SOME FORMULAS INVOLVING RAMANUJAN SUMS

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Introduction. The purpose of this note is to establish an identity involving the cyclotomic polynomial and a function of the Ramanujan sums. Some consequences are then derived from this identity.

For the reader desiring a background in cyclotomy, **(2)** is mentioned. Also, **(4)** is intimately connected with the following discussion and should be consulted.

Preliminaries. The cyclotomic polynomial $F_n(x)$ is defined as the monic polynomial whose roots are the primitive nth roots of unity. It is well known that

2.1 $$F_n(x) = \prod_{d\mid n} (x^d - 1)^{\mu(n/d)}.$$

For the proof of Corollary 3.2 it is mentioned that $F_n(0) = 1$ if $n > 1$ and that $F_n(x) > 0$ if $|x| < 1$ and $1 < n$.

The Ramanujan sums are defined by

2.2 $$C_n(k) = \sum_{(r,n)=1} \exp(2\pi irk/n)$$

where the sum is taken over all positive integers r less than or equal to n and relatively prime to n. It is also well known that

2.3 $$C_n(k) = \sum_{d\mid n, d\mid k} d\mu(n/d)$$

where the sum is taken over all positive divisors d common to n and k.

Hereafter p shall denote a prime.

THEOREM 3.1. *If $F_n'(x)$ is the derivative of $F_n(x)$, then*

3.1 $$\sum_{k=1}^{n} C_n(k)x^{k-1} = (x^n - 1)F_n'(x)/F_n(x).$$

Proof. Differentiating equation 2.1 we have

$$F_n'(x)/F_n(x) = \sum_{d\mid n} \mu(n/d)dx^{d-1}/(x^d - 1).$$

Multiplying both sides of this equation by $x^n - 1$ and expressing the right side as a polynomial we have

$$(x^n - 1)F_n'(x)/F_n(x) = \frac{1}{x}\sum_{d\mid n} d\mu(n/d) \sum_{h=0}^{n/d-1} x^{n-hd}.$$

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The coefficient of x^k will consist of all those terms for which $k = n - hd$ where $d|n$ and $0 \leqslant h < n/d$. Since $d|n$, then $d|k$ and as d ranges over all divisors of n only those d will appear which also divide k so that the coefficient of x^k will be simply

$$\sum_{d|k, d|n} d\mu(n/d).$$

But from 2.3 this is $C_n(k)$ and our proof is complete.

COROLLARY 3.1. *If x is an integer, then*

$$x(x^{p-1} - 1)F'_{p-1}(x)/F_{p-1}(x) \equiv \begin{cases} -1 \\ 0 \end{cases} \mod p \tag{3.2}$$

according as x is or is not a primitive root modulo p.

Proof. Vandiver and the author **(3)** have shown the following:
The only incongruent integral roots of the congruence

$$\sum_{t=1}^{p-1} x^t C_{p-1}(t) + 1 \equiv 0 \mod p$$

are the $\phi(p - 1)$ incongruent primitive roots mod p and the only incongruent integral roots of the congruence

$$\sum_{t=1}^{p-1} x^t C_{p-1}(t) \equiv 0 \mod p$$

are the integers in the least positive residue class modulo p which are not primitive roots of p.

This result with Theorem 3.1 finishes our proof.

COROLLARY 3.2. *If $|x| < 1$ and $1 < n$, then*

$$F_n(x) = \exp\left(-\sum_{k=1}^{\infty} C_n(k)x^k/k\right). \tag{3.3}$$

Proof. From equation 3.1 we write

$$-\sum_{k=1}^{n} C_n(k)x^{k-1}/(1 - x^n) = F'_n(x)/F_n(x).$$

If $|x| < 1$ and $1 < n$ expand $x^{k-1}/(1 - x^n)$ into a power series and integrate both sides of the last equation. This yields

$$-\log F_n(x) = \sum_{k=1}^{n} C_n(k) \sum_{h=0}^{\infty} x^{hn+k}/(hn + k).$$

Finally noting that $C_n(k) = C_n(hn + k)$ for $h = 0, 1, 2, \ldots,$ the proof is complete.

O. Hölder **(1)** has shown that the series $\sum_{k=1}^{\infty} C_n(k)x^k/k$ converges for

$x = 1$ and thus 3.3 is true for $x = 1$. Using the fact that $\log F_n(1) = \Lambda(n)$ a proof of Ramanujan's classical result **(5)**

$$\sum_{k=1}^{\infty} C_n(k)/k = -\Lambda(n) \tag{3.4}$$

for $n > 1$ is immediate.

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