EXTREME AND PERIODIC L_2 DISCREPANCY OF PLANE POINT SETS

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ABSTRACT. In this paper we study the extreme and the periodic L_2 discrepancy of plane point sets. The extreme discrepancy is based on arbitrary rectangles as test sets whereas the periodic discrepancy uses "periodic intervals", which can be seen as intervals on the torus. The periodic L_2 discrepancy is, up to a multiplicative factor, also known as diaphony. The main results are exact formulas for these kinds of discrepancies for the Hammersley point set and for rational lattices.

We also prove a general lower bound on the extreme L_2 discrepancy for arbitrary point sets in dimension d, which is of order of magnitude $(\log N)^{(d-1)/2}$, like the standard and periodic L_2 discrepancies, respectively. Our results confirm that the extreme and periodic L_2 discrepancies of the Hammersley point set are of best possible asymptotic order of magnitude. This is in contrast to the standard L_2 discrepancy of the Hammersley point set. Furthermore our exact formulas show that also the L_2 discrepancies of the Fibonacci lattice are of the optimal order.

We also prove that the extreme L_2 discrepancy is always dominated by the standard L_2 discrepancy, a result that was already conjectured by Morokoff and Caflisch when they introduced the notion of extreme L_2 discrepancy in 1994.

1. INTRODUCTION

We study several discrepancy notions of two well-known instances of plane point sets, namely the Hammersley point set and rational lattices. The discrepancies are considered with respect to the L_2 norm and a variety of test sets. We define the (standard) L_2 discrepancy, the extreme L_2 discrepancy and the periodic L_2 discrepancy.

Let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ be an arbitrary N-element point set in the unit square $[0, 1)^2$. For any measurable subset B of $[0, 1]^2$ we define the counting function

 $A(B, \mathcal{P}) := |\{n \in \{0, 1, \dots, N-1\} : \boldsymbol{x}_n \in B\}|,$

i.e., the number of elements from \mathcal{P} that belong to the set B. By the *local discrepancy* of \mathcal{P} with respect to a given measurable "test set" B one understands the expression

$$A(B,\mathcal{P}) - N\lambda(B)$$

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where λ denotes the Lebesgue measure of *B*. A global discrepancy measure is then obtained by considering a norm of the local discrepancy with respect to a fixed class of test sets. Here we restrict ourselves to the L_2 norm, but we variegate the class of test sets.

The (standard) L_2 discrepancy uses the class of axis-parallel squares anchored at the origin as test sets. The formal definition is

$$L_{2,N}(\mathcal{P}) := \left(\int_{[0,1]^2} |A([\mathbf{0},\boldsymbol{t}),\mathcal{P}) - N\lambda([\mathbf{0},\boldsymbol{t}))|^2 \, \mathrm{d}\boldsymbol{t} \right)^{\frac{1}{2}},$$

where for $t = (t_1, t_2) \in [0, 1]^2$ we set $[0, t) = [0, t_1) \times [0, t_2)$ with area $\lambda([0, t)) = t_1 t_2$.

The extreme L_2 discrepancy uses arbitrary axis-parallel rectangles contained in the unit square as test sets. For $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_2, y_2)$ in $[0, 1]^2$ and $\boldsymbol{x} \leq \boldsymbol{y}$ let $[\boldsymbol{x}, \boldsymbol{y}) = [x_1, y_1) \times [x_2, y_2)$, where $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_1 \leq y_1$ and $x_2 \leq y_2$. The extreme L_2 discrepancy of \mathcal{P} is then defined as

$$L_{2,N}^{\text{extr}}(\mathcal{P}) := \left(\int_{[0,1]^2} \int_{[0,1]^2, \boldsymbol{x} \leq \boldsymbol{y}} |A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P}) - N\lambda([\boldsymbol{x}, \boldsymbol{y}))|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{2}}$$

Note that the only difference between standard and extreme L_2 discrepancy is the use of anchored and arbitrary rectangles in $[0, 1]^2$, respectively. The term "extreme" is used in order to distinguish this notion of L_2 discrepancy from the standard L_2 discrepancy and refers to the corresponding nomenclature for L_{∞} discrepancies (see, e.g., [25, Definition 2.1 and 2.2]).

The *periodic* L_2 *discrepancy* uses periodic rectangles as test sets, which are defined as follows: For $x, y \in [0, 1]$ set

$$I(x,y) = \begin{cases} [x,y) & \text{if } x \le y, \\ [0,y) \cup [x,1) & \text{if } x > y, \end{cases}$$

and for $\boldsymbol{x}, \boldsymbol{y}$ as above we set $B(\boldsymbol{x}, \boldsymbol{y}) = I(x_1, y_1) \times I(x_2, y_2)$. We define the periodic L_2 discrepancy of \mathcal{P} as

$$L_{2,N}^{\mathrm{per}}(\mathcal{P}) := \left(\int_{[0,1]^2} \int_{[0,1]^2} |A(B(\boldsymbol{x},\boldsymbol{y}),\mathcal{P}) - N\lambda(B(\boldsymbol{x},\boldsymbol{y}))|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \right)^{\frac{1}{2}}$$

These discrepancy notions can also be defined for point sets in the *d*-dimensional unit cube $[0, 1)^d$ in an obvious way.

The standard L_2 discrepancy is a well known measure for the irregularity of distribution of point sets in the unit square with a close relation to the integration error of quasi-Monte Carlo rules via a Koksma-Hlawka type inequality (see, for example, [9, 26]). In contrast, the extreme and the periodic L_2 discrepancies are often not so familiar. For this reason we summarize a few facts about these discrepancy notions in the following.

According to [26], the extreme L_2 discrepancy was first considered by Morokoff and Caflisch in [23] since it is more symmetric than the standard L_2 discrepancy, which prefers the lower left vertex of the unit square. Morokoff and Caflisch could not state a Koksma-Hlawka type inequality for the extreme L_2 discrepancy, but later it has been shown that this quantity is the worst-case integration error of a certain space of periodic functions with a boundary condition (see [26] and the proof of Theorem 5 in Section 2).

The notion of periodic L_2 discrepancy is known from a paper by Lev [22], but as a matter of fact, it is just a geometric interpretation of the diaphony according to Zinterhof [31] (see Proposition 3 in Section 2). Its relation to the integration error of quasi-Monte Carlo rules is well-known, see, e.g., [16].

The celebrated lower bound of Roth [28] states that there exists a c > 0 such that for every N-element point set \mathcal{P} in $[0,1)^2$ the standard L_2 discrepancy satisfies $L_{2,N}(\mathcal{P}) \geq c\sqrt{1 + \log N}$. A general lower bound of the same order of magnitude also holds for the periodic L_2 discrepancy (see Corollary 2 in Section 2). In the present paper we adapt the proof of Roth to show that also the extreme L_2 discrepancy satisfies a lower bound $L_{2,N}^{\text{extr}}(\mathcal{P}) \geq c\sqrt{1 + \log N}$ (see Theorem 6 in Section 2).

For every \mathcal{P} it is obviously true that

(1)
$$L_{2,N}^{\text{per}}(\mathcal{P}) \ge L_{2,N}^{\text{extr}}(\mathcal{P}).$$

This is because when restricting the range of integration in the definition of periodic L_2 discrepancy to $x \leq y$, then the test sets are exactly those used for the extreme discrepancy. In [23] the authors further conjectured that the extreme L_2 discrepancy is smaller than the standard L_2 discrepancy. They could not prove a result in this direction, but their conjecture was supported by numerical experiments. We will show that this order relation indeed holds true (see Theorem 5 in Section 2).

We mention some further results about extreme and periodic L_2 discrepancy: The exact asymptotic behaviour of the average of standard, extreme and periodic L_2 discrepancy of random point sets is given in [14] and [17]. See also [12] for an upper bound in case of extreme L_2 discrepancy. Bounds on the periodic L_2 discrepancy for certain multi-dimensional point sets (Korobov's *p*-sets) can be found in [7]. There the dependence of the bounds on the dimension *d* is of particular interest.

In the present paper we prove exact formulas of the aforementioned L_2 discrepancies for Hammersley point sets and for rational lattices. In the next section we present some further information and new results about periodic and extreme L_2 discrepancy. There we also prove the already mentioned "Roth-type" lower bound on extreme L_2 discrepancy and the order relation between standard and extreme L_2 discrepancy that was already conjectured by Morokoff and Caflisch. The exact discrepancy formulas for Hammersley point sets (Theorem 8) and for rational lattices (Theorem 10) will then be presented in Section 3. Their proofs are given in Sections 4-7.

2. More results about periodic- and extreme L_2 discrepancy

For a point set $\mathcal{P} = \{x_0, x_1, \dots, x_{N-1}\}$ and a real vector $\boldsymbol{\delta} \in [0, 1]^d$ the shifted point set $\mathcal{P} + \boldsymbol{\delta}$ is defined as $\mathcal{P} + \boldsymbol{\delta} = \{\{x_0 + \boldsymbol{\delta}\}, \dots, \{x_{N-1} + \boldsymbol{\delta}\}\}$, where $\{x_j + \boldsymbol{\delta}\}$ means that the fractional-part-function $\{x\} = x - \lfloor x \rfloor$ for non-negative real numbers x is applied component-wise to the vector $x_j + \boldsymbol{\delta}$.

We call this kind of shift a geometric shift - in contrast to the digital shift as explained in Section 3. The root-mean-square L_2 discrepancy of a shifted (and weighted) point set \mathcal{P} with respect to all uniformly distributed shift vectors $\boldsymbol{\delta} \in [0, 1]^d$ is

(2)
$$\sqrt{\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\boldsymbol{\mathcal{P}}+\boldsymbol{\delta}))^2]} = \left(\int_{[0,1]^d} (L_{2,N}(\boldsymbol{\mathcal{P}}+\boldsymbol{\delta}))^2 \,\mathrm{d}\boldsymbol{\delta}\right)^{\frac{1}{2}}.$$

The following relation between periodic L_2 discrepancy and root-meansquare L_2 discrepancy of a shifted point set \mathcal{P} holds (see [7, 22] for proofs):

Proposition 1. For every N-element point set \mathcal{P} in $[0,1)^d$ we have

$$L_{2,N}^{\mathrm{per}}(\mathcal{P}) = \sqrt{\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{P}+\boldsymbol{\delta}))^2]}.$$

From this relation we can deduce the following general lower bound on the periodic L_2 discrepancy of point sets in $[0, 1)^d$:

Corollary 2. For every dimension d there exists a quantity $c_d > 0$ such that every N-element point set \mathcal{P} in the unit cube $[0,1)^d$ has periodic L_2 discrepancy bounded by

$$L_{2,N}^{\mathrm{per}}(\mathcal{P}) \ge c_d \left(1 + \log N\right)^{\frac{d-1}{2}}.$$

Proof. Let \mathcal{P} be an arbitrary N-element point sets \mathcal{P} in $[0,1)^d$. Then we have

$$L_{2,N}^{\mathrm{per}}(\mathcal{P}) = \sqrt{\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{P}+\boldsymbol{\delta}))^2]} \ge \inf_{\boldsymbol{\delta}\in[0,1]^d} L_{2,N}(\mathcal{P}+\boldsymbol{\delta}) \ge c_d \left(1 + \log N\right)^{\frac{d-1}{2}},$$

where we used Roth's lower bound on the standard L_2 discrepancy.

Another important fact is that the periodic L_2 discrepancy can be expressed in terms of exponential sums.

Proposition 3. For $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ in $[0, 1)^d$ we have

$$(L_{2,N}^{\mathrm{per}}(\mathcal{P}))^2 = rac{1}{3^d} \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}} rac{1}{r(\boldsymbol{k})^2} \left| \sum_{h=0}^{N-1} \exp(2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}_h) \right|^2,$$

where $i = \sqrt{-1}$ and where for $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ we set

(3)
$$r(\mathbf{k}) = \prod_{j=1}^{d} r(k_j) \quad and \quad r(k_j) = \begin{cases} 1 & \text{if } k_j = 0, \\ \frac{2\pi |k_j|}{\sqrt{6}} & \text{if } k_j \neq 0. \end{cases}$$

Proof. See [16, p. 390].

The above formula shows that the periodic L_2 discrepancy is - up to a multiplicative factor - exactly the diaphony which is a well-known measure for the irregularity of distribution of point sets and which was introduced by Zinterhof [31] in 1976 (see also [10]).

From this view point we immediately find an order relation between the standard and the periodic L_2 discrepancy in the one-dimensional case.

Corollary 4. For every N-element point set \mathcal{P} in the unit interval [0,1) we have

$$L_{2,N}^{\mathrm{per}}(\mathcal{P}) \leq \sqrt{2} L_{2,N}(\mathcal{P})$$

We have equality if N is even and \mathcal{P} is symmetric, i.e., with every x_n also $1 - x_n$ belongs to \mathcal{P} .

Proof. In the one-dimensional case the well-known formula of Koksma (see [21, p. 110]) establishes a connection between L_2 discrepancy and diaphony. This formula follows easily from an application of Parseval's identity to the local discrepancy. From this we have

$$(L_{2,N}(\mathcal{P}))^{2} = \left(\sum_{n=0}^{N-1} \left(\frac{1}{2} - x_{n}\right)\right)^{2} + \frac{1}{2\pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}} \left|\sum_{h=0}^{N-1} \exp(2\pi i k x_{h})\right|^{2}$$

$$(4) \geq \frac{1}{2\pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}} \left|\sum_{h=0}^{N-1} \exp(2\pi i k x_{h})\right|^{2}$$

$$= \frac{1}{2} (L_{2,N}^{\text{per}}(\mathcal{P}))^{2},$$

where we used Proposition 3 in the last step. The result follows from multiplying by two and taking the square root. For symmetric \mathcal{P} we have equality in (4), because then $\sum_{n=0}^{N-1} \left(\frac{1}{2} - x_n\right)$ equals 0.

We now show that the extreme L_2 discrepancy is indeed always smaller than the standard L_2 discrepancy as conjectured in [23]. This is actually implied by the known relationships of the extreme and the standard L_2 discrepancy to worst-case errors of quasi-Monte Carlo rules for numerical integration.

Theorem 5. For every N-element point set \mathcal{P} in $[0,1)^d$ we have

$$L_{2,N}^{\text{extr}}(\mathcal{P}) \leq L_{2,N}(\mathcal{P})$$

Proof. As already mentioned, we need the relationship between the extreme and the standard L_2 discrepancy, respectively, and worst-case errors of quasi-Monte Carlo rules for numerical integration. The quoted facts can all be found in [26].

Recall that the worst-case error $e(I, Q, H(K_d))$ of the quasi-Monte Carlo rule

$$Q(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(\boldsymbol{x}_k)$$

for the integration problem

$$I(f) = \int_{[0,1]^d} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

of functions $f : [0,1]^d \to \mathbb{R}$ in a reproducing kernel Hilbert space $H(K_d)$ with kernel $K_d : [0,1]^d \times [0,1]^d \to \mathbb{R}$ is given as

$$e(I, Q, H(K_d)) = \sup_{\|f\|_{H(K_d)} \le 1} |I(f) - Q(f)|.$$

A closed formula involving the kernel and the Riesz representer $h_d \in H(K_d)$ of the integration functional I is

$$e(I,Q,H(K_d))^2 = \|h_d\|_{H(K_d)}^2 - \frac{2}{N}\sum_{k=0}^{N-1}h_d(\boldsymbol{x}_k) + \frac{1}{N^2}\sum_{k,\ell=0}^{N-1}K(\boldsymbol{x}_k,\boldsymbol{x}_\ell)$$

see [26, (9.31)].

We now introduce the relevant reproducing kernel Hilbert spaces. They are Hilbert space tensor products of Sobolev spaces of univariate functions. Let $W_2^1([0,1])$ be the Sobolev space of absolutely continuous functions f: $[0,1] \to \mathbb{R}$ with weak first derivative $f' \in L_2([0,1])$. Let H be the subspace of all functions $f \in W_2^1([0,1])$ satisfying the boundary condition f(1) = 0equipped with the norm $||f||_H = ||f'||_{L_2}$. Let H^{extr} be the subspace of all functions $f \in W_2^1([0,1])$ satisfying the boundary conditions f(0) = f(1) = 0equipped with the norm $||f||_{H^{\text{extr}}} = ||f'||_{L_2}$. Obviously, H^{extr} is the subspace of the 1-periodic functions in H. Both H and H^{extr} are reproducing kernel Hilbert spaces. The kernels are given as $K(x, y) = \min\{1 - x, 1 - y\}$ for Hand $K^{\text{extr}}(x, y) = \min\{x, y\} - xy$ for H^{extr} . Denote the d-fold Hilbert space tensor products of these spaces by H_d and H_d^{extr} , respectively. Their kernels K_d and K_d^{extr} are the d-fold tensor products of the corresponding univariate kernels.

Now, using the above formula for the worst-case error of the integration problem and comparing to the formulas of the standard and extreme L_2 discrepancy in Proposition 13 in Section 4 below shows that

$$Ne(I,Q,H_d) = L_{2,N}(\mathcal{P})$$
 and $Ne(I,Q,H_d^{\text{extr}}) = L_{2,N}^{\text{extr}}(\mathcal{P})$

where $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ is the point set used by the quasi-Monte Carlo rule Q. A complete derivation of the first equation is given in [26, Section 9.5.1], for the second identity we refer to [26, Section 9.5.5].

But, since H_d^{extr} is a subspace of H_d (with the induced scalar product and norm), the inequality $e(I, Q, H_d^{\text{extr}}) \leq e(I, Q, H_d)$ is obvious from the definition of the worst-case error.

Next, we show how to adapt the proof of Roth's lower bound for the extreme L_2 discrepancy.

Theorem 6. For every dimension d there exists a quantity $c_d > 0$ such that every N-element point set \mathcal{P} in the unit cube $[0,1)^d$ has extreme L_2 discrepancy bounded by

$$L_{2,N}^{\operatorname{extr}}(\mathcal{P}) \ge c_d \left(1 + \log N\right)^{\frac{d-1}{2}}.$$

Proof. We assume some familiarity with the proof of Roth in the language of Haar functions as it can be found, e.g., in [2] or [6]. We only prove the case d = 2, the extension to general d is done as for Roth's lower bound.

A dyadic interval in [0, 1] is an interval of the form $I = [2^{-m}n, 2^{-m}(n+1))$ with nonnegative integers m, n satisfying $0 \le n < 2^m$. The Haar function supported on I is the function $h_I : [0, 1] \to \mathbb{R}$ which is +1 on the left and -1 on the right half of I and 0 outside of I. The Haar functions form an orthogonal system in $L_2([0, 1])$. The Haar functions in $[0, 1]^2$ are tensor products of the univariate Haar functions. A dyadic rectangle in $[0, 1]^2$ is a product $R = I \times J$ of two dyadic intervals I and J. The Haar function supported on R is the function $h_R : [0, 1]^2 \to \mathbb{R}$ given as $h_R(x, y) = h_I(x)h_J(y)$. The Haar functions form an orthogonal system in $L_2([0, 1]^2)$.

Roth's method for proving an order optimal lower bound for the standard L_2 dicrepancy uses the orthogonal expansion of the discrepancy function into a series of Haar functions. To adapt the proof for the extreme L_2 discrepancy, we first fix $\boldsymbol{x} \in [0, 1/2)^2$ and consider the discrepancy function

$$D(\boldsymbol{y}) = A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P}) - N\lambda([\boldsymbol{x}, \boldsymbol{y}))$$

just as a function of $\boldsymbol{y} \in [1/2, 1)^2$. For $\boldsymbol{y} \in [0, 1]^2 \setminus [1/2, 1)^2$, we define $D(\boldsymbol{y}) = 0$. The crucial point in Roth's proof as well as in this argument here is that the scalar product of the discrepancy function $D(\boldsymbol{y})$ with a Haar function $h_R(\boldsymbol{y})$ does not depend on the point set \mathcal{P} as long as R does not contain a point of \mathcal{P} . In fact, we have

$$\langle D, h_R \rangle = -2^{-4} N \lambda(R)^2$$
 if $R \subseteq [1/2, 1)^2$ and $\mathcal{P} \cap R = \emptyset$.

We now fix a natural number m satisfying $2^{m-3} \leq 2N \leq 2^{m-2}$ and consider all dyadic rectangles $R = I \times J$ of area 2^{-m} . They come in m + 1different shapes according to the side length of R, i.e., the lengths of I and J. There are 2^m dyadic rectangles of the same shape tiling the unit square. There are m - 1 shapes where both side length are at most 1/2, and one quarter, that is 2^{m-2} , of the dyadic rectangles R of such a shape satisfy $R \subseteq [1/2, 1)^2$. Since $2N \leq 2^{m-2}$, at least half of those rectangles also satisfy $\mathcal{P} \cap R = \emptyset$.

Now Bessel's inequality implies

$$\int_{[0,1]^2} D(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \ge \sum_R \frac{\langle D, h_R \rangle^2}{\|h_R\|_{L_2}^2},$$

where the sum is taken over all dyadic rectangles R. Using just the dyadic rectangles with area 2^{-m} and satisfying $R \subseteq [1/2, 1)^2$ as well as $\mathcal{P} \cap R = \emptyset$, of which there are at least $(m-1)2^{m-3}$, we obtain that

$$\int_{[0,1]^2} D(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \ge (m-1)2^{m-3} \frac{2^{-8}N^2 2^{-4m}}{2^{-m}} = 2^{-11}(m-1)2^{-2m}N^2.$$

Now using $2^{-m}N \ge 2^{-4}$ and $m-1 \ge 2 + \log_2 N$ we arrive at

$$\int_{[0,1]^2} D(\boldsymbol{y})^2 \, \mathrm{d}\boldsymbol{y} \ge 2^{-19} (2 + \log_2 N).$$

Since this holds for any fixed $x \in [0, 1/2)^2$, we can finally integrate over all these x and obtain

$$L_{2,N}^{\text{extr}}(\mathcal{P})^2 \ge 2^{-21}(2 + \log_2 N).$$

Hence the desired result follows.

In dimension one we have the following surprising relationship between periodic and extreme L_2 discrepancy. Whether a corresponding relation also holds in higher dimensions is an open question (see also the brief discussion at the end of Section 3).

Theorem 7. For every N-element point set \mathcal{P} in the unit interval [0, 1) we have

$$(L_{2,N}^{\text{per}}(\mathcal{P}))^2 = 2(L_{2,N}^{\text{extr}}(\mathcal{P}))^2.$$

Proof. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\}$. We may assume that the points are ordered, i.e., $x_0 \leq x_1 \leq \ldots \leq x_{N-1}$. Easy computation (see also [20, Eq. (1.3) shows that

$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = \frac{1}{12} + \frac{1}{2} \sum_{n,m=0}^{N-1} \left(x_n - x_m - \frac{n-m}{N} \right)^2$$

From this formula and since $\sum_{n,m=0}^{N-1} (n-m)^2 = N^2 (N^2 - 1)/6$ we obtain

$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = \frac{1}{2} \left(\frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - \frac{2}{N} \sum_{n,m=0}^{N-1} (x_n - x_m)(n - m) \right).$$

We have

$$\sum_{n,m=0}^{N-1} (x_n - x_m)(n - m) = \sum_{n,m=0}^{N-1} (nx_n - mx_n - nx_m + mx_m)$$
$$= 2N \sum_{n=0}^{N-1} nx_n - N(N-1) \sum_{n=0}^{N-1} x_n$$

and hence

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$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = \frac{1}{2} \left(\frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - 4 \sum_{n=0}^{N-1} nx_n + 2(N-1) \sum_{n=0}^{N-1} x_n \right).$$

For the periodic L_2 discrepancy in dimension one we know (see, e.g., the forthcoming Proposition 13 or [16, p. 389-390]) that

$$(L_{2,N}^{\mathrm{per}}(\mathcal{P}))^2 = \sum_{n,m=0}^{N-1} B_2(|x_n - x_m|),$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial. Inserting the formula for B_2 we obtain

$$(L_{2,N}^{\text{per}}(\mathcal{P}))^2 = \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - \sum_{n,m=0}^{N-1} |x_n - x_m|.$$

We have further

$$\sum_{n,m=0}^{N-1} |x_n - x_m|$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{n} (x_n - x_m) + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} (x_m - x_n)$$

$$= \sum_{n=0}^{N-1} x_n (n+1) - \sum_{n=0}^{N-1} \sum_{m=0}^{n} x_m + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} x_m - \sum_{n=0}^{N-1} x_n (N-1-n)$$

$$= 2 \sum_{n=0}^{N-1} x_n (n+1) - N \sum_{n=0}^{N-1} x_n - \sum_{m=0}^{N-1} x_m \sum_{\substack{n=m\\ =N-m}}^{N-1} 1 + \sum_{m=0}^{N-1} x_m \sum_{\substack{n=0\\ =m}}^{m-1} 1$$

$$= 4 \sum_{n=0}^{N-1} n x_n - 2(N-1) \sum_{n=0}^{N-1} x_n.$$

Hence

(6)
$$(L_{2,N}^{\text{per}}(\mathcal{P}))^2 = \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - 4 \sum_{n=0}^{N-1} nx_n + 2(N-1) \sum_{n=0}^{N-1} x_n.$$

A comparison of (5) and (6) shows the result.

Note that Theorem 7 in combination with Corollary 4 gives another proof of Theorem 5 for the one-dimensional case.

Summary. In this section we presented a number of inequalities and relations between the three types of L_2 discrepancy. We briefly summarize these relations here: For every N-element point set \mathcal{P} in $[0, 1)^d$ we have

$$L_{2,N}^{\text{extr}}(\mathcal{P}) \leq L_{2,N}^{\text{per}}(\mathcal{P}) \quad \text{and} \quad L_{2,N}^{\text{extr}}(\mathcal{P}) \leq L_{2,N}(\mathcal{P}).$$

Furthermore, there exists a quantity $c_d > 0$ such that for every N-element point set \mathcal{P} in $[0,1)^d$ we have

$$c_d(1+\log N)^{\frac{d-1}{2}} \le L_{2,N}^{\text{extr}}(\mathcal{P}).$$

In the one-dimensional case we even know that

$$L_{2,N}^{\text{per}}(\mathcal{P}) = \sqrt{2} L_{2,N}^{\text{extr}}(\mathcal{P}) \quad \text{and} \quad L_{2,N}^{\text{per}}(\mathcal{P}) \le \sqrt{2} L_{2,N}(\mathcal{P}).$$

3. Exact discrepancy formulas

In this section we present exact formulas for the L_2 discrepancies of Hammersley point sets and of rational lattices. Both of them are well established constructions of point sets in discrepancy theory.

Hammersley point set. We calculate the extreme and the periodic L_2 discrepancy of the 2-dimensional Hammersley point set in base 2, which for $m \in \mathbb{N}$ is given as the set of $N = 2^m$ points

$$\mathcal{H}_m = \left\{ \left(\frac{t_m}{2} + \dots + \frac{t_1}{2^m}, \frac{t_1}{2} + \dots + \frac{t_m}{2^m} \right) : t_1, \dots, t_m \in \{0, 1\} \right\}.$$

The Hammersley point set is the prototype of low-discrepancy point sets whose construction is based on digit representations. Its elements (x_k, y_k)

for $k = 0, 1, ..., 2^m - 1$ can be also written in the form

$$x_k = \frac{k}{2^m}$$
 and $y_k = \varphi_2(k)$,

where $\varphi_2(k)$ is the van der Corput digit reversal function $\varphi_2(k) = \frac{\kappa_0}{2} + \frac{\kappa_1}{2^2} + \cdots + \frac{\kappa_r}{2^{r+1}}$ whenever k has dyadic expansion $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_r 2^r$ with $\kappa_i \in \{0, 1\}$. Note that the Hammersley point set is symmetric with respect to the main diagonal in \mathbb{R}^2 . Another view point of Hammersley point sets as a special instance of digital nets will be used in Section 6.

We have the following exact result on the extreme and the periodic L_2 discrepancy of the Hammersley point set. For comparison only we also include the formula for the standard L_2 discrepancy.

Theorem 8. We have

$$(L_{2,2^m}(\mathcal{H}_m))^2 = \frac{m^2}{64} + \frac{29m}{192} + \frac{3}{8} - \frac{m}{2^{m+4}} + \frac{1}{2^{m+2}} - \frac{1}{9 \cdot 2^{2m+3}},$$

$$(L_{2,2^m}^{\text{extr}}(\mathcal{H}_m))^2 = \frac{m}{64} + \frac{1}{72} - \frac{1}{9 \cdot 4^{m+2}}, \text{ and}$$

$$(L_{2,2^m}^{\text{per}}(\mathcal{H}_m))^2 = \frac{m}{16} + \frac{1}{9} + \frac{1}{9 \cdot 4^{m+1}}.$$

The result for the standard L_2 discrepancy is well-known. A proof can be found, for example, in [13, 27]. The results for the extreme and periodic L_2 discrepancy are new. The proofs of these formulas - along with a new proof for the standard L_2 discrepancy - will be presented in Section 4.

An immediate consequence of Theorem 8 is that - in contrast to the standard L_2 discrepancy - the extreme and periodic L_2 discrepancy of the Hammersley point set are of the optimal order $\sqrt{\log N}$, respectively. The L_2 discrepancy of the Hammersley point set is only of order log N, which is not the optimal order according to the aforementioned lower bound of Roth [28]. Several modifications such as digital shifts or symmetrization are necessary to overcome this defect of the Hammersley point set (see e.g. [11, 13, 15, 19]), which for the other two notions of L_2 discrepancy can be understood as a root-mean-square L_2 discrepancy of shifted point sets (see Proposition 1 in Section 2) and with inequality (1) in mind, this result does not come unexpected.

Theorem 8 further demonstrates that the standard and the extreme L_2 discrepancy are not equivalent in general. This is in contrast to the L_{∞} extreme/star discrepancies $D_N(\mathcal{P})$ and $D_N^*(\mathcal{P})$, which are defined as

$$D_N(\mathcal{P}) = \sup_{\boldsymbol{x}, \boldsymbol{y} \in [0,1]^2, \, \boldsymbol{x} \leq \boldsymbol{y}} |A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P}) - N\lambda([\boldsymbol{x}, \boldsymbol{y}))|$$

and

$$D_N^*(\mathcal{P}) = \sup_{\boldsymbol{t} \in [0,1]^2} |A([\boldsymbol{0},\boldsymbol{t}),\mathcal{P}) - N\lambda([\boldsymbol{0},\boldsymbol{t}))|$$

for two-dimensional point sets. For these discrepancy notions we have the almost trivial inequalities $D_N^*(\mathcal{P}) \leq D_N(\mathcal{P}) \leq 4D_N^*(\mathcal{P})$.

Another obvious implication of Theorem 8 in conjunction with Proposition 1 is the fact that there exists a geometric shift $\boldsymbol{\delta} \in [0, 1]^2$ such that the point set $\mathcal{H}_m + \boldsymbol{\delta}$ achieves the optimal order of L_2 discrepancy. In fact, Roth [29] used geometric shifts (but only in one coordinate) to prove for the first time the existence of point sets in $[0, 1)^d$ with the optimal L_2 discrepancy rate $(\log N)^{\frac{d-1}{2}}$. He could show that the average of the L_2 discrepancy of higher dimensional versions of the Hammersley point set over all possible shifts achieves this bound; hence it was a probabilistic existence result. In dimension 2, Roth's result has later been derandomized by Bilyk [1] who could find an explicit geometric shift $\boldsymbol{\delta} = (\delta, 0) \in [0, 1]^2$ such that $\mathcal{H}_m + \boldsymbol{\delta}$ has the optimal order of L_2 discrepancy.

Since the periodic L_2 discrepancy equals the root-mean-square discrepancy with respect to geometric shifts, we would like to compare the result on $L_{2,2^m}^{\text{per}}(\mathcal{H}_m)$ with the root-mean-square L_2 discrepancy of the Hammersley point set with respect to *digital* shifts, which are often studied in this context.

These kind of shifts are based on digit-wise addition modulo 2. In more detail, for $x, y \in [0, 1)$ with dyadic expansions $x = \sum_{i=1}^{\infty} \frac{\xi_i}{2^i}$ and $y = \sum_{i=1}^{\infty} \frac{\eta_i}{2^i}$ with digits $\xi_i, \eta_i \in \{0, 1\}$ for all $i, j \ge 1$ we define

$$x \oplus y := \sum_{i=1}^{\infty} \frac{\xi_i + \eta_i \pmod{2}}{2^i}.$$

For vectors $\boldsymbol{x}, \boldsymbol{y} \in [0, 1)^d$ the digit-wise addition $\boldsymbol{x} \oplus \boldsymbol{y}$ is defined componentwise.

For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ and a real vector $\boldsymbol{\delta} \in [0, 1]^d$ we define the *digitally shifted point set* $\mathcal{P} \oplus \boldsymbol{\delta}$ as

$$\mathcal{P}\oplus oldsymbol{\delta} = \{oldsymbol{x}_0\oplus oldsymbol{\delta}, oldsymbol{x}_1\oplus oldsymbol{\delta}, \dots, oldsymbol{x}_{N-1}\oplus oldsymbol{\delta}\}.$$

The root-mean-square L_2 discrepancy of a digitally shifted point set \mathcal{P} with respect to all uniformly distributed (digital) shift vectors $\boldsymbol{\delta} \in [0, 1)^d$ is

(7)
$$\sqrt{\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{P}\oplus\boldsymbol{\delta}))^2]} = \left(\int_{[0,1]^d} (L_{2,N}(\mathcal{P}\oplus\boldsymbol{\delta}))^2 \,\mathrm{d}\boldsymbol{\delta}\right)^{\frac{1}{2}}.$$

This is the digital equivalent to the root-mean-square L_2 discrepancy of a geometrically shifted point set \mathcal{P} given in (2) and therefore to the periodic L_2 discrepancy.

We compute $\mathbb{E}_{\delta}[(L_{2,N}(\mathcal{H}_m \oplus \delta))^2]$ and obtain the following result:

Theorem 9. For the 2^m -element Hammersley point set \mathcal{H}_m we have

$$\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{H}_m \oplus \boldsymbol{\delta}))^2] = \frac{m}{24} + \frac{5}{36}$$

The proof of Theorem 9 will be presented in Section 6. Note that the root-mean-square L_2 discrepancy for digitally shifted Hammersley points is about a factor $\sqrt{2/3}$ lower than for geometrially shifted Hammersley points.

Rational lattices. We will also calculate the extreme and the periodic L_2 discrepancy of rational lattices. First we introduce irrational lattices. Let $\alpha \in \mathbb{R}$ be an irrational number. Then for $N \in \mathbb{N}$ we define the point set

$$\mathcal{A}_N(\alpha) := \left\{ \left(\frac{k}{N}, \{k\alpha\} \right) : k = 0, 1, \dots, N - 1 \right\},\$$

where $\{k\alpha\}$ denotes the fractional part of the real $k\alpha$. Let $\alpha = [a_0; a_1, a_2, ...]$ be the continued fraction expansion of α and $\frac{p_n}{q_n}$ for $n \in \mathbb{N}$ be the n^{th} convergent of α ; i.e. $\frac{p_n}{q_n} = [a_0; a_1, ..., a_n]$. Further we consider the sets

$$\mathcal{L}_n(\alpha) := \left\{ \left(\frac{k}{q_n}, \left\{ \frac{kp_n}{q_n} \right\} \right) : k = 0, 1, \dots, q_n - 1 \right\},\$$

which are an approximation of the set $\mathcal{A}_N(\alpha)$. We call a point set $\mathcal{L}_n(\alpha)$ a rational lattice. A special instance of a rational lattice is the Fibonacci lattice \mathcal{F}_n , which is obtained for $\alpha = \frac{1}{2}(\sqrt{5}+1)$; i.e. the golden ratio. Then $\alpha = [1; 1, 1, ...], (p_n, q_n) = (F_{n-1}, F_n)$ and

$$\mathcal{F}_n := \left\{ \left(\frac{k}{F_n}, \left\{ \frac{kF_{n-1}}{F_n} \right\} \right) : k = 0, 1, \dots, F_n - 1 \right\},\$$

where the Fibonacci numbers are defined recursively via $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-1}$ for $n \ge 2$.

We have the following formula for the L_2 discrepancies of rational lattices.

Theorem 10. Let α be given as above. Then we have

$$(L_{2,q_n}(\mathcal{L}_n(\alpha))^2 = \frac{1}{16q_n^2} \sum_{r=1}^{q_n-1} \frac{1+2\cos^2\left(\frac{\pi rp_n}{q_n}\right)}{\sin^2\left(\frac{\pi rp_n}{q_n}\right)} + \left(\mathcal{D}(p_n,q_n) + \frac{3}{4}\right)^2 + \frac{1}{18} - \frac{1}{144q_n^2},$$

$$(L_{2,q_n}^{\text{extr}}(\mathcal{L}_n(\alpha)))^2 = \frac{1}{16q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2\left(\frac{\pi r}{q_n}\right)\sin^2\left(\frac{\pi rp_n}{q_n}\right)} + \frac{1}{72} - \frac{1}{144q_n^2}, \text{ and }$$

$$(L_{2,q_n}^{\text{per}}(\mathcal{L}_n(\alpha))^2 = \frac{1}{4q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2\left(\frac{\pi r}{q_n}\right)\sin^2\left(\frac{\pi rp_n}{q_n}\right)} + \frac{1}{9} + \frac{1}{36q_n^2},$$

where in the first formula $\mathcal{D}(p,q)$ is the inhomogeneous Dedekind sum

$$\mathcal{D}(p,q) = \sum_{k=1}^{q-1} \rho\left(\frac{k}{q}\right) \rho\left(\frac{kp}{q}\right) \quad where \quad \rho(x) = \frac{1}{2} - \{x\}.$$

The first formula for the L_2 discrepancy is [3, Theorem 6]. The proofs of the formulas for the extreme and periodic L_2 discrepancy will be given in Section 7.

The case of Fibonacci lattices is a matter of particular interest. Hinrichs and Oetters-hagen [16] minimized the periodic L_2 discrepancy over *N*-element point sets in the unit square for small values of *N*. If $N \in \{1, 2, 3, 5, 8, 13\}$ (all of them Fibonacci numbers), then the obtained unique global minimizer of the periodic L_2 discrepancy (modulo geometric shifts and other torus symmetries; see [16, Section 3.2]) are Fibonacci lattices.

One can show that the term

$$\frac{1}{F_n^2} \sum_{r=1}^{F_n-1} \frac{1}{\sin^2\left(\frac{\pi r}{F_n}\right) \sin^2\left(\frac{\pi r F_{n-1}}{F_n}\right)}$$

is of order n. Numerical experiments in [3] indicate that

(8)
$$\frac{1}{F_n^2} \sum_{r=1}^{F_n - 1} \frac{1}{\sin^2\left(\frac{\pi r}{F_n}\right) \sin^2\left(\frac{\pi r F_{n-1}}{F_n}\right)} \approx 0.119257n.$$

A few years later the involved constant on the right hand side of (8) was identified to have the explicit expression $\frac{4}{15\sqrt{5}}$ (see [5]). Furthermore, it is well-known that $\log F_n$ is of order of magnitude n, i.e., $\log F_n \simeq n$. This shows that all considered L_2 discrepancies of the Fibonacci lattice are of optimal order of magnitude with respect to the corresponding Roth-type lower bounds. In fact it follows from [4, Lemma 7] that in case of extreme and periodic L_2 discrepancy the same is true for all irrational $\alpha = [a_0; a_1, a_2, ...]$ with bounded partial quotients (i.e. $a_k \leq M$ for some constant M and for all $k \geq 0$). Therefore every rational lattice connected to such an α can be shifted geometrically in a way such that the resulting point set achieves the optimal order of L_2 discrepancy. From the same paper it is known that the unshifted lattice $\mathcal{L}_n(\alpha)$ has the optimal order of L_2 discrepancy if and only if $\sum_{k=0}^{n} (-1)^k a_k \leq c\sqrt{n}$ for a constant c > 0.

Remark 11. It follows from Theorem 10 and (8) that

$$\liminf_{N \to \infty} \inf_{\#\mathcal{P}=N} \frac{L_{2,N}^{\text{extr}}(\mathcal{P})}{\sqrt{\log N}} \le \eta := \sqrt{\frac{1}{60\sqrt{5}\log(\frac{\sqrt{5}+1}{2})}} = 0.124455\dots,$$

and

$$\liminf_{N \to \infty} \inf_{\#\mathcal{P}=N} \frac{L_{2,N}^{\text{per}}(\mathcal{P})}{\sqrt{\log N}} \le 2\eta = 0.248910\dots$$

Note that the corresponding constants one can derive from the results on the Hammersley point set in Theorem 8 are larger. For the standard L_2 discrepancy we have

$$\liminf_{N \to \infty} \inf_{\#\mathcal{P}=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}} \le \sqrt{2}\eta = 0.176006\dots,$$

where this constant is attained by symmetrized Fibonacci lattices; see [3].

Brief discussion of possible relationships between L_2 discrepancies. We point out the following peculiarity, which follows from Theorems 8 and 10:

Remark 12. If \mathcal{P} is either the Hammersley point set \mathcal{H}_m or a rational lattice $\mathcal{L}_n(\alpha)$, then we have the relation

(9)
$$(L_{2,N}^{\text{per}}(\mathcal{P})^2 = 4(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 + \frac{1}{18} + \frac{1}{18N^2},$$

where $N = 2^m$ or $N = q_n$, respectively.

From Remark 12 and other observations (e.g. the one-element point set $\mathcal{P} = \{(0,0)\}$ satisfies (9) because, as easily checked, $(L_{2,N}^{\text{per}}(\mathcal{P})^2 = 5/36 \text{ and } (L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = 1/144)$ one might conjecture that (9) holds for arbitrary *N*-element point sets in the unit square.

However, let us consider the regular grid

$$\Gamma_{m,d} = \left\{0, \frac{1}{m}, \dots, \frac{m-1}{m}\right\}^d$$

consisting of $N = m^d$ points in $[0, 1)^d$, where $m \in \mathbb{N}$. For this point set the L_2 discrepancies are easily computed using formulas which were introduced by Koksma [18] and Warnock [30] (see the forthcoming Proposition 13). As a result one obtains

$$(L_{2,m^d}^{\text{per}}(\Gamma_{m,d}))^2 = \left(\frac{m^2}{3} + \frac{1}{6}\right)^d - \left(\frac{m^2}{3}\right)^d$$

and

$$(L_{2,m^d}^{\text{extr}}(\Gamma_{m,d}))^2 = \frac{m^{2d} - (m^2 - 1)^d}{12^d}.$$

For d = 1 we have

$$(L_{2,m}^{\text{per}}(\Gamma_{m,1}))^2 = \frac{1}{6}$$
 and $(L_{2,m}^{\text{extr}}(\Gamma_{m,1}))^2 = \frac{1}{12}$

and hence we nicely observe the relation from Theorem 7.

For d = 2 we have

$$(L_{2,m^2}^{\text{extr}}(\Gamma_{m,2}))^2 = \frac{2m^2 - 1}{144}$$
 and $(L_{2,m^2}^{\text{per}}(\Gamma_{m,2}))^2 = \frac{m^2}{9} + \frac{1}{36}$

If m = 1, then $\Gamma_{1,2} = \{(0,0)\}$ and (9) is still satisfied. But if m > 1, then the relation (9) does *not* hold anymore for $\Gamma_{m,2}$. Not even the implied multiplier 4 complies, because

$$\lim_{m \to \infty} \frac{(L_{2,m^2}^{\text{per}}(\Gamma_{m,2}))^2}{(L_{2,m^2}^{\text{extr}}(\Gamma_{m,2}))^2} = 8.$$

These observations raise some interesting questions about relationships between periodic and extreme L_2 discrepancy. In particular: Which plane point sets satisfy relation (9)? Are the periodic and extreme L_2 discrepancies in arbitrary dimension d equivalent (like for d = 1 according to Theorem 7)?

4. The proof of Theorem 8

We use the following well known formulas for the standard, extreme and periodic L_2 discrepancy of point sets. Although we only need the twodimensional versions of these formulas in our proofs, we state the results for arbitrary dimension d.

Proposition 13. Let $\mathcal{P} = \{ x_0, x_1, ..., x_{N-1} \}$ be a point set in $[0, 1)^d$, where we write $x_k = (x_{k,1}, ..., x_{k,d})$ for $k \in \{0, 1, ..., N-1\}$. Then we have

(10)

$$(L_{2,N}(\mathcal{P}))^2 = \frac{N^2}{3^d} - \frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^d (1 - x_{k,i}^2) + \sum_{k,l=0}^{N-1} \prod_{i=1}^d \min(1 - x_{k,i}, 1 - x_{l,i}),$$

(11)
$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = \frac{N^2}{12^d} - \frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^d x_{k,i} (1 - x_{k,i})$$
$$+ \sum_{k,l=0}^{N-1} \prod_{i=1}^d \left(\min(x_{k,i}, x_{l,i}) - x_{k,i} x_{l,i} \right).$$

and

(12)
$$(L_{2,N}^{\text{per}}(\mathcal{P}))^2 = -\frac{N^2}{3^d} + \sum_{k,l=0}^{N-1} \prod_{i=1}^d \left(\frac{1}{2} - |x_{k,i} - x_{l,i}| + (x_{k,i} - x_{l,i})^2\right).$$

Proof. The first formula is well known and easily proved by direct integration (see [18, 30]). Sometimes this formula is referred to Warnock [30] what is historically not entirely correct, since it was already provided by Koksma [18] in 1942 for d = 1, but using the same proof method as later Warnock [30] for arbitrary dimension (see also [24]). Also the second formula follows by simple direct integration and can be found in [30] and [23, 26], respectively. The last formula can be found in [16, 26], where it was derived in the context of the worst-case error in a certain reproducing kernel Hilbert space. This formula can also be derived more directly from Proposition 1 and Equation (10). To this end, we observe that for $x, y \in [0, 1]$ we have

$$\int_0^1 \{x+\delta\} \,\mathrm{d}\delta = \frac{1}{2}, \quad \int_0^1 \{x+\delta\}^2 \,\mathrm{d}\delta = \frac{1}{3},$$

and

$$\int_0^1 \max\{\{x+\delta\}, \{y+\delta\}\} \,\mathrm{d}\delta = \frac{1}{2} + |y-x| - (y-x)^2.$$

This is easy calculation. We just show the third formula. Assume without loss of generality that $0 \le x \le y \le 1$. Then we have

$$\int_{0}^{1} \max\{\{x+\delta\}, \{y+\delta\}\} d\delta$$

= $\int_{0}^{1-y} \{y+\delta\} d\delta + \int_{1-y}^{1-x} \{x+\delta\} d\delta + \int_{1-x}^{1} \{y+\delta\} d\delta$

A. HINRICHS, R. KRITZINGER, AND F. PILLICHSHAMMER

$$= \int_{y}^{1} u \, \mathrm{d}u + \int_{1-(y-x)}^{1} u \, \mathrm{d}u + \int_{1+(y-x)}^{1+y} (u-1) \, \mathrm{d}u.$$

Now the result follows from evaluating the elementary integrals. The formula (12) follows as well.

Remark 14. Using the formulas (10), (11) and (12) and regarding the fact that $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$ for $x, y \in \mathbb{R}$, we find that for the standard L_2 discrepancy of a two-dimensional point set $\mathcal{P} = \{(x_k, y_k) : k = 0, 1, \ldots, N-1\}$ we have

$$(L_{2,N}(\mathcal{P}))^2 = \frac{N^2}{9} - \frac{N}{2} \sum_{k=0}^{N-1} (1 - x_k^2)(1 - y_k^2) + \frac{1}{4} \sum_{k,l=0}^{N-1} (2 - x_k - x_l - |x_k - x_l|)(2 - y_k - y_l - |y_k - y_l|),$$

for its extreme L_2 discrepancy we have

$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^{2} = \frac{N^{2}}{144} - \frac{N}{2} \sum_{k=0}^{N-1} x_{k}(1-x_{k})y_{k}(1-y_{k}) + \frac{1}{4} \sum_{k,l=0}^{N-1} (x_{k}+x_{l}-2x_{k}x_{l}-|x_{k}-x_{l}|)(y_{k}+y_{l}-2y_{k}y_{l}-|y_{k}-y_{l}|)$$

and for its periodic L_2 discrepancy we have

$$(L_{2,N}^{\text{per}}(\mathcal{P}))^2 = -\frac{N^2}{9} + \sum_{k,l=0}^{N-1} \left(\frac{1}{2} - |x_k - x_l| + (x_k - x_l)^2\right) \left(\frac{1}{2} - |y_k - y_l| + (y_k - y_l)^2\right).$$

The following lemma giving the exact values of various sums involving the components of the Hammersley point set is crucial.

Lemma 15. Let $\mathcal{H}_m = \{(x_k, y_k) : k = 0, 1, \dots, 2^m - 1\}$ be the Hammersley point set. Then we have

$$S_{1} := \sum_{k=0}^{2^{m}-1} x_{k} = \sum_{k=0}^{2^{m}-1} y_{k} = \frac{2^{m}-1}{2},$$

$$S_{2} := \sum_{k=0}^{2^{m}-1} x_{k}^{2} = \sum_{k=0}^{2^{m}-1} y_{k}^{2} = \frac{(2^{m}-1)(2^{m+1}-1)}{6 \cdot 2^{m}},$$

$$S_{3} := \sum_{k=0}^{2^{m}-1} x_{k} y_{k} = 2^{m-2} + \frac{m}{8} - \frac{1}{2} + \frac{1}{2^{m+2}},$$

$$S_{4} := \sum_{k=0}^{2^{m}-1} x_{k} y_{k}^{2} = \sum_{k=0}^{2^{m}-1} x_{k}^{2} y_{k} = \frac{(2^{m}-1)(4^{m+1}+3 \cdot 2^{m}(m-2)+2)}{3 \cdot 2^{2m+3}},$$

16

$$\begin{split} S_5 &:= \sum_{k=0}^{2^{m-1}} x_k^2 y_k^2 \\ &= \frac{8(2^{2m+1} - 3 \cdot 2^m + 1)^2 + 9m2^m(4^{m+1} + 2^m(m-9) + 4)}{9 \cdot 2^{3m+5}}, \\ S_6 &:= \sum_{k,l=0}^{2^{m-1}} |x_k - x_l| = \sum_{k,l=0}^{2^{m-1}} |y_k - y_l| = \frac{4^m - 1}{3}, \\ S_7 &:= \sum_{k,l=0}^{2^{m-1}} x_k |y_k - y_l| = \sum_{k,l=0}^{2^{m-1}} y_k |x_k - x_l| = \frac{(2^m - 1)^2(2^m + 1)}{6 \cdot 2^m}, \\ S_8 &:= \sum_{k,l=0}^{2^{m-1}} x_k^2 |y_k - y_l| = \sum_{k,l=0}^{2^{m-1}} y_k^2 |x_k - x_l| \\ &= \frac{16(2^m - 1)^2(2^{2m+1} + 2^m - 1) + 9m(m-1)4^m}{9 \cdot 2^{2m+5}}, \\ S_9 &:= \sum_{k,l=0}^{2^{m-1}} x_k x_l |y_k - y_l| = \sum_{k,l=0}^{2^{m-1}} y_k y_l |x_k - x_l| \\ &= \frac{8(3 \cdot 16^m - 4^m - 6 \cdot 8^m + 3 \cdot 2^{m+1} - 2) - 3m4^m(3m+1))}{9 \cdot 2^{2m+5}}, \\ S_{10} &:= \sum_{k,l=0}^{2^{m-1}} |x_k - x_l| |y_k - y_l| = \frac{8(4^m - 1) + 9m^2 + 3m}{72}. \end{split}$$

We defer the technical proofs of these formulas to the next section. We are ready to prove the discrepancy formulas for the Hammersley point set:

Proof of Theorem 8. We expand the formulas for $(L_{2,2^m}(\mathcal{H}_m))^2$, $(L_{2,2^m}^{\text{extr}}(\mathcal{H}_m))^2$ and $(L_{2,2^m}^{\text{per}}(\mathcal{H}_m))^2$ as given in Remark 14 and express them in terms of the sums which appear in Lemma 15. We obtain

$$(L_{2,N}(\mathcal{H}_m))^2 = \frac{11 \cdot 4^m}{18} - \frac{2^m}{2} (S_5 - 2S_2) + \frac{1}{4} (-2^{m+3}S_1 + 2^{m+1}S_3 + 2S_1^2 - 4S_6 + 4S_7 + S_{10}),$$
$$(L_{2,N}^{\text{extr}}(\mathcal{H}_m))^2 = \frac{4^m}{144} - \frac{2^m}{2} (S_3 - 2S_4 + S_5) + \frac{1}{4} (2^{m+1}S_3 + 2S_1^2 - 8S_1S_3 + 4S_3^2 - 4S_7 + 4S_9 + S_{10}).$$

and

$$(L_{2,N}^{\text{per}}(\mathcal{H}_m))^2 = \frac{5 \cdot 4^m}{36} - 4S_8 + 4S_9 - S_6 + 2^{m+1}S_2 - 2S_1^2 + 2^{m+1}S_5 - 8S_1S_4 + 4S_3^2 + 2S_2^2 + S_{10}.$$

The remaining trivial task is to insert the expressions for the sums S_i , $1 \le i \le 10$, as given in Lemma 2.

5. The proof of Lemma 15

Calculation of S_1 , S_2 and S_6 . We have

$$S_1 = \sum_{k=0}^{2^m - 1} \frac{k}{2^m}$$

 $S_2 = \sum_{k=0}^{2^m - 1} \left(\frac{k}{2^m}\right)^2$

and

as well as

$$S_6 = \frac{2}{2^m} \sum_{k=1}^{2^m - 1} \sum_{l=0}^{k-1} (k-l),$$

which yields the results for these sums.

Calculation of S_3 , S_4 and S_5 . Since the proofs for the formulas of these sums are very similar, we only sketch the proof of the evaluation of the most complicated sum S_5 . We have

$$S_{5} = \sum_{t_{1},\dots,t_{m}=0}^{1} \left(\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}}{2^{m+1-j_{1}}}\right)^{2} \left(\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}}{2^{j_{2}}}\right)^{2}$$

$$= \sum_{a,b,c,d=1}^{m} \frac{1}{2^{2m+2-a-b+c+d}} \sum_{t_{1},\dots,t_{m}=0}^{1} t_{a}t_{b}t_{c}t_{d}$$

$$= \sum_{a,b,c,d=1, \text{ p.d.}}^{m} \frac{2^{m-4}}{2^{2m+2-a-b+c+d}} + \sum_{a,c,d=1, \text{ p.d.}}^{m} \frac{2^{m-3}}{2^{2m+2-2a+c+d}}$$

$$+ 4 \sum_{a,b,d=1, \text{ p.d.}}^{m} \frac{2^{m-3}}{2^{2m+2-b+d}} + \sum_{a,b,c=1, \text{ p.d.}}^{m} \frac{2^{m-3}}{2^{2m+2-a-b+2c}}$$

$$+ \sum_{a,b=1, \text{ p.d.}}^{m} \frac{2^{m-2}}{2^{2m+2-2a+2c}} + 2 \sum_{\substack{a,b=1, \text{ p.d.}\\a=c,b=d}}^{m} \frac{2^{m-2}}{2^{2m+2}}$$

$$+ 4 \sum_{\substack{a,d=1, \text{ p.d.}\\a=b,c=d}}^{m} \frac{2^{m-2}}{2^{2m+2-a+d}} + \sum_{\substack{a=1\\a=b=c=d}}^{m} \frac{2^{m-1}}{2^{2m+2}},$$

where "p.d." stands for "pairwise different". For the first sum in the last expression we obtain

$$\sum_{a,b,c,d=1, \text{ p.d.}}^{m} \frac{2^{m-4}}{2^{2m+2-a-b+c+d}}$$
$$= \frac{1}{2^{m+6}} \left(\sum_{a,b,c,d=0}^{1} 2^{a+b-c-d} - \sum_{\substack{a,c,d=1 \text{ p.d.}\\a=b}}^{m} 2^{2a-c-d} \right)$$

$$-\sum_{\substack{a,b,c=1 \text{ p.d.}\\c=d}}^{m} 2^{a+b-2c} - 4 \sum_{\substack{a,b,d=1 \text{ p.d.}\\a=c}}^{m} 2^{b-d} - \sum_{\substack{a,c=1 \text{ p.d.}\\a=b,c=d}}^{m} 2^{2a-2c}$$
$$-2\sum_{\substack{a,b=1 \text{ p.d.}\\a=c,b=d}}^{m} 1 - 4 \sum_{\substack{a,d=1 \text{ p.d.}\\a=b=c}}^{m} 2^{a-d} - \sum_{\substack{a=1\\a=b=c=d}}^{m} 1 \right).$$

The calculation of these sums is straight-forward. The remaining summands in the expression for S_5 can be computed analogously. This leads to the final result.

Calculation of S_7 , S_8 and S_9 . These sums can be treated similarly. Therefore we will only show how to evaluate the probably most complicated sum S_9 . We write this sum in the following way:

$$S_{9} = \sum_{\substack{t_{1}^{(k)}, \dots, t_{m}^{(k)}, t_{1}^{(l)}, \dots, t_{m}^{(l)} = 0 \\ t_{1}^{(k)}, \dots, t_{m}^{(k)}, t_{1}^{(l)}, \dots, t_{m}^{(l)} = 0 \\ = \sum_{r=0}^{m-1} \sum_{\substack{t_{1}^{(k)}, \dots, t_{m}^{(k)}, t_{1}^{(l)}, \dots, t_{m}^{(l)} = 0 \\ t_{i}^{(k)} = t_{i}^{(l)} \forall i = 1, \dots, r, t_{r+1}^{(k)} \neq t_{r+1}^{(l)}}} \left(\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}} \right) \left(\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}} \right) \\ \times \left| \sum_{j_{3}=r+1}^{m} \frac{t_{j_{3}}^{(k)} - t_{j_{3}}^{(l)}}{2^{j_{3}}} \right|.$$

We define

$$P_{0}(t_{r+1}^{(k)}) := \sum_{\substack{j_{1}=1\\j_{1}\neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}} + \frac{t_{r+1}^{(k)}}{2^{m-r}}, \quad T := \sum_{\substack{j_{3}=r+2\\j_{3}=r+2}}^{m} \frac{t_{j_{3}}^{(k)} - t_{j_{3}}^{(l)}}{2^{j_{3}}}$$
$$P_{1}(t_{r+1}^{(l)}) := \sum_{j_{1}=1}^{r} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}} + \frac{t_{r+1}^{(l)}}{2^{m-r}} + \sum_{j_{1}=r+2}^{m} \frac{t_{j_{1}}^{(l)}}{2^{m+1-j_{1}}}$$

to write (after summation over the indices $t_{r+1}^{(k)}$ and $t_{r+1}^{(l)}$ with $t_{r+1}^{(k)} \neq t_{r+1}^{(l)}$)

$$S_{9} = \sum_{r=0}^{m-1} \sum_{\substack{t_{1}^{(k)}, \dots, t_{r}^{(k)}, t_{r+2}^{(k)}, \dots, t_{m}^{(k)}, \\ t_{r+2}^{(l)}, \dots, t_{m}^{(l)} = 0}} \left(\frac{P_{0}(1)P_{1}(0) + P_{0}(0)P_{1}(1)}{2^{r+1}} + T(P_{0}(1)P_{1}(0) - P_{0}(0)P_{1}(1)) \right).$$

Since

$$P_0(1)P_1(0) - P_0(0)P_1(1) = -\frac{1}{2^{m-r}} \sum_{j=r+2}^m \frac{t_j^{(k)} - t_j^{(l)}}{2^{m+1-j}},$$

we obtain

$$\sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} T(P_0(1)P_1(0) - P_0(0)P_1(1)$$

$$= -\frac{1}{2^{m-r}} \sum_{j_1,j_3=r+2}^m \frac{1}{2^{m+1-j_1}} \frac{1}{2^{j_3}}$$

$$\times \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} (t_{j_1}^{(k)} - t_{j_1}^{(l)})(t_{j_3}^{(k)} - t_{j_3}^{(l)})$$

$$= -\frac{1}{2^{m-r}} \sum_{j=r+2}^m \frac{1}{2^{m+1}} 2^{2m-r-3}$$

$$= -\frac{m-r-1}{16}.$$

Observe that

$$P_{0}(1)P_{1}(0) + P_{0}(0)P_{1}(1)$$

$$= 2\left(\sum_{\substack{j_{1}=1\\j_{1}\neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}\right)\left(\sum_{\substack{j_{2}=1\\j_{2}\neq r+1}}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right)$$

$$+ \frac{1}{2^{m-r}}\sum_{\substack{j_{1}=1\\j_{1}\neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}} + \frac{1}{2^{m-r}}\sum_{\substack{j_{2}=1\\j_{2}\neq r+1}}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}$$

$$=: A + B + C.$$

It is straight-forward to prove

$$\sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}}^{1} B = \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}}^{1} C = \frac{1}{16} \sum_{\substack{j=1\\j\neq r+1}}^{m} 2^j.$$

Further we have

$$\sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} A = \frac{2}{4^{m+1}} \sum_{\substack{j_1,j_2=1\\j_1\neq r+1,j_2\leq r}}^m 2^{j_1+j_2} \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} A = \frac{2}{4^{m+1}} \sum_{\substack{j_1,j_2=1\\j_1\neq r+1,j_2\geq r+2}}^m 2^{j_1+j_2} \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(l)},\dots,t_m^{(l)}=0}} A = \frac{2}{4^{m+1}} \sum_{\substack{t_1^{(k)},\dots,t_r^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(k)},\dots,t_m^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(k)},\dots,t_m^{(k)},t_{r+2}^{(k)},\dots,t_m^{(k)},\\t_{r+2}^{(k)},\dots,t_m^{(k)},t_{r+2}^{(k)},$$

The second sum is easily computed to equal

$$\frac{2}{4^{m+1}}2^{2m-r-4}\sum_{\substack{j_1,j_2=1\\j_1\neq r+1,j_2\geq r+2}}^m 2^{j_1+j_2} = \frac{1}{2^{r+4}}\sum_{j_2=r+2}^m 2^{j_2}\left(\sum_{j_1=1}^m 2^{j_1} - 2^{r+1}\right),$$

while in the first sum it is necessary to distinguish between the cases $j_1 = j_2$ and $j_1 \neq j_2$. We obtain for this sum the result

$$\frac{2}{4^{m+1}}2^{m-r-1}\left(2^{m-3}\sum_{j_1=r+2}^m\sum_{j_2=1}^r2^{j_1+j_2}+2^{m-3}\left(\sum_{j_1,j_2=1}^r2^{j_1+j_2}-\sum_{j=1}^r2^{2j}\right)+2^{m-2}\sum_{j_1=1}^r2^{2j_1}\right).$$

We put everything together to find the claimed result for S_9 . Calculation of S_{10} . We have

$$\begin{split} S_{10} &= \sum_{t_1^{(k)}, \dots, t_m^{(k)}, t_1^{(l)}, \dots, t_m^{(l)} = 0} \left| \sum_{j_1=1}^m \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2^{j_1}} \right| \left| \sum_{j_2=1}^m \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2^{m+1-j_2}} \right| \\ &= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} \sum_{\substack{t_{r+1}^{(k)}, \dots, t_{m-s}^{(k)}, t_{r+1}^{(l)}, \dots, t_{m-s}^{(l)} = 0\\ t_i^{(k)} = t_i^{(l)} \forall i = 1, \dots, r, t_{r+1}^{(k)} \neq t_{r+1}^{(l)}}{t_{m+1-i}^{(k)} = t_{m+1-i}^{(l)} \forall i = 1, \dots, s, t_{m-s}^{(k)} \neq t_{m-s}^{(l)}} \left| \sum_{j_1=r+1}^{m-s} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2^{j_1}} \right| \\ &= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} 2^{r+s} \sum_{\substack{t_{r+1}^{(k)}, \dots, t_{m-s}^{(k)}, t_{r+1}^{(l)}, \dots, t_{m-s}^{(l)} = 0\\ t_{r+1}^{(k)} \neq t_{r+1}^{(l)}, t_{m-s}^{(k)} \neq t_{m-s}^{(l)}} \left| \sum_{j_1=r+1}^{m-s} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2^{j_1}} \right| \\ &\times \left| \sum_{j_2=r+1}^{m-s} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2^{j_1}} \right| \end{split}$$

We write $S_{10} = P_1 + P_2$, where P_1 is the part of the last expression where s = m - r - 1 and P_2 is the part where $s \le m - r - 2$. For P_1 we have

$$P_{1} = \sum_{r=0}^{m-1} 2^{m-1} \sum_{\substack{t_{r+1}^{(k)} = 0 \\ t_{r+1}^{(l)} = 1 - t_{r+1}^{(k)}}} \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2^{r+1}} \right| \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2^{m-r}} \right| = \sum_{r=0}^{m-1} \frac{1}{2} = \frac{m}{2}.$$

For the evaluation of P_2 we abbreviate

$$T_1 := \sum_{j_1=r+2}^{m-s-1} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2^{j_1}} \quad \text{and} \quad T_2 := \sum_{j_2=r+2}^{m-s-1} \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2^{m+1-j_2}}$$

(which are empty sums for s = m - r - 2). Then we sum the expression over $t_{r+1}^{(k)}$, $t_{r+1}^{(l)}$, $t_{m-s}^{(k)}$ and $t_{m-s}^{(l)}$, where the first and the latter two must be different, respectively. We get

$$P_{2} = \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \sum_{\substack{t_{r+1}^{(k)}, \dots, t_{m-s}^{(k)}, \\ t_{r+1}^{(k)}, \dots, t_{m-s}^{(k)} = 0}} \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2^{r+1}} + T_{1} + \frac{t_{m-s}^{(k)} - t_{m-s}^{(l)}}{2^{m-s}} \right|$$

$$\times \left| \frac{t_{m-s}^{(k)} - t_{m-s}^{(l)}}{2^{s+1}} + T_{2} + \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2^{r+1}} \right|$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s}$$

$$\times \sum_{\substack{t_{r+2}^{(k)}, \dots, t_{m-s-1}^{(k)}, \\ t_{r+2}^{(l)}, \dots, t_{m-s-1}^{(l)} = 0}} \left\{ \left(\frac{1}{2^{r+1}} + T_{1} + \frac{1}{2^{m-s}} \right) \left(\frac{1}{2^{s+1}} + T_{2} + \frac{1}{2^{m-r}} \right) + \left(\frac{1}{2^{r+1}} - T_{1} - \frac{1}{2^{m-s}} \right) \left(\frac{1}{2^{s+1}} - T_{2} - \frac{1}{2^{m-r}} \right) + \left(\frac{1}{2^{r+1}} - T_{1} - \frac{1}{2^{m-s}} \right) \left(\frac{1}{2^{s+1}} - T_{2} + \frac{1}{2^{m-r}} \right) + \left(\frac{1}{2^{r+1}} - T_{1} + \frac{1}{2^{m-s}} \right) \left(\frac{1}{2^{s+1}} - T_{2} + \frac{1}{2^{m-r}} \right) + \left(\frac{1}{2^{r+1}} - T_{1} + \frac{1}{2^{m-s}} \right) \left(\frac{1}{2^{s+1}} - T_{2} + \frac{1}{2^{m-r}} \right) \right\}.$$

The expression in curled brackets simplifies very nicely and we get

$$P_{2} = 4 \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \sum_{\substack{t_{r+2}^{(k)}, \dots, t_{m-s-1}^{(k)}, \\ t_{r+2}^{(l)}, \dots, t_{m-s-1}^{(l)} = 0}}^{1} \left(\frac{1}{2^{r+s+2}} + \frac{1}{2^{2m-r-s}} \right)$$
$$= 4^{m-1} \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{-r-s} \left(\frac{1}{2^{r+s+2}} + \frac{1}{2^{2m-r-s}} \right)$$
$$= \frac{8(4^{m}-1) + 9m^{2} - 33m}{72}.$$

The formula for S_{10} follows.

6. The proof of Theorem 9

In this proof we consider the Hammersley point set as digital net with generating matrices

$$C_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let $k \in \{0, 1, \dots, 2^m - 1\}$ with dyadic expansion $k = \kappa_0 + \kappa_1 2 + \dots + \kappa_{m-1} 2^{m-1}$ and corresponding digit vector $\vec{k} = (\kappa_0, \kappa_1, \dots, \kappa_{m-1})^{\top}$ over \mathbb{Z}_2 .

Then the
$$k^{\text{th}}$$
 element (x_k, y_k) of the Hammersley point set is given by $x_k = \frac{\xi_{k,1}}{2} + \frac{\xi_{k,2}}{2^2} + \dots + \frac{\xi_{k,m}}{2^m}$ and $y_k = \frac{\eta_{k,1}}{2} + \frac{\eta_{k,2}}{2^2} + \dots + \frac{\eta_{k,m}}{2^m}$, where
 $(\xi_{k,1}, \xi_{k,2}, \dots, \xi_{k,m})^{\top} = C_1 \vec{k}$ and $(\eta_{k,1}, \eta_{k,2}, \dots, \eta_{k,m})^{\top} = C_2 \vec{k}$.

Proof of Theorem 9. In [8] the analogous quantity, but for digital shifts of depth m was computed. The present case can be interpreted as digital shifts of depth $m = \infty$. Let (x_k, y_k) for $k = 0, 1, \ldots, 2^m - 1$ denote the elements of the Hammersley point set. A slight modification¹ of the proof in [8] shows that

$$\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{H}_{m} \oplus \boldsymbol{\delta}))^{2}] = -\frac{1}{4} \sum_{k=1}^{\infty} \tau(k) \sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n} \oplus x_{h}) - \frac{1}{4} \sum_{l=1}^{\infty} \tau(l) \sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{l}(y_{n} \oplus y_{h}) + \frac{1}{4} \sum_{k,l=0 \atop (k,l) \neq (0,0)}^{\infty} \tau(k)\tau(l) \sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n} \oplus x_{h}) \operatorname{wal}_{l}(y_{n} \oplus y_{h}),$$

where wal_k denotes the k^{th} dyadic Walsh function which is given by

$$\operatorname{wal}_{k}(x) = (-1)^{\kappa_{0}\xi_{1} + \kappa_{1}\xi_{2} + \dots + \kappa_{r-1}\xi_{r}}$$

whenever $k \in \mathbb{N}_0$ and $x \in [0,1)$ have dyadic expansions $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{r-1} 2^{r-1}$ and $x = \frac{\xi_1}{2} + \frac{\xi_2}{2^2} + \cdots$, respectively. Further $\tau(0) = \frac{1}{3}$ and $\tau(k) = -\frac{1}{6 \cdot 4^{r(k)}}$ for k > 0, where r(k) denotes the unique integer r such that $2^r \leq k < 2^{r+1}$.

We have

$$\sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n} \oplus x_{h}) = \left| \sum_{n=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n}) \right|^{2} = \begin{cases} 4^{m} & \text{if } C_{1}^{\top} \vec{k} = \vec{0}, \\ 0 & \text{otherwise}, \end{cases}$$

where we used a well-known relation between digital nets and Walsh-functions (see, for example, [9, Lemma 4.75] or [8, Lemma 2]). Although this relation is only stated for $0 \le k \le 2^m - 1$, it also holds for $k \ge 2^m$ with dyadic expansion $k = \sum_{i=0}^{s} \kappa_i 2^i$, where $s \ge m$, if we set $\vec{k} = (\kappa_0, \ldots, \kappa_{m-1})^{\top}$. Since C_1 is regular the condition $C_1^{\top}\vec{k} = \vec{0}$ is equivalent to $k = 2^m k'$ with $k' \in \mathbb{N}$. Therefore we obtain

$$\sum_{k=1}^{\infty} \tau(k) \sum_{n,h=0}^{2^m - 1} \operatorname{wal}_k(x_n \oplus x_h) = 4^m \sum_{k'=1}^{\infty} \tau(2^m k') = \sum_{u=0}^{\infty} \left(-\frac{1}{6 \cdot 4^u} \right) 2^u = -\frac{1}{3}.$$

Likewise we have

$$\sum_{l=1}^{\infty} \tau(l) \sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{l}(y_{n} \oplus y_{h}) = -\frac{1}{3}.$$

¹Set $m = \infty$ in [8, Lemma 3] and take care of the resulting consequences.

Furthermore,

$$\sum_{n,h=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n} \oplus x_{h}) \operatorname{wal}_{l}(y_{n} \oplus y_{h}) = \left| \sum_{n=0}^{2^{m}-1} \operatorname{wal}_{k}(x_{n}) \operatorname{wal}_{l}(y_{n}) \right|^{2}$$
$$= \begin{cases} 4^{m} & \text{if } C_{1}^{\top} \vec{k} + C_{2}^{\top} \vec{l} = \vec{0}, \\ 0 & \text{otherwise,} \end{cases}$$

where we used [9, Lemma 4.75] (or [8, Lemma 2]) again. Hence

$$\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{P}\oplus\boldsymbol{\delta}))^2] = \frac{1}{6} + 4^{m-1} \sum_{\substack{k,l=0\\(k,l)\neq(0,0)\\C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}}}^{\infty} \tau(k)\tau(l).$$

We have

$$\sum_{\substack{k,l=0\\C_1^\top \vec{k}+C_2^\top \vec{l}=\vec{0}}}^{\infty} \tau(k)\tau(l) = \sum_{\substack{k=1\\C_1^\top \vec{k}=\vec{0}}}^{\infty} \tau(k)\tau(0) + \sum_{\substack{l=1\\C_2^\top \vec{l}=\vec{0}}}^{\infty} \tau(0)\tau(l) + \sum_{\substack{k,l=1\\C_1^\top \vec{k}+C_2^\top \vec{l}=\vec{0}}}^{\infty} \tau(k)\tau(l)$$
$$= -\frac{2}{9\cdot 4^m} + \sum_{\substack{c_1^\top \vec{k}+C_2^\top \vec{l}=\vec{0}}}^{\infty} \tau(k)\tau(l).$$

Hence

$$\mathbb{E}_{\boldsymbol{\delta}}[(L_{2,N}(\mathcal{P}\oplus\boldsymbol{\delta}))^2] = \frac{1}{9} + 4^{m-1} \sum_{\substack{k,l=1\\C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}}}^{\infty} \tau(k)\tau(l).$$

We have

$$\Sigma := \sum_{\substack{k,l=1\\C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}}}^{\infty} \tau(k)\tau(l) = \frac{1}{36} \sum_{u,v=0}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{k=2^u}}_{C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}}^{2^{u+1}-1} 1.$$

Denote by e_1, \ldots, e_m the row vectors of C_1 and by d_1, \ldots, d_m the row vectors of C_2 . Set $e_i = d_i = \vec{0}$ for $i \ge m + 1$. The condition $C_1^{\top} \vec{k} + C_2^{\top} \vec{l} = \vec{0}$ can be rewritten as

$$e_1\kappa_0 + \dots + e_u\kappa_{u-1} + e_{u+1} + d_1\lambda_0 + \dots + d_v\lambda_{v-1} + d_{v+1} = \vec{0},$$

where $k = \kappa_0 + \kappa_1 2 + \dots + \kappa_{u-1} 2^{u-1} + 2^u$ and $l = \lambda_0 + \lambda_1 p + \dots + \lambda_{v-1} 2^{v-1} + 2^v$. Since $e_1, \dots, e_{u+1}, d_1, \dots, d_{v+1}$ are linearly independent as long as $u + 1 + \dots + 2^{u-1} 2^{u-1} + 2^{u$

Since $e_1, \ldots, e_{u+1}, a_1, \ldots, a_{v+1}$ are linearly independent as long as u+1 $v+1 \le m$ we must have $u+v \ge m-1$. Hence

$$\Sigma = \frac{1}{36} \sum_{\substack{u,v=0\\u+v \ge m-1}}^{\infty} \frac{1}{4^{u+v}} \qquad \sum_{\underbrace{\kappa_{u-1},\dots,\kappa_0=0}^{1} \lambda_{v-1},\dots,\lambda_0=0}^{1} 1.$$

 $e_1\kappa_0 + \dots + e_{u+1}\kappa_u + d_1\lambda_0 + \dots + d_{v+1}\lambda_v = \vec{0}$

Now we split the range of summation over u and v. We have

$$\Sigma = \frac{1}{36} \sum_{\substack{u,v=0\\u+v \ge m-1}}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1$$

$$+ \frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{\substack{k_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1$$

$$+ \frac{1}{36} \sum_{u,v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{\substack{k_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1.$$

We consider the first sum where $u, v \in \{0, 1, \dots, m-1\}$ and $\tau := u + v \ge v$ m-1. Then we have

$$e_1\kappa_0 + \dots + e_{u+1}\kappa_u + d_1\lambda_0 + \dots + d_{v+1}\lambda_v = \vec{0}$$

iff

$$\begin{pmatrix} \kappa_0 \\ \vdots \\ \kappa_{m-\tau+u-2} \\ \kappa_{m-\tau+u-1} \\ \vdots \\ \kappa_u = 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{\tau-u} = 1 \\ \vdots \\ \lambda_{m-u-1} \\ \lambda_{m-u-2} \\ \vdots \\ \lambda_0 \end{pmatrix} = \vec{0},$$

i.e., iff $\tau = m - 1$ and

• $\kappa_0 = \ldots = \kappa_{u-1} = 0$ and

•
$$\kappa_u = \lambda_v = 1$$
 and

• $\lambda_0 = \ldots = \lambda_{v-1} = 0$,

or $\tau \in \{m, \ldots, 2m-2\}$ and

- $\kappa_0 = \cdots = \kappa_{m-\tau+u-2} = 0$, $\kappa_{m-\tau+u-2} = 1$ and $\lambda_0 = \cdots = \lambda_{m-u-2} = 0$, $\lambda_{m-u-1} = 1$ and
- $\kappa_i = \lambda_{m-1-i}$ for $i = m \tau + u, \dots, u 1$.

Therefore we have

$$\frac{\frac{1}{36} \sum_{\substack{u,v=0\\u+v \ge m-1}}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{\substack{e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} = \frac{1}{36} \left[\frac{1}{4^{m-1}} \sum_{\substack{u,v=0\\u+v=m-1}}^{m-1} 1 + \sum_{\tau=m}^{2m-2} \frac{2^{\tau-m}}{4^{\tau}} \sum_{\substack{u,v=0\\u+v=\tau}}^{m-1} 1 \right]$$

For $m-1 \le \tau \le 2m-2$ we have

$$\sum_{\substack{u,v=0\\u+v=\tau}}^{m-1} 1 = 2m - \tau - 1.$$

Hence

$$\frac{1}{36} \sum_{\substack{u,v=0\\u+v\geq m-1}}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} = \frac{1}{36} \left[\frac{m}{4^{m-1}} + \frac{1}{2^m} \sum_{\tau=m}^{2m-2} \frac{2m-\tau-1}{2^\tau} \right].$$

Now we use

$$\sum_{\tau=m}^{2m-2} \frac{2m-\tau-1}{2^{\tau}} = \frac{2m}{2^m} + \frac{4(1-2^m)}{4^m}$$

and hence

$$\frac{1}{36} \sum_{\substack{u,v=0\\u+v\geq m-1}}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{\substack{e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} = \frac{1}{36} \left[\frac{m}{4^{m-1}} + \frac{2m}{4^m} + \frac{4(1-2^m)}{8^m} \right] \\ = \frac{m}{6\cdot 4^m} + \frac{1}{9\cdot 8^m} - \frac{1}{9\cdot 4^m}.$$

Next we consider the second sum where $u \in \{m, m + 1, ...\}$ and $v \in \{0, 1, ..., m - 1\}$. Then we have

$$e_1\kappa_0 + \dots + e_{u+1}\kappa_u + d_1\lambda_0 + \dots + d_{v+1}\lambda_v = \vec{0}$$

$$\begin{pmatrix} \kappa_0 \\ \vdots \\ \kappa_{m-v-2} \\ \kappa_{m-v-1} \\ \kappa_{m-v} \\ \vdots \\ \kappa_{m-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_v = 1 \\ \lambda_{v-1} \\ \vdots \\ \lambda_0 \end{pmatrix} = \vec{0},$$

i.e., iff

•
$$\kappa_0 = \ldots = \kappa_{m-v-2} = 0, \ \kappa_{m-v-1} = 1$$
, and
• $\kappa_{m-v} = \lambda_{v-1}, \ \ldots, \ \kappa_{m-1} = \lambda_0$.

The digits $\kappa_m, \ldots, \kappa_{u-1}$ are arbitrary. Hence

$$\sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1 = 2^{u-m}2^v = 2^{u+v-m}.$$

This yields for the second sum

$$\frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0 \ \lambda_{v-1},\dots,\lambda_0=0\\ e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1 = \frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} 2^{u+v-m}$$
$$= \frac{1}{36 \cdot 2^m} \sum_{u=m}^{\infty} \frac{1}{2^u} \sum_{v=0}^{m-1} \frac{1}{2^v}$$
$$= \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m}.$$

In the same way we can calculate the third sum and obtain

$$\frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0 \ \lambda_{v-1},\dots,\lambda_0=0\\e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1 = \frac{1}{9\cdot 4^m} - \frac{1}{9\cdot 8^m}.$$

It remains to evaluate the last sum where $u, v \in \{m, m + 1, \ldots\}$. Then we have

$$e_1\kappa_0 + \dots + e_{u+1}\kappa_u + d_1\lambda_0 + \dots + d_{v+1}\lambda_v = \vec{0}$$

 iff

$$\left(\begin{array}{c} \kappa_0\\ \vdots\\ \kappa_{m-1} \end{array}\right) + \left(\begin{array}{c} \lambda_{m-1}\\ \vdots\\ \lambda_0 \end{array}\right) = \vec{0},$$

i.e., iff $\kappa_i = \lambda_{m-i-1}$ for $i = 0, \ldots, m-1$. The digits $\kappa_m, \ldots, \kappa_{u-1}$ and $\lambda_m, \ldots, \lambda_{v-1}$ are arbitrary. Hence

$$\sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0 \ \lambda_{v-1},\dots,\lambda_0=0 \\ e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} 1 = 2^m 2^{u-m} 2^{v-m} = 2^{u+v-m}.$$

This yields for the last sum

$$\frac{1}{36} \sum_{u,v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\substack{\kappa_{u-1},\dots,\kappa_0=0 \ \lambda_{v-1},\dots,\lambda_0=0\\ e_1\kappa_0+\dots+e_{u+1}\kappa_u+d_1\lambda_0+\dots+d_{v+1}\lambda_v=\vec{0}}}^{1} \sum_{\substack{n=1\\ 36}}^{\infty} \frac{1}{2^{u+v}} 2^{u+v-m}$$
$$= \frac{1}{36 \cdot 2^m} \left(\sum_{u=m}^{\infty} \frac{1}{2^u}\right)^2$$
$$= \frac{1}{9 \cdot 8^m}.$$

Putting all four sums together we obtain

$$\Sigma = \frac{m}{6 \cdot 4^m} + \frac{1}{9 \cdot 8^m} - \frac{1}{9 \cdot 4^m} + \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m} + \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m} + \frac{1}{9 \cdot 8^m} + \frac{1}{9 \cdot 8^m} = \frac{m}{6 \cdot 4^m} + \frac{1}{9 \cdot 4^m}.$$

Finally this yields

$$\mathbb{E}_{\delta}[(L_{2,N}(\mathcal{P}\oplus \delta))^2] = \frac{1}{9} + 4^{m-1}\Sigma = \frac{m}{24} + \frac{5}{36}.$$

Remark 16. If we restrict to the average over all digital *m*-bit shifts $\delta = \frac{\delta^{(1)}}{2} + \frac{\delta^{(2)}}{2^2} + \cdots + \frac{\delta^{(m)}}{2^m}$ per coordinate, then it follows easily from [19, Theorem 1] that

$$\mathbb{E}_{\boldsymbol{\delta}_m}[(L_{2,N}(\mathcal{P} \oplus \boldsymbol{\delta}_m))^2] = \frac{m}{24} + \frac{3}{8} + \frac{1}{4 \cdot 2^m} - \frac{1}{72 \cdot 4^m}$$

Remark 17. It can be shown that Theorem 9 does not only hold for the Hammersley point set, but for all (0, m, 2)-nets over \mathbb{F}_2 . The proof is similar, but a bit more involved than for \mathcal{H}_m .

7. The proof of Theorem 10

We need the following lemma, which has essentially been proven in [3, 4] already. Since this result is crucial for the computation of the periodic and extreme L_2 discrepancy of rational lattices, we would like to repeat the short proof. Let $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

Lemma 18. With the notation explained in the lines before Theorem 10, we have

$$\sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n} \\ k_1+k_2p_n \equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} = \frac{\pi^4}{q_n^4} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2\left(\frac{\pi r}{q_n}\right) \sin^2\left(\frac{\pi r p_n}{q_n}\right)}.$$

Proof. We make use of the formula

$$\sum_{k \in \mathbb{Z}} \frac{1}{(k+x)^2} = \frac{\pi^2}{\sin^2(\pi x)} \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}.$$

For $k_1, k_2 \in \mathbb{Z}^*$ with $k_1, k_2 \not\equiv 0 \pmod{q_n}$ and $k_1 + k_2 p_n \equiv 0 \pmod{q_n}$ we write $k_1 + k_2 p_n = lq_n$ with $l \in \mathbb{Z}$, and $k_2 = mq_n + r$ for $m \in \mathbb{Z}$ and $r \in \{1, \ldots, q_n - 1\}$. Then

$$\sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \neq 0 \pmod{q_n} \\ k_1+k_2p_n \equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} = \sum_{\substack{k_2 \notin \mathbb{Z} \\ k_2 \neq 0 \pmod{q_n}}} \frac{1}{k_2^2} \sum_{\substack{l \in \mathbb{Z} \\ k_1 = lq_n - k_2p_n}} \frac{1}{(lq_n - k_2p_n)^2}$$
$$= \frac{1}{q_n^2} \sum_{\substack{k_2 \notin \mathbb{Z} \\ k_2 \neq 0 \pmod{q_n}}} \frac{1}{k_2^2} \sum_{l \in \mathbb{Z}} \frac{1}{(l - \frac{k_2p_n}{q_n})^2}$$
$$= \frac{1}{q_n^4} \sum_{r=1}^{q_n-1} \sum_{m \in \mathbb{Z}} \frac{1}{(m + \frac{r}{q_n})^2} \frac{\pi^2}{\sin^2(\frac{\pi rp_n}{q_n})}$$
$$= \frac{\pi^4}{q_n^4} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2(\frac{\pi r}{q_n}) \sin^2(\frac{\pi rp_n}{q_n})}.$$

Proof of Theorem 10. First we prove the result on the periodic L_2 discrepancy of $\mathcal{L}_n(\alpha)$. To this end we use the representation of the periodic L_2 discrepancy in terms of exponential sums as given in Proposition 3. Writing $\mathcal{L}_n(\alpha) = \{ \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{q_n-1} \}$, where $\boldsymbol{x}_h = \left(\frac{h}{q_n}, \left\{ \frac{hp_n}{q_n} \right\} \right)$ for $h = 0, 1, \ldots, q_n - 1$, we have

(13)
$$(L_{2,q_n}^{\mathrm{per}}(\mathcal{L}_n(\alpha)))^2 = \frac{1}{9} \sum_{\boldsymbol{k} \in \mathbb{Z}^2 \setminus \{\boldsymbol{0}\}} \frac{1}{r(\boldsymbol{k})^2} \left| \sum_{h=0}^{q_n-1} \exp(2\pi \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}_h) \right|^2,$$

where the $r(\mathbf{k})$ are defined according to (3). Note that the following arguments are similar to those used in the proof of [4, Theorem 3]. In order to study the sum (13) we need to distinguish different instances for the vector \mathbf{k} .

• The case $\mathbf{k} = (k, 0), k \neq 0$. Then we have

$$\sum_{\substack{k=1\\\mathbf{k}=(k,0)}}^{\infty} \frac{1}{r(\mathbf{k})^2} \left| \sum_{h=0}^{q_n-1} \exp\left(2\pi i k \frac{h}{q_n}\right) \right|^2 + \sum_{\substack{k=1\\\mathbf{k}=(-k,0)}}^{\infty} \frac{1}{r(\mathbf{k})^2} \left| \sum_{h=0}^{q_n-1} \exp\left(-2\pi i k \frac{h}{q_n}\right) \right|^2$$
$$= 2\sum_{\substack{k=1\\q_n\mid k}}^{\infty} \frac{q_n^2}{r(k)^2} = 2\frac{6}{4\pi^2} \sum_{l=1}^{\infty} \frac{q_n^2}{(lq_n)^2} = \frac{1}{2},$$

where we used the the well known identity $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and the fact that

$$\sum_{h=0}^{q_n-1} \exp\left(\pm 2\pi i k \frac{h}{q_n}\right) = \begin{cases} q_n & \text{if } k \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}$$

• The case $\mathbf{k} = (0, k), k \neq 0$. This case can be treated analogously as the previous one and yields the same result. One has to use that $gcd(p_n, q_n) = 1$, which is a well known fact from the theory of continued fractions. Therefore

$$\sum_{h=0}^{q_n-1} \exp\left(\pm 2\pi i k \frac{hp_n}{q_n}\right) = \begin{cases} q_n & \text{if } k \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}$$

• The case $\mathbf{k} = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \equiv 0 \pmod{q_n}$, but $k_2 \not\equiv 0 \pmod{q_n}$. In this case we find

$$\sum_{\substack{\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2\backslash\{\boldsymbol{0}\}\\k_1\equiv 0\pmod{q_n}\\k_2\not\equiv 0\pmod{q_n}}}^{\infty} \frac{1}{r(\boldsymbol{k})^2} \underbrace{\left|\sum_{h=0}^{q_n-1}\exp\left(2\pi \mathrm{i}k_2\frac{hp_n}{q_n}\right)\right|^2}_{=0} = 0$$

- The case $\mathbf{k} = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_2 \equiv 0 \pmod{q_n}$, but $k_1 \not\equiv 0 \pmod{q_n}$ can be treated analogously as the previous one and yields the same result.
- The case $\mathbf{k} = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \equiv 0 \pmod{q_n}$ as well as $k_2 \equiv 0 \pmod{q_n}$. In this case we find

$$\sum_{\substack{\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}\\k_1\equiv 0\pmod{q_n}\\k_2\equiv 0\pmod{q_n}}}^{\infty} \frac{q_n^2}{r(\boldsymbol{k})^2} = q_n^2 \left(\frac{6}{4\pi^2}\right)^2 \sum_{l_1,l_2\in\mathbb{Z}^*} \frac{1}{(q_n l_1)^2 (q_n l_2)^2} = \frac{1}{q_n^2} \left(\frac{6}{4\pi^2}\right)^2 \left(2\frac{\pi^2}{6}\right)^2 = \frac{1}{4q_n^2}.$$

• The case $\mathbf{k} = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \not\equiv 0 \pmod{q_n}$ as well as $k_2 \not\equiv 0 \pmod{q_n}$. In this case we have to evaluate the sum

$$q_n^2 \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \\ k_1, k_2 \not\equiv 0 \pmod{q_n} \\ k_1 + k_2 p_n \equiv 0 \pmod{q_n}}} \frac{1}{r(\boldsymbol{k})^2},$$

which equals

$$q_n^2 \left(\frac{6}{4\pi^2}\right)^2 \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \\ k_1, k_2 \not\equiv 0 \pmod{q_n} \\ k_1 + k_2 p_n \equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} = \frac{9}{4q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2\left(\frac{\pi r}{q_n}\right) \sin^2\left(\frac{\pi r p_n}{q_n}\right)}$$

by Lemma 18. The result on $(L_{2,q_n}^{\text{per}}(\mathcal{L}_n(\alpha)))^2$ follows.

Finally it remains to prove the result for the extreme L_2 discrepancy of $\mathcal{L}_n(\alpha)$. Recall from Remark 14 that the extreme L_2 discrepancy of a point set $\mathcal{P} = \{(x_h, y_h) : h = 0, 1, \ldots, N-1\}$ can be calculated via the formula

(14)
$$(L_{2,N}^{\text{extr}}(\mathcal{P}))^2 = \frac{N^2}{144} - \frac{N}{2} \sum_{h=0}^{N-1} f(x_h) f(y_h) + \frac{1}{4} \sum_{h,l=0}^{N-1} g(x_h, x_l) g(y_h, y_l),$$

where we define f(x) := x(1-x) and g(x,y) = x + y - 2xy - |x-y|. We compute the Fourier series of these two functions. Let $\widehat{f}(k)$ and $\widehat{g}(k_1, k_2)$ for $k, k_1, k_2 \in \mathbb{Z}$ be the Fourier coefficients of f and g; i.e.

$$\widehat{f}(k) = \int_0^1 f(x) \exp(-2\pi i kx) \, \mathrm{d}x$$

and

$$\widehat{g}(k_1, k_2) = \int_0^1 \int_0^1 g(x, y) \exp(-2\pi i (k_1 x + k_2 y)) \, \mathrm{d}x \, \mathrm{d}y.$$

It is not difficult to find that $\hat{f}(0) = \frac{1}{6}$ and $\hat{f}(k) = -\frac{1}{2\pi^2 k^2}$ for $k \in \mathbb{Z}^*$. Therefore

$$f(x) = \frac{1}{6} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(-2\pi i kx)}{2\pi^2 k^2} = \sum_{k \in \mathbb{Z}^*} \frac{1 - \exp(-2\pi i kx)}{2\pi^2 k^2}$$

For the function g we find

$$\widehat{g}(k_1, k_2) = \begin{cases} \frac{1}{6} & \text{if } k_1 = k_2 = 0, \\ -\frac{1}{2\pi^2 k_1^2} & \text{if } k_1 \in \mathbb{Z}^* \text{ and } k_2 = 0, \\ -\frac{1}{2\pi^2 k_2^2} & \text{if } k_1 = 0 \text{ and } k_2 \in \mathbb{Z}^*, \\ \frac{1}{2\pi^2 k_1^2} & \text{if } k_1 \in \mathbb{Z}^* \text{ and } k_2 = -k_1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$g(x,y) = \frac{1}{6} - \sum_{k_1 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_1 x)}{2\pi^2 k_1^2} - \sum_{k_2 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_2 y)}{2\pi^2 k_2^2} + \sum_{k_1 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_1 x) \exp(2\pi i k_1 y)}{2\pi^2 k_1^2} = \sum_{k \in \mathbb{Z}^*} \frac{1}{2\pi^2 k^2} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(-2\pi i k x)}{2\pi^2 k^2} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(2\pi i k y)}{2\pi^2 k^2}$$

$$+\sum_{k\in\mathbb{Z}^{*}}\frac{\exp(-2\pi i kx)\exp(2\pi i ky)}{2\pi^{2}k^{2}}$$
$$=\sum_{k\in\mathbb{Z}^{*}}\frac{(1-\exp(-2\pi i kx))(1-\exp(2\pi i ky))}{2\pi^{2}k^{2}}.$$

We insert the Fourier expansions of f and g into equation (14) and obtain after some simplifications

$$\begin{aligned} (L_{2,N}^{\text{extr}}(\mathcal{P}))^2 \\ &= \frac{N^2}{144} \\ &- \frac{N}{2} \sum_{k_1, k_2 \in \mathbb{Z}^*} \frac{1}{4\pi^4 k_1^2 k_2^2} \sum_{h=0}^{N-1} (1 - \exp(-2\pi i k_1 x_h))(1 - \exp(-2\pi i k_2 y_h)) \\ &+ \frac{1}{4} \sum_{k_1, k_2 \in \mathbb{Z}^*} \frac{1}{4\pi^4 k_1^2 k_2^2} \left| \sum_{h=0}^{N-1} (1 - \exp(-2\pi i k_1 x_h))(1 - \exp(-2\pi i k_2 y_h)) \right|^2. \end{aligned}$$

In order to find the exact formula for $L_{2,q_n}^{\text{extr}}(\mathcal{L}_n(\alpha))$, we need to investigate the expression

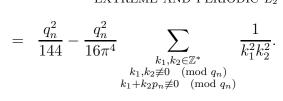
$$\Sigma_{k_1,k_2} := \sum_{h=0}^{q_n-1} \left(1 - \exp\left(-2\pi i k_1 \frac{h}{q_n}\right) \right) \left(1 - \exp\left(-2\pi i k_2 \frac{hp_n}{q_n}\right) \right)$$

for non-zero integers k_1 and k_2 . We observe that \sum_{k_1,k_2} can have the following values:

$$\Sigma_{k_1,k_2} = \begin{cases} q_n & \text{if } k_1, k_2 \not\equiv 0 \pmod{q_n} \text{ and } k_1 + k_2 p_n \not\equiv 0 \pmod{q_n}, \\ 2q_n & \text{if } k_1, k_2 \not\equiv 0 \pmod{q_n} \text{ and } k_1 + k_2 p_n \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to

$$\begin{aligned} (L_{2,q_n}^{\text{extr}}(\mathcal{L}_n(\alpha)))^2 &= \frac{q_n^2}{144} \\ &- \frac{q_n}{2} \left(\sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n} \\ k_1+k_2 p_n \not\equiv 0 \pmod{q_n}}} \frac{q_n}{4\pi^4 k_1^2 k_2^2} + \sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{2q_n}{4\pi^4 k_1^2 k_2^2} \right) \\ &+ \frac{1}{4} \left(\sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{q_n^2}{4\pi^4 k_1^2 k_2^2} + \sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{4q_n^2}{4\pi^4 k_1^2 k_2^2} \right) \\ \end{aligned}$$



We have

$$(15) \sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n} \\ k_1+k_2p_n \not\equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} = \sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} - \sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2}.$$

For the first sum on the right hand side we find

$$\sum_{\substack{k_1,k_2 \in \mathbb{Z}^* \\ k_1,k_2 \not\equiv 0 \pmod{q_n}}} \frac{1}{k_1^2 k_2^2} = \left(\sum_{\substack{k \in \mathbb{Z}^* \\ (\text{mod } q_n)}} \frac{1}{k^2}\right)^2$$
$$= \left(\sum_{k \in \mathbb{Z}^*} \frac{1}{k^2} - \sum_{k \in \mathbb{Z}^*} \frac{1}{(kq_n)^2}\right)^2$$
$$= \frac{\pi^4}{9} \left(1 - \frac{1}{q_n^2}\right)^2.$$

The value of the second sum in (15) is known by Lemma 18. Now the result follows. $\hfill \Box$

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