{matrix} & a & & b \\ & \diagdown & & \diagup \\ & c & & c \\ & \diagup & & \diagdown \\ a & & & b \end{matrix}.$$

For a fixed $d \in [a, b]$, the constant $C(d)$ is computed by composing this expression with $\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ d \end{array}$:

$$\begin{aligned} a \Upsilon_d^b &= \sum_{c \in [a, b]} C(c) \, a \Upsilon_c^b \circ a \text{ (loop on } d \text{)}^c_b = \sum_{c \in [a, b]} C(c) \left(\frac{\Theta(a, b, c)}{\Delta(c)} \right) a \Upsilon_c^b \circ \Big|_d^d \delta_{cd} \\ &= C(d) \left(\frac{\Theta(a, b, d)}{\Delta(d)} \right) a \Upsilon_d^b \implies C(d) = \frac{\Delta(d)}{\Theta(a, b, d)}. \end{aligned}$$

The second equation follows from the first and from Proposition 3.22 below:

$$\begin{aligned}
\begin{array}{c} a \\ \diagup \\ \diagdown \\ b \end{array} &= (-1)^b \begin{array}{c} a \\ \diagup \\ \diagdown \\ b \end{array} = \sum_{c \in [a, b]} (-1)^b \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c} a \\ \diagup \\ \diagdown \\ b \end{array} \\
&= \sum_{c \in [a, b]} (-1)^{\frac{1}{2}(a-b-c)} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c} a \\ \diagup \\ \diagdown \\ b \end{array} \\
&= \sum_{c \in [a, b]} (-1)^{\frac{1}{2}(a-b+c)} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c} a \\ \diagup \\ \diagdown \\ b \end{array}. \square
\end{aligned}$$

The identity $\begin{array}{c} \vdots \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \vdots \end{array}$ gives rise to the following compendium of sign changes through diagram manipulations:

Proposition 3.22.

$$\begin{array}{c} \vdots \\ \vdots \end{array}^n = (-1)^n \begin{array}{c} \vdots \\ \vdots \end{array}^n; \quad (18)$$

$$\begin{array}{c} c \\ \diagup \\ \diagdown \\ a \quad b \end{array} = (-1)^{\frac{1}{2}(a+b-c)} \begin{array}{c} c \\ \diagup \\ \diagdown \\ a \quad b \end{array}; \quad (19)$$

$$\begin{array}{c} c \\ \diagup \\ \diagdown \\ a \quad b \end{array} = (-1)^{\frac{1}{2}(-a+b+c)} \begin{array}{c} c \\ \diagup \\ \diagdown \\ a \quad b \end{array}; \quad (20)$$

$$\begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b = (-1)^{\frac{1}{2}(a+b+c+d-2e)} \begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b; \quad (21)$$

$$(-1)^{\frac{1}{2}(a+c)} \begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b = (-1)^{\frac{1}{2}(b+d)} \begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b; \quad (22)$$

$$\begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b = (-1)^{b+d-e} \begin{array}{c} a \\ \diagup \\ \diagdown \\ d \quad c \end{array}^b. \quad (23)$$

Proof. First, (18) is just a restatement of $\left(\begin{array}{c} \text{diagram} \end{array} \right)^n = (-1)^n \left(\begin{array}{c} \text{diagram} \end{array} \right)^n$, and (19)

follows directly from the Proposition 3.5, since $\begin{array}{c} c \\ \diagup \quad \diagdown \\ a \quad b \end{array}$ contains $\gamma = \frac{1}{2}(a+b-c)$

local extrema and $\begin{array}{c} c \\ \diagup \quad \diagdown \\ a \quad b \end{array} = \begin{array}{c} c \\ \diagup \quad \diagdown \\ b \quad a \end{array}$.

For (20), notice that in the simplest case

$$\begin{array}{c} \text{diagram} \end{array} = - \begin{array}{c} \text{diagram} \end{array},$$

the negative sign comes from the strand on top of the diagram. Similarly, the general case for transforming $\begin{array}{c} c \\ \diagup \quad \diagdown \\ a \quad b \end{array}$ into $\begin{array}{c} c \\ \diagup \quad \diagdown \\ a \quad b \end{array}$ has a sign for each strand between b and c , giving $(-1)^\alpha = (-1)^{\frac{1}{2}(-a+b+c)}$. This identity is used twice to give (21).

Finally, (22) follows from:

$$\begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array} = (-1)^e \begin{array}{c} a \quad e \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array} = (-1)^{e+\frac{1}{2}(d+e-a+b+e-c)} \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array},$$

and (23) is given by combining (21) and (22). \square

The above relations permit the definition of a “ $\frac{\pi}{4}$ -reflection” on certain types of diagrams, which will be important later:

Proposition 3.23. *If a relation consists entirely of terms of the form $\begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array}$*

and $\begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array}$, then one may “reflect about the line through a and c ” in the following sense:

$$\sum_e \alpha_e \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array} = \sum_f \beta_f \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ d \quad c \end{array} \iff \sum_e \alpha_e \begin{array}{c} a \quad d \\ \diagup \quad \diagdown \\ b \quad c \end{array} = \sum_f \beta_f \begin{array}{c} a \quad d \\ \diagup \quad \diagdown \\ b \quad c \end{array}.$$

Proof. By horizontally reflecting the first relation, using Theorem 3.6,

$$\begin{aligned}
\sum_e \alpha_e \text{ (strand } a \text{ from } d \text{ to } c \text{ via } e) &= \sum_f \beta_f \text{ (strand } a \text{ from } d \text{ to } c \text{ via } f) \\
\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(a+b+c+d-2e)} \text{ (strand } b \text{ from } c \text{ to } d \text{ via } e) &= \sum_f \beta_f (-1)^{\frac{1}{2}(a+b+c+d-2f)} \text{ (strand } b \text{ from } c \text{ to } d \text{ via } f) \\
\iff \sum_e \alpha_e \text{ (strand } b \text{ from } c \text{ to } d \text{ via } e) &= \sum_f \beta_f \text{ (strand } b \text{ from } c \text{ to } d \text{ via } f),
\end{aligned}$$

where the signs cancel due to the admissibility conditions.

Now, add strands to both sides, so that the right side $\text{ (strand } b \text{ from } c \text{ to } d \text{ via } f)$ becomes

$$\text{ (strand } a \text{ from } d \text{ to } c \text{ via } f) = (-1)^{b+d-f} \text{ (strand } a \text{ from } d \text{ to } c \text{ via } f).$$

Likewise, on the left side, $\text{ (strand } b \text{ from } c \text{ to } d \text{ via } e)$ becomes $(-1)^{b+d-e} \text{ (strand } b \text{ from } c \text{ to } d \text{ via } e)$. Once again, admissibility implies that e and f must have the same parity, so these signs cancel. \square

Two alternate versions of this proposition follow (see [25]).

Corollary 3.24.

$$\begin{aligned}
\sum_e \alpha_e \text{ (strand } a \text{ from } d \text{ to } c \text{ via } e) &= \sum_f \beta_f \text{ (strand } a \text{ from } d \text{ to } c \text{ via } f) \iff \sum_e \alpha_e \text{ (strand } a \text{ from } b \text{ to } c \text{ via } e) = \sum_f \beta_f \text{ (strand } a \text{ from } b \text{ to } c \text{ via } f) \\
\sum_e \alpha_e \text{ (strand } a \text{ from } d \text{ to } c \text{ via } e) &= \sum_f \beta_f \text{ (strand } a \text{ from } d \text{ to } c \text{ via } f) \\
\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(e-b)} \text{ (strand } a \text{ from } d \text{ to } c \text{ via } e) &= \sum_f \beta_f (-1)^{\frac{1}{2}(d-f)} \text{ (strand } a \text{ from } d \text{ to } c \text{ via } f).
\end{aligned}$$

4 Decomposition of $\mathbb{C}[G]$

The following theorem is a consequence of the “unitary trick” [10], the Peter-Weyl Theorem, and the fact that the set of matrix coefficients of G is exactly its coordinate ring [7]. We offer a self-contained constructive proof in Section 4.2, since it gives an explicit correspondence between regular functions and spin networks.

Theorem 4.1. *There is a G -module isomorphism*

$$\mathbb{C}[G] \cong \sum_{n \geq 0} V_n^* \otimes V_n.$$

4.1 Central Functions

Theorem 4.1 allows $\mathbb{C}[G \times G]^G$ to be described in terms of an additive basis of class functions that have an elegant realization as spin networks. Indeed, together with the Clebsch-Gordan decomposition, it implies

$$\begin{aligned} \mathbb{C}[G \times G] &\cong \mathbb{C}[G] \otimes \mathbb{C}[G] \\ &\cong \left(\sum_{a \geq 0} V_a^* \otimes V_a \right) \otimes \left(\sum_{b \geq 0} V_b^* \otimes V_b \right) \\ &\cong \sum_{a \geq 0} \sum_{b \geq 0} V_a^* \otimes V_a \otimes V_b^* \otimes V_b \\ &\cong \sum_{0 \leq a, b < \infty} (V_a^* \otimes V_b^*) \otimes (V_a \otimes V_b) \\ &\cong \sum_{0 \leq a, b < \infty} \left(\sum_{i=0}^{\min(a,b)} V_{a+b-2i}^* \right) \otimes \left(\sum_{j=0}^{\min(a,b)} V_{a+b-2j} \right) \\ &\cong \sum_{\substack{0 \leq a, b < \infty \\ 0 \leq i, j \leq \min(a,b)}} V_{a+b-2i}^* \otimes V_{a+b-2j}. \end{aligned}$$

Since the above maps are G -equivariant,

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a, b < \infty \\ 0 \leq i, j \leq \min(a,b)}} (V_{a+b-2i}^* \otimes V_{a+b-2j})^G. \quad (1)$$

By Schur’s Lemma (Proposition 2.4),

$$\dim_{\mathbb{C}} (V_{a+b-2i}^* \otimes V_{a+b-2j})^G = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

so

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a, b < \infty \\ 0 \leq j \leq \min(a, b)}} \mathrm{End}(V_{a+b-2j})^G.$$

Definition 4.2. Given the above isomorphism, for each $c \in [a, b]$ (see Definition 3.15), there exists a class function $\chi^{a,b,c} \in \mathbb{C}[G \times G]^G$ which corresponds to a generating homothety (unique up to scalar) in $\mathrm{End}(V_c)^G$. We refer to the functions $\chi^{a,b,c}$ as *central functions*.

Denote by $\mathbb{C}\chi^{a,b,c} \subset \mathbb{C}[G \times G]^G$ the linear span over \mathbb{C} of $\chi^{a,b,c}$. Then (1) may be rewritten as

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a, b < \infty \\ c \in [a, b]}} \mathbb{C}\chi^{a,b,c}.$$

Thus, the central functions $\chi^{a,b,c}$ form an additive basis for the ring of regular functions on $\mathfrak{X} = \mathrm{Spec}_{\max}(\mathbb{C}[\mathcal{R}]^G) = \mathcal{R} // G$. In Section 5, we describe the multiplicative structure of $\mathbb{C}[G \times G]^G$ in terms of this basis.

The central functions may be described using the Clebsch-Gordan injection $\iota_c^{a,b} : V_c \hookrightarrow V_a \otimes V_b$:

$$\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{tr} \left(\iota(\mathbf{c}_i^*) \left((\mathbf{x}_1, \mathbf{x}_2) \cdot \iota(\mathbf{c}_j) \right) \right)_{ij},$$

where $\{\mathbf{c}_j\}$ is a basis for V_c . We will omit indices on ι when they are clear from context.

The functions $\chi^{a,b,c}$ take a natural diagrammatic form. If the matrix \mathbf{x} is represented diagrammatically by $\bigoplus : V \rightarrow V$, then its action on V_a can be

represented by $\bigoplus^a \equiv \bigoplus^a_a$. A closed spin network with r different matrices

is an invariant regular function $G^{\times r} \rightarrow \mathbb{C}$. In particular, since \bigvee and \bigwedge are the Clebsch-Gordan injection and projection, respectively,

$$\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) = \bigvee^a_{\bigwedge^b} = a \bigvee^a_{\bigwedge^b} b.$$

As a special case, setting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbb{I}$, where \mathbb{I} is the identity matrix in G , gives $\chi^{a,b,c}(\mathbb{I}, \mathbb{I}) = \Theta(a, b, c)$.

4.2 Proof of $\mathbb{C}[G]$ Decomposition Theorem

Define

$$\Upsilon : \sum_{n \geq 0} V_n^* \otimes V_n \longrightarrow \mathbb{C}[G]$$

by linear extension of the mapping

$$\mathfrak{n}_{n-k}^* \otimes \mathfrak{n}_{n-l} \mapsto \mathfrak{n}_{n-k}^*(\mathbf{x} \cdot \mathfrak{n}_{n-l}),$$

where $\mathbf{x} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ is a matrix variable.

Proposition 4.3. Υ is a well-defined G -equivariant morphism.

Proof. The image of Υ consists of regular functions since

$$\begin{aligned} \mathfrak{n}_{n-k}^*(\mathbf{x} \cdot \mathfrak{n}_{n-l}) &= \mathfrak{n}_{n-k}^* \left((x_{11}e_1 + x_{21}e_2)^{n-l} (x_{12}e_1 + x_{22}e_2)^l \right) \\ &= \sum_{\substack{i+j=k \\ 0 \leq i \leq n-l \\ 0 \leq j \leq l}} \binom{n}{k}^{-1} \binom{n-l}{i} \binom{l}{j} x_{11}^{n-l-i} x_{12}^{l-j} x_{21}^i x_{22}^j. \end{aligned}$$

Equivariance is verified by the calculation:

$$\begin{aligned} \Upsilon(g \cdot (\mathfrak{n}_{n-k}^* \otimes \mathfrak{n}_{n-l})) &= \Upsilon((g \cdot \mathfrak{n}_{n-k}^*) \otimes (g \cdot \mathfrak{n}_{n-l})) \\ &= (g \cdot \mathfrak{n}_{n-k}^*)(\mathbf{x} \cdot (g \cdot \mathfrak{n}_{n-l})) = \mathfrak{n}_{n-k}^*((g^{-1}\mathbf{x}g) \cdot \mathfrak{n}_{n-l}) \\ &= g \cdot \mathfrak{n}_{n-k}^*(\mathbf{x} \cdot \mathfrak{n}_{n-l}) = g \cdot \Upsilon(\mathfrak{n}_{n-k}^* \otimes \mathfrak{n}_{n-l}). \square \end{aligned}$$

There is a right action of G on $\mathbb{C}[G]$ given by $f \cdot g(\mathbf{x}) = f(\mathbf{x}g)$. Denote by $\mathbb{C}[G]_{\text{right}}$ the ring $\mathbb{C}[G]$ with this right action, to distinguish it from the conjugation action already imposed on $\mathbb{C}[G]$. Additionally, G acts on the left of $\text{Hom}_G(V_n, \mathbb{C}[G]_{\text{right}})$ by

$$(g \cdot \gamma)(v)(\mathbf{x}) = \gamma_v(g^{-1}\mathbf{x}),$$

where $\gamma_v = \gamma(v)$. This action is well-defined since

$$(g \cdot \gamma)(g' \cdot v)(\mathbf{x}) = \gamma_{g' \cdot v}(g^{-1}\mathbf{x}) = \gamma_v(g^{-1}\mathbf{x}g') = ((g \cdot \gamma)(v)) \cdot g'(\mathbf{x}).$$

The next two lemmas, whose proofs are deferred, define two additional maps which will be used to prove the theorem.

Lemma 4.4. *The map*

$$\Phi : \sum_{n \geq 0} \text{Hom}_G(V_n, \mathbb{C}[G]_{\text{right}}) \otimes V_n \longrightarrow \mathbb{C}[G]$$

defined by linearly extending the mappings $\gamma \otimes v \mapsto \gamma(v)$ is an isomorphism of G -modules.

Lemma 4.5. Define $\Psi_n : V_n^* \rightarrow \mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}})$ by $w^* \mapsto \mathbf{F}_{w^*}$, where $\mathbf{F}_{w^*}(v)(\mathbf{x}) = w^*(\mathbf{x} \cdot v)$. Then the map

$$\Psi : \sum_{n \geq 0} V_n^* \otimes V_n \longrightarrow \sum_{n \geq 0} \mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}}) \otimes V_n$$

given by $\Psi = \sum (\Psi_n \otimes \mathrm{id})$ is an isomorphism of G -modules.

Assuming the above lemmas, Theorem 4.1 is equivalent to showing that the following diagram commutes:

$$\begin{array}{ccc} \sum_{n \geq 0} V_n^* \otimes V_n & \xrightarrow{\Upsilon} & \mathbb{C}[G] \\ & \searrow \Psi \quad \nearrow \Phi & \\ & \sum_{n \geq 0} \mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}}) \otimes V_n & \end{array}$$

The proof of commutativity follows:

$$\Phi \circ \Psi(w^* \otimes v) = \Phi(\mathbf{F}_{w^*} \otimes v) = \mathbf{F}_{w^*}(v) = w^*(\mathbf{x} \cdot v) = \Upsilon(w^* \otimes v). \quad \square$$

It remains to establish Lemmas 4.4 and 4.5. The proof of Lemma 4.4 requires some preliminary technical results.

Lemma 4.6. Every regular function is contained in a finite-dimensional subrepresentation of $\mathbb{C}[G]$.

Proof of Lemma 4.6. The following $G \times G$ -action encompasses both the right and diagonal G -actions defined above. Let

$$\alpha : G \times G \times G \longrightarrow G$$

be defined by $(g_1, g_2, \mathbf{x}) \mapsto g_1 \mathbf{x} g_2^{-1}$, and further let

$$\alpha^* : \mathbb{C}[G] \longrightarrow \mathbb{C}[G \times G \times G] \cong \mathbb{C}[G]^{\otimes 3} \quad (2)$$

be defined by $f \mapsto f \circ \alpha$, the *pull-back* of regular functions on G to regular functions on $G \times G \times G$. For $f \in \mathbb{C}[G]$, (2) implies that there exist $n_f \in \mathbb{N}$ and regular functions f_i, f'_i, f''_i for $1 \leq i \leq n_f$ such that

$$\alpha^*(f) = \sum_{i=1}^{n_f} f_i \otimes f'_i \otimes f''_i.$$

Therefore

$$\alpha^*(f)(g_1^{-1}, g_2^{-1}, \mathbf{x}) = \sum_{i=1}^{n_f} f_i(g_1^{-1}) f'_i(g_2^{-1}) f''_i(\mathbf{x}).$$

On the other hand,

$$\alpha^*(f)(g_1^{-1}, g_2^{-1}, \mathbf{x}) = f(\alpha(g_1^{-1}, g_2^{-1}, \mathbf{x})) = f(g_1^{-1} \mathbf{x} g_2) = ((g_1, g_2) \cdot f)(\mathbf{x}),$$

which implies

$$(g_1, g_2) \cdot f = \sum_{i=1}^{n_f} f_i(g_1^{-1}) f'_i(g_2^{-1}) f''_i. \quad (3)$$

Let $(G \times G)f = \{(g_1, g_2) \cdot f : f \in G\}$ be the $G \times G$ -orbit of f , and let W_f be the linear subspace spanned over \mathbb{C} by $(G \times G)f$ in $\mathbb{C}[G]$. By (3), $\{f''_i\}$ is a spanning set for W_f , and so W_f is finite-dimensional. Clearly W_f is $G \times G$ -invariant, and so invariant with respect to the diagonal and right G -actions. Thus, it is a finite-dimensional sub-representation containing f . \square

Lemma 4.7. $\mathbb{C}[G]$ is completely $G \times G$ -reducible.

Proof of Lemma 4.7. Let \mathcal{I} be the set of direct sums of irreducible finite-dimensional sub-representations of $\mathbb{C}[G]$. \mathcal{I} is partially ordered by set inclusion and is nonempty. Thus, by Zorn's lemma there exists a maximal element $M \in \mathcal{I}$. If $M \neq \mathbb{C}[G]$, then consider any $f \notin M$. By Lemma 4.6, there exists a finite-dimensional sub-representation W_f that contains f . Let $K = \mathrm{SU}(2)$ be the maximal compact subgroup of G . Restrict the action of $G \times G$ to $K \times K$ to find an invariant orthogonal complement to W_f in $M \cup W_f$. Denote this complement by M^\perp . Then $M^\perp \oplus W_f \in \mathcal{I}$, since $K \times K$ representations extend to $G \times G$ representations. Hence M is not maximal, which is a contradiction. Therefore $\mathbb{C}[G]$ is completely reducible with respect to the $G \times G$ -action, and so

$$\mathbb{C}[G] \cong \sum_{j \geq 0} c_j V_j,$$

where c_j is the (possibly infinite) multiplicity of V_j in $\mathbb{C}[G]$. This decomposition also holds for $\mathbb{C}[G]$ with both the right and diagonal actions since they are restrictions of the same $G \times G$ -action. \square

Proof of Lemma 4.4. By Lemma 4.7,

$$\Phi : \sum_{n \geq 0} (\mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}}) \otimes V_n) \longrightarrow \mathbb{C}[G]$$

is an isomorphism if and only if

$$\sum_{n \geq 0} \left(\sum_{j \geq 0} \mathrm{Hom}_G(V_n, c_j V_j) \otimes V_n \right) \longrightarrow \sum_{j \geq 0} c_j V_j$$

is an isomorphism. By Schur's Lemma, this reduces to

$$\sum_{n \geq 0} (c_n \mathbb{C} \otimes V_n) \cong \sum_{n \geq 0} (\mathrm{Hom}_G(V_n, c_n V_n) \otimes V_n) \longrightarrow \sum_{n \geq 0} c_n V_n.$$

However, this is the map sending $\sum \lambda \otimes v \mapsto \sum \lambda v$ for $\lambda \in \mathbb{C}$ and $v \in V_n$, which is canonically an isomorphism. \square

The final task is to show that Ψ is an isomorphism:

Proof of Lemma 4.5. Recall that

$$\Psi_n : V_n^* \longrightarrow \mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}})$$

was defined by $w^* \mapsto \mathbf{F}_{w^*}$, where $\mathbf{F}_{w^*}(v)(\mathbf{x}) = w^*(\mathbf{x} \cdot v)$. Ψ_n is well-defined since

$$\mathbf{F}_{w^*}(g \cdot v)(\mathbf{x}) = w^*(\mathbf{x} \cdot (g \cdot v)) = w^*((\mathbf{x}g) \cdot v) = \mathbf{F}_{w^*}(v)(\mathbf{x}g) = (\mathbf{F}_{w^*}(v)) \cdot g(\mathbf{x}),$$

and is G -equivariant because

$$\begin{aligned} \Psi_n(g \cdot w^*)(v)(\mathbf{x}) &= \mathbf{F}_{g \cdot w^*}(v)(\mathbf{x}) = (g \cdot w^*)(\mathbf{x} \cdot v) = w^*((g^{-1}\mathbf{x}) \cdot v) \\ &= \mathbf{F}_{w^*}(v)(g^{-1}\mathbf{x}) = (g \cdot \mathbf{F}_{w^*})(v)(\mathbf{x}) = g \cdot \Psi_n(w^*)(v)(\mathbf{x}). \end{aligned}$$

Since V_n^* is irreducible, Schur's Lemma implies Ψ_n is injective. We now show surjectivity. Consider $\gamma \in \mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}})$. For $\mathbb{I} \in G$, $\gamma(v)(\mathbb{I})$ is a linear functional on V_n . Hence there exists $w^* \in V_n^*$ such that $w^*(v) = \gamma(v)(\mathbb{I})$ for all $v \in V_n$. The following computation establishes that $\Psi_n(w^*) = \gamma$:

$$\mathbf{F}_{w^*}(v)(\mathbf{x}) = w^*(\mathbf{x} \cdot v) = \gamma(\mathbf{x} \cdot v)(\mathbb{I}) = (\gamma(v)) \cdot \mathbf{x}(\mathbb{I}) = \gamma(v)(\mathbb{I}\mathbf{x}) = \gamma(v)(\mathbf{x}).$$

Therefore Ψ_n is an isomorphism and so is $\Psi = \sum (\Psi_n \otimes \mathrm{id})$:

$$\sum_{n \geq 0} V_n^* \otimes V_n \cong \sum_{n \geq 0} (\mathrm{Hom}_G(V_n, \mathbb{C}[G]_{\mathrm{right}}) \otimes V_n). \quad \square$$

4.3 Ring Structure of $\mathbb{C}[G]^G$

We have established

$$\mathbb{C}[G] \cong \sum_{n \geq 0} V_n^* \otimes V_n.$$

By Schur's Lemma and the fact that $V_n^* \otimes V_n \cong \mathrm{End}(V_n)$,

$$\mathbb{C}[G]^G \cong \sum_{n \geq 0} (V_n^* \otimes V_n)^G \cong \sum_{n \geq 0} \mathbb{C} \chi^n,$$

where $\chi^n \in \mathrm{End}(V_n)^G$ is a multiple of the identity.

The isomorphism $\text{End}(V_n) \rightarrow V_n^* \otimes V_n$ is given by

$$\mathbf{n}_{n-l}(\mathbf{n}_{n-k})^T \mapsto \binom{n}{k} \mathbf{n}_{n-k}^* \otimes \mathbf{n}_{n-l}.$$

Therefore, the central function χ^n corresponds to an invariant function in $\mathbb{C}[G]^G$ by

$$\chi^n = \sum_{i=0}^n \mathbf{n}_i(\mathbf{n}_i)^T \mapsto \sum_{i=0}^n \binom{n}{i} \mathbf{n}_i^* \otimes \mathbf{n}_i \xrightarrow{\Upsilon} \sum_{i=0}^n \binom{n}{i} \mathbf{n}_i^*(\mathbf{x} \cdot \mathbf{n}_i).$$

We will freely identify χ^n with its image in $\mathbb{C}[G]^G$.

For example, the trivial representation V_0 gives $\chi^0 = 1$. The standard representation V_1 has diagonal matrix coefficients x_{11} and x_{22} , hence

$$\chi^1 = x_{11} + x_{22} = \text{tr}(\mathbf{x}).$$

The remaining functions may be computed directly, or by using the following product formula:

Theorem 4.8 (Product Formula).

$$\chi^a \chi^b = \sum_{c \in [a, b]} \chi^c \tag{4}$$

Proof. From the Clebsch-Gordan decomposition,

$$(V_a \otimes V_b)^* \otimes (V_a \otimes V_b) \cong \sum_{c, d \in [a, b]} V_c^* \otimes V_d.$$

Hence

$$\text{End}(V_a \otimes V_b)^G \cong \sum_{c \in [a, b]} \text{End}(V_c)^G$$

and the characters satisfy

$$\chi^a \chi^b = \chi_{(V_a \otimes V_b)} = \chi_{\oplus_c V_c} = \sum_{c \in [a, b]} \chi^c.$$

There is an alternate diagrammatic proof of this statement, which uses the fusion and bubble identities in Propositions 3.20 and 3.21. If the matrix \mathbf{x} is

represented by \otimes , then:

$$\begin{aligned}
 \chi^a \chi^b &= \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^a_b = \sum_{c \in [a, b]} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^a_c \\
 &= \sum_{c \in [a, b]} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^c_a = \sum_{c \in [a, b]} \left(\frac{\Delta(c)}{\Theta(a, b, c)} \right) \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^c_b \\
 &= \sum_{c \in [a, b]} \left(\frac{\Delta(c)\Theta(a, b, c)}{\Theta(a, b, c)\Delta(c)} \right) \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^c_c = \sum_{c \in [a, b]} \left(\text{diagram of two circles with } \otimes \text{ in the middle} \right)^c_c = \sum_{c \in [a, b]} \chi^c. \square
 \end{aligned}$$

The product formula (4) and the initial calculations of χ^0 and χ^1 may be used to show:

Theorem 4.9. $\mathbb{C}[G]^G \cong \mathbb{C}[t]$.

Proof. Consider the ring homomorphism $\Phi : \mathbb{C}[t] \rightarrow \mathbb{C}[G]^G$ defined by $f \mapsto f \circ \mathrm{tr}$. Suppose $f(\mathrm{tr}(g)) = 0$ for all $g \in G$. If $f \neq 0$, then since f has a finite number of zeros, $\mathrm{tr}(g)$ must have a finite number of values. However,

$$\begin{bmatrix} t & 1 \\ -1 & 0 \end{bmatrix} \in G$$

for all values of t . Hence, $f = 0$ and Φ is injective. It remains to establish surjectivity. We have already shown $t \mapsto \chi^1$ and $1 \mapsto \chi^0$. Suppose $a \geq 2$ and χ^b is in the image of Φ for all $b < a$. Equation (4) implies $\chi^1 \chi^{a-1} = \chi^a + \chi^{a-2}$. Thus, by induction,

$$t\Phi^{-1}(\chi^{a-1}) - \Phi^{-1}(\chi^{a-2}) \mapsto \chi^a. \quad \square$$

The following closed formula for χ^n is given in [25]:

$$\chi^n(t) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} t^{n-2r}.$$

The characters χ^n may also be expressed as functions of eigenvalues, since χ^n is determined by its values on normal forms

$$\begin{bmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{bmatrix} \in G.$$

Explicitly, $\begin{bmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{bmatrix}$ acts on V_n by the matrix

$$\begin{bmatrix} \lambda^n & * & * & \cdots & * \\ 0 & \lambda^{n-2} & * & \cdots & * \\ \vdots & 0 & \ddots & * & * \\ 0 & \vdots & 0 & \lambda^{2-n} & * \\ 0 & 0 & \cdots & 0 & \lambda^{-n} \end{bmatrix}.$$

Hence,

$$\chi^n = \lambda^n + \lambda^{n-2} + \cdots + \lambda^{2-n} + \lambda^{-n} = \frac{\lambda^{n+1} - \lambda^{-n-1}}{\lambda - \lambda^{-1}} = [n+1]_\lambda,$$

where $[n+1]_\lambda$ is the quantized integer for $q = \lambda$.

5 Structure of $\mathbb{C}[G \times G]^G$

Recall the decomposition

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{a,b \in \mathbb{N} \\ c \in [a,b]}} \mathbb{C} \chi^{a,b,c},$$

where $\chi^{a,b,c}$ corresponds by Υ to the image of

$$\sum_{k=0}^c \mathbf{c}_k (\mathbf{c}_k)^T \mapsto \sum_{k=0}^c \binom{c}{k} \mathbf{c}_k^* \otimes \mathbf{c}_k$$

under the injection $V_c^* \otimes V_c \hookrightarrow V_a^* \otimes V_b^* \otimes V_a \otimes V_b$. This inclusion is determined by the Clebsch-Gordan injection $\iota : V_c \hookrightarrow V_a \otimes V_b$. Hence, an explicit formula for ι provides a means to compute $\chi^{a,b,c}$ directly. We freely use $\chi^{a,b,c}$ to denote its image in $\mathbb{C}[G \times G]^G$.

A few simple examples will motivate the construction of ι . For $k = 1, 2$, let $\mathbf{x}_k = [x_{ij}^k]$ be 2×2 matrix variables, and let

$$\begin{aligned} x &= \text{tr}(\mathbf{x}_1) = x_{11}^1 + x_{22}^1, \\ y &= \text{tr}(\mathbf{x}_2) = x_{11}^2 + x_{22}^2, \\ z &= \text{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1}) = (x_{11}^1 x_{22}^2 + x_{22}^1 x_{11}^2) - (x_{12}^1 x_{21}^2 + x_{21}^1 x_{12}^2). \end{aligned}$$

Recall that the map $\cup : V_0 \hookrightarrow V_1 \otimes V_1$ given by

$$\mathbf{c}_0 \mapsto \mathbf{a}_1 \otimes \mathbf{b}_0 - \mathbf{a}_0 \otimes \mathbf{b}_1$$

is invariant, using the notation defined in section 2.3. More generally, the injection $V_0 \hookrightarrow V_a \otimes V_a$ is given by

$$\bigcup^a : c_0 \mapsto \sum_{m=0}^a (-1)^m \binom{a}{m} a_{a-m} \otimes b_m. \quad (1)$$

Hence, $\chi^{0,0,0} = 1$ and $\chi^{1,1,0}$ may be computed by:

$$\begin{aligned} \chi^{1,1,0} &\mapsto c_0^* \otimes c_0 \\ &\mapsto (a_1^* \otimes b_0^* - a_0^* \otimes b_1^*) \otimes (a_1 \otimes b_0 - a_0 \otimes b_1) \\ &\mapsto (a_1^* \otimes a_1) \otimes (b_0^* \otimes b_0) - (a_0^* \otimes a_1) \otimes (b_1^* \otimes b_0) \\ &\quad - (a_1^* \otimes a_0) \otimes (b_0^* \otimes b_1) + (a_0^* \otimes a_0) \otimes (b_1^* \otimes b_1) \\ &\mapsto x_{11}^1 \otimes x_{22}^2 - x_{12}^1 \otimes x_{21}^2 - x_{21}^1 \otimes x_{12}^2 + x_{22}^1 \otimes x_{11}^2 \\ &\mapsto (x_{11}^1 x_{22}^2 + x_{22}^1 x_{11}^2) - (x_{12}^1 x_{21}^2 + x_{21}^1 x_{12}^2) = z. \end{aligned}$$

The representation V_c may be identified with a subset of $V^{\otimes c}$ via the equivariant maps

$$\begin{array}{ccc} & \text{Sym} & \\ V_c & \xrightarrow{\quad} & V^{\otimes c} \\ & \text{Proj} & \end{array}$$

where $\text{Proj} \circ \text{Sym} = \text{id}$. Thus, when $c = a + b$, ι is given by the commutative diagram

$$\begin{array}{ccc} V^{\otimes c} & \xlongequal{\quad} & V^{\otimes a} \otimes V^{\otimes b} \\ \text{Sym} \uparrow & \circlearrowright & \downarrow \text{Proj} \otimes \text{Proj} \\ V_c & \xrightarrow{\quad \iota \quad} & V_a \otimes V_b. \end{array}$$

In particular,

$$\binom{c}{k} c_k \xrightarrow{\iota} \sum_{\substack{0 \leq i \leq a \\ 0 \leq j \leq b \\ i+j=k}} \binom{a}{i} a_i \otimes \binom{b}{j} b_j. \quad (2)$$

For example, consider $\chi^{1,0,1}$. In this case, $c_0 \mapsto a_0 \otimes b_0$ and $c_1 \mapsto a_1 \otimes b_0$. Hence,

$$\begin{aligned} \chi^{1,0,1} &\mapsto c_0^* \otimes c_0 + c_1^* \otimes c_1 \mapsto (a_0^* \otimes a_0) \otimes (b_0^* \otimes b_0) + (a_1^* \otimes a_1) \otimes (b_0^* \otimes b_0) \\ &\mapsto x_{11}^1 \otimes 1 + x_{22}^1 \otimes 1 \mapsto x_{11}^1 + x_{22}^1 = x. \end{aligned}$$

A similar computation shows that $\chi^{0,1,1} \mapsto y$.

The general form of ι is determined by combining (1) and (2) in the following diagram:

$$\begin{array}{ccc}
 V_c & \xrightarrow{\iota} & V_\beta \otimes V_\alpha \\
 \downarrow \iota & \circlearrowleft & \downarrow \text{id} \otimes \bigcup \gamma \otimes \text{id} \\
 V_a \otimes V_b & \xleftarrow{\quad} & V_\beta \otimes V_\gamma \otimes V_\gamma \otimes V_\alpha
 \end{array}$$

It follows that the mapping $\iota : V_c \rightarrow V_a \otimes V_b$ is explicitly given by:

$$\begin{aligned}
 \binom{c}{k} c_k &\mapsto \sum_{\substack{0 \leq i \leq \beta \\ 0 \leq j \leq \alpha \\ 0 \leq m \leq \gamma \\ i+j=k}} \binom{\beta}{i} a_i \otimes [(-1)^m \binom{\gamma}{m} a_{\gamma-m} \otimes b_m] \otimes \binom{\alpha}{j} b_j \\
 &\mapsto \sum_{\substack{0 \leq i \leq \beta \\ 0 \leq j \leq \alpha \\ 0 \leq m \leq \gamma \\ i+j=k}} (-1)^m \binom{\beta}{i} \binom{\alpha}{j} \binom{\gamma}{m} a_{i+\gamma-m} \otimes b_{j+m}.
 \end{aligned}$$

5.1 Symmetry of Central Functions

Our first theorem regarding central functions is a symmetry property that is essentially trivial in diagram form, despite being highly nontrivial algebraically. A portion of the Fricke-Klein-Vogt Theorem (5.12) is required to state the theorem. We begin with a diagrammatic proof of this classical result, in which the binor identity plays the role of the characteristic equation in the classical proof.

Lemma 5.1. *Each central function $\chi^{a,b,c}$ is associated to a unique polynomial $p_{a,b,c}$, denoted for all pairs $(\mathbf{x}_1, \mathbf{x}_2) \in G \times G$ by*

$$\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) = p_{a,b,c}(\text{tr}(\mathbf{x}_2), \text{tr}(\mathbf{x}_1), \text{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1})).$$

Proof. Expanding the symmetrizers in $\chi^{a,b,c}$ gives a collection of circles with matrix elements, each of which correspond to a product of traces of words in \mathbf{x}_1 and \mathbf{x}_2 , so it suffices to show that every loop can be reduced to a collection of loops containing one of \mathbf{x}_1 , \mathbf{x}_2 , or $\mathbf{x}_1 \mathbf{x}_2^{-1}$.

This reduction depends entirely on the binor identity, which when com-

posed with $\mathbf{x}_1 \otimes \mathbf{x}_2 = \bigcirc \bigcirc$ gives:

$$\begin{array}{c} \bigcirc \bigcirc \end{array} = \begin{array}{c} \bigcirc \bigcirc \end{array} - \begin{array}{c} \bigcirc \bigcirc \end{array}. \quad (3)$$

Denote \mathbf{x}_1^{-1} by $\bar{\mathbf{x}}_1$. Two special cases of (3) follow:

$$\begin{array}{c} \text{Diagram 1: A loop with two vertices labeled } \mathbf{x}_1 \text{ and } \mathbf{x}_1^{-1} \text{ is equal to the difference of two diagrams: a vertical line with a loop on the left and a vertical line with a loop on the right.} \\ \text{Diagram 2: A loop with two vertices labeled } \mathbf{x}_1 \text{ and } \mathbf{x}_1^{-1} \text{ is equal to the difference of two diagrams: a vertical line with a loop on the left and a vertical line with a loop on the right.} \end{array}$$

The first relation allows us to assume no loop has both \mathbf{x}_1 and \mathbf{x}_1^{-1} , while the second allows us to assume no loop has more than one of any matrix. The remaining cases are the traces $\mathrm{tr}(\mathbf{x}_1)$, $\mathrm{tr}(\mathbf{x}_2)$, $\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$, and $\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$. Finally, closing off (3) permits the reduction of $\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$:

$$\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2) = \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2) - \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}). \quad \square$$

We can now prove the symmetry result. In the statement and proof below, $\sigma(\diamond_1, \diamond_2, \diamond_3)$ denotes the ordered triple $(\diamond_{\sigma(1)}, \diamond_{\sigma(2)}, \diamond_{\sigma(3)})$ obtained by applying a given permutation $\sigma \in \Sigma_3$ to the triple $(\diamond_1, \diamond_2, \diamond_3)$. This result was first outlined in [29].

Theorem 5.2 (Symmetry of Central Functions). *The family of polynomials $\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) = \mathfrak{p}_{a,b,c}(\mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}))$ possesses the following symmetry:*

$$\mathfrak{p}_{\sigma(a,b,c)}(y, x, z) = \mathfrak{p}_{a,b,c}(\sigma^{-1}(y, x, z)).$$

Proof. Define the following function $G \times G \times G \rightarrow \mathbb{C}$:

$$\chi_{\alpha,\beta,\gamma}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Diagram: A horizontal line with three loops labeled } \alpha, \beta, \gamma \text{ above them. Each loop contains a vertex labeled } \mathbf{x}, \mathbf{y}, \text{ or } \mathbf{z} \text{ respectively. The line continues to the right, indicating a wrap-around.}$$

where the symmetrizer on the right is assumed to ‘wrap around’ to the one on the left (imagine this diagram being drawn on a cylinder). By construction this function is symmetric, in the sense that:

$$\chi_{\sigma(\alpha,\beta,\gamma)} \left(\sigma \left(\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{array} \right) \right) = \chi_{\alpha,\beta,\gamma} \left(\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{array} \right).$$

A central function $\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2)$ may be drawn as:

$$\text{Diagram: A loop with two vertices labeled } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \text{ is equal to the difference of two diagrams: a vertical line with a loop on the left and a vertical line with a loop on the right. The diagrams are labeled with } \frac{a-b+c}{2}, \frac{a+b-c}{2}, \text{ and } \frac{-a+b+c}{2} \text{ above them.}$$

with the symmetrizers in the last two diagrams assumed to wrap around as before. Thus, $\mathfrak{p}_{a,b,c}(y, x, z) = \chi_{\alpha,\beta,\gamma}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1 \mathbf{x}_2^{-1})$ and so:

$$\begin{aligned} \mathfrak{p}_{\sigma(a,b,c)}(y, x, z) &= \chi_{\sigma(\alpha,\beta,\gamma)}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1 \mathbf{x}_2^{-1}) \\ &= \chi_{\alpha,\beta,\gamma}(\sigma^{-1}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1 \mathbf{x}_2^{-1})) \\ &= \mathfrak{p}_{a,b,c}(\sigma^{-1}(y, x, z)). \square \end{aligned}$$

Table 5.1 contains six central functions illustrating this symmetry.

$\chi^{1,2,3} = xy^2 - \frac{2}{3}(yz + x)$	$\chi^{3,2,1} = xz^2 - \frac{2}{3}(yz + x)$
$\chi^{2,3,1} = yz^2 - \frac{2}{3}(xz + y)$	$\chi^{1,3,2} = y^2z - \frac{2}{3}(xy + z)$
$\chi^{3,1,2} = x^2z - \frac{2}{3}(xy + z)$	$\chi^{2,1,3} = x^2y - \frac{2}{3}(xz + y)$

Table 1. Rank Two Central Function Symmetry.

5.2 A Recurrence Relation for Central Functions

Define the *degree* of a central function to be:

$$\delta = \deg(\chi^{a,b,c}) = \frac{1}{2}(a + b + c).$$

We will obtain a recurrence relation for an arbitrary central function $\chi^{a,b,c}$ by manipulating diagrams to express the product

$$\text{tr}(\mathbf{x}_1) \cdot \chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2)$$

as a sum of central functions. This formula can be rearranged to write $\chi^{a,b,c}$ as a linear combination of central functions *with lower degree*. There are three main ingredients to the diagram manipulations: the *bubble identity* and the *fusion identity* from Section 3.5, and two *recoupling formulae* which we prove in the following lemma.

Lemma 5.3. *For $i = \frac{1}{2}(a + 1 - b + c)$ and appropriate triples admissible,*

$$\begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ a-1 \\ b \end{array} = \begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ a+1 \\ b \end{array} - (-1)^i \left(\frac{a+b-c+1}{2(a+1)} \right) \begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ a-1 \\ b \end{array}; \quad (4)$$

$$\begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ c+1 \\ b \end{array} = (-1)^i \left(\frac{-a+b+c+1}{2(c+1)} \right) \begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ a+1 \\ b \end{array} + \left(\frac{(a+b+c+3)(a-b+c+1)}{4(a+1)(c+1)} \right) \begin{array}{c} 1 \\ \diagup \\ c \end{array} \begin{array}{c} \diagdown \\ a-1 \\ b \end{array}. \quad (5)$$

Proof. Note that i is just the number of strands connecting $\begin{smallmatrix} | & \cdots & | \\ \hline & & \\ | & \cdots & | \end{smallmatrix}^{a+1}$ to $\begin{smallmatrix} | & \cdots & | \\ \hline & & \\ | & \cdots & | \end{smallmatrix}^c$ in

$$\begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} = \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix}. \text{ For (4), use } n = a + 1 \text{ and } i \text{ in recurrence relation (9) to get:}$$

$$\begin{smallmatrix} | & \cdots & | \\ \hline & & \\ | & \cdots & | \end{smallmatrix}^{a+1} = \begin{smallmatrix} | & \cdots & | \\ \hline & & \\ | & \cdots & | \end{smallmatrix}^{a+1-i} \begin{smallmatrix} i \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1-i} + (-1)^i \left(\frac{a+1-i}{a+1} \right) \begin{smallmatrix} | & \cdots & | \\ \hline & & \\ | & \cdots & | \end{smallmatrix}^{a+1-i} \begin{smallmatrix} i \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1-i}.$$

Compose this equation with $\begin{smallmatrix} i & a+1-i \\ \diagdown & \diagup \\ \text{Y} \\ \diagup & \diagdown \\ c & b \end{smallmatrix}$ to get, via the *stacking relation*:

$$\begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} = \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} = \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} + (-1)^i \left(\frac{a+1-i}{a+1} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix},$$

which is the desired result.

To prove (5), notice that if we switch a and c in the previous relation, and apply a $\frac{\pi}{4}$ -reflection to the relation about the $1 \leftrightarrow b$ axis as in Proposition 3.23, then i is unchanged and the equation becomes:

$$\begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} = \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} + (-1)^i \left(\frac{c+1-i}{c+1} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix}.$$

Rearrange this equation, and use (4) in its exact form to get:

$$\begin{aligned} \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} &= \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} + (-1)^i \left(\frac{c+1-i}{c+1} \right) \left(\begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} - (-1)^i \left(\frac{a+1-i}{a+1} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} \right) \\ &= (-1)^i \left(\frac{c+1-i}{c+1} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} + \left(1 - \frac{(a+1-i)(c+1-i)}{(a+1)(c+1)} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} \\ &= (-1)^i \left(\frac{-a+b+c+1}{2(c+1)} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix} + \left(\frac{(a+b+c+3)(a-b+c+1)}{4(a+1)(c+1)} \right) \begin{smallmatrix} 1 \\ \diagdown \\ \text{Y} \\ \diagup \\ c \end{smallmatrix}^{a+1} \begin{smallmatrix} a \\ \diagdown \\ \text{Y} \\ \diagup \\ b \end{smallmatrix}. \end{aligned}$$

To show the last computation, note that $a+1-i = \frac{1}{2}(a+b-c+1)$ and $c+1-i = \frac{1}{2}(-a+b+c+1)$, so the numerator of the last term is:

$$\begin{aligned} &4((a+1)(c+1) - (a+1-i)(c+1-i)) \\ &= 4(a+1)(c+1) - ((b+1) + (c-a))((b+1) - (c-a)) \\ &= 4(a+1)(c+1) - (b+1)^2 + (a-c)^2 \\ &= ((a+1) - (c+1))^2 + 4(a+1)(c+1) - (b+1)^2 \\ &= ((a+1) + (c+1))^2 - (b+1)^2 \\ &= (a+1+c+1+b+1)(a+1+c+1-b-1) \\ &= (a+b+c+3)(a-b+c+1). \square \end{aligned}$$

The coefficients we have computed are examples of *6j-symbols*, most easily defined to be the coefficients $\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right]'$ in the following *change of basis* equation:

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ e \\ \diagup \quad \diagdown \\ d \quad c \end{array} = \sum_{f \in [a,b] \cap [c,d]} \left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right]' \cdot \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ f \\ \diagup \quad \diagdown \\ d \quad c \end{array}.$$

We use a prime because we will need an alternate version later:

Definition 5.4. The *6j-symbols* $\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right]$ are the coefficients given by

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ e \\ \diagup \quad \diagdown \\ d \end{array} = \sum_{f \in [a,b] \cap [c,d]} \left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right] \cdot \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ f \\ \diagup \quad \diagdown \\ d \end{array}.$$

Both versions given here differ from those in the literature [5, 21]. It is not hard to show, using Corollary 3.24, that

$$\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right]' = (-1)^{\frac{1}{2}(b+d-e-f)} \left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix} \right].$$

Thus, as a corollary to the above lemma we have the following *6j-symbols*, given by replacing c with $c+1$ or $c-1$, which will be used to prove the next theorem:

Corollary 5.5.

$$\begin{aligned} \left[\begin{smallmatrix} 1 & a & a+1 \\ c+1 & b & c \end{smallmatrix} \right] &= 1; & \left[\begin{smallmatrix} 1 & a & a-1 \\ c+1 & b & c \end{smallmatrix} \right] &= (-1)^{\frac{1}{2}(a-b+c+2)} \frac{(a+b-c)}{2(a+1)}; \\ \left[\begin{smallmatrix} 1 & a & a+1 \\ c-1 & b & c \end{smallmatrix} \right] &= (-1)^{\frac{1}{2}(a-b+c)} \frac{(-a+b+c)}{2c}; & \left[\begin{smallmatrix} 1 & a & a-1 \\ c-1 & b & c \end{smallmatrix} \right] &= \frac{(a+b+c+2)(a-b+c)}{4(a+1)c}. \end{aligned}$$

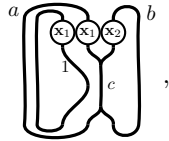
We can now prove the “multiplication by x ” formula.

Theorem 5.6. *The product $x \cdot \chi^{a,b,c}(x, y, z)$ can be expressed by:*

$$\begin{aligned} x \cdot \chi^{a,b,c} &= \chi^{a+1,b,c+1} + \frac{(a+b-c)^2}{4a(a+1)} \chi^{a-1,b,c+1} + \frac{(-a+b+c)^2}{4c(c+1)} \chi^{a+1,b,c-1} \\ &\quad + \frac{(a+b+c+2)^2(a-b+c)^2}{16a(a+1)c(c+1)} \chi^{a-1,b,c-1}. \end{aligned} \quad (6)$$

This equation still holds for $a = 0$ or $c = 0$, provided we exclude the terms with a or c in the denominator.

Proof. Diagrammatically, $x \cdot \chi^{a,b,c}(x, y, z)$ is represented by



since $x = \mathrm{tr}(\mathbf{x}_1) = \bigcirc \begin{smallmatrix} \mathbf{x}_1 \end{smallmatrix}$ and multiplication is automatic on disjoint diagrams.

Now manipulate the diagram to obtain a sum over χ 's with the following three steps.

First, apply the fusion identity to connect the lone $\bigcirc \begin{smallmatrix} \mathbf{x}_1 \end{smallmatrix}$ strand to the $\chi^{a,b,c}$:

$$\begin{array}{c} a \quad b \\ \text{Diagram 1} \end{array} = \begin{array}{c} a \quad b \\ \text{Diagram 2} \end{array} + \frac{c}{c+1} \begin{array}{c} a \quad b \\ \text{Diagram 3} \end{array}, \quad (7)$$


where the coefficients are evaluated from

$$\frac{\Delta(c \pm 1)}{\Theta(1, c, c \pm 1)} = \frac{c \pm 1 + 1}{c + \frac{3}{2} \pm \frac{1}{2}}.$$

Second, use the $6j$ -symbols computed in Corollary 5.5 above to move the a strand from one side of the diagram to the other:

$$\begin{array}{c} a \quad b \\ \text{Diagram 4} \end{array} = \begin{array}{c} a+1 \quad b \\ \text{Diagram 5} \end{array} + \frac{(a+b-c)^2}{4(a+1)^2} \begin{array}{c} a \quad b \\ \text{Diagram 6} \end{array} \quad (8)$$

$$\begin{array}{c} a \quad b \\ \text{Diagram 7} \end{array} = \frac{(-a+b+c)^2}{4c^2} \begin{array}{c} a+1 \quad b \\ \text{Diagram 8} \end{array} + \frac{(a+b+c+2)^2(a-b+c)^2}{16(a+1)^2c^2} \begin{array}{c} a \quad b \\ \text{Diagram 9} \end{array}. \quad (9)$$

In each case, we are recoupling twice: once for the top piece  and once for the corresponding bottom piece. In doing this, we would actually get four terms, but since the $a \pm 1$ labels *must be the same* on both the top and the bottom (a consequence of Schur's Lemma or the bubble identity), two of the terms vanish.

In the final step, use the bubble identity to collapse the final pieces:

$$\begin{aligned}
& \begin{array}{c} a \quad b \\ \text{Diagram 1} \\ c \pm 1 \end{array} = \left(\frac{\Theta(1, a, a+1)}{\Delta(a+1)} \right) \begin{array}{c} a+1 \quad b \\ \text{Diagram 2} \\ c \pm 1 \end{array} = \chi^{a+1, b, c \pm 1}; \\
& \begin{array}{c} a \quad b \\ \text{Diagram 3} \\ c \pm 1 \end{array} = \left(\frac{\Theta(1, a, a-1)}{\Delta(a-1)} \right) \begin{array}{c} a-1 \quad b \\ \text{Diagram 4} \\ c \pm 1 \end{array} = \left(\frac{a+1}{a} \right) \chi^{a-1, b, c \pm 1}.
\end{aligned}$$

At this point, obtaining (6) is simply a matter of multiplying the coefficients obtained in the previous formulae.

Now consider the special cases. For $a = 0$, since $b = c$ and consequently $\frac{c}{c+1} = \frac{(-a+b+c)^2}{4c(c+1)}$, the desired formula is exactly (7). Similarly, for $c = 0$, the desired formula is (8). \square

We find it interesting that, for all our discussion of signs introduced by non-topological invariance, all signs introduced are eventually squared and thus do not show up in this result.

We can rearrange the terms in (6) and re-index to get:

Corollary 5.7 (Central Function Recurrence). *Provided $a > 1$ and $c > 1$, we can write*

$$\begin{aligned}
\chi^{a, b, c} = x \cdot \chi^{a-1, b, c-1} & - \frac{(a+b-c)^2}{4a(a-1)} \chi^{a-2, b, c} - \frac{(-a+b+c)^2}{4c(c-1)} \chi^{a, b, c-2} \\
& - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)} \chi^{a-2, b, c-2}.
\end{aligned}$$

The relation still holds for $a = 1$ or $c = 1$, provided we exclude the terms with $a - 1$ or $c - 1$ in the denominator.

The condition $a > 1$, $c > 1$ arises because decrementing a and c in (6) means $(a - 1, b, c - 1)$ must now be admissible. Also, note that formulae for multiplication by y and z may be obtained by applying the symmetry relation of Theorem 5.2. This fact is indispensable in our proof of Theorem 5.12.

5.3 Graded Structure of the Central Function Basis

The majority of the content in this section was suggested to us by Carlos Florentino [12] after he read an early draft of this chapter.

Recall the α, β, γ notation used earlier, and the notation

$$\chi_{\alpha, \beta, \gamma}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1 \mathbf{x}_2^{-1}) = \chi^{a, b, c}(\mathbf{x}_1, \mathbf{x}_2)$$

introduced in the proof of Theorem 5.2. The recurrence in Corollary 5.7 may be rewritten as

$$\begin{aligned} \chi_{\alpha, \beta, \gamma} = & \chi_{0,1,0} \chi_{\alpha, \beta-1, \gamma} - \frac{\gamma^2}{a(a-1)} \chi_{\alpha+1, \beta-1, \gamma-1} - \frac{\alpha^2}{c(c-1)} \chi_{\alpha-1, \beta-1, \gamma+1} \\ & - \frac{\delta^2(\beta-2)^2}{a(a-1)c(c-1)} \chi_{\alpha, \beta-2, \gamma}. \end{aligned}$$

The interchangeability of (a, α) and (c, γ) is guaranteed by the symmetry theorem.

Proposition 5.8. *The polynomial $\chi^{a,b,c} = \chi_{\alpha, \beta, \gamma}$ is monic, with highest degree monomial $x^\beta y^\alpha z^\gamma$.*

Proof. Induct on the degree $\delta = \alpha + \beta + \gamma$ of central functions. The statement is clearly true for the base cases, since $\chi_{0,0,0} = 1, \chi_{0,1,0} = x, \chi_{1,0,0} = y$, and $\chi_{0,0,1} = z$. The recurrence relation implies that the highest order term of $\chi_{\alpha, \beta, \gamma}$ is x times the highest order term of $\chi_{\alpha, \beta-1, \gamma}$, hence $x(x^{\beta-1} y^\alpha z^\gamma) = x^\beta y^\alpha z^\gamma$. This fact, together with the appropriate symmetric facts for y and z , completes the induction. \square

The basis also preserves a certain grading on $\mathbb{C}[x, y, z]$. To define this grading, partition the standard basis $\mathcal{B} = \{x^a y^b z^c\}$ of this space as follows. Let $\mathrm{gr} : \mathcal{B} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ be defined by:

$$\mathrm{gr}(x^a y^b z^c) = (a + c, b + c) \bmod 2.$$

If \mathcal{B} is considered as a semigroup under multiplication, then gr is a homomorphism since

$$\begin{aligned} \mathrm{gr}(x^a y^b z^c) + \mathrm{gr}(x^{a'} y^{b'} z^{c'}) &= (a + c, b + c) + (a' + c', b' + c') \bmod 2 \\ &= (a + a' + c + c', b + b' + c + c') \bmod 2 \\ &= \mathrm{gr}(x^{a+a'} y^{b+b'} z^{c+c'}) \bmod 2. \end{aligned}$$

Therefore, gr defines a grading on this basis.

Proposition 5.9. *The basis $\{\chi^{a,b,c}\}$ respects the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading on $\mathbb{C}[x, y, z]$ defined by gr , in the sense that*

$$\chi^{a,b,c} \in \mathbb{C}(\mathrm{gr}^{-1}(a, b)).$$

Proof. This is another proof by induction on the degree δ . Clearly, $\chi^{0,0,0} = 1 \in \mathrm{gr}^{-1}(0, 0)$, and likewise $\chi^{1,0,1} = x \in \mathrm{gr}^{-1}(1, 0)$, $\chi^{0,1,1} = y \in \mathrm{gr}^{-1}(0, 1)$, and $\chi^{1,1,0} = z \in \mathrm{gr}^{-1}(1, 1)$. In the induction step, note that

$$(a, b) = (1, 0) + (a - 1, b) = (a - 2, b) \bmod 2,$$

so all terms on the righthand side of the recurrence relation in Corollary 5.7 have the same grading. Thus $\chi^{a,b,c} \in \mathrm{gr}^{-1}(a, b)$. \square

5.4 Multiplication of Central Functions

It is not difficult to write down the formula for the product of two central functions, although the formula is by no means simple. The proof that follows was motivated by [29]. We begin with a lemma which encapsulates the most tedious diagram manipulations:

Lemma 5.10.

$$\begin{array}{c} a' \quad b \\ \diagdown \quad \diagup \\ c \quad c' \\ \diagup \quad \diagdown \\ a' \quad b \end{array} = \sum_{i,j,k,l,m} C_{j_1 k_1 l_1, j_2 k_2 l_2, m}^{abc, a' b' c'} \begin{array}{c} a' \quad b \\ \diagdown \quad \diagup \\ k_1 \quad l_1 \\ \diagup \quad \diagdown \\ m \\ \diagdown \quad \diagup \\ k_2 \quad l_2 \\ \diagup \quad \diagdown \\ a' \quad b \end{array},$$

where the coefficients are given by the formula

$$C_{j_1 k_1 l_1, j_2 k_2 l_2, m}^{abc, a' b' c'} = \frac{\Theta(c, c', m)}{\Delta(m)} \prod_{i=1,2} \frac{\Delta(j_i)}{\Theta(a', b, j_i)} \cdot \begin{bmatrix} a & a' & k_i \\ c & j_i & b \end{bmatrix} \begin{bmatrix} b' & b & l_i \\ c' & j_i & a' \end{bmatrix} \begin{bmatrix} k_i & l_i & m \\ c & c' & j_i \end{bmatrix},$$

and the following 15 triples are assumed to be admissible:

$$(a, a', k_i), (b, b', l_i), (c, c', m), (a', b, j_i), (c, j_i, k_i), (c', j_i, l_i), (b, j_i, l_i), (k_i, l_i, m).$$

Proof. We will just demonstrate the diagram manipulation for the top half of the diagram, which by symmetry must be the same as for the bottom half. Combining these two manipulations and applying a bubble identity will give the desired result. We will save enumeration of admissible triples until after the manipulation, but keep a close eye on signs in the meantime.

$$\begin{aligned}
& \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array} = \sum_j (-1)^{\frac{1}{2}(a'-b+j)} \frac{\Delta(j)}{\Theta(a', b, j)} \begin{array}{c} a \quad j \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array} \\
& = \sum_{j,k} (-1)^{\frac{1}{2}(a'-b+j)+j} \frac{\Delta(j)}{\Theta(a', b, j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{array}{c} a \quad j \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array} \\
& = \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)} \frac{\Delta(j)}{\Theta(a', b, j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ c' & j & a' \end{bmatrix} \begin{array}{c} a \quad j \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array} \\
& = \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)+\frac{1}{2}(j+l-c')} \frac{\Delta(j)}{\Theta(a', b, j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ c' & j & a' \end{bmatrix} \begin{array}{c} a \quad j \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array} \\
& = \sum_{j,k,l,m} (-1)^{\frac{1}{2}(a'-b+c-j-m)+l} \frac{\Delta(j)}{\Theta(a', b, j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ c' & j & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c & c' & j \end{bmatrix} \begin{array}{c} a \quad j \quad b \\ \diagdown \quad \diagup \\ c \quad c' \end{array}
\end{aligned}$$

The (-1) terms all cancel in the end, a consequence of the fact that the following triples must be admissible:

$$(a, a', k), (b, b', l), (c, c', m), (a', b, j), (c, j, k), (c', j, l), (b, j, l), (k, l, m).$$

One computes the 13-parameter coefficients $C_{j_1 k_1 l_1, j_2 k_2 l_2, m}^{abc, a' b' c'}$ above by reflecting this result vertically, taking two sets of indices for the variables j, k, l, m on the two halves, and noting that the resulting bubble in the middle collapses with a factor of $\frac{\Theta(c, c', m)}{\Delta(m)}$ for $m = m_1 = m_2$. \square

With that out of the way, we can describe the central function multiplication table explicitly. Note the symmetry with respect to k, l, m , which is guaranteed by Theorem 5.2.

Theorem 5.11 (Multiplication of Central Functions). *The product of two central functions $\chi^{a,b,c}$ and $\chi^{a',b',c'}$ is given by:*

$$\chi^{a,b,c} \chi^{a',b',c'} = \sum_{j_1, j_2, k, l, m} C_{j_1 k l m} C_{j_2 k l m} \frac{\Theta(a, a', k) \Theta(b, b', l) \Theta(c, c', m)}{\Delta(k) \Delta(l) \Delta(m)} \chi^{k, l, m},$$

where the sum is taken over admissible triples

$$(a, a', k), (b, b', l), (c, c', m), (a', b, j_i), (c, j_i, k), (c', j_i, l), (b, j_i, l), (k, l, m)$$

and the coefficients are given by:

$$C_{j_i k l m} = \frac{\Delta(j_i)}{\Theta(a', b, j_i)} \begin{bmatrix} a & a' & k \\ c & j_i & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ c' & j_i & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c & c' & j_i \end{bmatrix}.$$

Proof. By the previous lemma and the *bubble identity*, we have:

$$\begin{aligned} \text{Diagram 1} &= \sum_{j_1, k_1, l_1, j_2, k_2, l_2, m} C_{j_1 k_1 l_1, j_2 k_2 l_2, m}^{abc, a' b' c'} \text{Diagram 2} \\ &= \sum_{j_1, j_2, k, l, m} C_{j_1 k l, j_2 k l, m}^{abc, a' b' c'} \left(\frac{\Theta(a, a', k) \Theta(b, b', l)}{\Delta(k) \Delta(l)} \right) \text{Diagram 3} \\ &= \sum_{i, j, k, l} C_{j_1 k l m} C_{j_2 k l m} \frac{\Theta(a, a', k) \Theta(b, b', l) \Theta(c, c', m)}{\Delta(k) \Delta(l) \Delta(m)} \text{Diagram 4} \quad \square \end{aligned}$$

5.5 Applications

Spin networks offer a novel approach to a classical theorem of Fricke, Klein, and Vogt [14, 32]. We give here a new *constructive* proof which depends on the symmetry, recurrence, and multiplication formulae for central functions.

Theorem 5.12 (Fricke-Klein-Vogt Theorem). *Let $G = \text{SL}(2, \mathbb{C})$ act on $G \times G$ by simultaneous conjugation. Then*

$$\mathbb{C}[G \times G]^G \cong \mathbb{C}[t_x, t_y, t_z],$$

the complex polynomial ring in three indeterminates. In particular, every regular function $f : \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \rightarrow \mathbb{C}$ satisfying

$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1g^{-1}, g\mathbf{x}_2g^{-1}) \quad \text{for all } g \in \text{SL}(2, \mathbb{C}),$$

can be written uniquely as a polynomial in the three trace variables $x = \text{tr}(\mathbf{x}_1)$, $y = \text{tr}(\mathbf{x}_2)$, and $z = \text{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$.

Proof. Define the ring homomorphism

$$\Gamma : \mathbb{C}[t_x, t_y, t_z] \rightarrow \mathbb{C}[G \times G]^G$$

by $f(t_x, t_y, t_z) \mapsto f(\text{tr}(\mathbf{x}_1), \text{tr}(\mathbf{x}_2), \text{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}))$.

We first show that Γ is injective. Suppose $f(\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1})) = 0$ for all pairs $(\mathbf{x}_1, \mathbf{x}_2) \in G \times G$. Let $(\tau_x, \tau_y, \tau_z) \in \mathbb{C}^3$, $\epsilon_x = \begin{bmatrix} \tau_x & 1 \\ -1 & 0 \end{bmatrix}$, and $\eta_{y,z} = \begin{bmatrix} \tau_y & \frac{1}{\zeta} \\ -\zeta & 0 \end{bmatrix}$, where $\zeta + \zeta^{-1} = \tau_z$. Then

$$(\tau_x, \tau_y, \tau_z) = (\mathrm{tr}(\epsilon_x), \mathrm{tr}(\eta_{y,z}), \mathrm{tr}(\epsilon_x \eta_{y,z}^{-1})).$$

Hence $f = 0$ on \mathbb{C}^3 , $\mathrm{Ker}(\Gamma) = \{0\}$, and Γ is injective. This is the “Fricke slice” given by Goldman in [18].

It remains to show that Γ is surjective. Theorem 4.1 implies that the central functions form a basis for $\mathbb{C}[G \times G]^G$. Since $t_x \mapsto x$, $t_y \mapsto y$, and $t_z \mapsto z$, it suffices to show that every $\chi^{a,b,c}$ may be written as a polynomial in x, y , and z . This was already done via Lemma 5.1, but we provide here a *constructive* proof.

Proceed by induction on the degree $\delta = \frac{1}{2}(a + b + c)$ of a central function $\chi^{a,b,c}$. For the base cases $\delta = 0, 1$ recall our earlier computations demonstrating

$$\chi^{0,0,0} = 1, \chi^{1,0,1} = x, \chi^{0,1,1} = y, \chi^{1,1,0} = z.$$

For $\delta > 0$, we may inductively assume that all central functions with degree less than δ are in $\mathbb{C}[x, y, z]$. The admissibility conditions imply that at least two out of the triple (a, b, c) are positive. Without loss of generality, using Theorem 5.2, we may assume that a and c are positive. In this case, the recurrence given by Corollary 5.7,

$$\begin{aligned} \chi^{a,b,c} &= x \cdot \chi^{a-1,b,c-1} - \frac{(a+b-c)^2}{4a(a-1)} \chi^{a-2,b,c} \\ &\quad - \frac{(-a+b+c)^2}{4c(c-1)} \chi^{a,b,c-2} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)} \chi^{a-2,b,c-2}, \end{aligned}$$

allows us to write $\chi^{a,b,c}$ in terms of central functions of lower degree, which by induction must be in $\mathbb{C}[x, y, z]$. Thus, $\chi^{a,b,c} \in \mathbb{C}[x, y, z]$, and we have established surjectivity. \square

The recursion relations provide an algorithm for writing any $\chi^{a,b,c}$ as a polynomial in $\{x, y, z\}$. Conversely, in [25] the following formula is established, which may be used to express any polynomial in $\mathbb{C}[x, y, z]$ in terms of central functions:

$$\begin{aligned} x^a y^b z^c &= \sum_{\substack{r,s,t=0 \\ k,l,m}}^{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor, \lfloor \frac{c}{2} \rfloor} \left(\binom{a}{r} - \binom{a}{r-1} \right) \left(\binom{b}{s} - \binom{b}{s-1} \right) \left(\binom{c}{t} - \binom{c}{t-1} \right) \\ &\quad \left(\frac{\Delta(l)\Delta(m)\Theta(a-2r,c-2t,k)}{\Delta(k)\Theta(a-2r,b-2s,m)\Theta(b-2s,c-2t,l)} \right) \begin{bmatrix} a-2r & c-2t & k \\ m & l & b-2s \end{bmatrix}^2 \chi^{k,l,m}. \end{aligned}$$

Table 5.5 lists several central functions that were computed with Mathematica using Corollary 5.7. Only one function per triple of indices is listed; the others follow directly from Theorem 5.2.

δ	$\chi^{a,b,c}$	$\chi_{\alpha,\beta,\gamma}$	$\mathbf{p}_{a,b,c}(y, x, z)$
0	$\chi^{0,0,0}$	$\chi_{0,0,0}$	1
1	$\chi^{1,0,1}$	$\chi_{0,1,0}$	x
2	$\chi^{2,0,2}$	$\chi_{0,2,0}$	$x^2 - 1$
	$\chi^{1,1,2}$	$\chi_{1,1,0}$	$xy - \frac{1}{2}z$
3	$\chi^{3,0,3}$	$\chi_{0,3,0}$	$x^3 - 2x$
	$\chi^{2,1,3}$	$\chi_{1,2,0}$	$x^2y - \frac{2}{3}(xz + y)$
	$\chi^{2,2,2}$	$\chi_{1,1,1}$	$xyz - \frac{1}{2}(x^2 + y^2 + z^2) + 1$
4	$\chi^{4,0,4}$	$\chi_{0,4,0}$	$x^4 - 3x^2 + 1$
	$\chi^{3,1,4}$	$\chi_{1,3,0}$	$x^3y - \frac{3}{4}x^2z - \frac{1}{2}(3xy - z)$
	$\chi^{2,2,4}$	$\chi_{2,2,0}$	$x^2y^2 - xyz + \frac{1}{6}z^2 - \frac{1}{2}(x^2 + y^2) + \frac{1}{3}$
	$\chi^{3,2,3}$	$\chi_{1,2,1}$	$x^2yz - \frac{2}{3}(xz^2 + xy^2) - \frac{1}{2}x^3 - \frac{1}{9}(2yz - 13x)$

Table 2. $\mathrm{SL}(2, \mathbb{C})$ -Central Functions.

References

- [1] M. Artin, On Azumaya algebras and finite dimensional representations of rings, *J. Algebra* **11** (1969), pp. 532-536.
- [2] J. Baez, Spin networks in gauge theory, *Adv. Math.* **117** (1996), pp. 253-272.
- [3] T. Bröcker and T. tom Dieck, Representations of Compact Lie Groups, *Grad. Texts in Math.* **No. 98**, Springer-Verlag, New York (1985).
- [4] D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, Understanding the Kauffman Bracket Skein Module, *J. Knot Theory Ramifications* **8** (1999), pp. 265-277.
- [5] J. Carter, D. Flath, and M. Saito, The Classical and Quantum $6j$ -Symbols, *Mathematical Notes* **No. 43**, Princeton University Press, New Jersey (1995).
- [6] M. Culler and P. Shalen, Varieties of group representations and splittings of 3-manifolds, *Ann. of Math. (2)* **117** (1983), no. 1, pp. 109-146.
- [7] R. Carter, G. Segal, and I. MacDonald, Lectures on Lie Groups and Lie Algebras, *Lond. Math. Soc. Student Texts* **No. 32**, Cambridge University Press, Cambridge (1995).
- [8] P. Cvitanović, Group Theory: Birdtracks, Lie's, and Exceptional Groups, Princeton University Press (2008), available online at <http://birdtracks.eu/>.
- [9] P. Cvitanović, H. Elvang, and A. D. Kennedy, Diagrammatic Young Projection Operators for $U(n)$, *J. Math. Phys.* **46** (2005).
- [10] I. Dolgachev, Lectures on Invariant Theory, *Lond. Math. Soc. Lecture Notes Series* **296**, Cambridge University Press (2003).
- [11] V. Drensky, Defining relations for the algebra of invariants of 2×2 matrices, *Algebr. Represent. Theory* **6 no. 2** (2003), pp. 193-214.
- [12] C. Florentino, personal correspondence.
- [13] C. Florentino, J. Mourão, and J. P. Nunes, Theta functions, geometric quantization and unitary Schottky bundles, *The Geometry of Riemann Surfaces and Abelian Varieties, Contemp. Math.* **397** (2006), pp. 55-72.
- [14] R. Fricke and F. Klein, Über die theorie der automorphen modulgruppen, *Kgl. Ges. d. W. Nachrichten*, Math-Phys. Klasse (1896), pp. 91-93.
- [15] W. Fulton, and J. Harris, Representation Theory: A First Course, *Grad. Texts in Math.* **No. 129**, Springer-Verlag, New York (1991).
- [16] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Invent. Math.* **85** (1986), pp. 263-302.
- [17] W. Goldman, Introduction to character varieties, unpublished notes (2003).
- [18] W. Goldman, Trace coordinates on Fricke spaces of some simple hyperbolic surfaces, in *Handbook of Teichmüller theory* (A. Papadopoulos, editor), Volume II, EMS Publishing House, Zürich, 2008, ???.

- [19] F. González-Acuña and J. Montesinos-Amilibia, On the character variety of group representations in $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$, *Math. Z.* **214** (1993), pp. 627-652.
- [20] R. Horowitz, Characters of free groups represented in the two-dimensional special linear group, *Comm. Pure Appl. Math.* **25** (1972), pp. 635-649.
- [21] L. Kauffman, *Knots and Physics*, World Scientific, Singapore (1991).
- [22] W. Magnus, Rings of Fricke characters and automorphism groups of free groups, *Math. Z.* **170** no. 1 (1980), pp. 91-103.
- [23] R. Penrose, Applications of negative dimensional tensors, *Combinatorial Mathematics and its Applications*, Academic Press (1971).
- [24] R. Penrose, letter to P. Cvitanović (1981).
- [25] E. Peterson, Trace Diagrams, Representation Theory, and Low-Dimensional Topology, PhD Thesis, University of Maryland (2006).
- [26] C. Procesi, Invariant theory of $n \times n$ matrices, *Adv. in Math.* **19** (1976), pp. 306-381.
- [27] C. Procesi, Finite dimensional representations of algebras, *Israel J. Math.* **19** (1974), pp. 169-182.
- [28] Y. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), pp. 723-756 (in Russian).
- [29] N. Reshetikhin, personal correspondence.
- [30] A. Sikora, $SL(n)$ -character varieties as spaces of graphs, *Trans. Amer. Math. Soc.* **353** (2001), pp. 2773-2804.
- [31] G. E. Stedman, *Diagram Techniques in Group Theory*, Cambridge University Press (1990).
- [32] M. Vogt, Sur les invariants fondamentaux des equations differentielles linéaires du second ordre, *Ann. Sci. Écol. Norm. Supér. Troi.* **6** (1889).