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Vector fields tangent to foliations*

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§0. Introduction

For constrained differential equation of the form

$$\text{(0.1)} \qquad \begin{cases} \dot{x} = f(x, y) \\ 0 = g(x, y) \end{cases}$$

$x \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^n$, the equation

$$\text{(0.2)} \qquad \begin{cases} \dot{x} = f(x, y) \\ \varepsilon y = g(x, y) \end{cases}$$

is considered in singular perturbation theory. As an attempt to globalize the local product structure $\boldsymbol{R}^m \times \boldsymbol{R}^n$, Takens [15] considered fiber bundle structures. For the generalization of equation (0.2), it is natural to consider the vector field $X+(1/\varepsilon)Y$. To the equation $0=g(x, y)$ of (0.1), correspond the set, Σ_Y, of the equilibria of Y.

In this paper, we study the generic properties of Y on the neighborhood of Σ_Y. Let $\mathscr{Y}^r(M, \mathscr{F})$ be the space of all C^r vector fields Y tangent to $\mathscr{F}$ with Whitney C^r topology. In section 1, properties G0, G1, and G2 are defined (Definition 1.3), and it is shown that there is an open dense subset of $Y \in \mathscr{Y}^r(M, \mathscr{F})$ which satisfies G0, G1, and G2 (Theorem A).

In section 2, the singularity theory of Thom-Boardman is translated into, so called situation, *jet spaces modulo foliation* $\mathscr{F}$ of mappings from a manifold into M. This jet is defined by contant of mappings modulo leaves (definition 2.1).

In section 3, we show the genericity in $\mathscr{Y}^r(M, \mathscr{F})$ of vector fields Y such that the jet of the injection $\iota: \Sigma_Y \to M$ is transverse to the Thom-Boardman submanifolds with respect to jets modulo $\mathscr{F}$. (Theorem B and Theorem C).

In Definition 1.3 of G2, a stratification $\tilde{S}$ of Σ_Y is defined. Another stratification $\tilde{\tilde{S}}$ of Σ_Y is induced from Thom-Boardam stratification of order two, if Y has the property GB$_2$: the jet of $\Sigma_Y \hookrightarrow M$ is transverse to Tohm-

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Boardman submanifolds of length one and length 2.

Saddle-node bifurcation and Hopf bifurcation are well known as typical codimension one bifurcations of equilibria (e.g. [6]). Theorem D in § 4 shows how these bifurcations arise in our global situation with respect to the stratifications $\mathcal{S}$ and $\tilde{\mathcal{S}}$. $\mathcal{S}$ is defined by using only the first derivatives of Y. But, saddle-node bifurcation does not occur under the condition stated only in terms of the first derivatives. As another condition we take the second derivatives modulo $\mathscr{F}$ of the inclusion map $\Sigma_Y \hookrightarrow M$: while J. Guckenheimer-P. Holmes [6, Theorem 3.4.1] has taken the assumption for the second derivative of the vector field Y. For this purpose, we use the stratification $\tilde{\mathcal{S}}$. In the study of constrained equations or constraint systems, it is natural to consider Thom-Boardman singularities, (e.g. [17], [15], or [8]).

Let Σ_Y^s be the normally stable domain in Σ_Y^h. Theorem E determines the qualitative structure of Y near point $p \in \partial \Sigma_Y^s$ at which Y has a saddle-node bifurcation. Especially, the unstable set $W^u(p)$ of p is the image of an injective immersion of the half line $[0, \infty)$. This property is used in the study of singular perturbations in higher dimensional spaces [8].

§ 1. Generic properties

In this paper M is a smooth (C^∞) manifold with dimension $m+n$, and $\mathscr{F}$ is a smooth foliation on M with codimension m. $\mathscr{F}$ is a disjoint decomposition of M into n dimensional injectively immersed connected smooth submanifolds (leaves) such that M is covered by C^∞ charts (foliation boxes)

$$\alpha_1 \times \alpha_2 \colon U \longrightarrow D^m \times D^n \tag{1.1}$$

and $(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n) \subset$ the leaf through $(\alpha_1 \times \alpha_2)^{-1}(x, y)$, $y \in D^n$, where D^m, D^n are open sets in $\boldsymbol{R}^m$, $\boldsymbol{R}^n$, respectively, and $\alpha_1 \times \alpha_2$ is a smooth diffeomorphism. We call $(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n)$ a *plaque*.

Let $\tau \colon T\mathscr{F} \to M$ be the subbundle of the tangent bundle $TM \to M$ such that the fiber $\tau^{-1}(x)$ is n-dimensional vector space which is tangent to the leaf of $\mathscr{F}$ through x. A natural vector bundle chart on τ is a triple $(\alpha, \alpha_1 \times \alpha_2, U)$ where $(\alpha_1 \times \alpha_2, U)$ is a C^∞ chart on $(M, \mathscr{F})$, $\alpha \colon \tau^{-1}(U) \to (\alpha_1 \times \alpha_2)(U) \times \boldsymbol{R}^n$ is a bijection (C^∞ diffeomorphism), and the diagram

$$\begin{array}{ccc} \tau^{-1}(U) & \xrightarrow{\ \alpha\ } & (\alpha_1 \times \alpha_2)(U) \times \boldsymbol{R}^n \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow \\ U & \xrightarrow{\ \alpha_1 \times \alpha_2\ } & (\alpha_1 \times \alpha_2)(U) \end{array}$$

commutes. Here, the right-hand map is the natural projection. We sometimes denote $\tau^{-1}(x)$ by $T_x\mathscr{F}$.

Let $Y: M \to T\mathscr{F}$ be a C^r section of the vector bundle τ. Y is also a C^r-section of the tangent bundle $TM \to M$. We call such a section a C^r vector field on M *tangent to the foliation* $\mathscr{F}$. Denote by $\mathscr{Y}^r(M, \mathscr{F})$ the space of all C^r vector fields on M tangent to $\mathscr{F}$ with the Whitney C^r topology; if M is compact it is equivalent to the C^r topology. (See e.g. [14].) For the vector bundle chart $(\alpha, \alpha_1 \times \alpha_2, U)$ the local representative of Y

$$\alpha \circ Y \circ (\alpha_1 \times \alpha_2)^{-1}: (\alpha_1 \times \alpha_2)(U) \longrightarrow (\alpha_1 \times \alpha_2)(U) \times \mathbf{R}^n$$

has the form

$$\alpha \circ Y \circ (\alpha_1 \times \alpha_2)^{-1}(x) = (x, Y_\alpha(x))$$

for $x \in (\alpha_1 \times \alpha_2)(U)$. The map $Y_\alpha: (\alpha_1 \times \alpha_2)(U) \to \mathbf{R}^n$ is called the *principal part of the local representative* of Y.

Let Σ_Y be a subset of M such that every $x \in \Sigma_Y$ is an equilibrium point of a vector field $Y \in \mathscr{Y}^r(M, \mathscr{F})$. For $x \in \Sigma_Y$ let

$$(DY_\alpha)(x): \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^n \tag{1.2}$$

be the differential of the principal part of Y_α of the local representative of Y.

Definition 1.1. We say that Y is *regular at* $p \in \Sigma_Y$ if the dimension of the image of $(DY_\alpha)(x)$ is n, where $x = (\alpha_1 \times \alpha_2)(p)$.

Since $(DY_\alpha)(x)$ is linear, this mapping is devided as $(DY_\alpha)(x) = ((DY_\alpha)_1(x), (DY_\alpha)_2(x))$;

$$\begin{aligned} (DY_\alpha)_1(x)&: \mathbf{R}^m \longrightarrow \mathbf{R}^n \\ (DY_\alpha)_2(x)&: \mathbf{R}^n \longrightarrow \mathbf{R}^n \end{aligned} \tag{1.3}$$

Definition 1.2. We say that Y is *normally hyperbolic* at p, if $(DY_\alpha)_2(x)$ has no eigenvalue with real part zero. If all the eigenvalues have negative real part, Y is said to be normally stable at p. (These do not depend on the choice of the vector bundle chart $(\alpha, \alpha_1 \times \alpha_2, U)$.)

A *stratification* $\mathcal{S}$ of a topological space N is a partition of N into subsets, which will be called the *strata* of $\mathcal{S}$, such that the following conditions are satisfied:

(a) Each stratum S is locally closed, i.e. each point $s \in S$ has a neighborhood U such that $U \cap S$ is closed in U.

(b) $\mathcal{S}$ is locally finite.

(c) If S_1 and S_2 are strata and $\bar{S}_1 \cap S_2 \neq \phi$, then $S_2 \subset \bar{S}_1$.

The relation $S_2 < S_1$ defined by $S_2 \subset \bar{S}_1$, $S_2 \neq S_1$, is an order on $\mathcal{S}$. It is transitive and one cannot have both $S_2 < S_1$ and $S_1 < S_2$ ([12, p. 200]).

Let $\tilde{N}$ be a C^1 manifold, let $N \subset \tilde{N}$, and let $\mathcal{S}$ be a stratification of N.

We will say that $\mathcal{S}$ is a *Whitney stratification* if each stratum is a C^1 submanifold, and if S_1, S_2 are two strata with $S_2 < S_1$, then for all $x \in S_2$ the triple (S_1, S_2, x) satisfies the following Whitney's regularity condition (b).

Condition (b): For any sequences $\{x_i\}$ of points in S_2 and $\{y_i\}$ of points in S_1, such that $x_i \to x$, $y_i \to x$, $x_i \neq y_i$, segment $\overline{x_i y_i}$ converges (in projective space), and the tangent space $T_{x_i}S^1$ converges (in Grassmanian of $(\dim S_1)$-plane in $\boldsymbol{R}^n$, $n = \dim N$), we have $l \subset T_\infty$, where $l = \lim \overline{x_i y_i}$ and $T_\infty = \lim T_{x_i}S^1$.

Condition (b) implies the following condition (a), (see e.g. Wall [16] or Mather [12, p. 203]).

Condition (a): If x_i is a sequence of points in S_1 such that $x_i \to x \in S_2$ and $T_{x_i}S_1$ converges to T_∞, then $T_x S_2 \subset T_\infty$.

Let $\mathcal{S}^i$ denote the substratification of stratification $\mathcal{S}$ such that $\mathcal{S}^i$ consists of all strata of dimension $\leqq i$ of $\mathcal{S}$. $\mathcal{S}^i$ is called the *i-skeleton* (or *codim* $(n-i)$ *skeleton*) of $\mathcal{S}$. Here, $n = \max\{\dim S : S \in \mathcal{S}\}$. Let Σ_Y^h be the set of all points of Σ_Y, where Y is normally hyperbolic. Let $\partial\Sigma_Y^h$ be the set of all frontiers of Σ_Y^h; $\partial\Sigma_Y^h = \overline{\Sigma_Y^h} - \Sigma_Y^h$.

DEFINITION 1.3. What follows are the properties of the vector field $Y \in \mathcal{Y}^r(M, \mathcal{F})$.

G0: The set Σ_Y of all equilibrium points of Y is, if nonempty, an m dimensional C^r manifold.

G1: Y is regular at every equilibrium point of Y.

G2: Y has the property G0 and there is a Whitney stratification $\mathcal{S}$ on Σ_Y having the following properties:

(i) If the differential $(DY_\alpha)_2(x)$ at $x = (\alpha_1 \times \alpha_2)(p)$ (see Definition 1.2) has l eigenvalues of zero and $2(k-l)$ non-zero pure imaginary eigenvalues

$$0, \cdots, 0, ib_1, -ib_1, \cdots, ib_{k-l}, -ib_{k-l},$$

then the point p is contained in the $(m-k)$ skeleton $\mathcal{S}^{m-k}$.

(ii) The union of all $(m-1)$ dimensional strata $\cup S^{m-1}$ is a dense subset of $\partial\Sigma_Y^h$.

(iii) $\cup S^{m-1}$ is divided into two parts, $(\partial\Sigma_Y^h)_0$ and $(\partial\Sigma_Y^h)_{\mathrm{img}}$, of unions of strata such that

$p \in (\partial\Sigma_Y^h)_0 \Rightarrow 0$ is an eigenvalue of $(DY_\alpha)_2(x)$,

$p \in (\partial\Sigma_Y^h)_{\mathrm{img}} \Rightarrow$ the eigenvalues of $(DY_\alpha)_2(x)$ include a pair of non-zero pure imaginary numbers.

THEOREM A. *Let M be a smooth manifold of dimension $m+n$ and $\mathcal{F}$ be a smooth foliation on M with codimension m. Let $\mathcal{Y}^r(M, \mathcal{F})$ be the space of all C^r vector fields on M tangent to $\mathcal{F}$. Let $\mathcal{Y}_j^r$ denote the set of all $Y \in \mathcal{Y}^r(M, \mathcal{F})$ satisfying the property Gj, $j = 0, 1, 2$, respectively. Then,*

(i) $\mathscr{Y}_0^r$ *is open dense in* $\mathscr{Y}^r(M, \mathscr{F})$, *if* $1 \leqq r \leqq \infty$.
(ii) $\mathscr{Y}_1^r$ *is open dense in* $\mathscr{Y}^r(M, \mathscr{F})$, *if* $2 \leqq r \leqq \infty$.
(iii) $\mathscr{Y}_2^r$ *is open dense in* $\mathscr{Y}^r(M, \mathscr{F})$, *if* $3 \leqq r \leqq \infty$.

Denote by $\Gamma^r(\pi)$ the space of all C^r sections of a vector bundle π with the C^r topology. $\Gamma^r(\pi)$ is a separable Banach space. Especially for the bundle $\tau\colon T\mathscr{F} \to M$ the space $\Gamma^r(\tau)$ has been denoted by $\mathscr{Y}^r(M, \mathscr{F})$.

Let $\xi_1, \xi_2\colon U \to T_U\mathscr{F}$ be partial sections of τ where $T_U\mathscr{F} = \tau^{-1}(U) \subset T\mathscr{F}$. Let $(\alpha, \alpha_1 \times \alpha_2, U)$ be a vector bundle chart on τ and $x_1, x_2 \in U$. Let $\xi_{1\alpha}, \xi_{2\alpha}\colon \alpha(U) \to \boldsymbol{R}^n$ be the principal parts of local representatives of ξ_1, ξ_2 respectively. We denote $(\xi_1, x_1) \sim_0 (\xi_2, x_2)$ if $x_1 = x_2$ and $\xi_{1\alpha}(x_1) = \xi_{2\alpha}(x_2)$. We denote $(\xi_1, x_1) \sim_1 (\xi_2, x_2)$ if $x_1 = x_2$, $\xi_{1\alpha}(x_1) = \xi_{2\alpha}(x_2)$, and $D\xi_{1\alpha}(x_1) = D\xi_{2\alpha}(x_2)$. Here $D\xi_{i\alpha}$ is the derivative of $\xi_{i\alpha}$ which is a mapping $(\alpha_1 \times \alpha_2)(U) \to L(\boldsymbol{R}^{m+n}, \boldsymbol{R}^n)$, where $L(\boldsymbol{R}^{m+n}, \boldsymbol{R}^n)$ is the set of all linear mappings $\boldsymbol{R}^{m+n} \to \boldsymbol{R}^n$. This definition of $\sim_0$ and $\sim_1$ is independent of the choice of vector bundle chart $(\alpha, \alpha_1 \times \alpha_2, U)$. $\sim_0$ and $\sim_1$ are equivalent relations. The equivalence classes of the pair (ξ, x) are denoted by $j^0\xi(x)$ and $j^1\xi(x)$, respectively. Let $J^0(\tau)$ and $J^1(\tau)$ be the sets of all $j^0\xi(x)$ and $j^1\xi(x)$, respectively. For each C^r section $Y\colon M \to T\mathscr{F}$ the map $j^iY\colon M \to J^i(\tau)$ given by $x \to j^iY(x)$ is called the *i-jet extension* (or *i-jet section*) of Y, $i = 0, 1$. The map

$$\tau^i\colon J^i(\tau) \longrightarrow M$$

given by $\tau^i(j^i\xi(x)) = x$ is a C^∞ vector bundle, which is called the *i-jet bundle of sections of* τ, $i = 0, 1$. For each vector bundle chart $(\alpha, \alpha_1 \times \alpha_2, U)$ on τ the natural *i-jet* chart on τ^i, $i = 0, 1$, is given by

$$\text{(1.4)} \qquad \begin{cases} \alpha^0(j^0\xi(x)) = (y, \xi_\alpha(y)) \in (D^m \times D^n) \times \boldsymbol{R}^n, \\ \alpha^1(j^1\xi(x)) = (y, \xi_\alpha(y), D\xi_\alpha(y)) \\ \qquad\qquad \in (D^m \times D^n) \times \boldsymbol{R}^n \times L(\boldsymbol{R}^m \times \boldsymbol{R}^n, \boldsymbol{R}^n), \end{cases}$$

where $y = (\alpha_1 \times \alpha_2)(x)$.

Let

$$\text{(1.5)} \qquad \tau_0^1\colon J^1(\tau) \longrightarrow J^0(\tau)$$

be a mapping defined by $\tau_0^1(j^1\xi(x)) = j^0\xi(x)$ for every $j^1\xi(x) \in J^1(\tau)$. Then $\tau^0 \circ \tau_0^1 = \tau^1\colon J^1(\tau) \to M$ and τ_0^1 are vector bundle projections.

Lemma 1.4 (Abraham-Robbin [1, Theorem 12, 4]). *Suppose M is compact, $r \geqq 1$, and $i = 0, 1$. Let*

$$\mathrm{ev}_i\colon \Gamma^r(\tau) \times M \longrightarrow J^i(\tau)$$

be given by $\mathrm{ev}_i(Y, x) = j^i Y(x)$. *Then,* (i) ev_i *is of class* C^{r-i}; *and* (ii) ev_i *is a submersion, if* $r-i>0$.

LEMMA 1.5. *Let* $q=\min\{m, n\}$ *and* $L_r \subset L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *be the set of all linear mappings with rank* r, $r=0, \cdots, q$. *Then,* L_r *is a submanifold of* $L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *with codimension* $(m-r)(n-r)$, *and the subdivision* $\{L_r\}$ *of* $L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *induces a Whitney stratification of* $L(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *such that each point of* L_r *is a frontier of every* $L_{r+1}, \cdots, L_q$.

Proof is given in the same manner as that of Golubitsky-Guillemin [5, p. 60 Proposition 5.3].

Let $R(k)$ be the set of all elements $(c_1, \cdots, c_n) \in \boldsymbol{C}^n$ such that (i) there are h factors of zeros; $c_{j_1} = \cdots = c_{j_h} = 0$, $0 \leqq h \leqq k$; and (ii) there are $2(k-h)$ factors of non-zero pure imaginary numbers;

$$\pm ib_{h+1}, \cdots, \pm ib_k \qquad (b_{h+1}, \cdots, b_k \in \boldsymbol{R}).$$

LEMMA 1.6. *Let* K *be the set of all elements of* $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ *having at least one eigenvalue with real part zero. Then,* K *is a closed semialgebraic set. Furthermore, there is a Whitney stratification* $\mathscr{K}$ *of* K *satisfying the following*:

(i) *If the set of eigenvalues* $(\lambda_1, \cdots, \lambda_n)$ *of* $A \in L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ *is contained in* $R(k)$, *then* A *is contained in the codim* $(k-1)$ *skeleton,* $\mathscr{K}^{k-1}$, *of* $\mathscr{K}$.

(ii) *The union* $\cup S^{\max}$ *of all strata with maximal dimension is dense in* K.

(iii) $\cup S^{\max}$ *is divided into two parts,* $\cup S_0$ *and* $\cup S_{\mathrm{img}}$, *consisting of unions of strata denoted by* S_0 *and* S_{img} *where*

$A \in S_0 \Rightarrow 0$ *is an eigenvalue of* A,

$A \in S_{\mathrm{img}} \Rightarrow$ *the eigenvalues of* A *includes a pair of non-zero pure imaginary numbers.*

PROOF. By Abraham-Robbin [1, § 30], K is a closed semialgebraic set, and is a union of submanifolds of $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ with codimension $\geqq 1$. Let ν: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ be the Newton map. It is an algebraic map defined by $\nu(c_1, \cdots, c_n) = (a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ are the coefficients of the unique monic polynomial $a_1 + a_2 z + \cdots + a_n z^{n-1} + z^n$ whose roots are $c_1, \cdots, c_n$. Let $\gamma: L(\boldsymbol{R}^n, \boldsymbol{R}^n) \to \boldsymbol{R}^n$ be the map which assigns to a linear map A the coefficients of the characteristic polynomial of A, i.e. $\gamma(A) = (a_1, \cdots, a_n)$ where

$$\det(Ix - A) = a_1 + a_2 x + \cdots + a_n x^{n-1} + x^n.$$

Let ι: $\boldsymbol{R}^n \to \boldsymbol{C}^n$ be the embedding obtained by

$$\iota(a_1, \cdots, a_n) = (a_1 + i0, \cdots, a_n + i0) \in \boldsymbol{C}^n.$$

Since $R(h)$ is a semialgebraic set, by Tarski-Seidenberg theorem $H_h=(\iota\Upsilon)^{-1}\nu(R(h))$ is a closed semialgebraic set in $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$, and thus, H_h is a finite union of submanifolds (see Abraham-Robbin [1, § 30]). Hence, we have a sequence of semialgebraic sets $K=H_1\supset H_2\supset\cdots\supset H_n$. Since, by an arbitrarily small perturbation, any linear map in H_h can be changed into one contained in H_{h-1} $(h>0)$, H_h must have local codimension $\geqq 1$ in H_{h-1}; i.e. if a point $x\in H_h$ is contained in a submanifold $S_h\subset H_h$, then x is contained in the frontier of some submanifold S_{h-1} in H_{h-1} with $\dim S_h<\dim S_{h-1}$.

Next, we construct a Whitney stratification of K similar to that given by Mather [12, Theorem (4.9)]. (He uses the word "prestratification" meaning our stratification.) Let $m=\dim K$, i.e. the maximal dimension of manifolds which constitute K is m. We construct, by decreasing induction, a sequence $K_m, K_{m-1}, \cdots$ of semialgebraic subset of K, closed in K, where $\dim K_k\leqq k$, such that K_k-K_{k-1} is an algebraic manifold and that $K_k\supset H_{m-k+1}$. Here, we recall that $H_j=0$ if $j>n$.

We begin with $K_m=K=H_1$. We suppose inductively that K_k has been constructed. Let K_{k-1} be the closure in K of the set of points x in K_k such that one of the following conditions holds:

(0) $x\in H_{m-k+2}$.

(1) x is not a regular point of K_k or the local dimension of K_k at x is smaller than k. (A *regular point* of a subset Z of an algebraic manifold N is a point which has a neighborhood N such that $N\cap Z$ is a closed algebraic submanifold of N.)

(2) x is a regular point of K_k and the local dimension of K_k at x is k, but there exists $l>k$ scuh that the triple $(K_l-K_{l-1}, K_{k,\mathrm{reg}}, x)$ does not satisfy Whitney's regularity condition (b).

As mentioned before, H_{m-k+2} is a finite union of submanifolds. Moreover the dimensions of these submanifolds are $\leqq(k-1)$, since $\dim H_1\leqq m$ and the local codimension of H_k in H_{k-1} is $\geqq 1$, (see the following diagram).

$$\begin{array}{ccccccccc} K= & K_m & \supset K_{m-1} & \supset\cdots\supset & K_k & & \supset K_{k-1} & \supset\cdots \\ & \| & \cup & & \cup & & \cup & \\ & H_1 & \supset H_2 & \supset\cdots\supset & H_{m-k+1} & & \supset H_{m-k+2} & \supset\cdots \end{array}$$

From (1), it follows that K_k-K_{k-1} is an algebraic manifold and $\dim(K_k-K_{k-1})=k$ everywhere. Each of the sets defined by one of the conditions (i) or (ii) is semialgebraic and its dimension is $\leqq k-1$ ([12, Proof of Theorem (4.9)]). Since $\dim H_{m-k+2}\leqq k-1$, it follows that K_{k-1} is a semialgebraic set with dimension $\leqq k-1$.

Let $\mathscr{K}$ denote the collection of connected components of the K_k-K_{k-1}, $k=0,\cdots,m$. By Mather [12, Proof of Theorem (4.9) and Addendum (4.10)], $\mathscr{K}$ is a Whitney stratification of K satisfying (i).

Next, we show that $\mathscr{K}$ satisfies (ii). Suppose that there exists a stratum S of $\mathscr{K}$ such that there is no stratum T of $\mathscr{K}$ satisfying $S \subset \bar{T}$ and such that $\dim S < m$. Since $\operatorname{codim} K \leqq 1$ in $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$, then $\operatorname{codim} S \leqq 2$ in $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$. Since $K = H_1$ and $H_{k-1} - H_k$ is dense in H_{k-1} for all $k \geqq 1$, it follows that any element of S is approximated by $A \in S \cap (H_1 - H_2)$. One of the following holds for such A.

(a) One eigenvalue of A is zero and the real part of every other eigenvalue is non-zero.

(b) Two eigenvalues of A are ib and $-ib$ $(0 \neq b \in \boldsymbol{R})$ and the real part of every other eigenvalue of A is non-zero.

Element A satisfies (a) if and only if $\operatorname{corank} A = 1$. By Lemma 1.5, the set of all such elements is a codimension one submanifold L_{n-1} of $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$. This contradicts the assumption that codimension of S is $\leqq 2$.

In case of (b), let $\lambda_1 = ib$, $\lambda_2 = -ib$, and $\lambda_3, \cdots, \lambda_n$ be all the other eigenvalues of A. We take disjoint open sets N_1, N_2, N_+, and N_- in the plane $\boldsymbol{C}$ such that N_1 and N_2 do not intersect with the real line, N_+ and N_- do not intersect with the pure imaginary line, and such that

$$\lambda_1 \in N_1, \qquad \lambda_2 \in N_2,$$
$$\{\lambda_3, \cdots, \lambda_n\} \subset N_+ \cup N_-.$$

Since the characteristic polynomial $\det(Ix - A)$ of A is holomorphic, there is a neighborhood U of A in $L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ such that, if $B \in U$, then $\lambda_1(B) \in N_1$, $\lambda_2(B) \in N_2$, and $\lambda_3(B), \cdots, \lambda_n(B) \in N_+ \cup N_-$, where $\lambda_1(B), \cdots, \lambda_n(B)$ denote the eigenvalues of B. Let $\mathscr{R}\lambda$ be the real part of λ. There are A_0 and A_1 in U such that

$$\lambda_1(A_0) = \overline{\lambda_2(A_0)}, \qquad \lambda_1(A_1) = \overline{\lambda_2(A_1)},$$
$$\mathscr{R}(\lambda_1(A_0)) = \mathscr{R}(\lambda_2(A_0)) > 0,$$

and

$$\mathscr{R}(\lambda_1(A_1)) = \mathscr{R}(\lambda_2(A_1)) < 0.$$

Since $\operatorname{codim} S \leqq 2$ in U, there is an arc A_γ, $0 \leqq \gamma \leqq 1$, connecting A_0 and A_1 in $U - S = U - K$. By the assumptions for N_1, N_2, N_+, N_- and U, it holds that the arc $\lambda_1(A_\gamma)$, $0 \leqq \gamma \leqq 1$, is included in N_1. Since $\mathscr{R}(\lambda_1(A_0)) > 0$ and $\mathscr{R}(\lambda_1(A_1)) < 0$, there is c such that $\mathscr{R}(\lambda_1(A_c)) = 0$. This contradicts the assumption that $A_\gamma \notin K$ for each γ. Therefore $\mathscr{K}$ satisfies (ii).

Finally, we show that $\mathscr{K}$ satisfies (iii). By the above, it is clear that the subset K_a (K_b, resp.) of all the elements in $K - K_{m-1}$ satisfying the condition (a) ((b), respectively) is open in $K - K_{m-1}$. Hence, for an m-dimensional stratum S, both $K_a \cap S$ and $K_b \cap S$ are open in S. Assuming S connected, we have $S=K_a \cap S=S_0$ or $S=K_b \cap S=S_{\text{img}}$. □

For an element $C \in L(\boldsymbol{R}^m \times \boldsymbol{R}^n, \boldsymbol{R}^n)$, let $C=(A, B) \in L(\boldsymbol{R}^m, \boldsymbol{R}^n) \times L(\boldsymbol{R}^n, \boldsymbol{R}^n)$.

DEFINITION 1.7. We define subsets $E \subset J^0(\tau)$, $V \subset J^1(\tau)$, and $W \subset J^1(\tau)$ as follows: In some (and hence every) natural vector bundle chart $(\alpha, \alpha_1 \times \alpha_2, U)$ with $\tau^i(\sigma) \in U$ $(i=0$ or $1)$,

$\sigma \in E \Longleftrightarrow \alpha^0(\sigma)=(x, 0) \in (D^m \times D^n) \times \boldsymbol{R}^n$,

$\sigma \in V \Longleftrightarrow \alpha^1(\sigma)=(x, 0, C)$ and the rank of $C \in L(\boldsymbol{R}^m \times \boldsymbol{R}^n, \boldsymbol{R}^n) \leqq n-1$,

$\sigma \in W \Longleftrightarrow \alpha^1(\sigma)=(x, 0, (A, B))$ and $B \in L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ has at least one eigenvalue with real part zero.

LEMMA 1.8. (i) *E is a closed submanifold of $J^0(\tau)$ of codimension n.*

(ii) *V is a closed subset of $J^1(\tau)$. Furthermore, $V=V_0 \cup V_1 \cup \cdots \cup V_{n-1}$ where $V_0, \cdots, V_{n-1}$ are submanifolds of $J^1(\tau)$ of codimension $\geqq m+n+1$.*

(iii) *W is a closed subset of $J^1(\tau)$. Furthermore, there is a Whitney stratification $\mathscr{W}$ of W such that* (a) *the union $\cup S^{\max}$ of all strata with maximal dimension is a dense subset of W;* (b) *if $\alpha^1(\sigma)=(x, 0, (A, B))$ and the set of eigenvalues of B is contained in $R(k)$, then σ is contained in the codim $(k-1)$-skeleton; and* (c) *$\cup S^{\max}$ is divided into two parts, $\cup S_0$ and $\cup S_{\text{img}}$ of unions of strata such that, for $\alpha'(\sigma)=(x, 0, (A, B))$*

$\sigma \in S_0 \Longrightarrow 0$ is an eigenvalue of B

$\sigma \in S_{\text{img}} \Longrightarrow$ the eigenvalues of B includes a pair of non-zero pure imaginary numbers.

PROOF. (i) is trivial. (ii) and (iii) are obtained from Lemma 1.5 and Lemma 1.6, respectively, by choosing a vector bundle atlas on τ^1. □

For a section $Y \in \mathscr{Y}^r(M, \mathscr{F})=\Gamma^r(\tau)$, define a map $\rho_Y^i\colon M \to J^i(\tau)$, $i=0, 1$, by $\rho_Y^i(x)=j^i Y(x)$, $x \in M$. Then, the map

$$\rho^i\colon \Gamma^r(\tau) \longrightarrow \Gamma^{r-i}(\tau^i)$$

given by $Y \mapsto \rho_Y^i$ for $Y \in \Gamma^r(\tau)$ is a C^{r-i} representation of mappings by Lemma 1.4 (i).

LEMMA 1.9. (i) *If ρ_Y^0 is transverse to E in $J^0(\tau)$ then condition* G0 *is satisfied.*

(ii) *If ρ_Y^1 is transverse to $V=V_0 \cup \cdots \cup V_{n-1} \subset J^1(\tau)$ (i.e. $\rho_Y^1 \pitchfork V_r$ for $r=0, \cdots, n-1$), then condition* G1 *is satisfied.*

(iii) *If ρ_Y^1 is of calss C^2 and transverse to the stratification $\mathscr{W}$ (i.e. transverse to each stratum of $\mathscr{W}$), then condition* G2 *is satisfied.*

PROOF. (i) If C^r map ρ_Y^0 is transverse to E, in symbol: $\rho_Y^0 \pitchfork E$, then $(\rho_Y^0)^{-1}(E)=\Sigma_Y$ is a C^r submanifold of M and the codimension of Σ_Y is $(\dim M)+(\dim E)-(\dim J^0(\tau))=m$, by [1, Corollary of 17.2]. Hence, G0 is satisfied.

(ii) Since $\operatorname{codim} V_r \geqq m+n+1$ and $\operatorname{codim}(\rho_Y^1)^{-1}(V_r)\geqq m+n+1>\dim M$, it follows that $\rho_Y^1(M)\cap V_r=\phi$ if and only if $\rho_Y^1 \pitchfork V_r$. Hence, G1 is satisfied.

(iii) Let ρ_Y^1 be of class C^2 and $\rho_Y^1 \pitchfork \mathscr{W}$. By Mather [11, Corollary (8.8) and the proof], $(\rho_Y^1)^*\mathscr{W}=\mathcal{S}$ is a Whitney stratification of $(\rho_Y^1)^{-1}(W)\subset\Sigma_Y$. Here, $(\rho_Y^1)^*\mathscr{W}$ is the stratification which consists of strata $\{(\rho_Y^1)^{-1}(W_i)\}$ for every strata $\{W_i\}$ of $\mathscr{W}$. Then, by Lemma 1.8, $\mathcal{S}$ is the desired stratification in G2. □

PROOF OF THEOREM A. (i) Put $\mathscr{Y}_E=\{Y\in\Gamma^r(\tau);\ \rho_Y^0 \pitchfork E\}$. We only need, by Lemma 1.9, to show $\mathscr{Y}_E$ is open dense in $\Gamma^r(\tau)$. Since $\rho^0\colon \Gamma^r(\tau)\to\Gamma^r(\tau^0)$ is a C^r representation of mappings and E is a closed manifold in $J^0(\tau)$, the openness of $\mathscr{Y}_E$ is obtained by Abraham-Robbin [1, Theorem 18.2]. To prove the density of $\mathscr{Y}_E$, we consider the evaluation map

$$\mathrm{ev}_0\colon \Gamma^r(\tau)\times M \longrightarrow J^0(\tau)$$

defined by $(Y,x)\mapsto j^0Y(x)$. It is of class C^r and transverse to any submanifold of $J^0(t)$, and hence to E, by Lemma 1.4. By [1, Theorem 18.2], $\mathscr{Y}_E$ is residual if $r>\max(0,\dim M-q)$ where $q=$(codimension of E in $J^0(\tau)$). Since $q=n$ in our case, $\mathscr{Y}_E$ is residual if $r\geqq m$. Even if $1\leqq r<m+1$ we can prove the density of $\mathscr{Y}_E$ by the same argument of [1. p. 98, Proof of 30.1] using the densities of $\mathscr{Y}_E$ in $\Gamma^{m+1}(\tau)$ and $\Gamma^{m+1}(\tau)$ in $\Gamma^r(\tau)$.

(ii) We recall that $V=V_0\cup\cdots\cup V_{n-1}$ by Lemma 1.8. Put $\mathscr{Y}_V=\{Y\in\Gamma^r(\pi);\ \rho_Y^1 \pitchfork V_j,\ j=0,\cdots,n-1\}$. We only need, by Lemma 1.9, to show $\mathscr{Y}_V$ is open dense in $\Gamma^r(\tau)$. The evaluation map $\mathrm{ev}_1\colon\Gamma^r(\tau)\times M\to J^1(\tau)$ is C^{r-1} and transversal to any submanifold of $J^1(\tau)$. We have $\operatorname{codim} V_0\geqq m+n+1$ in $J^1(\tau)$. Hence, we can show the density of $\mathscr{Y}_V$ in $J^r(\tau)$ as above if $r-1>\max(0,m+n-\operatorname{codim} V)$, i.e. $r\geqq 2$. Since V is closed and $j_Y^1(M)\cap V=\phi$ if and only if $j_Y^1(M)\pitchfork V$, then the openness of $\mathscr{Y}_V$ holds by [1, Theorem 18.1].

(iii) Let $\mathscr{W}$ be the Whitney stratification on W obtained by Lemma 1.8 (iii). Put $\mathscr{Y}_W=\{Y\in\mathscr{Y}^r(M;\mathscr{F})\colon \rho_Y^1 \pitchfork \mathscr{W}\}$. Since $r\geqq 3$ is assumed, we have $\mathscr{Y}_W\subset\mathscr{Y}_2$ by Lemma 1.9, (iii). Now we prove that $\mathscr{Y}_W$ is open dense in $\mathscr{Y}^r$. Since the strata of $\mathscr{W}$ are submanifolds of $J^1(\tau)$ with codimension $\geqq(\operatorname{codim} E)+1=n+1$ we can show as above that $\mathscr{Y}_W$ is residual in $\mathscr{Y}^r$ if $r-1>\max(0,m-1)$, i.e. $r\geqq m+1$. Even if $3\leqq r\leqq m$ $\mathscr{Y}_W$ is residual as above.

To show the openness of $\mathscr{Y}_W$, let $Y\in\mathscr{Y}_W$. For $x\in M$, either of the following is satisfied.

(1) $j^1Y(x)\notin W$,

(2) $j^1Y(x)\in W$ and $\rho_Y^1 \pitchfork W_i$ at $\rho=j^1Y(x)$, where W_i is the stratum of $\mathscr{W}$

containing σ. Let Y' be an element of $\mathscr{Y}^r$ which is sufficiently close to Y. If case (1) is satisfied then $j^1Y'(x) \notin W$, since W is a closed subset of $\mathscr{Y}^r$ by Lemma 1.8 (iii). In case (2), let W_2 be the stratum of $\mathscr{W}$ containing σ. Let $j^1Y'(x)=\sigma'$. Then we may assume that $\sigma' \in W_2$ or $\sigma' \in W_1$ where W_1 is a stratum such that $W_2 < W_1$. If $\sigma' \in W_2$, then $\rho^1_{Y'} \pitchfork W_2$ at σ' since W_2 is a manifold and Y' is sufficiently C^r close to Y. In case of $\sigma' \in W_1$, let $\sigma \in W_2 \subset \overline{W}_1$. For an open neighborhood U of σ in $\overline{W}_1$ let

$$TU = \{T_{\sigma'}W_1 : \sigma' \in U\}.$$

T_U is a subset of Grassmanian of (dim W_2)-planes. If $U_1 \supset U_2 \supset \cdots \supset U_i \supset \cdots \ni \sigma$ is a sequence converging to σ, then $TU_1 \supset TU_2 \supset \cdots \supset TU_i \supset \cdots$. The set $\bigcap_i TU_i$ is nonempty since for a sequence $\{\sigma_i\}_{i=1,2,\dots}$ with $\sigma_i \in U_i$ (hence $\sigma_i \to \sigma$) there is a subsequence $\{\sigma_j\}$ such that $\{T_{\sigma_j}W_i\}$ converges to a plane. Let T be any element in $\bigcap_i TU_i$. By Whitney's Condition (a) the plane T includes the tangent plane $T_\sigma W_2$ of W_2 at σ. It follows that, if $\rho^1_Y \pitchfork W_2$ at σ and Y' is sufficiently close to Y, then $\rho^1_Y \pitchfork W_1$ at $\sigma' = j^1Y'(x)$, where W_1 is the stratum containing σ'. Therefore, if $j^1Y(x)$ satisfies (1) or (2) then $j^1Y'(x)$ does also.

By the well known argument (e.g. [1, 18.2]), the openness of $\mathscr{Y}_W$ is shown if M is compact. We can extend this to noncompact case by the argument of Peixoto [14, § 5].

§ 2. Thom-Boardman singularities modulo foliation

In this section, we will define a jet space modulo foliation. After this we will explain Thom-Boardman's singularities by the translation into our jet spaces modulo foliations.

Definition 2.1. Let $\mathscr{F}$ be a smooth foliation on M. Suppose $f, g: L \to M$ are C^k maps with $f(p)=g(p)=q$. f is said to have *kth order contact modulo* $\mathscr{F}$ with g at p if, for some (and hence for any) chart $(U, \alpha_1 \times \alpha_2)$ of $\mathscr{F}$ with $q \in U$ given by (1,1), $\alpha_1 \circ f: L \to D^m$ has *kth* order contact with $\alpha_1 \circ g$ at p. This is written as $f \sim_k g \bmod \mathscr{F}$ at p. Let $J^k(L, M; \mathscr{F})_{p,q}$, $k \geqq 1$, denote the set of equivalence classes under "$\sim_k \bmod \mathscr{F}$ at p" of mappings $f: L \to M$ where $f(p)=q$. Let

$$J^0(L, M; \mathscr{F})_{p,q} = J^0(L, M)_{p,q} = \{(p, q)\}.$$

Let

$$J^k(L, M; \mathscr{F}) = \bigcup_{(p,q)\in L\times M} J^k(L, M; \mathscr{F})_{p,q}$$

(disjoint union). We call $J^k(L, M; \mathscr{F})$ a *jet space modulo* $\mathscr{F}$. An element σ

in $J^k(L, M; \mathscr{F})$ is called *a k-jet modulo $\mathscr{F}$* of mapping from L to M. Let σ be a k-jet modulo $\mathscr{F}$, then $\sigma \in J^k(L, M; \mathscr{F})_{p,q}$ for a $(p, q) \in L \times M$.

For manifolds L, M and a foliation $\mathscr{F}$ on M, $J^k(L, M; \mathscr{F})$ has a smooth manifold structure. Moreover, *the mapping*

$$\pi_0^k\colon J^k(L, M; \mathscr{F}) \longrightarrow L \times M = J^0(L, M; \mathscr{F}) \tag{2.1}$$

defined by $\pi_0^k(\sigma) = $(source of σ, target of σ) *is a smooth fiber bundle.*

Let $\pi_L\colon L \times M \to L$ and $\pi_M\colon L \times M \to M$ be natural projections. Then $\pi_L^k = \pi_L \circ \pi_0^k$ and $\pi_M^k = \pi_M \circ \pi_0^k$ are bundle projections;

$$\begin{aligned} \pi_L^k\colon J^k(L, M; \mathscr{F}) &\longrightarrow L \\ \pi_M^k\colon J^k(L, M; \mathscr{F}) &\longrightarrow M. \end{aligned}$$

If $k \geqq h$, we have the canonical bundle projection

$$\pi_h^k\colon J^k(L, M; \mathscr{F}) \longrightarrow J^h(L, M; \mathscr{F}) \tag{2.2}$$

by restricting the order of jets.

The bundle atlas for a jet space modulo foliation is essentially same as the usual jet space of mappings (see [5]). In fact, a jet space modulo foliation is locally same as a usual jet space of mappings in the following sense. Let $J^k(U, V, \mathscr{F}_V)$ be a local subbundle of $J^k(L, M; \mathscr{F})$ and $\beta\colon V \to D^m \times D^n$ a foliation chart. For a point $(x, y) \in D^m \times D^n$, let $v^m = \beta^{-1}(D^m \times \{y\})$ and $V^n = \beta^{-1}(\{x\} \times D^n)$. V^n is a plaque of $\mathscr{F}$ and $V = V^m \times V^n$. Recall that $J^k(L, N) = \bigcup_{(x,y) \in L \times N} J^k(L, N)_{x,y}$ and $J^k(L, M; \mathscr{F}) = \bigcup_{(x,y) \in L \times M} J^k(L, M; \mathscr{F})_{x,y}$. There is a bijection, $J^k(U, V; \mathscr{F}_V)_{x,y} \approx J^k(U, V^m)_{x,p(y)}$. Here, the latter jet space is the space of jet of the mappings $U \to V$ composed by the projection $p\colon V = V^m \times V^n \to V^m$. Furthermore, this is a bundle isomorphism between the following bundles

$$\begin{cases} \pi_U^k\colon J^k(U, V; \mathscr{F}_V) \longrightarrow U, \\ \pi_U^k\colon J^k(U, V^m) \times V^n \longrightarrow U \text{(naturally defined)} \end{cases}$$

or

$$\begin{cases} \pi_h^k\colon J^k(U, V; \mathscr{F}_V) \longrightarrow J^h(U, V; \mathscr{F}_V) \\ \pi_h^k\colon J^k(U, V^m) \times V^n \longrightarrow J^h(U, V^m) \times V^n. \end{cases}$$

This remark indicates that our *jet spaces modulo foliations follow the J. M. Boardman's theory* [2] on the usual jet spaces of mappings, because the essential part of [2] is the discussion on the local jet bundles. [2] is a generalization of Levine [10].

We will translate the definitions and the main theorems of [2] into our situation.

Let π_h^k be the bundle projection of (2.2). We have the inverse limit of the finite jet spaces

$$J(L, M; \mathscr{F}) = \varprojlim J^k(L, M; \mathscr{F})$$

and the projection

$$\pi_h\colon J(L, M; \mathscr{F}) \longrightarrow J^h(L, M; \mathscr{F}).$$

We give $J(L, M; \mathscr{F})$ the inverse limit topology, which has $(\pi_h)^{-1}(U)$ as the basis, where h is finite and U is open in $J^h(L, M; \mathscr{F})$. We give $J^h(L, M; \mathscr{F})$ the limit differential structure as follows:

A function $\Phi\colon U \to \boldsymbol{R}$, where U is open in $J(L, M; \mathscr{F})$, is called smooth if it is locally of the form $\Psi \circ \pi_h$, where Ψ is a smooth function on some open subset of $J^h(L, M; \mathscr{F})$. By this definition of smoothness we have a differential manifold structure on $J(L, M; \mathscr{F})$, (see Chevalley [3, Chap. III, § 1]).

For a C^h mapping $f\colon L \to M$, a *jet extension* (or *jet section*)

$$j^h f\colon L \longrightarrow J^h(L, M; \mathscr{F})$$

is defined by stipulating that $j^h f(x)$ is the h-jet mod $\mathscr{F}$ of f at $x \in L$. The mapping

$$jf\colon L \longrightarrow J(L, M; \mathscr{F})$$

is naturally defined and is smooth, if f is smooth.

Let $f\colon L \to M$, and let U be an open neighborhood of $f(p) \in U$ such that $\alpha_1 \times \alpha_2\colon U \to D^m \times D^n$ is a chart of $\mathscr{F}$. We define the *kernel* of 1-jet, $j^1 f(p) \in J^1(L, M; \mathscr{F})$, by

$$\operatorname{Ker} j^1 f(p) = \operatorname{Ker} d_p(\alpha_i \circ f),$$

where $d_p(\alpha_1 \circ f)$ is the differential of the mapping $\alpha_1 \circ f\colon V \to D^m$ from a neighborhood V of p in L. $\operatorname{Ker} j^1 f(p)$ does not depend on the choice of chart $(\alpha_1 \times \alpha_2, U)$.

Let Q be a submanifold of $J^h(L, M; \mathscr{F})$ and h be finite. The only *submanifold* of $J(L, M; \mathscr{F})$ we consider are those of the form $\pi_h^{-1}(Q)$. These submanifolds have finite codimensions. The *transversality* of a jet section jf to such a submanifold means that of $j^h f$ to Q.

We take fixed manifolds L^l, M^{m+n} of dimensions l, $m+n$ respectively and a foliation $\mathscr{F}$ on M of codimension m.

Proposition 2.2 (J. M. Boardman [2, Theorem (6.1)]). *For each sequence $I = (i_1, i_2, \cdots, i_k)$ of integers, the submanifold (not necessarily closed) $\tilde{\Sigma}^I$ of the jet space modulo foliation $J(L, M; \mathscr{F})$ is defined. $\tilde{\Sigma}^I$ has codimension ν_I,* *where the number ν_I is defined below* (2.3). *In fact $\tilde{\Sigma}^I$ is the inverse image of a submanifold of $J^k(L, M; \mathscr{F})$ having codimension ν_I. The set $\tilde{\Sigma}^I$ is empty unless I satisfies*

a) $i_1 \geqq i_2 \geqq \cdots \geqq i_{k-1} \geqq i_k \geqq 0$,

b) $l \geqq i_1 \geqq l-m$,

c) *if* $i_1 = l-m$, *then* $i_1 = i_2 = \cdots = i_k$.

PROPOSITION 2.3 (J. M. Boardman [2, Theorem (6.2)]). *If $f: L \to M$ is a map whose jet section $jf: L \to J(L, M; \mathscr{F})$ is transverse to $\tilde{\Sigma}^I$, then $\tilde{\Sigma}^I(f)$, which is defined as $(jf)^{-1}(\tilde{\Sigma}^I)$, is a submanifold of L having codimension ν_I. If I, j denotes the extended sequence $(i_1, i_2, \cdots, i_k, j)$, we have*

$$\tilde{\Sigma}^{I,j}(f) = \tilde{\Sigma}^j(f \mid \tilde{\Sigma}^I(f)).$$

Also, when $I = \phi$, $\tilde{\Sigma}^j(f) = \{p \in L: \dim \operatorname{Ker} j^1 f(p) = j\}$.

PROPOSITION 2.4 (J. M. Boardman [2, Theorem (6.3)]). *Any map $f: L \to M$ may be approximated in the C^∞ sense by a map $g: L \to M$, whose jet section jg: $L \to J(L, M; \mathscr{F})$ is transverse to all the submanifolds $\tilde{\Sigma}^I$.*

This proposition can be slightly modified as follows by observing the proof of [2, Theorem (6.3)].

PROPOSITION 2.4′. *Any map $f: L \to M$ of class C^{r+1} may be C^{r+1} approximated in the C^{r+1} sense by a map $g: L \to M$ whose r-jet section $j^r g: L \to J^r(L, M; \mathscr{F})$ is transverse to all the submanifolds $\tilde{\Sigma}^{i_1, \cdots, i_s}$, $1 \leqq s \leqq r$.*

The number ν_I is defined in [2] as follows for the sequence $I = (i_1, i_2, \cdots, i_k)$ satisfying $i_1 \geqq i_2 \geqq \cdots \geqq i_k \geqq 0$. (We need consider only this case, by a) of Proposition 2.2) Define $\mu(I)$ as the number of sequences $(j_1, j_2, \cdots, j_k)$ of integers that satisfy

a) $j_1 \geqq j_2 \geqq \cdots \geqq j_k \geqq 0$

b) $i_r \geqq j_r \geqq 0$ for all r $(1 \leqq r \leqq k)$, and $j_1 > 0$;

then define

$$\begin{aligned} \nu_I &= (m-l+i_1)\mu(i_1, \cdots, i_k) \\ &\quad -(i_1-i_2)\mu(i_2, \cdots, i_k) - \cdots - (i_{k-1}-i_k)\mu(i_k). \end{aligned} \tag{2.3}$$

For example, in the case $k=1$ we have $\mu(i) = i$ and hence the codimension of $\tilde{\Sigma}^i$ in $J(L, M; \mathscr{F})$ is $(m-l+i)i$, which agrees with the codimension of L_{m-i} in $L(\boldsymbol{R}^l, \boldsymbol{R}^m)$ obtained in Lemma 1.5. In the case $k=2$ we have $\mu(i, j) = i(j+1) - j(j-1)/2$, so the codimension of $\tilde{\Sigma}^{i,j}$ in $J(L, M; \mathscr{F})$ is given by

$$(m-l+i)i + \frac{j}{2}[(m-l+i)(2i-j+1) - 2i + 2j].$$

We call $\tilde{\Sigma}^I$ the *Thom-Boardman submanifold* of $J(L, M; \mathscr{F})$ associated with *Thom-Boardman symbol* I.

Let $\mathscr{C}^r(L, M)$ be the space of all smooth mappings with Whitney C^r topology.

PROPOSITION 2.5. *The maps $f: L \to M$ whose jet section $jf: L \to J(L, M; \mathscr{F})$ is transverse to all the submanifolds $\tilde{\Sigma}^i$, $l-m \leqq i \leqq l$, make up an open and dense subset of $\mathscr{C}^r(L, M)$, $2 \leqq r \leqq \infty$.*

PROOF. If $r=\infty$ the density is mentioned in Proposition 2.4. Since $\mathscr{C}^\infty(L, N)$ is embedded in $\mathscr{C}^r(L, M)$ as a dense subset, the density holds for all $2 \leqq r \leqq \infty$. Let

$$\tilde{\Sigma}_1^i = \{\sigma \in J^1(L, M; \mathscr{F}) : \dim \operatorname{Ker} \sigma = i\}.$$

By the definition of $\tilde{\Sigma}^I$ ([2, (2.7)]), we see that $\tilde{\Sigma}^i$ is defined by $\tilde{\Sigma}^i = (\pi_1)^{-1} \tilde{\Sigma}_1^i$. By Lemma 1.5 the subdivision of $J^1(L, M; \mathscr{F})$ by $\tilde{\Sigma}_1^i$ induces a Whitney stratification $\tilde{S}$. Then, by a similar argument as (iii) in the proof of Theorem A, we see that the set of f with $j^1 f \pitchfork \tilde{S}$ is open dense in $C^r(L, M)$. $\square$

§ 3. Another generic property

In this section we will show that the property of Σ_Y having a fine position in the sense of Thom-Boardman is generic in $\mathscr{Y}^r(M, \mathscr{F})$.

Let $(M, \mathscr{F})$ be the pair consisting of a manifold and a foliation on it as before. Let Σ_Y be the set of all equilibrium points of Y and $\iota: \Sigma_Y \to M$ be the inclusion map.

DEFINITION 3.1. The following is a property for $Y \in \mathscr{Y}^r(M, \mathscr{F})$.

G2′: The vector field Y has the property G0, and the 1-jet section $j^1\iota$: $\Sigma_Y \to J^1(\Sigma_Y, M; \mathscr{F})$ is transverse to $(\pi_1)^{-1}\Sigma^i$ for all Thom-Boardman submanifolds $\tilde{\Sigma}^i$ of length 1 symbol. Here, π_1: $J(\Sigma_Y, M; \mathscr{F}) \to J^1(\Sigma_Y, M; \mathscr{F})$ is the natural projection.

THEOREM B. *Let $\mathscr{Y}^r_{2'}$ be the set of all C^r vector fields tangent to $\mathscr{F}$ satisfying* G2′. *Then, $\mathscr{Y}^r_{2'}$ is an open subset of $\mathscr{Y}^r(M, \mathscr{F})$, if $2 \leqq r \leqq \infty$.*

We will show a lemma for the proof of Theorem B.

Let M and N be smooth manifolds with finite dimensions, and W a closed submanifold of N. Let $f: M \to N$ be of class C^r, $r \geqq 1$, satisfying $f \pitchfork W$. Then $W_f = f^{-1}(W)$ is a closed C^r submanifold of M. There is a total tubular neighborhood of class C^r of W_f in M by Munkres [13, Theorem 5.5] and Lang [9, IV § 5, VII § 3, 4]; this implies that we have an open neighborhood T of W_f in M, a surjective C^r map $\tau: T \to W_f$, and a vector bundle structure on τ.

LEMMA 3.2. *There is an open neighborhood $\mathcal{N}$ of f in $\mathcal{C}^r(M, N)$, $1 \leqq r \leqq \infty$, such that, for $g \in \mathcal{N}$, $W_g = g^{-1}(W)$ is the image of a C^r section ξ_g of $\bar{\tau}$; i.e., $W_g = \xi_g(W_f)$. Moreover, ξ_g depends C^r continuously on g. That is, if $g' \to g$ in $\mathcal{C}^r(M, N)$, then $\xi_{g'} \to \xi_g$ in the section space $\Gamma^r(\bar{\tau})$.*

PROOF. The first statement is included in a lemma of Abraham-Robbin [1, Lemma 20.3], in case of compact M; but the non-compact case follows from the proof of [1]. For the last statement, we recall the proof of this lemma: Let $\|\cdot\|$ be the Finsler of T associated with a Riemannian metric on $\bar{\tau}$. There is an atlas of $\bar{\tau}$ which consists of vector bundle charts $\{(\alpha, \alpha_0, U)\}$ of $\bar{\tau}$ where U is an open subset of W_f and $\alpha: \bar{\tau}^{-1}(U) \to \alpha_0(U) \times F_\alpha$ (F_α is a normed space with the norm $\|\cdot\|_\alpha$) such that $\|v\| = \|\alpha(v)\|_\alpha$ for $v \in \bar{\tau}^{-1}(U)$. Let $t: M \to \boldsymbol{R}$ be a real valued function such that $t(x) > 0$ for every $x \in M$. Define

$$B_t = \{e \in E: \|v\| < t(x),\ x = \bar{\tau}(e)\}$$

and define $B_t(U) = B_t \cap \bar{\tau}^{-1}(U)$. We may assume that U is closure compact, then there is a real number $d > 0$ such that the set

$$B_d(U) = \{e \in E: \|e\| < d,\ x = \bar{\tau}(e) \in U\}$$

is included in $B_t(U)$. Then, for the above chart (α, α_0, U), we have $\alpha(B_d(U)) = \alpha_0(U) \times B_{\alpha d}$, where $B_{\alpha d}$ is the open ball in F_α about the origin with radius d. Choose a chart (V, β) in the manifold N at $f(x)$ such that $f(x) \in V$, $\beta(V) = V_1 \times V_2$ where V_1 and V_2 are open neighborhoods of the origin in Banach spaces F_1 and F_2 respectively, $\beta(f(x)) = (0, 0)$, and $\beta(V \cap W) = V_1 \times \{0\}$. Suppose $f(\bar{U}) \subset V$. Let $\mathcal{N}$ be a sufficiently small neighborhood of f in $\mathcal{C}^\infty(M, N)$. For $g \in \mathcal{N}$ we define a map

$$\varphi_g\colon \alpha_0(U) \times B_{\alpha d} \longrightarrow V_2$$

by

$$\varphi_g(u, e) = pr \circ \beta \circ g \circ \alpha^{-1}(u, e)$$

for $u \in \alpha_0(U)$, $e \in B_{\alpha d}$, where pr is the projection $V_1 \times V_2 \to V_2$ on the second factor. Since $f \pitchfork W$, the differential $D\varphi_f(u, e)$ is surjective. Hence for each $g \in \mathcal{N}$, $D\varphi_g(u, e)$ is surjective. Define

$$\Phi_g\colon \alpha_0(U) \times B_{\alpha d} \longrightarrow \alpha_0(U) \times V_2$$

by $\Phi_g(u, e) = (u, \varphi_g(u, e))$ for $u \in \alpha_0(U)$ and $v \in B_{\alpha d}$. Then $D\Phi_g(u, e)$ is a linear isomorphism. By the inverse function theorem (Lang [9, p. 12]), we have an inverse map Φ_g^{-1}: image $\Phi_g \to \alpha_0(U) \times B_{\alpha d}$ of Φ_g (making $\alpha_0(U)$ and $B_{\alpha d}$ smaller if necessary). For $(w, v) \in$ image $\Phi_g \subset \alpha_0(U) \times V_2$ let $\Phi_g^{-1}(w, v) = (h_1(w, v),$

$h_2(w, v)) \in \alpha_0(U)\times B_{\alpha d}$. Since $\Phi_g(u, e)\in\alpha_0(U)\times\{0\}$ if and only if $g\circ\alpha^{-1}(u, e)\in W$, we may assume that the image of $h_2(w, \cdot)$ contains the origin of $B_{\alpha d}$ for each $w\in\alpha_0(U)$. Define $h\colon\alpha_0(U)\to B_{\alpha d}$ by $h(w)=h_2(w, 0)$, for $w\in\alpha_0(U)$. Then $\Phi_g^{-1}(w, 0)=(w, h(w))$. Define $\xi_g\colon W_f\to E$ by

$$\xi_g(x) = \alpha^{-1}(\alpha_0^{-1}(x),\ h\circ\alpha_0^{-1}(x))$$

for sufficiently small vector bundle chart (α, α_0, U) with $x\in U$. ξ_g is the desired section.

If $g'\to g$ in $\mathscr{C}^r(M, N)$, the $\Phi_{g'}\to\Phi_g$ and $\Phi_{g'}^{-1}\to\Phi_g^{-1}$. Therefore we have $\xi_{g'}\to\xi_g$. □

Proof of Theorem B. Let $\mathscr{C}_*^r(\Sigma, M; \mathscr{F})$ denote the subset of $\mathscr{C}^r(\Sigma, M)$ which consists of all mappings $f\colon\Sigma\to M$ such that $j^1f \pitchfork \tilde{\Sigma}^i$ for the Thom-Boardman manifold $\tilde{\Sigma}^i$ for every symbol i of length one. Let $E\subset J^0(\tau)$ be the submanifold of Definition 1.7. By Theorem A and the proof, the set

$$\mathscr{Y}_0 = \{Y\in\mathscr{Y}^r(M; \mathscr{F})\colon \rho_Y^0 \pitchfork E\}$$

is open and dense in $\mathscr{Y}^r(M; \mathscr{F})$, and if $Y\in\mathscr{Y}_0$, then $\Sigma_Y=(\rho_Y^0)^{-1}(E)$ is a closed smooth submanifold of M. Suppose $Y\in\mathscr{Y}_0$. If Y' converges to Y in $\mathscr{Y}^r(M; \mathscr{F})$, then $\rho_{Y'}^0$ converges to ρ_Y^0 in $\Gamma^r(\tau^0)$ which is the space of the sections of $\tau^0\colon J^0(\tau)\to M$. It implies that $\rho_{Y'}^0$ converges to ρ_Y^0 as mappings $M\to J^0(\tau)$. Let $\bar{\tau}\colon T\to\Sigma_Y$ be a tubular neighborhood of $\Sigma_Y\subset M$. By Lemma 3.2, there is an open neighborhood $\mathscr{N}$ of ρ_Y^0 in $\Gamma^r(\tau^0)$ such that if $\rho_{Y'}^0\in\mathscr{N}$, then $\Sigma_{Y'}$ is the image of a C^r section $\xi_{Y'}$ of $\bar{\tau}$. Suppose $Y\in\mathscr{Y}_3$. Since $\mathscr{C}_3^r(\Sigma_Y, M; \mathscr{F})$ is open in $\mathscr{C}^r(\Sigma_Y, M)$ by Proposition 2.5, and the inclusion map $\iota\colon\Sigma_Y\to M$ is contained in $\mathscr{C}_*^r(\Sigma_Y, M; \mathscr{F})$, then we have a neighborhood $\mathscr{U}$ of Y in $\mathscr{Y}^r(M; \mathscr{F})$ such that, if $Y'\in\mathscr{U}$, then $\xi_{Y'}\in\mathscr{C}_*^r(\Sigma_Y, M; \mathscr{F})$, by the last statement of Lemma 3.2. This implies that the map $\xi_{Y'}\colon\Sigma_Y\to M$ satisfies $j^1\xi_{Y'}\pitchfork\tilde{\Sigma}^i$ for every i. Since image $\xi_{Y'}=\Sigma_{Y'}$, then by the construction of $\tilde{\Sigma}^i$ in [2], we see that the inclusion map $\iota'\colon\Sigma_{Y'}\to M$ satisfies $j^1\iota'\pitchfork\tilde{\Sigma}^i$ if $r\geqq 2$, and we have $Y'\in\mathscr{Y}_{2'}$. Therefore, $\mathscr{Y}_{2'}$ is open.

We consider another property for Y.

Definition 3.3. **GB**$_s$: Y has the property G0, and the k-jet section $j^k\iota\colon \Sigma_Y\to J^k(\Sigma_Y, M; \mathscr{F})$ is transverse to $(\pi_k)^{-1}\tilde{\Sigma}^I$ for all Thom-Boardman submanifold $\tilde{\Sigma}^I$ of length k symbol I, where π_k is the natural projection of $J(\Sigma_Y, M; \mathscr{F})$ onto $J^k(\Sigma_Y, M; \mathscr{F})$, $k=1, \cdots, s$.

Theorem C (Density theorem). *Let $\mathscr{Y}_{Bs}^r$ be the set of all C^r vector fields tangent to $\mathscr{F}$ satisfying* GB$_s$. *Then, $\mathscr{Y}_{Bs}^r$ is a dense subset of $\mathscr{Y}^r(M, \mathscr{F})$ for $s+1\leqq r<\infty$. Especially $\mathscr{Y}_{2'}^r$ is a dense subset for $2\leqq r<\infty$.*

For the proof, we show a lemma. Let $\mathscr{D}^r(M)$ be the space of all C^r diffeomorphisms on M with Whitney C^r topology.

LEMMA 3.4. *Let W be a closed submanifold of M and let $\tau: T\to W$ be a tubular neighborhood of class C^r of W in M, $1\leqq r<\infty$. For any neighborhood $\mathscr{N}$ of the identity in $\mathscr{D}^r(M)$ there exists a neighborhood $\mathscr{N}_0$ of the zero section of $\bar{\tau}$ in $\Gamma^r(\bar{\tau})$ and a continuous mapping $\Phi: \mathscr{N}_0\to\mathscr{N}$ such that $\Phi(f)\,|\,W=f$ and $\Phi(f)=id$ on $M-T$ for every section f in $\mathscr{N}_0$.*

PROOF. Let $\{(\alpha, \alpha_0, U)\}$ be the atlas of $\bar{\tau}$, associated with the Finsler, defined in the proof of Lemma 3.2. Let $\tau^*: T^*\to W$ be an open disk bundle such that $T^*\subset T$ and $\tau^*=\bar{\tau}\,|\,T^*$. Moreover we assume that

$$\alpha((\tau^*)^{-1}(x)) = \{\alpha_0(x)\times B(t_{\alpha x})\} \tag{3.1}$$

where $B(t_{\alpha x})$ is the open ball in F_α about the origin with radius $t_{\alpha x}$ and that the mapping $x\mapsto t_{\alpha x}$ is of class C^r. We take a continuous function $\varepsilon: T\to \boldsymbol{R}$ such that $\varepsilon(x)>0$ if $x\in T^*$ and $\varepsilon(x)=0$ if $x\notin T^*$. Let $g: T\to T$ be a bundle map such that $\bar{\tau}\circ g=\bar{\tau}$. We can take ε such that it satisfies the following: If

$$\|\alpha\circ g\circ\alpha^{-1}\|_r < \varepsilon(x)$$

for every chart (α, α_0, U) and every x with $\bar{\tau}(x)\in U$, where $\|\cdot\|_r$ is the C^r norm of $\alpha_0(U)\times F_\alpha$, then the trivially extended map $\tilde{g}$ is contained in $\mathscr{N}$. Here, $\tilde{g}$ is defined by $\tilde{g}(x)=g(x)$ if $x\in T$ and, $=x$ if $x\notin T$. For a continuous function $\delta: W\to\boldsymbol{R}$ with $\delta(x)>0$ for any $x\in W$, we define $\mathscr{N}_0(\delta)\subset\Gamma^r(\bar{\tau})$ to be the set of all C^r sections $f: W\to T$ satisfying

$$\|pr\circ\alpha\circ f\circ\alpha_0^{-1}\|_r < \delta(x)$$

for every chart (α, α_0, U) and every $x\in U$, where pr is the natural projection $\alpha_0(U)\times F_\alpha\to F_\alpha$ and $\|\cdot\|_r$ is the C^r norm of the sections of $\alpha_0(U)\times F_\alpha\to\alpha_0(U)$.

Let $\varphi: \boldsymbol{R}\to\boldsymbol{R}$ be a bump function such that

(1) $0\leqq\varphi(t)\leqq 1$, $\forall t\in\boldsymbol{R}$,
$\varphi(0)=1$; $\varphi(t)=0$ if $t\notin(-1, 1)$

(2) φ is of class C^r, $1\leqq r<\infty$, and there is a constant $b>0$ such that

$$|\varphi^{(s)}(t)| < b, \qquad 1\leqq s\leqq r.$$

(If $r=\infty$, there is no such bump function.)

We define Φ as follows. For a chart (α, α_0, U), let $x\in U$ and $y\in\tau^{-1}(x)$. For $f\in\mathscr{N}_0(\delta)$, we have

$$\alpha\circ f(x) = (\alpha_0(x), v_x), \qquad \text{where } v_x = pr\circ\alpha\circ f(x)\in F_\alpha,$$
$$\alpha(y) = (\alpha_0(x), v), \qquad v\in F_\alpha.$$

Let $t_{\alpha x}$ be the radius of $\alpha((\tau^*)^{-1}(x))$ in (3.1). Then, Φ is defined by

$$\Phi(f)(y)=\begin{cases}\alpha^{-1}\left(\alpha_0(x), \quad v+\varphi\left(\dfrac{\|v\|}{t_x}\right)v_x\right), & y \in T \\ y & , \ y \notin T.\end{cases}$$

$\Phi(f)$ is well-defined and of class C^r, since $\varphi(\|v\|/t_{\alpha x})=0$ for every $v \in T-T^*$. Since the derivatives of φ are bounded, we can take δ such that, for any x, y with $y=\bar{\tau}(x)$, if

$$\|D^r v_x\| < \delta(x),$$

then

$$\|D^r\varphi(\|v\|/t_{\alpha\tau(y)})v_{\tau(y)}\| < \varepsilon(y).$$

This implies that if $f \in \mathscr{N}_0(\delta)$, then $\Phi(f) \in \mathscr{N}$. Obviously, if $f' \in \Gamma^r(\tau)$ is a C^r approximation of f, then $\Phi(f')$ is a C^r approximation of $\Phi(f)$; so that Φ is continuous. Therefore, $\mathscr{N}_0=\mathscr{N}_0(\delta)$ is the desired one. □

PROOF OF THEOREM C. Let $Y \in \mathscr{Y}^r(M, \mathscr{F})$. Y is C^r approximated by $\tilde{Y}$ of class C^∞. Then, $\Sigma_{\tilde{Y}}$ is a C^∞ manifold. By Proposition 2.4, the inclusion map $\iota: \Sigma_{\tilde{Y}} \to M$ is C^∞ approximated by $f: \Sigma_{\tilde{Y}} \to M$ such that the jet section jf: $\Sigma_{\tilde{Y}} \to J(\Sigma_{\tilde{Y}}, M; \mathscr{F})$ is transverse to all the Thom-Boardman submanifolds $\tilde{\Sigma}^I$. By the lemma 3.4, there is a C^{r+1} diffeomorphism $F: M \to M$ such that $F|\Sigma_{\tilde{Y}}=f$ and that, if $f \to \iota$ in $C^{r+1}(\Sigma_{\tilde{Y}}, M)$, then $F \to$identity in $\mathscr{D}^{r+1}(M)$. Let π_F: $TM \to T\mathscr{F}$ be the orthogonal projection. We define a vector field $Y' \in \mathscr{Y}^r(M, \mathscr{F})$ by

$$Y'(x) = \pi_F \circ df \circ Y(f^{-1}(x)).$$

Let $\iota': \Sigma_{Y'} \to M$ be the inclusion map. Since $\Sigma_{Y'}=f(\tilde{\Sigma}_{\tilde{Y}})$, then $j\iota' \pitchfork \tilde{\Sigma}^I$ for all Thom-Boardman manifolds $\tilde{\Sigma}^I$. Hence, by the definition of this transversality in section 2, Y' satisfies GB_s. Clearly, we can take Y' arbitrarily C^r near to Y. □

§ 4. Boundaries of normally hyperbolic domains

Let $(M, \mathscr{F})$ be as before. We define a Whitney stratification of the jet bundle $J^1(\tau)$ of sections of $\tau: T(\mathscr{F}) \to M$, as follows. Let $(\alpha^i, \alpha_1 \times \alpha_2, U)$ be a i-jet chart on $\tau^i: J^i(\tau) \to M$ given by (1.4), $i=0, 1$. For each $\sigma \in J^1(\tau)$, let

$$(4.1) \qquad \begin{cases}\alpha^1(\sigma) = (y, v, (A, B)) \in D^{m+n} \times \boldsymbol{R}^n \times L(\boldsymbol{R}^m \times \boldsymbol{R}^n, \boldsymbol{R}^n), \\ \qquad A \in L(\boldsymbol{R}^m, \boldsymbol{R}^n) \quad \text{and} \quad B \in L(\boldsymbol{R}^n, \boldsymbol{R}^n),\end{cases}$$

similarly as the representation of (1.4). Define

$$\tilde{\Sigma}^i_\tau = \{\sigma \in J^1(\tau): v=0, \text{rank } B=n-i\}. \tag{4.2}$$

By Lemma 1.5 we have; $\tilde{\Sigma}^i_\tau$ is a submanifold of $J^1(\tau)$ with codimension $n+i^2$, and $\{\tilde{\Sigma}^i_\tau\}_{i=0,\dots,n}$ induces a Whitney stratification of $J^1(\tau)$.

Denote $\tilde{\Sigma}_\tau(Y)=(j^1Y)^{-1}(\tilde{\Sigma}^i_\tau)$ and $\tilde{\Sigma}^I(Y)=(j^1\iota)^{-1}(\tilde{\Sigma}^I)$ for Thom-Boardman symbol I.

THEOREM 4.1. *Let $Y \in \mathscr{Y}^r(M, \mathscr{F})$, $r \geqq 2$, and $\iota: \Sigma_Y \to M$ be the inclusion map. Then, we have the following.*

(i) *Under* G0 *and* G1, $\tilde{\Sigma}^i_\tau(Y)=\tilde{\Sigma}^i(Y)$.

(ii) *Let $p \in \tilde{\Sigma}^{1,0}(Y)$. Under* BG_2, *there exist coordinates of class C^{r-1} $x_1, \cdots, x_m$ centered at p in Σ_Y and $y_1, \cdots, y_m, z_1, \cdots, z_n$ centered at p in M, such that* (a) *$z_1, \cdots, z_n$ is the coordinates of a leaf of $\mathscr{F}$,* (b) *in these coordinates $\iota: \Sigma_Y \to M$ is given by*

$$y_1 = x_1, \cdots, \quad y_{m-1} = x_{m-1}, \quad y_m = x_m^2;$$
$$z_1 = x_m, \quad z_2 = \cdots = z_n = 0.$$

PROOF. (i) Let $p \in \Sigma_Y$. Since $\dim \Sigma_Y = \operatorname{codim} \mathscr{F}$ by G0, the condition G1 implies

$$\operatorname{Ker} dY_p = T_p\Sigma_Y.$$

Let $B \in L(\boldsymbol{R}^n, \boldsymbol{R}^n)$ be the one given by (3.1). Then we have

$$\begin{aligned} n-\text{rank } B &= \dim (\operatorname{Ker} dY_p) \cap T_p\mathscr{F} \\ &= \dim T_p\Sigma_Y \cap T_p\mathscr{F} \\ &= \dim \operatorname{Ker} \text{ of } j^1\iota \text{ at } p \text{ modulo } \mathscr{F}. \end{aligned}$$

Therefore, $p \in \tilde{\Sigma}_\tau(Y)$ if and only if $p \in \tilde{\Sigma}^i(Y)$.

(ii) Since $j^1\iota \pitchfork \tilde{\Sigma}^1$ and $j^1(\iota \,|\, \tilde{\Sigma})$ has full rank at p by Proposition 2.3, then p is a fold point (Golubitsky-Guillemin [5, p. 87 Definition 4.1]). Then, by [5, p. 88 Theorem 4.5], we have (iii) obviously. □

Next, we study the bifurcations of Y at $\partial\Sigma^h_Y$. Suppose that $\dim M = m+n$, $\operatorname{codim} \mathscr{F} = m$, and Y is class C^r, $r \geqq 3$. Let p be a point in the boundary $\partial\Sigma^h_Y$ of Σ^h_Y. Assume that there is a neighborhood N of p in $\partial\Sigma^h_Y$ such that N is an $(m-1)$ dimensional manifold. Let $\alpha_1 \times \alpha_2: U \to D^m \times D^n$ be a chart of $\mathscr{F}$ such that $(\alpha_1 \times \alpha_2)(p)=(0, 0)$, (see (1.1)). Let I be a segment in D^m parametrized by μ such that $\mu=0$ indicates the origin of D^m.

Assumption: $L \equiv (\alpha_1 \times \alpha_2)^{-1}(I \times D^n)$ is transverse to both Σ_Y and N in M.

DEFINITION 4.2. Under the above assumption we say that Y has *saddle-node bifurcation* at $p \in \partial\Sigma_Y^h$, if there is an segment I as above satisfying the following: The smooth curve $L \cap \Sigma_Y$ is tangent to L_0 at p, $\Sigma_Y \cap L_\mu = \phi$ if $\mu < 0$, and $\Sigma_Y \cap L_\mu$ consists of two points, p_μ^s and p_μ^u if $\mu > 0$. Furthermore, Y is hyperbolic at p_μ^s and p_μ^u. The dimensions of the stable manifolds at p_μ^s and p_μ^u are k and $k-1$, respectively, $1 \leqq k \leqq m$. See Figure 1.

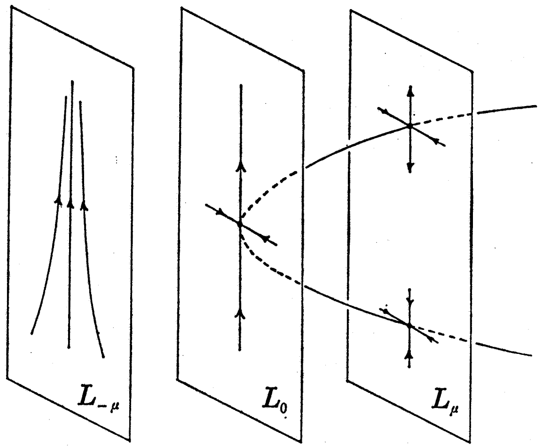

Figure 1.

DEFINITION 4.3. Under the above assumption we say that Y has *Hopf bifurcation* at $p \in \partial\Sigma_Y^h$, if the following holds for every segment $I \subset D^m$ as above: There is a unique 3-dimensional center manifold C (see Guckenheimer-Holmes [6, p. 127]) containing $L \cap \Sigma_Y = (\cup_\mu L_\mu) \cap \Sigma_Y$ and a system of coordinates (x, y, μ) on C (with $(x, y, \mu) \in L_\mu$ for a fixed μ) for which the Taylor expansion of degree 3 of Y on C is given by

$$\begin{cases} \dot{x} = (d\mu + a(x^2+y^2))x - (\omega + c\mu + b(x^2+y^2))y \\ \dot{y} = (\omega + c\mu + b(x^2+y^2))x + (d\mu + a(x^2+y^2))y, \end{cases}$$

which is expressed in polar coordinates as

$$\begin{cases} \dot{r} = (d\mu + ar^2)r \\ \dot{\theta} = (\omega + c\mu + br^2). \end{cases}$$

See Figure 2. Consequently, if $a \neq 0$, there is a surface of periodic solutions in C which has quadratic tangency with the eigenspace of $\lambda(0)$, $\bar{\lambda}(0)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2+y^2)$. If $a < 0$, then these solutions are stable limit cycles, while if $a > 0$, these are repelling. (See [6, Theorem 3.4.2].)

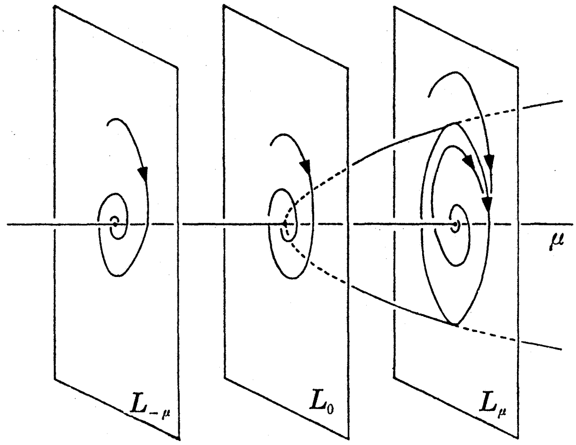

Figure 2.

We want to see how these bifurcations arise in our global situation with respect to the stratifications which we have defined.

Let $\mathcal{S}^k$ be the k-skeleton of the stratification $\mathcal{S}$ defined in G2. Let $\tilde{\mathcal{S}}^k$ be the k-skeleton of the stratification $\tilde{\mathcal{S}}$ on Σ_Y induced from Thom-Boardman singularities $\tilde{\Sigma}^i(Y)=(j\iota)^{-1}(\tilde{\Sigma}^i)$, $i=0, 1, \cdots, m$. We have $\tilde{\mathcal{S}}^k=\tilde{\Sigma}^{m-k}(Y)\cup \tilde{\Sigma}^{m-k+1}(Y)\cup\cdots\cup\tilde{\Sigma}^m(Y)$. Under G1, we have $\mathcal{S}^k\supset\tilde{\mathcal{S}}^k$ and $\partial\Sigma_h$ by Theorem 4.1 (i) and the definition of $\mathcal{S}$. Moreover, we have that an $(m-1)$ dimensional stratum of $\mathscr{S}$ is included in an $(m-1)$ dimensional stratum of $\tilde{\mathcal{S}}$. For the sets defined in G2, we observe

$$(\partial\Sigma_Y^h)_0\subset\tilde{\mathcal{S}}^{m-1}$$

and

$$(\partial\Sigma_Y^h)_{\mathrm{img}}\cap\tilde{\mathcal{S}}^{m-1}=\phi.$$

Denote by $(\partial\Sigma_Y^h)_f$ the set of fold points in $\partial\Sigma_Y^h$;

$$(\partial\Sigma_Y^h)_f\equiv(\partial\Sigma_Y^h)_0\cap\tilde{\Sigma}^{1,0}(Y).$$

THEOREM D. *Let $Y\in\mathscr{Y}^r(\mathscr{F})$, $r\geqq 3$. Under* G1, G2, *and* GB_2, *there is an open dense subset $(\partial\Sigma_Y^h)_f\cup(\partial\Sigma_Y^h)_{\mathrm{img}}$ of the boundary $\partial\Sigma_Y^h$ of the normally hyperbolic domain Σ_Y^h such that Y has saddle-node bifurcation at each point of $(\partial\Sigma_Y^h)_f$ and has Hopf bifurcation at each point of $(\partial\Sigma_Y^h)_{\mathrm{img}}$.*

PROOF. By definition of $(\partial\Sigma_Y^h)_0$, we have $(\partial\Sigma_Y^h)_0\subset\tilde{\Sigma}_\tau^1(Y)$. By Theorem 4.1. (i), we have $\tilde{\Sigma}_\tau^1(Y)=\tilde{\Sigma}^1(Y)$. Since $\tilde{\Sigma}^{1,0}$ is open dense in $\tilde{\Sigma}^1$ by Proposition 2.2 and (2.3), then $(\partial\Sigma_Y^h)_f$ is open dense in $(\partial\Sigma_Y^h)_0$.

Let $p\in(\partial\Sigma_Y^h)_f$. Let $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m, z_1, \cdots, z_n)$ be the coordinate systems centered at p for Σ_Y and M, respectively, given by Theorem 4.1.

(iii). For an interval $I\subset \boldsymbol{R}$ containing 0, let $\mu \in I$. We define

$$(4.3)\qquad L_\mu = \{(y_1, \cdots, y_m, z_1, \cdots, z_n)\in M: y_1=\cdots=y_{m-1}=0, y_m=\mu\}.$$

$L=\bigcup_{\mu\in I} L_\mu$ is coordinated by $(\mu, z_1, \cdots, z_n)$, and for these coordinates, we have

$$(4.4)\qquad L\cap \Sigma_Y = \{(\mu, z_1, \cdots, z_n): \mu=z_1^2, z_2=\cdots=z_n=0\}.$$

Hence, $L\cap\Sigma_Y$ is tangent to L_0 at p, $\Sigma_Y\cap L_\mu=\phi$ if $\mu<0$, and $\Sigma_Y\cap L_\mu$ consists of two points, p_1 and p_2 if $\mu>0$. $Y_\mu=Y|L_\mu$ is hyperbolic at both of p_1 and p_2, since these points are contained in Σ_Y^h. By the definition of $\tilde{\Sigma}_\tau^1$ and the transversality of j^1Y with $\tilde{\Sigma}_\tau^1$, it is ovbious that the difference of the stable dimensions of p_1 and p_2 is just one. Therefore, Y has saddle-node bifurcation at any point of $(\partial\Sigma_Y^h)_f$, which is open dense in $(\partial\Sigma_Y^h)_0$.

Let $p\in(\partial\Sigma_Y^h)_{\text{img}}$ and L_0 be the leaf of $\mathscr{F}$ containing p. Since the differential of $Y|L_0$ at p does not have zero eigenvalue by the definition of $(\partial\Sigma_Y^h)_{\text{img}}$ in G2, it follows that $p\notin\tilde{\Sigma}_\tau^i(Y)=\tilde{\Sigma}^i(Y)$ (Theorem 4.1) for any $i\neq 0$. Hence, for a small neighborhood U of p in Σ_Y, the composition $(pr)\circ(\alpha_1\times\alpha_2)$: $U\to D^m\times D^n\to D^m$ is a diffeomorphism. This implies that we can take a segment I in U instead of a segment $I\subset D^m$ (in Definition 4.3). Point p is contained in an $(m-1)$ dimensional stratum $S_{\text{img}}\subset(\partial\Sigma_Y^h)_{\text{img}}$ of $\mathcal{S}$. Let $I\subset\Sigma_Y$ be an open segment which is transverse to S_{img} at p. Let μ be a parameter of I such that $\mu=0$ at p. Let L_μ be the leaf of $\mathscr{F}|U$ passing through $\mu\in I$. The derivative of $Y|L_0$ at p has a smiple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of the differential of $Y|L_\mu$ at $\mu\in I$ which are pure imaginary at $\mu=0$. $\lambda(\mu)$, $\bar{\lambda}(\mu)$ vary smoothly with μ. Moreover,

$$\frac{d}{d\mu}(\mathscr{R}\lambda(\mu))|_{\mu=0}\neq 0$$

by the transversality $I\pitchfork S_{\text{img}}$ and the definition of $(\partial\Sigma_Y^h)_{\text{img}}$, (cf. the proof of Lemma 1.6). Then, by Guckenheimer-Holmes [6, Theorem 3.4.2], Y has Hopf bifurcation at p.

Since $(\partial\Sigma_Y^h)_f\cup(\partial\Sigma_Y^h)_{\text{img}}$ is open dense in $\partial\Sigma_Y^h$, the theorem is proved. □

Let X be a C^r vector field on an open set U in $\boldsymbol{R}^n$, let φ_t be the flow of X, and let $p\in U$ be an equilibrium point of X. Suppose that the eigenvalues $\lambda_0, \cdots, \lambda_{n-1}$ of $DX(p)$ satisfy that $\lambda_0=0$, $\mathscr{R}\lambda_1, \cdots, \mathscr{R}\lambda_{n-1}<0$. Let E^c and E^s be the generalized eigen spaces of λ_0 and $\lambda_1, \cdots, \lambda_{n-1}$, respectively. By center manifold theorem (Chow-Hale [4, Theorem 2.2] and Guckenheimer-Holmes [6, Theorem 3.2.1]), there are an invariant C^r manifold $W^s(p)$ (called the *stable manifold*) tangent to E^s at p and a C^r manifold $W^c(p)$ (called the (*local*) *center manifold*) tangent to E^c at p. W^c is locally invariant in the sense that, if $q \in W^c$ and $\varphi_t(q) \in U$, then $\varphi_t(q) \in W^c$. W^s is unique, but W^c need not be. W^c is asymptotically stable in the sense that if $t>0$ and $\varphi_t(q)$ remains defined in U for all $t>0$, then $\varphi_t(q) \to W^c$ as $t\to\infty$ (Chow-Hale [4, Theorem 2.13]).

Let φ_t be the flow associated to a vector field on a manifold. The subsets

$$V^s(p) = \{q : \varphi_t(q) \to p \text{ as } t\to\infty\},$$

and

$$V^u(p) = \{q : \varphi_t(q) \to P \text{ as } t\to -\infty\}$$

are called the *stable set* and the *unstable set* of p, respectively.

Let Σ_Y^s denote the normally stable domain of Σ_Y (see Definition 1.4). Let

$$(\partial\Sigma_Y^s)_f \equiv (\partial\Sigma_Y^h)_f \cap (\partial\Sigma_Y^s) \quad \text{and} \quad (\partial\Sigma_Y^s)_{\mathrm{img}} \equiv (\partial\Sigma_Y^h)_{\mathrm{img}} \cap (\partial\Sigma_Y^s).$$

THEOREM E. *Suppose a vector field $Y \in \mathscr{Y}^r(M, \mathscr{F})$ satisfies* G1, G2 *and* GB$_2$. *Let $p \in (\partial\Sigma_Y^s)_f$. Then, there is an open neighborhood U of p in M, and, denoting by L_p the connected component of a leaf of $\mathscr{F}|U$ containing p (i.e. a plaque of $\mathscr{F}$), there is a C^r embedding h_p: $L_p \to \boldsymbol{R}^1 \times \boldsymbol{R}^{n-1}$ such that the following are satisfied.*

(i) *$W^s(p) \cap L_p \subset h_p^{-1}(\{0\} \times \boldsymbol{R}^{n-1})$ and $W^c(p) \cap L_p \subset h_p^{-1}(\boldsymbol{R}^1 \times \{0\})$, where $W^s(p)$ and $W^c(p)$ are the stable and center manifolds of Y restricted in a leaf, respectively.*

(ii) *$V^s(p) \cap L_p \subset h_p^{-1}([0, \infty) \times \boldsymbol{R}^{n-1})$ and $V^u(p) \cap L_p \subset h_p^{-1}((-\infty, 0] \times \{0\})$ $\subset W^c(p)$, where $V^s(p)$ and $V^u(p)$ are the stable and unstable sets of p, respectively (Figure 3).*

(iii) *The C^r embedding h_p depends C^{r-1} continuously on $p \in (\partial\Sigma_Y^s)_f$. So that, the unstable set*

$$V^u = \{q \in V^u(p) : p \in (\partial\Sigma_Y^s)_f \cap U\}$$

is an injectively C^{r-1} immersed submanifold of M, where $V^u(p)$ is an injectively C^r immersed submanifold of M.

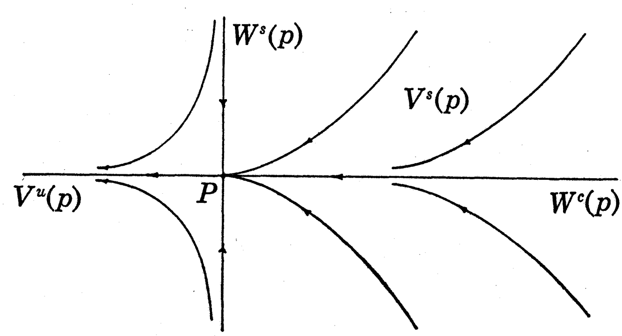

Figure 3.

PROOF. (i) is obtained easily from center manifold theorem. To show (ii) let L_μ be the plaque defined by (4.3) and let I be an interval with $0 \in I \subset \boldsymbol{R}$. Then, $L = \bigcup_{\mu \in I} L_\mu$ is a C^{r-1} manifold. By center manifold theorem, we have a two-dimensional center manifold $\tilde{W}^c(p)$ at p of $Y|L$. Let $\tilde{U}$ be a small neighborhood of p in L. Let $\tilde{h}\colon \tilde{U} \to \boldsymbol{R}^1 \times \boldsymbol{R}^{n-1} \times I$ be an embedding such that $\tilde{h}(q) = h(q) \times \{0\}$ if $q \in L_0$ $(=L_p)$ and $\tilde{h}(q) \in \boldsymbol{R}^1 \times \boldsymbol{R}^{n-1} \times \{\mu\}$ if $q \in L_\mu$.

Since $Y|L$ has saddle-node bifurcation at p, there are two hyperbolic points p_μ^s and p_μ^u of $Y|L$ near p. We assert here that these points are contained in $\tilde{W}^c(p)$. In fact, let φ_t be the flow of Y. Since $\tilde{W}^c(p)$ is asymptotically stable, we have $\varphi_t(q) \to \tilde{W}^c(p)$ as $t \to \infty$, if $\varphi_t(q) \in \tilde{U}$ for any $t > 0$. Especially, if $q \in L_\mu$, then $\varphi_t(q) \to \tilde{W}^c(p) \cap L_\mu$. Let $\tilde{W}^s(p_\mu^u)$ and $\tilde{W}^u(p_\mu^u)$ be the stable and unstable manifolds at p_μ^u of $Y|L_\mu$, respectively. If $p_\mu^u \notin \tilde{W}^c(p)$, then we have $\varphi_t(p_\mu^u) \to \tilde{W}^c(p) \cap L_\mu$ for $t \to 0$. Since p_μ^u is a fixed point of φ_t, this is a contradiction. Therefore, we have $p_\mu^u \in \tilde{W}^c(q)$, and similarly for p_μ^s.

We identify $\tilde{U}$ by $\tilde{h}$ with an open set of $\boldsymbol{R}^1 \times \boldsymbol{R}^{n-1} \times I$ containing the origin. Let E_p^s and E_p^c be the generalized eigenspaces of the eigenvalues with negative real parts and zero of $(DY)_p$, respectively. Since $E_p^s = \{0\} \times \boldsymbol{R}^{n-1} \times \{0\}$ and $E_p^c \supset \boldsymbol{R}^1 \times \{0\} \times \{0\}$, we have $\tilde{W}^c(p) \pitchfork L_\mu$ for each $\mu \in I$ by taking I smaller if necessary. Hence, $\tilde{W}^c(p) \cap L$ is a Y-invariant C^r curve.

The unstable manifold $\tilde{W}^u(p_\mu^u)$ of $Y|L_\mu$ is included locally in $\tilde{W}^c(p) \cap L_\mu$. In fact, this is obtained from the fact that the Y-invariant 1-dimensional manifold $\tilde{W}^c(p) \cap L_\mu$ is transverse to the $(n-1)$ dimensional stable manifold $\tilde{W}^s(p_\mu^u)$ of $Y|L$ at p_μ^u. We can easily see that p_μ^s is a sink in $\tilde{W}^c(p) \cap L_\mu$.

For each point q in one component $W_+^c(p)$ of $W^c(p) - \{p\}$, we have $\varphi_t(q) \to p$ as $t \to \infty$; and for each point q in the other component $W_-^c(p)$, we have $\varphi_t(q) \to p$ as $t \to -\infty$. This is shown as follows. The 2-dimensional manifold $\tilde{W}^c(p)$ is φ_t-invariant. If $\mu > 0$, there are only two equilibrium points p_μ^s and p_μ^u in

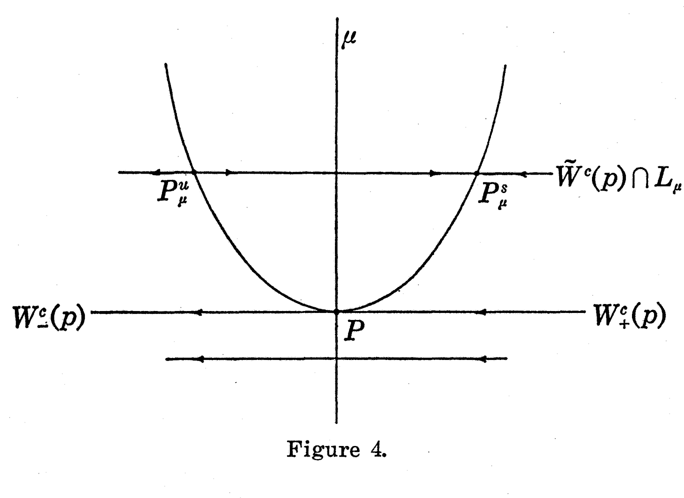

Figure 4.

$\tilde{W}^c(p) \cap L_\mu$ such that p_μ^s is a sink and p_μ^u is a source of $Y \mid \tilde{W}^c(p) \cap L_\mu$. If $\mu < 0$, there is no equilibrium. By the continuity of φ_t the above facts are obtained. (See Figure 4.) Hence, by Chow-Hale [4, p. 324], we have the property (iii) of the lemma.

For (iv), we recall that $(\partial \Sigma_Y^s)_t$ is a manifold of class C^{r-1}. Then, the C^{r-1} dependence of $W^c(p)$ on p is obvious by Chow-Hale [4, Theorem 2.1]. □

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