# EXTREME AND PERIODIC $L_{2}$ DISCREPANCY OF PLANE POINT SETS 


#### Abstract

In this paper we study the extreme and the periodic $L_{2}$ discrepancy of plane point sets. The extreme discrepancy is based on arbitrary rectangles as test sets whereas the periodic discrepancy uses "periodic intervals", which can be seen as intervals on the torus. The periodic $L_{2}$ discrepancy is, up to a multiplicative factor, also known as diaphony. The main results are exact formulas for these kinds of discrepancies for the Hammersley point set and for rational lattices.

We also prove a general lower bound on the extreme $L_{2}$ discrepancy for arbitrary point sets in dimension $d$, which is of order of magnitude $(\log N)^{(d-1) / 2}$, like the standard and periodic $L_{2}$ discrepancies, respectively. Our results confirm that the extreme and periodic $L_{2}$ discrepancies of the Hammersley point set are of best possible asymptotic order of magnitude. This is in contrast to the standard $L_{2}$ discrepancy of the Hammersley point set. Furthermore our exact formulas show that also the $L_{2}$ discrepancies of the Fibonacci lattice are of the optimal order.

We also prove that the extreme $L_{2}$ discrepancy is always dominated by the standard $L_{2}$ discrepancy, a result that was already conjectured by Morokoff and Caflisch when they introduced the notion of extreme $L_{2}$ discrepancy in 1994.


## 1. Introduction

We study several discrepancy notions of two well-known instances of plane point sets, namely the Hammersley point set and rational lattices. The discrepancies are considered with respect to the $L_{2}$ norm and a variety of test sets. We define the (standard) $L_{2}$ discrepancy, the extreme $L_{2}$ discrepancy and the periodic $L_{2}$ discrepancy.

Let $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ be an arbitrary $N$-element point set in the unit square $[0,1)^{2}$. For any measurable subset $B$ of $[0,1]^{2}$ we define the counting function

$$
A(B, \mathcal{P}):=\left|\left\{n \in\{0,1, \ldots, N-1\}: \boldsymbol{x}_{n} \in B\right\}\right|,
$$

i.e., the number of elements from $\mathcal{P}$ that belong to the set $B$. By the local discrepancy of $\mathcal{P}$ with respect to a given measurable "test set" $B$ one understands the expression

$$
A(B, \mathcal{P})-N \lambda(B)
$$

[^0]where $\lambda$ denotes the Lebesgue measure of $B$. A global discrepancy measure is then obtained by considering a norm of the local discrepancy with respect to a fixed class of test sets. Here we restrict ourselves to the $L_{2}$ norm, but we variegate the class of test sets.

The (standard) $L_{2}$ discrepancy uses the class of axis-parallel squares anchored at the origin as test sets. The formal definition is

$$
L_{2, N}(\mathcal{P}):=\left(\int_{[0,1]^{2}}|A([\mathbf{0}, \boldsymbol{t}), \mathcal{P})-N \lambda([\mathbf{0}, \boldsymbol{t}))|^{2} \mathrm{~d} \boldsymbol{t}\right)^{\frac{1}{2}}
$$

where for $\boldsymbol{t}=\left(t_{1}, t_{2}\right) \in[0,1]^{2}$ we set $[\mathbf{0}, \boldsymbol{t})=\left[0, t_{1}\right) \times\left[0, t_{2}\right)$ with area $\lambda([\mathbf{0}, \boldsymbol{t}))=t_{1} t_{2}$.

The extreme $L_{2}$ discrepancy uses arbitrary axis-parallel rectangles contained in the unit square as test sets. For $\boldsymbol{x}=\left(x_{1}, x_{2}\right)$ and $\boldsymbol{y}=\left(y_{2}, y_{2}\right)$ in $[0,1]^{2}$ and $\boldsymbol{x} \leq \boldsymbol{y}$ let $[\boldsymbol{x}, \boldsymbol{y})=\left[x_{1}, y_{1}\right) \times\left[x_{2}, y_{2}\right)$, where $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_{1} \leq y_{1}$ and $x_{2} \leq y_{2}$. The extreme $L_{2}$ discrepancy of $\mathcal{P}$ is then defined as

$$
L_{2, N}^{\operatorname{extr}}(\mathcal{P}):=\left(\int_{[0,1]^{2}} \int_{[0,1]^{2}, \boldsymbol{x} \leq \boldsymbol{y}}|A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P})-N \lambda([\boldsymbol{x}, \boldsymbol{y}))|^{2} \mathrm{~d} \boldsymbol{x} \mathrm{~d} \boldsymbol{y}\right)^{\frac{1}{2}}
$$

Note that the only difference between standard and extreme $L_{2}$ discrepancy is the use of anchored and arbitrary rectangles in $[0,1]^{2}$, respectively. The term "extreme" is used in order to distinguish this notion of $L_{2}$ discrepancy from the standard $L_{2}$ discrepancy and refers to the corresponding nomenclature for $L_{\infty}$ discrepancies (see, e.g., [25, Definition 2.1 and 2.2]).

The periodic $L_{2}$ discrepancy uses periodic rectangles as test sets, which are defined as follows: For $x, y \in[0,1]$ set

$$
I(x, y)= \begin{cases}{[x, y)} & \text { if } x \leq y \\ {[0, y) \cup[x, 1)} & \text { if } x>y\end{cases}
$$

and for $\boldsymbol{x}, \boldsymbol{y}$ as above we set $B(\boldsymbol{x}, \boldsymbol{y})=I\left(x_{1}, y_{1}\right) \times I\left(x_{2}, y_{2}\right)$. We define the periodic $L_{2}$ discrepancy of $\mathcal{P}$ as

$$
L_{2, N}^{\mathrm{per}}(\mathcal{P}):=\left(\int_{[0,1]^{2}} \int_{[0,1]^{2}}|A(B(\boldsymbol{x}, \boldsymbol{y}), \mathcal{P})-N \lambda(B(\boldsymbol{x}, \boldsymbol{y}))|^{2} \mathrm{~d} \boldsymbol{x} \mathrm{~d} \boldsymbol{y}\right)^{\frac{1}{2}}
$$

These discrepancy notions can also be defined for point sets in the $d$ dimensional unit cube $[0,1)^{d}$ in an obvious way.

The standard $L_{2}$ discrepancy is a well known measure for the irregularity of distribution of point sets in the unit square with a close relation to the integration error of quasi-Monte Carlo rules via a Koksma-Hlawka type inequality (see, for example, [9, 26]). In contrast, the extreme and the periodic $L_{2}$ discrepancies are often not so familiar. For this reason we summarize a few facts about these discrepancy notions in the following.

According to [26], the extreme $L_{2}$ discrepancy was first considered by Morokoff and Caflisch in [23] since it is more symmetric than the standard $L_{2}$ discrepancy, which prefers the lower left vertex of the unit square. Morokoff and Caflisch could not state a Koksma-Hlawka type inequality for the
extreme $L_{2}$ discrepancy, but later it has been shown that this quantity is the worst-case integration error of a certain space of periodic functions with a boundary condition (see [26] and the proof of Theorem 5 in Section [2).

The notion of periodic $L_{2}$ discrepancy is known from a paper by Lev [22], but as a matter of fact, it is just a geometric interpretation of the diaphony according to Zinterhof [31] (see Proposition 3 in Section 24). Its relation to the integration error of quasi-Monte Carlo rules is well-known, see, e.g., 16.

The celebrated lower bound of Roth [28] states that there exists a $c>$ 0 such that for every $N$-element point set $\mathcal{P}$ in $[0,1)^{2}$ the standard $L_{2}$ discrepancy satisfies $L_{2, N}(\mathcal{P}) \geq c \sqrt{1+\log N}$. A general lower bound of the same order of magnitude also holds for the periodic $L_{2}$ discrepancy (see Corollary 2 in Section (2). In the present paper we adapt the proof of Roth to show that also the extreme $L_{2}$ discrepancy satisfies a lower bound $L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \geq c \sqrt{1+\log N}$ (see Theorem 6 in Section (2)).

For every $\mathcal{P}$ it is obviously true that

$$
\begin{equation*}
L_{2, N}^{\mathrm{per}}(\mathcal{P}) \geq L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \tag{1}
\end{equation*}
$$

This is because when restricting the range of integration in the definition of periodic $L_{2}$ discrepancy to $\boldsymbol{x} \leq \boldsymbol{y}$, then the test sets are exactly those used for the extreme discrepancy. In [23] the authors further conjectured that the extreme $L_{2}$ discrepancy is smaller than the standard $L_{2}$ discrepancy. They could not prove a result in this direction, but their conjecture was supported by numerical experiments. We will show that this order relation indeed holds true (see Theorem 5 in Section (2).

We mention some further results about extreme and periodic $L_{2}$ discrepancy: The exact asymptotic behaviour of the average of standard, extreme and periodic $L_{2}$ discrepancy of random point sets is given in [14] and [17]. See also [12] for an upper bound in case of extreme $L_{2}$ discrepancy. Bounds on the periodic $L_{2}$ discrepancy for certain multi-dimensional point sets (Korobov's $p$-sets) can be found in [7]. There the dependence of the bounds on the dimension $d$ is of particular interest.

In the present paper we prove exact formulas of the aforementioned $L_{2}$ discrepancies for Hammersley point sets and for rational lattices. In the next section we present some further information and new results about periodic and extreme $L_{2}$ discrepancy. There we also prove the already mentioned "Roth-type" lower bound on extreme $L_{2}$ discrepancy and the order relation between standard and extreme $L_{2}$ discrepancy that was already conjectured by Morokoff and Caflisch. The exact discrepancy formulas for Hammersley point sets (Theorem (8) and for rational lattices (Theorem (10) will then be presented in Section 3. Their proofs are given in Sections 4.7.

## 2. More results about periodic- and extreme $L_{2}$ discrepancy

For a point set $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ and a real vector $\boldsymbol{\delta} \in[0,1]^{d}$ the shifted point set $\mathcal{P}+\boldsymbol{\delta}$ is defined as $\mathcal{P}+\boldsymbol{\delta}=\left\{\left\{\boldsymbol{x}_{0}+\boldsymbol{\delta}\right\}, \ldots,\left\{\boldsymbol{x}_{N-1}+\boldsymbol{\delta}\right\}\right\}$, where $\left\{\boldsymbol{x}_{j}+\boldsymbol{\delta}\right\}$ means that the fractional-part-function $\{x\}=x-\lfloor x\rfloor$ for non-negative real numbers $x$ is applied component-wise to the vector $\boldsymbol{x}_{j}+\boldsymbol{\delta}$.

We call this kind of shift a geometric shift - in contrast to the digital shift as explained in Section 3, The root-mean-square $L_{2}$ discrepancy of a shifted (and weighted) point set $\mathcal{P}$ with respect to all uniformly distributed shift vectors $\boldsymbol{\delta} \in[0,1]^{d}$ is

$$
\begin{equation*}
\sqrt{\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P}+\boldsymbol{\delta})\right)^{2}\right]}=\left(\int_{[0,1]^{d}}\left(L_{2, N}(\mathcal{P}+\boldsymbol{\delta})\right)^{2} \mathrm{~d} \boldsymbol{\delta}\right)^{\frac{1}{2}} \tag{2}
\end{equation*}
$$

The following relation between periodic $L_{2}$ discrepancy and root-meansquare $L_{2}$ discrepancy of a shifted point set $\mathcal{P}$ holds (see [7, 22] for proofs):
Proposition 1. For every $N$-element point set $\mathcal{P}$ in $[0,1)^{d}$ we have

$$
L_{2, N}^{\text {per }}(\mathcal{P})=\sqrt{\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P}+\boldsymbol{\delta})\right)^{2}\right]} .
$$

From this relation we can deduce the following general lower bound on the periodic $L_{2}$ discrepancy of point sets in $[0,1)^{d}$ :

Corollary 2. For every dimension $d$ there exists a quantity $c_{d}>0$ such that every $N$-element point set $\mathcal{P}$ in the unit cube $[0,1)^{d}$ has periodic $L_{2}$ discrepancy bounded by

$$
L_{2, N}^{\mathrm{per}}(\mathcal{P}) \geq c_{d}(1+\log N)^{\frac{d-1}{2}}
$$

Proof. Let $\mathcal{P}$ be an arbitrary $N$-element point sets $\mathcal{P}$ in $[0,1)^{d}$. Then we have

$$
L_{2, N}^{\mathrm{per}}(\mathcal{P})=\sqrt{\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P}+\boldsymbol{\delta})\right)^{2}\right]} \geq \inf _{\boldsymbol{\delta} \in[0,1]^{d}} L_{2, N}(\mathcal{P}+\boldsymbol{\delta}) \geq c_{d}(1+\log N)^{\frac{d-1}{2}}
$$

where we used Roth's lower bound on the standard $L_{2}$ discrepancy.
Another important fact is that the periodic $L_{2}$ discrepancy can be expressed in terms of exponential sums.
Proposition 3. For $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ in $[0,1)^{d}$ we have

$$
\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})\right)^{2}=\frac{1}{3^{d}} \sum_{\boldsymbol{k} \in \mathbb{Z}^{d} \backslash\{\mathbf{0}\}} \frac{1}{r(\boldsymbol{k})^{2}}\left|\sum_{h=0}^{N-1} \exp \left(2 \pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}_{h}\right)\right|^{2},
$$

where $i=\sqrt{-1}$ and where for $\boldsymbol{k}=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}^{d}$ we set

$$
r(\boldsymbol{k})=\prod_{j=1}^{d} r\left(k_{j}\right) \quad \text { and } \quad r\left(k_{j}\right)= \begin{cases}1 & \text { if } k_{j}=0  \tag{3}\\ \frac{2 \pi\left|k_{j}\right|}{\sqrt{6}} & \text { if } k_{j} \neq 0 .\end{cases}
$$

Proof. See [16, p. 390].
The above formula shows that the periodic $L_{2}$ discrepancy is - up to a multiplicative factor - exactly the diaphony which is a well-known measure for the irregularity of distribution of point sets and which was introduced by Zinterhof [31] in 1976 (see also [10]).

From this view point we immediately find an order relation between the standard and the periodic $L_{2}$ discrepancy in the one-dimensional case.

Corollary 4. For every $N$-element point set $\mathcal{P}$ in the unit interval $[0,1)$ we have

$$
L_{2, N}^{\text {per }}(\mathcal{P}) \leq \sqrt{2} L_{2, N}(\mathcal{P})
$$

We have equality if $N$ is even and $\mathcal{P}$ is symmetric, i.e., with every $x_{n}$ also $1-x_{n}$ belongs to $\mathcal{P}$.

Proof. In the one-dimensional case the well-known formula of Koksma (see [21, p. 110]) establishes a connection between $L_{2}$ discrepancy and diaphony. This formula follows easily from an application of Parseval's identity to the local discrepancy. From this we have

$$
\begin{align*}
\left(L_{2, N}(\mathcal{P})\right)^{2} & =\left(\sum_{n=0}^{N-1}\left(\frac{1}{2}-x_{n}\right)\right)^{2}+\frac{1}{2 \pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}}\left|\sum_{h=0}^{N-1} \exp \left(2 \pi \mathrm{i} k x_{h}\right)\right|^{2} \\
& \geq \frac{1}{2 \pi^{2}} \sum_{k=1}^{\infty} \frac{1}{k^{2}}\left|\sum_{h=0}^{N-1} \exp \left(2 \pi \mathrm{i} k x_{h}\right)\right|^{2}  \tag{4}\\
& =\frac{1}{2}\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})\right)^{2},
\end{align*}
$$

where we used Proposition 3 in the last step. The result follows from multiplying by two and taking the square root. For symmetric $\mathcal{P}$ we have equality in (4), because then $\sum_{n=0}^{N-1}\left(\frac{1}{2}-x_{n}\right)$ equals 0 .

We now show that the extreme $L_{2}$ discrepancy is indeed always smaller than the standard $L_{2}$ discrepancy as conjectured in [23]. This is actually implied by the known relationships of the extreme and the standard $L_{2}$ discrepancy to worst-case errors of quasi-Monte Carlo rules for numerical integration.

Theorem 5. For every $N$-element point set $\mathcal{P}$ in $[0,1)^{d}$ we have

$$
L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \leq L_{2, N}(\mathcal{P})
$$

Proof. As already mentioned, we need the relationship between the extreme and the standard $L_{2}$ discrepancy, respectively, and worst-case errors of quasi-Monte Carlo rules for numerical integration. The quoted facts can all be found in [26].

Recall that the worst-case error $e\left(I, Q, H\left(K_{d}\right)\right)$ of the quasi-Monte Carlo rule

$$
Q(f)=\frac{1}{N} \sum_{k=0}^{N-1} f\left(\boldsymbol{x}_{k}\right)
$$

for the integration problem

$$
I(f)=\int_{[0,1]^{d}} f(\boldsymbol{x}) \mathrm{d} \boldsymbol{x}
$$

of functions $f:[0,1]^{d} \rightarrow \mathbb{R}$ in a reproducing kernel Hilbert space $H\left(K_{d}\right)$ with kernel $K_{d}:[0,1]^{d} \times[0,1]^{d} \rightarrow \mathbb{R}$ is given as

$$
e\left(I, Q, H\left(K_{d}\right)\right)=\sup _{\|f\|_{H\left(K_{d}\right)} \leq 1}|I(f)-Q(f)|
$$

A closed formula involving the kernel and the Riesz representer $h_{d} \in H\left(K_{d}\right)$ of the integration functional $I$ is

$$
e\left(I, Q, H\left(K_{d}\right)\right)^{2}=\left\|h_{d}\right\|_{H\left(K_{d}\right)}^{2}-\frac{2}{N} \sum_{k=0}^{N-1} h_{d}\left(\boldsymbol{x}_{k}\right)+\frac{1}{N^{2}} \sum_{k, \ell=0}^{N-1} K\left(\boldsymbol{x}_{k}, \boldsymbol{x}_{\ell}\right),
$$

see [26, (9.31)].
We now introduce the relevant reproducing kernel Hilbert spaces. They are Hilbert space tensor products of Sobolev spaces of univariate functions. Let $W_{2}^{1}([0,1])$ be the Sobolev space of absolutely continuous functions $f$ : $[0,1] \rightarrow \mathbb{R}$ with weak first derivative $f^{\prime} \in L_{2}([0,1])$. Let $H$ be the subspace of all functions $f \in W_{2}^{1}([0,1])$ satisfying the boundary condition $f(1)=0$ equipped with the norm $\|f\|_{H}=\left\|f^{\prime}\right\|_{L_{2}}$. Let $H^{\text {extr }}$ be the subspace of all functions $f \in W_{2}^{1}([0,1])$ satisfying the boundary conditions $f(0)=f(1)=0$ equipped with the norm $\|f\|_{H^{\text {extr }}}=\left\|f^{\prime}\right\|_{L_{2}}$. Obviously, $H^{\text {extr }}$ is the subspace of the 1-periodic functions in $H$. Both $H$ and $H^{\text {extr }}$ are reproducing kernel Hilbert spaces. The kernels are given as $K(x, y)=\min \{1-x, 1-y\}$ for $H$ and $K^{\text {extr }}(x, y)=\min \{x, y\}-x y$ for $H^{\text {extr }}$. Denote the $d$-fold Hilbert space tensor products of these spaces by $H_{d}$ and $H_{d}^{\text {extr }}$, respectively. Their kernels $K_{d}$ and $K_{d}^{\text {extr }}$ are the $d$-fold tensor products of the corresponding univariate kernels.

Now, using the above formula for the worst-case error of the integration problem and comparing to the formulas of the standard and extreme $L_{2}$ discrepancy in Proposition 13 in Section 4 below shows that

$$
N e\left(I, Q, H_{d}\right)=L_{2, N}(\mathcal{P}) \quad \text { and } \quad N e\left(I, Q, H_{d}^{\text {extr }}\right)=L_{2, N}^{\text {extr }}(\mathcal{P})
$$

where $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ is the point set used by the quasi-Monte Carlo rule $Q$. A complete derivation of the first equation is given in [26, Section 9.5.1], for the second identity we refer to [26, Section 9.5.5].

But, since $H_{d}^{\text {extr }}$ is a subspace of $H_{d}$ (with the induced scalar product and norm), the inequality $e\left(I, Q, H_{d}^{\text {extr }}\right) \leq e\left(I, Q, H_{d}\right)$ is obvious from the definition of the worst-case error.

Next, we show how to adapt the proof of Roth's lower bound for the extreme $L_{2}$ discrepancy.

Theorem 6. For every dimension d there exists a quantity $c_{d}>0$ such that every $N$-element point set $\mathcal{P}$ in the unit cube $[0,1)^{d}$ has extreme $L_{2}$ discrepancy bounded by

$$
L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \geq c_{d}(1+\log N)^{\frac{d-1}{2}}
$$

Proof. We assume some familiarity with the proof of Roth in the language of Haar functions as it can be found, e.g., in [2] or [6]. We only prove the case $d=2$, the extension to general $d$ is done as for Roth's lower bound.

A dyadic interval in $[0,1]$ is an interval of the form $I=\left[2^{-m} n, 2^{-m}(n+1)\right)$ with nonnegative integers $m, n$ satisfying $0 \leq n<2^{m}$. The Haar function supported on $I$ is the function $h_{I}:[0,1] \rightarrow \mathbb{R}$ which is +1 on the left and -1 on the right half of $I$ and 0 outside of $I$. The Haar functions form an orthogonal system in $L_{2}([0,1])$.

The Haar functions in $[0,1]^{2}$ are tensor products of the univariate Haar functions. A dyadic rectangle in $[0,1]^{2}$ is a product $R=I \times J$ of two dyadic intervals $I$ and $J$. The Haar function supported on $R$ is the function $h_{R}:[0,1]^{2} \rightarrow \mathbb{R}$ given as $h_{R}(x, y)=h_{I}(x) h_{J}(y)$. The Haar functions form an orthogonal system in $L_{2}\left([0,1]^{2}\right)$.

Roth's method for proving an order optimal lower bound for the standard $L_{2}$ dicrepancy uses the orthogonal expansion of the discrepancy function into a series of Haar functions. To adapt the proof for the extreme $L_{2}$ discrepancy, we first fix $\boldsymbol{x} \in[0,1 / 2)^{2}$ and consider the discrepancy function

$$
D(\boldsymbol{y})=A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P})-N \lambda([\boldsymbol{x}, \boldsymbol{y}))
$$

just as a function of $\boldsymbol{y} \in[1 / 2,1)^{2}$. For $\boldsymbol{y} \in[0,1]^{2} \backslash[1 / 2,1)^{2}$, we define $D(\boldsymbol{y})=0$. The crucial point in Roth's proof as well as in this argument here is that the scalar product of the discrepancy function $D(\boldsymbol{y})$ with a Haar function $h_{R}(\boldsymbol{y})$ does not depend on the point set $\mathcal{P}$ as long as $R$ does not contain a point of $\mathcal{P}$. In fact, we have

$$
\left\langle D, h_{R}\right\rangle=-2^{-4} N \lambda(R)^{2} \quad \text { if } R \subseteq[1 / 2,1)^{2} \text { and } \mathcal{P} \cap R=\emptyset
$$

We now fix a natural number $m$ satisfying $2^{m-3} \leq 2 N \leq 2^{m-2}$ and consider all dyadic rectangles $R=I \times J$ of area $2^{-m}$. They come in $m+1$ different shapes according to the side length of $R$, i.e., the lengths of $I$ and $J$. There are $2^{m}$ dyadic rectangles of the same shape tiling the unit square. There are $m-1$ shapes where both side length are at most $1 / 2$, and one quarter, that is $2^{m-2}$, of the dyadic rectangles $R$ of such a shape satisfy $R \subseteq[1 / 2,1)^{2}$. Since $2 N \leq 2^{m-2}$, at least half of those rectangles also satisfy $\mathcal{P} \cap R=\emptyset$.

Now Bessel's inequality implies

$$
\int_{[0,1]^{2}} D(\boldsymbol{y})^{2} \mathrm{~d} \boldsymbol{y} \geq \sum_{R} \frac{\left\langle D, h_{R}\right\rangle^{2}}{\left\|h_{R}\right\|_{L_{2}}^{2}}
$$

where the sum is taken over all dyadic rectangles $R$. Using just the dyadic rectangles with area $2^{-m}$ and satisfying $R \subseteq[1 / 2,1)^{2}$ as well as $\mathcal{P} \cap R=\emptyset$, of which there are at least $(m-1) 2^{m-3}$, we obtain that

$$
\int_{[0,1]^{2}} D(\boldsymbol{y})^{2} \mathrm{~d} \boldsymbol{y} \geq(m-1) 2^{m-3} \frac{2^{-8} N^{2} 2^{-4 m}}{2^{-m}}=2^{-11}(m-1) 2^{-2 m} N^{2}
$$

Now using $2^{-m} N \geq 2^{-4}$ and $m-1 \geq 2+\log _{2} N$ we arrive at

$$
\int_{[0,1]^{2}} D(\boldsymbol{y})^{2} \mathrm{~d} \boldsymbol{y} \geq 2^{-19}\left(2+\log _{2} N\right)
$$

Since this holds for any fixed $\boldsymbol{x} \in[0,1 / 2)^{2}$, we can finally integrate over all these $\boldsymbol{x}$ and obtain

$$
L_{2, N}^{\operatorname{extr}}(\mathcal{P})^{2} \geq 2^{-21}\left(2+\log _{2} N\right)
$$

Hence the desired result follows.

In dimension one we have the following surprising relationship between periodic and extreme $L_{2}$ discrepancy. Whether a corresponding relation also holds in higher dimensions is an open question (see also the brief discussion at the end of Section (3).

Theorem 7. For every $N$-element point set $\mathcal{P}$ in the unit interval $[0,1)$ we have

$$
\left(L_{2, N}^{\text {per }}(\mathcal{P})\right)^{2}=2\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2} .
$$

Proof. Let $\mathcal{P}=\left\{x_{0}, x_{1}, \ldots, x_{N-1}\right\}$. We may assume that the points are ordered, i.e., $x_{0} \leq x_{1} \leq \ldots \leq x_{N-1}$. Easy computation (see also [20, Eq. (1.3)]) shows that

$$
\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2}=\frac{1}{12}+\frac{1}{2} \sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}-\frac{n-m}{N}\right)^{2}
$$

From this formula and since $\sum_{n, m=0}^{N-1}(n-m)^{2}=N^{2}\left(N^{2}-1\right) / 6$ we obtain

$$
\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2}=\frac{1}{2}\left(\frac{N^{2}}{6}+\sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)^{2}-\frac{2}{N} \sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)(n-m)\right)
$$

We have

$$
\begin{aligned}
\sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)(n-m) & =\sum_{n, m=0}^{N-1}\left(n x_{n}-m x_{n}-n x_{m}+m x_{m}\right) \\
& =2 N \sum_{n=0}^{N-1} n x_{n}-N(N-1) \sum_{n=0}^{N-1} x_{n}
\end{aligned}
$$

and hence

$$
\begin{equation*}
\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2}=\frac{1}{2}\left(\frac{N^{2}}{6}+\sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)^{2}-4 \sum_{n=0}^{N-1} n x_{n}+2(N-1) \sum_{n=0}^{N-1} x_{n}\right) \tag{5}
\end{equation*}
$$

For the periodic $L_{2}$ discrepancy in dimension one we know (see, e.g., the forthcoming Proposition 13 or [16, p. 389-390]) that

$$
\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})\right)^{2}=\sum_{n, m=0}^{N-1} B_{2}\left(\left|x_{n}-x_{m}\right|\right),
$$

where $B_{2}(x)=x^{2}-x+\frac{1}{6}$ is the second Bernoulli polynomial. Inserting the formula for $B_{2}$ we obtain

$$
\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})\right)^{2}=\frac{N^{2}}{6}+\sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)^{2}-\sum_{n, m=0}^{N-1}\left|x_{n}-x_{m}\right|
$$

We have further

$$
\sum_{n, m=0}^{N-1}\left|x_{n}-x_{m}\right|
$$

$$
\begin{aligned}
& =\sum_{n=0}^{N-1} \sum_{m=0}^{n}\left(x_{n}-x_{m}\right)+\sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1}\left(x_{m}-x_{n}\right) \\
& =\sum_{n=0}^{N-1} x_{n}(n+1)-\sum_{n=0}^{N-1} \sum_{m=0}^{n} x_{m}+\sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} x_{m}-\sum_{n=0}^{N-1} x_{n}(N-1-n) \\
& =2 \sum_{n=0}^{N-1} x_{n}(n+1)-N \sum_{n=0}^{N-1} x_{n}-\sum_{m=0}^{N-1} x_{m} \underbrace{\sum_{n=m}^{N-1}}_{=N-m} 1+\sum_{m=0}^{N-1} x_{m} \underbrace{\sum_{n=0}^{m-1} 1}_{=m} \\
& =4 \sum_{n=0}^{N-1} n x_{n}-2(N-1) \sum_{n=0}^{N-1} x_{n} .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})\right)^{2}=\frac{N^{2}}{6}+\sum_{n, m=0}^{N-1}\left(x_{n}-x_{m}\right)^{2}-4 \sum_{n=0}^{N-1} n x_{n}+2(N-1) \sum_{n=0}^{N-1} x_{n} \tag{6}
\end{equation*}
$$

A comparison of (5) and (6) shows the result.
Note that Theorem7 in combination with Corollary 4 gives another proof of Theorem 5 for the one-dimensional case.

Summary. In this section we presented a number of inequalities and relations between the three types of $L_{2}$ discrepancy. We briefly summarize these relations here: For every $N$-element point set $\mathcal{P}$ in $[0,1)^{d}$ we have

$$
L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \leq L_{2, N}^{\text {per }}(\mathcal{P}) \quad \text { and } \quad L_{2, N}^{\operatorname{extr}}(\mathcal{P}) \leq L_{2, N}(\mathcal{P})
$$

Furthermore, there exists a quantity $c_{d}>0$ such that for every $N$-element point set $\mathcal{P}$ in $[0,1)^{d}$ we have

$$
c_{d}(1+\log N)^{\frac{d-1}{2}} \leq L_{2, N}^{\operatorname{extr}}(\mathcal{P})
$$

In the one-dimensional case we even know that

$$
L_{2, N}^{\mathrm{per}}(\mathcal{P})=\sqrt{2} L_{2, N}^{\text {extr }}(\mathcal{P}) \quad \text { and } \quad L_{2, N}^{\text {per }}(\mathcal{P}) \leq \sqrt{2} L_{2, N}(\mathcal{P})
$$

## 3. Exact discrepancy formulas

In this section we present exact formulas for the $L_{2}$ discrepancies of Hammersley point sets and of rational lattices. Both of them are well established constructions of point sets in discrepancy theory.
Hammersley point set. We calculate the extreme and the periodic $L_{2}$ discrepancy of the 2-dimensional Hammersley point set in base 2, which for $m \in \mathbb{N}$ is given as the set of $N=2^{m}$ points

$$
\mathcal{H}_{m}=\left\{\left(\frac{t_{m}}{2}+\cdots+\frac{t_{1}}{2^{m}}, \frac{t_{1}}{2}+\cdots+\frac{t_{m}}{2^{m}}\right): t_{1}, \ldots, t_{m} \in\{0,1\}\right\}
$$

The Hammersley point set is the prototype of low-discrepancy point sets whose construction is based on digit representations. Its elements $\left(x_{k}, y_{k}\right)$
for $k=0,1, \ldots, 2^{m}-1$ can be also written in the form

$$
x_{k}=\frac{k}{2^{m}} \quad \text { and } \quad y_{k}=\varphi_{2}(k)
$$

where $\varphi_{2}(k)$ is the van der Corput digit reversal function $\varphi_{2}(k)=\frac{\kappa_{0}}{2}+\frac{\kappa_{1}}{2^{2}}+$ $\cdots+\frac{\kappa_{r}}{2^{r+1}}$ whenever $k$ has dyadic expansion $k=\kappa_{0}+\kappa_{1} 2+\cdots+\kappa_{r} 2^{r}$ with $\kappa_{i} \in\{0,1\}$. Note that the Hammersley point set is symmetric with respect to the main diagonal in $\mathbb{R}^{2}$. Another view point of Hammersley point sets as a special instance of digital nets will be used in Section 6 .

We have the following exact result on the extreme and the periodic $L_{2}$ discrepancy of the Hammersley point set. For comparison only we also include the formula for the standard $L_{2}$ discrepancy.

Theorem 8. We have

$$
\begin{aligned}
\left(L_{2,2^{m}}\left(\mathcal{H}_{m}\right)\right)^{2} & =\frac{m^{2}}{64}+\frac{29 m}{192}+\frac{3}{8}-\frac{m}{2^{m+4}}+\frac{1}{2^{m+2}}-\frac{1}{9 \cdot 2^{2 m+3}} \\
\left(L_{2,2^{m}}^{\mathrm{extr}}\left(\mathcal{H}_{m}\right)\right)^{2} & =\frac{m}{64}+\frac{1}{72}-\frac{1}{9 \cdot 4^{m+2}}, \text { and } \\
\left(L_{2,2^{m}}^{\mathrm{per}}\left(\mathcal{H}_{m}\right)\right)^{2} & =\frac{m}{16}+\frac{1}{9}+\frac{1}{9 \cdot 4^{m+1}}
\end{aligned}
$$

The result for the standard $L_{2}$ discrepancy is well-known. A proof can be found, for example, in [13, 27]. The results for the extreme and periodic $L_{2}$ discrepancy are new. The proofs of these formulas - along with a new proof for the standard $L_{2}$ discrepancy - will be presented in Section 4 .

An immediate consequence of Theorem 8 is that - in contrast to the standard $L_{2}$ discrepancy - the extreme and periodic $L_{2}$ discrepancy of the Hammersley point set are of the optimal order $\sqrt{\log N}$, respectively. The $L_{2}$ discrepancy of the Hammersley point set is only of order $\log N$, which is not the optimal order according to the aforementioned lower bound of Roth [28]. Several modifications such as digital shifts or symmetrization are necessary to overcome this defect of the Hammersley point set (see e.g. [11, 13, 15, [19]), which for the other two notions of $L_{2}$ discrepancy are not necessary. Considering the fact the periodic $L_{2}$ discrepancy can be understood as a root-mean-square $L_{2}$ discrepancy of shifted point sets (see Proposition 1 in Section (2) and with inequality (11) in mind, this result does not come unexpected.

Theorem 8 further demonstrates that the standard and the extreme $L_{2}$ discrepancy are not equivalent in general. This is in contrast to the $L_{\infty}$ extreme/star discrepancies $D_{N}(\mathcal{P})$ and $D_{N}^{*}(\mathcal{P})$, which are defined as

$$
D_{N}(\mathcal{P})=\sup _{\boldsymbol{x}, \boldsymbol{y} \in[0,1]^{2}, \boldsymbol{x} \leq \boldsymbol{y}}|A([\boldsymbol{x}, \boldsymbol{y}), \mathcal{P})-N \lambda([\boldsymbol{x}, \boldsymbol{y}))|
$$

and

$$
D_{N}^{*}(\mathcal{P})=\sup _{\boldsymbol{t} \in[0,1]^{2}}|A([\mathbf{0}, \boldsymbol{t}), \mathcal{P})-N \lambda([\mathbf{0}, \boldsymbol{t}))|
$$

for two-dimensional point sets. For these discrepancy notions we have the almost trivial inequalities $D_{N}^{*}(\mathcal{P}) \leq D_{N}(\mathcal{P}) \leq 4 D_{N}^{*}(\mathcal{P})$.

Another obvious implication of Theorem 8 in conjunction with Proposition 1 is the fact that there exists a geometric shift $\boldsymbol{\delta} \in[0,1]^{2}$ such that the point set $\mathcal{H}_{m}+\boldsymbol{\delta}$ achieves the optimal order of $L_{2}$ discrepancy. In fact, Roth [29] used geometric shifts (but only in one coordinate) to prove for the first time the existence of point sets in $[0,1)^{d}$ with the optimal $L_{2}$ discrepancy rate $(\log N)^{\frac{d-1}{2}}$. He could show that the average of the $L_{2}$ discrepancy of higher dimensional versions of the Hammersley point set over all possible shifts achieves this bound; hence it was a probabilistic existence result. In dimension 2, Roth's result has later been derandomized by Bilyk [1] who could find an explicit geometric shift $\boldsymbol{\delta}=(\delta, 0) \in[0,1]^{2}$ such that $\mathcal{H}_{m}+\boldsymbol{\delta}$ has the optimal order of $L_{2}$ discrepancy.

Since the periodic $L_{2}$ discrepancy equals the root-mean-square discrepancy with respect to geometric shifts, we would like to compare the result on $L_{2,2^{m}}^{\mathrm{per}}\left(\mathcal{H}_{m}\right)$ with the root-mean-square $L_{2}$ discrepancy of the Hammersley point set with respect to digital shifts, which are often studied in this context.

These kind of shifts are based on digit-wise addition modulo 2 . In more detail, for $x, y \in[0,1)$ with dyadic expansions $x=\sum_{i=1}^{\infty} \frac{\xi_{i}}{2^{i}}$ and $y=\sum_{i=1}^{\infty} \frac{\eta_{i}}{2^{i}}$ with digits $\xi_{i}, \eta_{i} \in\{0,1\}$ for all $i, j \geq 1$ we define

$$
x \oplus y:=\sum_{i=1}^{\infty} \frac{\xi_{i}+\eta_{i}(\bmod 2)}{2^{i}}
$$

For vectors $\boldsymbol{x}, \boldsymbol{y} \in[0,1)^{d}$ the digit-wise addition $\boldsymbol{x} \oplus \boldsymbol{y}$ is defined componentwise.

For a point set $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ and a real vector $\boldsymbol{\delta} \in[0,1]^{d}$ we define the digitally shifted point set $\mathcal{P} \oplus \boldsymbol{\delta}$ as

$$
\mathcal{P} \oplus \boldsymbol{\delta}=\left\{\boldsymbol{x}_{0} \oplus \boldsymbol{\delta}, \boldsymbol{x}_{1} \oplus \boldsymbol{\delta}, \ldots, \boldsymbol{x}_{N-1} \oplus \boldsymbol{\delta}\right\}
$$

The root-mean-square $L_{2}$ discrepancy of a digitally shifted point set $\mathcal{P}$ with respect to all uniformly distributed (digital) shift vectors $\boldsymbol{\delta} \in[0,1)^{d}$ is

$$
\begin{equation*}
\sqrt{\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P} \oplus \boldsymbol{\delta})\right)^{2}\right]}=\left(\int_{[0,1]^{d}}\left(L_{2, N}(\mathcal{P} \oplus \boldsymbol{\delta})\right)^{2} \mathrm{~d} \boldsymbol{\delta}\right)^{\frac{1}{2}} \tag{7}
\end{equation*}
$$

This is the digital equivalent to the root-mean-square $L_{2}$ discrepancy of a geometrically shifted point set $\mathcal{P}$ given in (22) and therefore to the periodic $L_{2}$ discrepancy.

We compute $\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}\left(\mathcal{H}_{m} \oplus \boldsymbol{\delta}\right)\right)^{2}\right]$ and obtain the following result:
Theorem 9. For the $2^{m}$-element Hammersley point set $\mathcal{H}_{m}$ we have

$$
\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}\left(\mathcal{H}_{m} \oplus \boldsymbol{\delta}\right)\right)^{2}\right]=\frac{m}{24}+\frac{5}{36} .
$$

The proof of Theorem 9 will be presented in Section 6. Note that the root-mean-square $L_{2}$ discrepancy for digitally shifted Hammersley points is about a factor $\sqrt{2 / 3}$ lower than for geometrially shifted Hammersley points.

Rational lattices. We will also calculate the extreme and the periodic $L_{2}$ discrepancy of rational lattices. First we introduce irrational lattices. Let $\alpha \in \mathbb{R}$ be an irrational number. Then for $N \in \mathbb{N}$ we define the point set

$$
\mathcal{A}_{N}(\alpha):=\left\{\left(\frac{k}{N},\{k \alpha\}\right): k=0,1, \ldots, N-1\right\}
$$

where $\{k \alpha\}$ denotes the fractional part of the real $k \alpha$. Let $\alpha=\left[a_{0} ; a_{1}, a_{2}, \ldots\right]$ be the continued fraction expansion of $\alpha$ and $\frac{p_{n}}{q_{n}}$ for $n \in \mathbb{N}$ be the $n^{\text {th }}$ convergent of $\alpha$; i.e. $\frac{p_{n}}{q_{n}}=\left[a_{0} ; a_{1}, \ldots, a_{n}\right]$. Further we consider the sets

$$
\mathcal{L}_{n}(\alpha):=\left\{\left(\frac{k}{q_{n}},\left\{\frac{k p_{n}}{q_{n}}\right\}\right): k=0,1, \ldots, q_{n}-1\right\}
$$

which are an approximation of the set $\mathcal{A}_{N}(\alpha)$. We call a point set $\mathcal{L}_{n}(\alpha)$ a rational lattice. A special instance of a rational lattice is the Fibonacci lattice $\mathcal{F}_{n}$, which is obtained for $\alpha=\frac{1}{2}(\sqrt{5}+1)$; i.e. the golden ratio. Then $\alpha=[1 ; 1,1, \ldots],\left(p_{n}, q_{n}\right)=\left(F_{n-1}, F_{n}\right)$ and

$$
\mathcal{F}_{n}:=\left\{\left(\frac{k}{F_{n}},\left\{\frac{k F_{n-1}}{F_{n}}\right\}\right): k=0,1, \ldots, F_{n}-1\right\}
$$

where the Fibonacci numbers are defined recursively via $F_{0}=F_{1}=1$ and $F_{n}=F_{n-1}+F_{n-1}$ for $n \geq 2$.

We have the following formula for the $L_{2}$ discrepancies of rational lattices.

Theorem 10. Let $\alpha$ be given as above. Then we have

$$
\begin{aligned}
\left(L_{2, q_{n}}\left(\mathcal{L}_{n}(\alpha)\right)^{2}=\right. & \frac{1}{16 q_{n}^{2}} \sum_{r=1}^{q_{n}-1} \frac{1+2 \cos ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)}+\left(\mathcal{D}\left(p_{n}, q_{n}\right)+\frac{3}{4}\right)^{2} \\
& +\frac{1}{18}-\frac{1}{144 q_{n}^{2}}, \\
\left(L_{2, q_{n}}^{\operatorname{extr}}\left(\mathcal{L}_{n}(\alpha)\right)\right)^{2}= & \frac{1}{16 q_{n}^{2}} \sum_{r=1}^{q_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)}+\frac{1}{72}-\frac{1}{144 q_{n}^{2}}, \text { and } \\
\left(L_{2, q_{n}}^{\mathrm{per}}\left(\mathcal{L}_{n}(\alpha)\right)^{2}=\right. & \frac{1}{4 q_{n}^{2}} \sum_{r=1}^{q_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)}+\frac{1}{9}+\frac{1}{36 q_{n}^{2}},
\end{aligned}
$$

where in the first formula $\mathcal{D}(p, q)$ is the inhomogeneous Dedekind sum

$$
\mathcal{D}(p, q)=\sum_{k=1}^{q-1} \rho\left(\frac{k}{q}\right) \rho\left(\frac{k p}{q}\right) \quad \text { where } \quad \rho(x)=\frac{1}{2}-\{x\} .
$$

The first formula for the $L_{2}$ discrepancy is [3, Theorem 6]. The proofs of the formulas for the extreme and periodic $L_{2}$ discrepancy will be given in Section 7

The case of Fibonacci lattices is a matter of particular interest. Hinrichs and Oetters-hagen [16] minimized the periodic $L_{2}$ discrepancy over
$N$-element point sets in the unit square for small values of $N$. If $N \in$ $\{1,2,3,5,8,13\}$ (all of them Fibonacci numbers), then the obtained unique global minimizer of the periodic $L_{2}$ discrepancy (modulo geometric shifts and other torus symmetries; see [16, Section 3.2]) are Fibonacci lattices.

One can show that the term

$$
\frac{1}{F_{n}^{2}} \sum_{r=1}^{F_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{F_{n}}\right) \sin ^{2}\left(\frac{\pi r F_{n-1}}{F_{n}}\right)}
$$

is of order $n$. Numerical experiments in [3] indicate that

$$
\begin{equation*}
\frac{1}{F_{n}^{2}} \sum_{r=1}^{F_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{F_{n}}\right) \sin ^{2}\left(\frac{\pi r F_{n-1}}{F_{n}}\right)} \approx 0.119257 n \tag{8}
\end{equation*}
$$

A few years later the involved constant on the right hand side of (8) was identified to have the explicit expression $\frac{4}{15 \sqrt{5}}$ (see [5]). Furthermore, it is well-known that $\log F_{n}$ is of order of magnitude $n$, i.e., $\log F_{n} \asymp n$. This shows that all considered $L_{2}$ discrepancies of the Fibonacci lattice are of optimal order of magnitude with respect to the corresponding Roth-type lower bounds. In fact it follows from [4, Lemma 7] that in case of extreme and periodic $L_{2}$ discrepancy the same is true for all irrational $\alpha=\left[a_{0} ; a_{1}, a_{2}, \ldots\right]$ with bounded partial quotients (i.e. $a_{k} \leq M$ for some constant $M$ and for all $k \geq 0$ ). Therefore every rational lattice connected to such an $\alpha$ can be shifted geometrically in a way such that the resulting point set achieves the optimal order of $L_{2}$ discrepancy. From the same paper it is known that the unshifted lattice $\mathcal{L}_{n}(\alpha)$ has the optimal order of $L_{2}$ discrepancy if and only if $\sum_{k=0}^{n}(-1)^{k} a_{k} \leq c \sqrt{n}$ for a constant $c>0$.

Remark 11. It follows from Theorem 10 and (8) that

$$
\liminf _{N \rightarrow \infty} \inf _{\# \mathcal{P}=N} \frac{L_{2, N}^{\operatorname{extr}}(\mathcal{P})}{\sqrt{\log N}} \leq \eta:=\sqrt{\frac{1}{60 \sqrt{5} \log \left(\frac{\sqrt{5}+1}{2}\right)}}=0.124455 \ldots
$$

and

$$
\liminf _{N \rightarrow \infty} \inf _{\# \mathcal{P}=N} \frac{L_{2, N}^{\text {per }}(\mathcal{P})}{\sqrt{\log N}} \leq 2 \eta=0.248910 \ldots
$$

Note that the corresponding constants one can derive from the results on the Hammersley point set in Theorem 8 are larger. For the standard $L_{2}$ discrepancy we have

$$
\liminf _{N \rightarrow \infty} \inf _{\# \mathcal{P}=N} \frac{L_{2, N}(\mathcal{P})}{\sqrt{\log N}} \leq \sqrt{2} \eta=0.176006 \ldots
$$

where this constant is attained by symmetrized Fibonacci lattices; see [3].
Brief discussion of possible relationships between $L_{2}$ discrepancies. We point out the following peculiarity, which follows from Theorems 8 and 10 .

Remark 12. If $\mathcal{P}$ is either the Hammersley point set $\mathcal{H}_{m}$ or a rational lattice $\mathcal{L}_{n}(\alpha)$, then we have the relation

$$
\begin{equation*}
\left(L_{2, N}^{\mathrm{per}}(\mathcal{P})^{2}=4\left(L_{2, N}^{\mathrm{extr}}(\mathcal{P})\right)^{2}+\frac{1}{18}+\frac{1}{18 N^{2}}\right. \tag{9}
\end{equation*}
$$

where $N=2^{m}$ or $N=q_{n}$, respectively.
From Remark 12 and other observations (e.g. the one-element point set $\mathcal{P}=\{(0,0)\}$ satisfies (9) because, as easily checked, $\left(L_{2, N}^{\text {per }}(\mathcal{P})^{2}=5 / 36\right.$ and $\left.\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2}=1 / 144\right)$ one might conjecture that (9) holds for arbitrary $N$ element point sets in the unit square.

However, let us consider the regular grid

$$
\Gamma_{m, d}=\left\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\right\}^{d}
$$

consisting of $N=m^{d}$ points in $[0,1)^{d}$, where $m \in \mathbb{N}$. For this point set the $L_{2}$ discrepancies are easily computed using formulas which were introduced by Koksma [18] and Warnock [30] (see the forthcoming Proposition [13). As a result one obtains

$$
\left(L_{2, m^{d}}^{\mathrm{per}}\left(\Gamma_{m, d}\right)\right)^{2}=\left(\frac{m^{2}}{3}+\frac{1}{6}\right)^{d}-\left(\frac{m^{2}}{3}\right)^{d}
$$

and

$$
\left(L_{2, m^{d}}^{\mathrm{extr}}\left(\Gamma_{m, d}\right)\right)^{2}=\frac{m^{2 d}-\left(m^{2}-1\right)^{d}}{12^{d}}
$$

For $d=1$ we have

$$
\left(L_{2, m}^{\text {per }}\left(\Gamma_{m, 1}\right)\right)^{2}=\frac{1}{6} \quad \text { and } \quad\left(L_{2, m}^{\text {extr }}\left(\Gamma_{m, 1}\right)\right)^{2}=\frac{1}{12}
$$

and hence we nicely observe the relation from Theorem 7 .
For $d=2$ we have

$$
\left(L_{2, m^{2}}^{\mathrm{extr}}\left(\Gamma_{m, 2}\right)\right)^{2}=\frac{2 m^{2}-1}{144} \quad \text { and } \quad\left(L_{2, m^{2}}^{\mathrm{per}}\left(\Gamma_{m, 2}\right)\right)^{2}=\frac{m^{2}}{9}+\frac{1}{36} .
$$

If $m=1$, then $\Gamma_{1,2}=\{(0,0)\}$ and (9) is still satisfied. But if $m>1$, then the relation (9) does not hold anymore for $\Gamma_{m, 2}$. Not even the implied multiplier 4 complies, because

$$
\lim _{m \rightarrow \infty} \frac{\left(L_{2, m^{2}}^{\mathrm{per}}\left(\Gamma_{m, 2}\right)\right)^{2}}{\left(L_{2, m^{2}}^{\mathrm{exxtr}}\left(\Gamma_{m, 2}\right)\right)^{2}}=8
$$

These observations raise some interesting questions about relationships between periodic and extreme $L_{2}$ discrepancy. In particular: Which plane point sets satisfy relation (9))? Are the periodic and extreme $L_{2}$ discrepancies in arbitrary dimension $d$ equivalent (like for $d=1$ according to Theorem[7)?

## 4. The proof of Theorem 8

We use the following well known formulas for the standard, extreme and periodic $L_{2}$ discrepancy of point sets. Although we only need the twodimensional versions of these formulas in our proofs, we state the results for arbitrary dimension $d$.

Proposition 13. Let $\mathcal{P}=\left\{\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N-1}\right\}$ be a point set in $[0,1)^{d}$, where we write $\boldsymbol{x}_{k}=\left(x_{k, 1}, \ldots, x_{k, d}\right)$ for $k \in\{0,1, \ldots, N-1\}$. Then we have

$$
\begin{align*}
&\left(L_{2, N}(\mathcal{P})\right)^{2}= \frac{N^{2}}{3^{d}}-\frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^{d}\left(1-x_{k, i}^{2}\right)+\sum_{k, l=0}^{N-1} \prod_{i=1}^{d} \min \left(1-x_{k, i}, 1-x_{l, i}\right),  \tag{10}\\
&(11)\left(L_{2, N}^{\mathrm{extr}}(\mathcal{P})\right)^{2}=  \tag{11}\\
& \frac{N^{2}}{12^{d}}-\frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^{d} x_{k, i}\left(1-x_{k, i}\right) \\
&+\sum_{k, l=0}^{N-1} \prod_{i=1}^{d}\left(\min \left(x_{k, i}, x_{l, i}\right)-x_{k, i} x_{l, i}\right) .
\end{align*}
$$

and

$$
\begin{equation*}
\left(L_{2, N}^{\text {per }}(\mathcal{P})\right)^{2}=-\frac{N^{2}}{3^{d}}+\sum_{k, l=0}^{N-1} \prod_{i=1}^{d}\left(\frac{1}{2}-\left|x_{k, i}-x_{l, i}\right|+\left(x_{k, i}-x_{l, i}\right)^{2}\right) . \tag{12}
\end{equation*}
$$

Proof. The first formula is well known and easily proved by direct integration (see [18, 30]). Sometimes this formula is referred to Warnock [30] what is historically not entirely correct, since it was already provided by Koksma [18] in 1942 for $d=1$, but using the same proof method as later Warnock [30] for arbitrary dimension (see also [24]). Also the second formula follows by simple direct integration and can be found in [30] and [23, 26], respectively. The last formula can be found in [16, 26], where it was derived in the context of the worst-case error in a certain reproducing kernel Hilbert space. This formula can also be derived more directly from Proposition 1 and Equation (10). To this end, we observe that for $x, y \in[0,1]$ we have

$$
\int_{0}^{1}\{x+\delta\} \mathrm{d} \delta=\frac{1}{2}, \quad \int_{0}^{1}\{x+\delta\}^{2} \mathrm{~d} \delta=\frac{1}{3},
$$

and

$$
\int_{0}^{1} \max \{\{x+\delta\},\{y+\delta\}\} \mathrm{d} \delta=\frac{1}{2}+|y-x|-(y-x)^{2}
$$

This is easy calculation. We just show the third formula. Assume without loss of generality that $0 \leq x \leq y \leq 1$. Then we have

$$
\begin{aligned}
& \int_{0}^{1} \max \{\{x+\delta\},\{y+\delta\}\} \mathrm{d} \delta \\
& \quad=\int_{0}^{1-y}\{y+\delta\} \mathrm{d} \delta+\int_{1-y}^{1-x}\{x+\delta\} \mathrm{d} \delta+\int_{1-x}^{1}\{y+\delta\} \mathrm{d} \delta
\end{aligned}
$$

$$
=\int_{y}^{1} u \mathrm{~d} u+\int_{1-(y-x)}^{1} u \mathrm{~d} u+\int_{1+(y-x)}^{1+y}(u-1) \mathrm{d} u .
$$

Now the result follows from evaluating the elementary integrals. The formula (12) follows as well.

Remark 14. Using the formulas (10), (11) and (12) and regarding the fact that $\min \{x, y\}=\frac{1}{2}(x+y-|x-y|)$ for $x, y \in \mathbb{R}$, we find that for the standard $L_{2}$ discrepancy of a two-dimensional point set $\mathcal{P}=\left\{\left(x_{k}, y_{k}\right): k=\right.$ $0,1, \ldots, N-1\}$ we have

$$
\begin{aligned}
\left(L_{2, N}(\mathcal{P})\right)^{2}= & \frac{N^{2}}{9}-\frac{N}{2} \sum_{k=0}^{N-1}\left(1-x_{k}^{2}\right)\left(1-y_{k}^{2}\right) \\
& +\frac{1}{4} \sum_{k, l=0}^{N-1}\left(2-x_{k}-x_{l}-\left|x_{k}-x_{l}\right|\right)\left(2-y_{k}-y_{l}-\left|y_{k}-y_{l}\right|\right)
\end{aligned}
$$

for its extreme $L_{2}$ discrepancy we have

$$
\begin{aligned}
& \left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2} \\
& =\frac{N^{2}}{144}-\frac{N}{2} \sum_{k=0}^{N-1} x_{k}\left(1-x_{k}\right) y_{k}\left(1-y_{k}\right) \\
& \quad+\frac{1}{4} \sum_{k, l=0}^{N-1}\left(x_{k}+x_{l}-2 x_{k} x_{l}-\left|x_{k}-x_{l}\right|\right)\left(y_{k}+y_{l}-2 y_{k} y_{l}-\left|y_{k}-y_{l}\right|\right)
\end{aligned}
$$

and for its periodic $L_{2}$ discrepancy we have

$$
\begin{aligned}
\left(L_{2, N}^{\text {per }}(\mathcal{P})\right)^{2}= & -\frac{N^{2}}{9} \\
& +\sum_{k, l=0}^{N-1}\left(\frac{1}{2}-\left|x_{k}-x_{l}\right|+\left(x_{k}-x_{l}\right)^{2}\right)\left(\frac{1}{2}-\left|y_{k}-y_{l}\right|+\left(y_{k}-y_{l}\right)^{2}\right) .
\end{aligned}
$$

The following lemma giving the exact values of various sums involving the components of the Hammersley point set is crucial.

Lemma 15. Let $\mathcal{H}_{m}=\left\{\left(x_{k}, y_{k}\right): k=0,1, \ldots, 2^{m}-1\right\}$ be the Hammersley point set. Then we have

$$
\begin{aligned}
& S_{1}:=\sum_{k=0}^{2^{m}-1} x_{k}=\sum_{k=0}^{2^{m}-1} y_{k}=\frac{2^{m}-1}{2} \\
& S_{2}:=\sum_{k=0}^{2^{m}-1} x_{k}^{2}=\sum_{k=0}^{2^{m}-1} y_{k}^{2}=\frac{\left(2^{m}-1\right)\left(2^{m+1}-1\right)}{6 \cdot 2^{m}} \\
& S_{3}:=\sum_{k=0}^{2^{m}-1} x_{k} y_{k}=2^{m-2}+\frac{m}{8}-\frac{1}{2}+\frac{1}{2^{m+2}} \\
& S_{4}:=\sum_{k=0}^{2^{m}-1} x_{k} y_{k}^{2}=\sum_{k=0}^{2^{m}-1} x_{k}^{2} y_{k}=\frac{\left(2^{m}-1\right)\left(4^{m+1}+3 \cdot 2^{m}(m-2)+2\right)}{3 \cdot 2^{2 m+3}}
\end{aligned}
$$

$$
\begin{aligned}
& S_{5}:=\sum_{k=0}^{2^{m}-1} x_{k}^{2} y_{k}^{2} \\
&=\frac{8\left(2^{2 m+1}-3 \cdot 2^{m}+1\right)^{2}+9 m 2^{m}\left(4^{m+1}+2^{m}(m-9)+4\right)}{9 \cdot 2^{3 m+5}}, \\
& S_{6}:=\sum_{k, l=0}^{2^{m}-1}\left|x_{k}-x_{l}\right|=\sum_{k, l=0}^{2^{m}-1}\left|y_{k}-y_{l}\right|=\frac{4^{m}-1}{3}, \\
& S_{7}:=\sum_{k, l=0}^{2^{m}-1} x_{k}\left|y_{k}-y_{l}\right|=\sum_{k, l=0}^{2^{m}-1} y_{k}\left|x_{k}-x_{l}\right|=\frac{\left(2^{m}-1\right)^{2}\left(2^{m}+1\right)}{6 \cdot 2^{m}}, \\
& S_{8}:=\sum_{k, l=0}^{2^{m}-1} x_{k}^{2}\left|y_{k}-y_{l}\right|=\sum_{k, l=0}^{2^{m}-1} y_{k}^{2}\left|x_{k}-x_{l}\right| \\
&=\frac{16\left(2^{m}-1\right)^{2}\left(2^{2 m+1}+2^{m}-1\right)+9 m(m-1) 4^{m}}{9 \cdot 2^{2 m+5}}, \\
& S_{9}:=\sum_{k, l=0}^{2^{m}-1} x_{k} x_{l}\left|y_{k}-y_{l}\right|=\sum_{k, l=0}^{2^{m}-1} y_{k} y_{l}\left|x_{k}-x_{l}\right| \\
&=\frac{\left.8\left(3 \cdot 16^{m}-4^{m}-6 \cdot 8^{m}+3 \cdot 2^{m+1}-2\right)-3 m 4^{m}(3 m+1)\right)}{9 \cdot 2^{2 m+5}}, \\
& S_{10}:=\sum_{k, l=0}^{2^{m}-1}\left|x_{k}-x_{l}\right|\left|y_{k}-y_{l}\right|=\frac{8\left(4^{m}-1\right)+9 m^{2}+3 m}{72} .
\end{aligned}
$$

We defer the technical proofs of these formulas to the next section. We are ready to prove the discrepancy formulas for the Hammersley point set:

Proof of Theorem 8. We expand the formulas for $\left(L_{2,2^{m}}\left(\mathcal{H}_{m}\right)\right)^{2},\left(L_{2,2^{m}}^{\operatorname{extr}}\left(\mathcal{H}_{m}\right)\right)^{2}$ and $\left(L_{2,2^{m}}^{\text {per }}\left(\mathcal{H}_{m}\right)\right)^{2}$ as given in Remark 14 and express them in terms of the sums which appear in Lemma 15. We obtain

$$
\begin{aligned}
\left(L_{2, N}\left(\mathcal{H}_{m}\right)\right)^{2}= & \frac{11 \cdot 4^{m}}{18}-\frac{2^{m}}{2}\left(S_{5}-2 S_{2}\right) \\
& +\frac{1}{4}\left(-2^{m+3} S_{1}+2^{m+1} S_{3}+2 S_{1}^{2}-4 S_{6}+4 S_{7}+S_{10}\right) \\
\left(L_{2, N}^{\operatorname{extr}}\left(\mathcal{H}_{m}\right)\right)^{2}= & \frac{4^{m}}{144}-\frac{2^{m}}{2}\left(S_{3}-2 S_{4}+S_{5}\right) \\
& +\frac{1}{4}\left(2^{m+1} S_{3}+2 S_{1}^{2}-8 S_{1} S_{3}+4 S_{3}^{2}-4 S_{7}+4 S_{9}+S_{10}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(L_{2, N}^{\mathrm{per}}\left(\mathcal{H}_{m}\right)\right)^{2}= & \frac{5 \cdot 4^{m}}{36}-4 S_{8}+4 S_{9}-S_{6}+2^{m+1} S_{2}-2 S_{1}^{2} \\
& +2^{m+1} S_{5}-8 S_{1} S_{4}+4 S_{3}^{2}+2 S_{2}^{2}+S_{10}
\end{aligned}
$$

The remaining trivial task is to insert the expressions for the sums $S_{i}$, $1 \leq i \leq 10$, as given in Lemma 2 .

## 5. The proof of Lemma 15

Calculation of $S_{1}, S_{2}$ and $S_{6}$. We have

$$
S_{1}=\sum_{k=0}^{2^{m}-1} \frac{k}{2^{m}}
$$

and

$$
S_{2}=\sum_{k=0}^{2^{m}-1}\left(\frac{k}{2^{m}}\right)^{2}
$$

as well as

$$
S_{6}=\frac{2}{2^{m}} \sum_{k=1}^{2^{m}-1} \sum_{l=0}^{k-1}(k-l)
$$

which yields the results for these sums.
Calculation of $S_{3}, S_{4}$ and $S_{5}$. Since the proofs for the formulas of these sums are very similar, we only sketch the proof of the evaluation of the most complicated sum $S_{5}$. We have

$$
\begin{aligned}
S_{5}= & \sum_{t_{1}, \ldots, t_{m}=0}^{1}\left(\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}}{2^{m+1-j_{1}}}\right)^{2}\left(\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}}{2^{j_{2}}}\right)^{2} \\
= & \sum_{\substack{a, b, c, d=1}}^{m} \frac{1}{2^{2 m+2-a-b+c+d}} \sum_{t_{1}, \ldots, t_{m}=0}^{1} t_{a} t_{b} t_{c} t_{d} \\
= & \sum_{a, b, c, d=1, \text { p.d. }}^{m} \frac{2^{m-4}}{2^{2 m+2-a-b+c+d}}+\sum_{\substack{a, c, d=1, \text { p.d. } \\
a=b}}^{m} \frac{2^{m-3}}{2^{2 m+2-2 a+c+d}} \\
& +4 \sum_{\substack{a, b, d=1, \text { p.d. } \\
a=c}}^{m} \frac{2^{m-3}}{2^{2 m+2-b+d}}+\sum_{\substack{a, b, c=1, \text { p.d. } \\
c=d}}^{m} \frac{2^{m-3}}{2^{2 m+2-a-b+2 c}} \\
& +\sum_{\substack{a, b=1, \text { p.d. } \\
a=b, c=d}}^{m} \frac{2^{m-2}}{2^{2 m+2-2 a+2 c}}+2 \sum_{\substack{a, b=1, \text { p.d. } \\
a=c, b=d}}^{m} \frac{2^{m-2}}{2^{2 m+2}} \\
& +4 \sum_{\substack{a, d=1, \text { p.d. } \\
a=b=c}}^{m} \frac{2^{m-2}}{2^{2 m+2-a+d}}+\sum_{\substack{a=1 \\
a=b=c=d}}^{m} \frac{2^{m-1}}{2^{2 m+2}},
\end{aligned}
$$

where "p.d." stands for "pairwise different". For the first sum in the last expression we obtain

$$
\begin{aligned}
& \sum_{a, b, c, d=1, \text { p.d. }}^{m} \frac{2^{m-4}}{2^{2 m+2-a-b+c+d}} \\
& =\frac{1}{2^{m+6}}\left(\sum_{a, b, c, d=0}^{1} 2^{a+b-c-d}-\sum_{\substack{a, c, d=1 \\
a=b}}^{m} 2^{2 a-c-d}\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{\substack{a, b, c=1 \text { p.d. } \\
c=d}}^{m} 2^{a+b-2 c}-4 \sum_{\substack{a, b, d=1 \\
a=c}}^{m} 2^{b-d}-\sum_{\substack{a, c=1 \text { p.d. } \\
a=b, c=d}}^{m} 2^{2 a-2 c} \\
& \left.-2 \sum_{\substack{a, b=1 \text { p.d. } \\
a=c, b=d}}^{m} 1-4 \sum_{\substack{a, d=1 \text { p.d. } \\
a=b=c}}^{m} 2^{a-d}-\sum_{\substack{a=1 \\
a=b=c=d}}^{m} 1\right) .
\end{aligned}
$$

The calculation of these sums is straight-forward. The remaining summands in the expression for $S_{5}$ can be computed analogously. This leads to the final result.
Calculation of $S_{7}, S_{8}$ and $S_{9}$. These sums can be treated simililarly. Therefore we will only show how to evaluate the probably most complicated sum $S_{9}$. We write this sum in the following way:

$$
\begin{aligned}
& S_{9}= \sum_{t_{1}^{(k)}, \ldots, t_{m}^{(k)}, t_{1}^{(l)}, \ldots, t_{m}^{(l)}=0}^{1}\left(\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}\right)\left(\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right)\left|\sum_{j_{3}=1}^{m} \frac{t_{j_{3}}^{(k)}-t_{j_{3}}^{(l)}}{2^{j_{3}}}\right| \\
&=\sum_{r=0}^{m-1} \sum_{\substack{t_{1}^{(k)}, \ldots, t_{m}^{(k)}, t_{1}^{(l)}, \ldots, t_{m}^{(l)}=0 \\
t_{i}^{(k)}=t_{i}^{(l)} \forall i=1, \ldots, r, t_{r+1}^{(k)} \neq t_{r+1}^{(l)}}}^{1}\left(\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}\right)\left(\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right) \\
& \times\left|\sum_{j_{3}=r+1}^{m} \frac{t_{j_{3}}^{(k)}-t_{j_{3}}^{(l)}}{2^{j_{3}}}\right|
\end{aligned}
$$

We define

$$
\begin{aligned}
& P_{0}\left(t_{r+1}^{(k)}\right):=\sum_{\substack{j_{1}=1 \\
j_{1} \neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}+\frac{t_{r+1}^{(k)}}{2^{m-r}}, \quad T:=\sum_{j_{3}=r+2}^{m} \frac{t_{j_{3}}^{(k)}-t_{j_{3}}^{(l)}}{2^{j_{3}}} \\
& P_{1}\left(t_{r+1}^{(l)}\right):=\sum_{j_{1}=1}^{r} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}+\frac{t_{r+1}^{(l)}}{2^{m-r}}+\sum_{j_{1}=r+2}^{m} \frac{t_{j_{1}}^{(l)}}{2^{m+1-j_{1}}}
\end{aligned}
$$

to write (after summation over the indices $t_{r+1}^{(k)}$ and $t_{r+1}^{(l)}$ with $\left.t_{r+1}^{(k)} \neq t_{r+1}^{(l)}\right)$

$$
\begin{aligned}
& S_{9}=\sum_{r=0}^{m-1} \sum_{\substack{(k) \\
t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+2}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{(l)}=0}}^{1}\left(\frac{P_{0}(1) P_{1}(0)+P_{0}(0) P_{1}(1)}{2^{r+1}}\right. \\
&\left.+T\left(P_{0}(1) P_{1}(0)-P_{0}(0) P_{1}(1)\right)\right) .
\end{aligned}
$$

Since

$$
P_{0}(1) P_{1}(0)-P_{0}(0) P_{1}(1)=-\frac{1}{2^{m-r}} \sum_{j=r+2}^{m} \frac{t_{j}^{(k)}-t_{j}^{(l)}}{2^{m+1-j}}
$$

we obtain

$$
\begin{aligned}
& \sum^{1} T\left(P_{0}(1) P_{1}(0)-P_{0}(0) P_{1}(1)\right. \\
& \begin{array}{c}
t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+1}^{(k)}, \ldots, t_{m}^{(k)}, \\
t_{r+2}^{(l)}, \ldots, t_{m}^{(L)}=0
\end{array} \\
& =-\frac{1}{2^{m-r}} \sum_{j_{1}, j_{3}=r+2}^{m} \frac{1}{2^{m+1-j_{1}}} \frac{1}{2^{j_{3}}} \\
& \times \underbrace{\sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{l(l)}=0}}^{1}\left(t_{j_{1}}^{(k)}-t_{j_{1}}^{(l)}\right)\left(t_{j_{3}}^{(k)}-t_{j_{3}}^{(l)}\right)}_{=0 \text { for } j_{1} \neq j_{3}} \\
& =-\frac{1}{2^{m-r}} \sum_{j=r+2}^{m} \frac{1}{2^{m+1}} 2^{2 m-r-3} \\
& =-\frac{m-r-1}{16} \text {. }
\end{aligned}
$$

Observe that

$$
\begin{aligned}
P_{0}(1) & P_{1}(0)+P_{0}(0) P_{1}(1) \\
= & 2\left(\sum_{\substack{j_{1}=1 \\
j_{1} \neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}\right)\left(\sum_{\substack{j_{2}=1 \\
j_{2} \neq r+1}}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right) \\
& +\frac{1}{2^{m-r}} \sum_{\substack{j_{1}=1 \\
j_{1} \neq r+1}}^{m} \frac{t_{j_{1}}^{(k)}}{2^{m+1-j_{1}}}+\frac{1}{2^{m-r}} \sum_{\substack{j_{2}=1 \\
j_{2} \neq r+1}}^{m} \frac{t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}} \\
= & A+B+C .
\end{aligned}
$$

It is straight-forward to prove

$$
\sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{(l)}=0}}^{1} B=\sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+2}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{l(2)}=0}}^{1} C=\frac{1}{16} \sum_{\substack{j=1 \\ j \neq r+1}}^{m} 2^{j} .
$$

Further we have

$$
\begin{aligned}
\sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+2}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{(l)}=0}}^{1} A= & \frac{2}{4^{m+1}} \sum_{\substack{j_{1}, j_{2}=1 \\
j_{1} \neq r+1, j_{2} \leq r}}^{m} 2^{j_{1}+j_{2}} \sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+2}^{(k)}, \ldots, t_{m}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m}^{(l)}=0}}^{1} t_{j_{1}}^{(k)} t_{j_{2}}^{(k)} \\
& +\frac{2}{4^{m+1}} \sum_{\substack{j_{1}, j_{2}=1 \\
j_{1} \neq r+1, j_{2} \geq r+2}}^{m} 2^{j_{1}+j_{2}} \sum_{\substack{t_{1}^{(k)}, \ldots, t_{r}^{(k)}, t_{r+2}^{(k)}, \ldots, t_{m}^{(k)} \\
t_{r+2}^{(l)}, \ldots, t_{m}^{(l)}=0}}^{1} t_{j_{1}}^{(k)} t_{j_{2}}^{(l)}
\end{aligned}
$$

The second sum is easily computed to equal

$$
\frac{2}{4^{m+1}} 2^{2 m-r-4} \sum_{\substack{j_{1}, j_{2}=1 \\ j_{1} \neq r+1, j_{2} \geq r+2}}^{m} 2^{j_{1}+j_{2}}=\frac{1}{2^{r+4}} \sum_{j_{2}=r+2}^{m} 2^{j_{2}}\left(\sum_{j_{1}=1}^{m} 2^{j_{1}}-2^{r+1}\right)
$$

while in the first sum it is necessary to distinguish between the cases $j_{1}=j_{2}$ and $j_{1} \neq j_{2}$. We obtain for this sum the result

$$
\begin{aligned}
& \frac{2}{4^{m+1}} 2^{m-r-1}\left(2^{m-3} \sum_{j_{1}=r+2}^{m} \sum_{j_{2}=1}^{r} 2^{j_{1}+j_{2}}\right. \\
& \\
& \left.\quad+2^{m-3}\left(\sum_{j_{1}, j_{2}=1}^{r} 2^{j_{1}+j_{2}}-\sum_{j=1}^{r} 2^{2 j}\right)+2^{m-2} \sum_{j_{1}=1}^{r} 2^{2 j_{1}}\right)
\end{aligned}
$$

We put everything together to find the claimed result for $S_{9}$.
Calculation of $S_{10}$. We have

$$
\begin{aligned}
& S_{10}=\sum_{t_{1}^{(k)}, \ldots, t_{m}^{(k)}, t_{1}^{(l)}, \ldots, t_{m}^{(l)}=0}^{1}\left|\sum_{j_{1}=1}^{m} \frac{t_{j_{1}}^{(k)}-t_{j_{1}}^{(l)}}{2^{j_{1}}}\right|\left|\sum_{j_{2}=1}^{m} \frac{t_{j_{2}}^{(k)}-t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right| \\
& =\sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} \sum_{t_{r+1}^{(k)}, \ldots, t_{m-s}^{(k)}, t_{r+1}^{(l)}, \ldots, t_{m-s}^{(l)}=0}^{1}\left|\sum_{j_{1}=r+1}^{m-s} \frac{t_{j_{1}}^{(k)}-t_{j_{1}}^{(l)}}{2^{j_{1}}}\right| \\
& t_{i}^{(k)}=t_{i}^{(l)} \forall i=1, \ldots, r, t_{r+1}^{(k)} \neq t_{r+1}^{(l)} \\
& t_{m+1-i}^{(k)}=t_{m+1-i}^{(l)} \forall i=1, \ldots, s, t_{m-s}^{(k)} \neq t_{m-s}^{(l)} \\
& \times\left|\sum_{j_{2}=r+1}^{m-s} \frac{t_{j_{2}}^{(k)}-t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right| \\
& =\sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} 2^{r+s} \sum_{\substack{t_{r}^{(k)}, \ldots, t_{m-1}^{(k)}, t_{r+1}^{(l)}, \ldots, t_{m-s}^{(l)}=0 \\
t_{r+1}^{(k)} \neq t_{r+1}^{(l)}, t_{m-s}^{(k)} \neq t_{m-s}^{(l)}}}^{1}\left|\sum_{j_{1}=r+1}^{m-s} \frac{t_{j_{1}}^{(k)}-t_{j_{1}}^{(l)}}{2^{j_{1}}}\right| \\
& \times\left|\sum_{j_{2}=r+1}^{m-s} \frac{t_{j_{2}}^{(k)}-t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}\right| .
\end{aligned}
$$

We write $S_{10}=P_{1}+P_{2}$, where $P_{1}$ is the part of the last expression where $s=m-r-1$ and $P_{2}$ is the part where $s \leq m-r-2$. For $P_{1}$ we have

$$
P_{1}=\sum_{r=0}^{m-1} 2^{m-1} \sum_{t_{r+1}^{(k)}=0}^{1} \sum_{t_{r+1}^{(l)}=1-t_{r+1}^{(k)}}\left|\frac{t_{r+1}^{(k)}-t_{r+1}^{(l)}}{2^{r+1}}\right|\left|\frac{t_{r+1}^{(k)}-t_{r+1}^{(l)}}{2^{m-r}}\right|=\sum_{r=0}^{m-1} \frac{1}{2}=\frac{m}{2} .
$$

For the evaluation of $P_{2}$ we abbreviate

$$
T_{1}:=\sum_{j_{1}=r+2}^{m-s-1} \frac{t_{j_{1}}^{(k)}-t_{j_{1}}^{(l)}}{2^{j_{1}}} \quad \text { and } \quad T_{2}:=\sum_{j_{2}=r+2}^{m-s-1} \frac{t_{j_{2}}^{(k)}-t_{j_{2}}^{(l)}}{2^{m+1-j_{2}}}
$$

(which are empty sums for $s=m-r-2$ ). Then we sum the expression over $t_{r+1}^{(k)}, t_{r+1}^{(l)}, t_{m-s}^{(k)}$ and $t_{m-s}^{(l)}$, where the first and the latter two must be
different, respectively. We get

$$
\begin{aligned}
& P_{2}=\sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \sum_{\substack{t_{r+1}^{(k)}, \ldots, t_{m-s}^{(k)}, t_{r+1}^{(l)}, \ldots, t_{m-s}^{(l)}=0}}^{1}\left|\frac{t_{r+1}^{(k)}-t_{r+1}^{(l)}}{2^{r+1}}+T_{1}+\frac{t_{m-s}^{(k)}-t_{m-s}^{(l)}}{2^{m-s}}\right| \\
& \times\left|\frac{t_{m-s}^{(k)}-t_{m-s}^{(l)}}{2^{s+1}}+T_{2}+\frac{t_{r+1}^{(k)}-t_{r+1}^{(l)}}{2^{r+1}}\right| \\
& =\sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \\
& \times \sum_{\substack{(k) \\
t_{r+2}^{(k)}, \ldots, t_{m-s-1}^{(k)}, t_{r+2}^{(l)}, \ldots, t_{m-s-1}^{(l)}=0}}^{1}\left\{\left(\frac{1}{2^{r+1}}+T_{1}+\frac{1}{2^{m-s}}\right)\left(\frac{1}{2^{s+1}}+T_{2}+\frac{1}{2^{m-r}}\right)\right. \\
& +\left(\frac{1}{2^{r+1}}+T_{1}-\frac{1}{2^{m-s}}\right)\left(\frac{1}{2^{s+1}}-T_{2}-\frac{1}{2^{m-r}}\right) \\
& +\left(\frac{1}{2^{r+1}}-T_{1}-\frac{1}{2^{m-s}}\right)\left(\frac{1}{2^{s+1}}+T_{2}-\frac{1}{2^{m-r}}\right) \\
& \left.+\left(\frac{1}{2^{r+1}}-T_{1}+\frac{1}{2^{m-s}}\right)\left(\frac{1}{2^{s+1}}-T_{2}+\frac{1}{2^{m-r}}\right)\right\} .
\end{aligned}
$$

The expression in curled brackets simplifies very nicely and we get

$$
\begin{aligned}
P_{2} & =4 \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \sum_{\substack{(k) \\
t_{l+2}^{(r), \ldots, t_{k-s-1}^{(k)},} \\
t_{r+2}^{l\left(2, \ldots, t_{m-s-1}^{l}=0\right.}}}^{1}\left(\frac{1}{2^{r+s+2}}+\frac{1}{2^{2 m-r-s}}\right) \\
& =4^{m-1} \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{-r-s}\left(\frac{1}{2^{r+s+2}}+\frac{1}{2^{2 m-r-s}}\right) \\
& =\frac{8\left(4^{m}-1\right)+9 m^{2}-33 m}{72} .
\end{aligned}
$$

The formula for $S_{10}$ follows.

## 6. The proof of Theorem 9

In this proof we consider the Hammersley point set as digital net with generating matrices

$$
C_{1}=\left(\begin{array}{ccccc}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{array}\right) \quad \text { and } \quad C_{2}=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{array}\right) .
$$

Let $k \in\left\{0,1, \ldots, 2^{m}-1\right\}$ with dyadic expansion $k=\kappa_{0}+\kappa_{1} 2+\cdots+$ $\kappa_{m-1} 2^{m-1}$ and corresponding digit vector $\vec{k}=\left(\kappa_{0}, \kappa_{1}, \ldots, \kappa_{m-1}\right)^{\top}$ over $\mathbb{Z}_{2}$.

Then the $k^{\text {th }}$ element $\left(x_{k}, y_{k}\right)$ of the Hammersley point set is given by $x_{k}=$ $\frac{\xi_{k, 1}}{2}+\frac{\xi_{k, 2}}{2^{2}}+\cdots+\frac{\xi_{k, m}}{2^{m}}$ and $y_{k}=\frac{\eta_{k, 1}}{2}+\frac{\eta_{k, 2}}{2^{2}}+\cdots+\frac{\eta_{k, m}}{2^{m}}$, where

$$
\left(\xi_{k, 1}, \xi_{k, 2}, \ldots, \xi_{k, m}\right)^{\top}=C_{1} \vec{k} \quad \text { and } \quad\left(\eta_{k, 1}, \eta_{k, 2}, \ldots, \eta_{k, m}\right)^{\top}=C_{2} \vec{k}
$$

Proof of Theorem [9. In [8] the analogous quantity, but for digital shifts of depth $m$ was computed. The present case can be interpreted as digital shifts of depth $m=\infty$. Let $\left(x_{k}, y_{k}\right)$ for $k=0,1, \ldots, 2^{m}-1$ denote the elements of the Hammersley point set. A slight modification ${ }^{1}$ of the proof in [8] shows that

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}\left(\mathcal{H}_{m} \oplus \boldsymbol{\delta}\right)\right)^{2}\right] \\
& = \\
& -\frac{1}{4} \sum_{k=1}^{\infty} \tau(k) \sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n} \oplus x_{h}\right)-\frac{1}{4} \sum_{l=1}^{\infty} \tau(l) \sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{l}\left(y_{n} \oplus y_{h}\right) \\
& \quad+\frac{1}{4} \sum_{\substack{k, l=0 \\
(k, l)=(0,0)}}^{\infty} \tau(k) \tau(l) \sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n} \oplus x_{h}\right) \operatorname{wal}_{l}\left(y_{n} \oplus y_{h}\right),
\end{aligned}
$$

where wal $_{k}$ denotes the $k^{\text {th }}$ dyadic Walsh function which is given by

$$
\operatorname{wal}_{k}(x)=(-1)^{\kappa_{0} \xi_{1}+\kappa_{1} \xi_{2}+\cdots+\kappa_{r-1} \xi_{r}}
$$

whenever $k \in \mathbb{N}_{0}$ and $x \in[0,1)$ have dyadic expansions $k=\kappa_{0}+\kappa_{1} 2+$ $\cdots+\kappa_{r-1} 2^{r-1}$ and $x=\frac{\xi_{1}}{2}+\frac{\xi_{2}}{2^{2}}+\cdots$, respectively. Further $\tau(0)=\frac{1}{3}$ and $\tau(k)=-\frac{1}{6 \cdot 4 r(k)}$ for $k>0$, where $r(k)$ denotes the unique integer $r$ such that $2^{r} \leq k<2^{r+1}$.

We have

$$
\sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n} \oplus x_{h}\right)=\left|\sum_{n=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n}\right)\right|^{2}= \begin{cases}4^{m} & \text { if } C_{1}^{\top} \vec{k}=\overrightarrow{0} \\ 0 & \text { otherwise }\end{cases}
$$

where we used a well-known relation between digital nets and Walsh-functions (see, for example, [9, Lemma 4.75] or [8, Lemma 2]). Although this relation is only stated for $0 \leq k \leq 2^{m}-1$, it also holds for $k \geq 2^{m}$ with dyadic expansion $k=\sum_{i=0}^{s} \kappa_{i} 2^{i}$, where $s \geq m$, if we set $\vec{k}=\left(\kappa_{0}, \ldots, \kappa_{m-1}\right)^{\top}$. Since $C_{1}$ is regular the condition $C_{1}^{\top} \vec{k}=\overrightarrow{0}$ is equivalent to $k=2^{m} k^{\prime}$ with $k^{\prime} \in \mathbb{N}$. Therefore we obtain
$\sum_{k=1}^{\infty} \tau(k) \sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n} \oplus x_{h}\right)=4^{m} \sum_{k^{\prime}=1}^{\infty} \tau\left(2^{m} k^{\prime}\right)=\sum_{u=0}^{\infty}\left(-\frac{1}{6 \cdot 4^{u}}\right) 2^{u}=-\frac{1}{3}$.
Likewise we have

$$
\sum_{l=1}^{\infty} \tau(l) \sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{l}\left(y_{n} \oplus y_{h}\right)=-\frac{1}{3}
$$

[^1]Furthermore,

$$
\begin{aligned}
\sum_{n, h=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n} \oplus x_{h}\right) \operatorname{wal}_{l}\left(y_{n} \oplus y_{h}\right) & =\left|\sum_{n=0}^{2^{m}-1} \operatorname{wal}_{k}\left(x_{n}\right) \operatorname{wal}_{l}\left(y_{n}\right)\right|^{2} \\
& = \begin{cases}4^{m} & \text { if } C_{1}^{\top} \vec{k}+C_{2}^{\top} \vec{l}=\overrightarrow{0} \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

where we used [9, Lemma 4.75] (or [8, Lemma 2]) again. Hence

$$
\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P} \oplus \boldsymbol{\delta})\right)^{2}\right]=\frac{1}{6}+4^{m-1} \sum_{\substack{k, l=0 \\
\left(\begin{array}{l}
(k, l) \neq 0,0) \\
C_{1}^{T}, \vec{k}+C_{2}^{T} \vec{T}=\overrightarrow{0}
\end{array}\right.}}^{\infty} \tau(k) \tau(l)
$$

We have

$$
\begin{aligned}
& \sum_{\substack { k, l=0 \\
\begin{subarray}{c}{(k, l)=(0,0) \\
C_{1}^{\top} \vec{k}+C_{2}^{+} T=\overrightarrow{0}{ k , l = 0 \\
\begin{subarray} { c } { ( k , l ) = ( 0 , 0 ) \\
C _ { 1 } ^ { \top } \vec { k } + C _ { 2 } ^ { + } T = \vec { 0 } } }\end{subarray}}^{\infty} \tau(k) \tau(l)=\sum_{\substack{k=1 \\
C_{1}^{\top}=\vec{k}=\overrightarrow{0}}}^{\infty} \tau(k) \tau(0)+\sum_{\substack{l=1 \\
C_{2}^{T} \vec{l}=\overrightarrow{0}}}^{\infty} \tau(0) \tau(l)+\sum_{\substack{k, l=1 \\
C_{1}^{\top}, l+C_{2}^{T} \\
l=\overrightarrow{0}}}^{\infty} \tau(k) \tau(l) \\
& =-\frac{2}{9 \cdot 4^{m}}+\sum_{\substack{k, l=1 \\
C_{1}^{\top} \vec{k}+C_{2}^{T} \vec{l}=\overrightarrow{0}}}^{\infty} \tau(k) \tau(l) .
\end{aligned}
$$

Hence

$$
\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P} \oplus \boldsymbol{\delta})\right)^{2}\right]=\frac{1}{9}+4^{m-1} \sum_{\substack{k, l=1 \\ C_{1}^{T}+C_{2}^{T} \vec{l}=0}}^{\infty} \tau(k) \tau(l) .
$$

We have

$$
\Sigma:=\sum_{\substack{k, l=1 \\ C_{1}^{\top} \vec{k}+C_{2}^{\top} \\ l}}^{\infty} \tau(k) \tau(l)=\frac{1}{36} \sum_{u, v=0}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{k=2^{u}}^{2^{u+1}-1} \sum_{l=2^{v}}^{2^{v+1}-1}}_{C_{1}^{\top} \vec{k}+C_{2}^{\top} \vec{l}=\overrightarrow{0}} 1 .
$$

Denote by $e_{1}, \ldots, e_{m}$ the row vectors of $C_{1}$ and by $d_{1}, \ldots, d_{m}$ the row vectors of $C_{2}$. Set $e_{i}=d_{i}=\overrightarrow{0}$ for $i \geq m+1$. The condition $C_{1}^{\top} \vec{k}+C_{2}^{\top} \vec{l}=\overrightarrow{0}$ can be rewritten as

$$
e_{1} \kappa_{0}+\cdots+e_{u} \kappa_{u-1}+e_{u+1}+d_{1} \lambda_{0}+\cdots+d_{v} \lambda_{v-1}+d_{v+1}=\overrightarrow{0}
$$

where $k=\kappa_{0}+\kappa_{1} 2+\cdots+\kappa_{u-1} 2^{u-1}+2^{u}$ and $l=\lambda_{0}+\lambda_{1} p+\cdots+\lambda_{v-1} 2^{v-1}+2^{v}$.
Since $e_{1}, \ldots, e_{u+1}, d_{1}, \ldots, d_{v+1}$ are linearly independent as long as $u+1+$ $v+1 \leq m$ we must have $u+v \geq m-1$. Hence

$$
\Sigma=\frac{1}{36} \sum_{\substack{u, v=0 \\ u+v \geq m-1}}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1 .
$$

Now we split the range of summation over $u$ and $v$. We have

$$
\begin{aligned}
& \Sigma=\frac{1}{36} \sum_{\substack{u, v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1 \\
& +\frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} 1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \\
& +\frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=m}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} 1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \\
& \frac{1}{36} \sum_{u, v=m}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1 .
\end{aligned}
$$

We consider the first sum where $u, v \in\{0,1, \ldots, m-1\}$ and $\tau:=u+v \geq$ $m-1$. Then we have

$$
e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}
$$

iff

$$
\left(\begin{array}{l}
\kappa_{0} \\
\vdots \\
\kappa_{m-\tau+u-2} \\
\kappa_{m-\tau+u-1} \\
\vdots \\
\kappa_{u}=1 \\
0 \\
\vdots \\
0
\end{array}\right)+\left(\begin{array}{l}
0 \\
\vdots \\
0 \\
\lambda_{\tau-u}=1 \\
\vdots \\
\lambda_{m-u-1} \\
\lambda_{m-u-2} \\
\vdots \\
\lambda_{0}
\end{array}\right)=\overrightarrow{0}
$$

i.e., iff $\tau=m-1$ and

- $\kappa_{0}=\ldots=\kappa_{u-1}=0$ and
- $\kappa_{u}=\lambda_{v}=1$ and
- $\lambda_{0}=\ldots=\lambda_{v-1}=0$,
or $\tau \in\{m, \ldots, 2 m-2\}$ and
- $\kappa_{0}=\cdots=\kappa_{m-\tau+u-2}=0, \kappa_{m-\tau+u-2}=1$ and
- $\lambda_{0}=\cdots=\lambda_{m-u-2}=0, \lambda_{m-u-1}=1$ and
- $\kappa_{i}=\lambda_{m-1-i}$ for $i=m-\tau+u, \ldots, u-1$.

Therefore we have

$$
\begin{aligned}
& \frac{1}{36} \sum_{\substack{u, v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4^{u+v}} \underbrace{1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \sum_{\kappa_{u-1, \ldots, \kappa_{0}=0} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}} 1 \\
& =\frac{1}{36}\left[\frac{1}{4^{m-1}} \sum_{\substack{u, v=0 \\
u+v=m-1}}^{m-1} 1+\sum_{\tau=m}^{2 m-2} \frac{2^{\tau-m}}{4^{\tau}} \sum_{\substack{u, v=0 \\
u+v=\tau}}^{m-1} 1\right]
\end{aligned}
$$

For $m-1 \leq \tau \leq 2 m-2$ we have

$$
\sum_{\substack{u, v=0 \\ u+v=\tau}}^{m-1} 1=2 m-\tau-1
$$

Hence

$$
\begin{aligned}
& \frac{1}{36} \sum_{\substack{u, v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4^{u+v}} \underbrace{1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \sum_{\kappa_{u-1, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}} 1 \\
& =\frac{1}{36}\left[\frac{m}{4^{m-1}}+\frac{1}{2^{m}} \sum_{\tau=m}^{2 m-2} \frac{2 m-\tau-1}{2^{\tau}}\right]
\end{aligned}
$$

Now we use

$$
\sum_{\tau=m}^{2 m-2} \frac{2 m-\tau-1}{2^{\tau}}=\frac{2 m}{2^{m}}+\frac{4\left(1-2^{m}\right)}{4^{m}}
$$

and hence

$$
\begin{aligned}
& \frac{1}{36} \sum_{\substack{u, v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4^{u+v}} \underbrace{1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \sum_{\kappa_{u-1, \ldots, \kappa_{0}=0}^{1}}^{\sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}} 1 \\
& =\frac{1}{36}\left[\frac{m}{4^{m-1}}+\frac{2 m}{4^{m}}+\frac{4\left(1-2^{m}\right)}{8^{m}}\right] \\
& =\frac{m}{6 \cdot 4^{m}}+\frac{1}{9 \cdot 8^{m}}-\frac{1}{9 \cdot 4^{m}}
\end{aligned}
$$

Next we consider the second sum where $u \in\{m, m+1, \ldots\}$ and $v \in$ $\{0,1, \ldots, m-1\}$. Then we have

$$
e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}
$$

iff

$$
\left(\begin{array}{l}
\kappa_{0} \\
\vdots \\
\kappa_{m-v-2} \\
\kappa_{m-v-1} \\
\kappa_{m-v} \\
\vdots \\
\kappa_{m-1}
\end{array}\right)+\left(\begin{array}{l}
0 \\
\vdots \\
0 \\
\lambda_{v}=1 \\
\lambda_{v-1} \\
\vdots \\
\lambda_{0}
\end{array}\right)=\overrightarrow{0}
$$

i.e., iff

- $\kappa_{0}=\ldots=\kappa_{m-v-2}=0, \kappa_{m-v-1}=1$, and
- $\kappa_{m-v}=\lambda_{v-1}, \ldots, \kappa_{m-1}=\lambda_{0}$.

The digits $\kappa_{m}, \ldots, \kappa_{u-1}$ are arbitrary. Hence

$$
\underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1=2^{u-m} 2^{v}=2^{u+v-m} .
$$

This yields for the second sum

$$
\begin{aligned}
\frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} \underbrace{1}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} \sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1} & =\frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^{u+v}} 2^{u+v-m} \\
& =\frac{1}{36 \cdot 2^{m}} \sum_{u=m}^{\infty} \frac{1}{2^{u}} \sum_{v=0}^{m-1} \frac{1}{2^{v}} \\
& =\frac{1}{9 \cdot 4^{m}}-\frac{1}{9 \cdot 8^{m}}
\end{aligned}
$$

In the same way we can calculate the third sum and obtain

$$
\frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=m}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1=\frac{1}{9 \cdot 4^{m}}-\frac{1}{9 \cdot 8^{m}}
$$

It remains to evaluate the last sum where $u, v \in\{m, m+1, \ldots\}$. Then we have

$$
e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}
$$

iff

$$
\left(\begin{array}{l}
\kappa_{0} \\
\vdots \\
\kappa_{m-1}
\end{array}\right)+\left(\begin{array}{l}
\lambda_{m-1} \\
\vdots \\
\lambda_{0}
\end{array}\right)=\overrightarrow{0}
$$

i.e., iff $\kappa_{i}=\lambda_{m-i-1}$ for $i=0, \ldots, m-1$. The digits $\kappa_{m}, \ldots, \kappa_{u-1}$ and $\lambda_{m}, \ldots, \lambda_{v-1}$ are arbitrary. Hence

$$
\underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1=2^{m} 2^{u-m} 2^{v-m}=2^{u+v-m} .
$$

This yields for the last sum

$$
\begin{aligned}
\frac{1}{36} \sum_{u, v=m}^{\infty} \frac{1}{4^{u+v}} \underbrace{\sum_{\kappa_{u-1}, \ldots, \kappa_{0}=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_{0}=0}^{1}}_{e_{1} \kappa_{0}+\cdots+e_{u+1} \kappa_{u}+d_{1} \lambda_{0}+\cdots+d_{v+1} \lambda_{v}=\overrightarrow{0}} 1 & =\frac{1}{36} \sum_{u, v=m}^{\infty} \frac{1}{4^{u+v}} 2^{u+v-m} \\
& =\frac{1}{36 \cdot 2^{m}}\left(\sum_{u=m}^{\infty} \frac{1}{2^{u}}\right)^{2} \\
& =\frac{1}{9 \cdot 8^{m}}
\end{aligned}
$$

Putting all four sums together we obtain

$$
\begin{aligned}
\Sigma & =\frac{m}{6 \cdot 4^{m}}+\frac{1}{9 \cdot 8^{m}}-\frac{1}{9 \cdot 4^{m}}+\frac{1}{9 \cdot 4^{m}}-\frac{1}{9 \cdot 8^{m}}+\frac{1}{9 \cdot 4^{m}}-\frac{1}{9 \cdot 8^{m}}+\frac{1}{9 \cdot 8^{m}} \\
& =\frac{m}{6 \cdot 4^{m}}+\frac{1}{9 \cdot 4^{m}}
\end{aligned}
$$

Finally this yields

$$
\mathbb{E}_{\boldsymbol{\delta}}\left[\left(L_{2, N}(\mathcal{P} \oplus \boldsymbol{\delta})\right)^{2}\right]=\frac{1}{9}+4^{m-1} \Sigma=\frac{m}{24}+\frac{5}{36} .
$$

Remark 16. If we restrict to the average over all digital $m$-bit shifts $\delta=\frac{\delta^{(1)}}{2}+\frac{\delta^{(2)}}{2^{2}}+\cdots+\frac{\delta^{(m)}}{2^{m}}$ per coordinate, then it follows easily from [19, Theorem 1] that

$$
\mathbb{E}_{\boldsymbol{\delta}_{m}}\left[\left(L_{2, N}\left(\mathcal{P} \oplus \boldsymbol{\delta}_{m}\right)\right)^{2}\right]=\frac{m}{24}+\frac{3}{8}+\frac{1}{4 \cdot 2^{m}}-\frac{1}{72 \cdot 4^{m}}
$$

Remark 17. It can be shown that Theorem 9 does not only hold for the Hammersley point set, but for all $(0, m, 2)$-nets over $\mathbb{F}_{2}$. The proof is similar, but a bit more involved than for $\mathcal{H}_{m}$.

## 7. The proof of Theorem 10

We need the following lemma, which has essentially been proven in [3, 4] already. Since this result is crucial for the computation of the periodic and extreme $L_{2}$ discrepancy of rational lattices, we would like to repeat the short proof. Let $\mathbb{Z}^{*}:=\mathbb{Z} \backslash\{0\}$.

Lemma 18. With the notation explained in the lines before Theorem 10, we have

$$
\sum_{\substack{\left.k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, q_{n}\right) \\ k_{1}+k_{2} \not p_{n} \equiv 0\left(\bmod q_{n}\right) \\\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}}=\frac{\pi^{4}}{q_{n}^{4}} \sum_{r=1}^{q_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)} .
$$

Proof. We make use of the formula

$$
\sum_{k \in \mathbb{Z}} \frac{1}{(k+x)^{2}}=\frac{\pi^{2}}{\sin ^{2}(\pi x)} \quad \text { for } x \in \mathbb{R} \backslash \mathbb{Z}
$$

For $k_{1}, k_{2} \in \mathbb{Z}^{*}$ with $k_{1}, k_{2} \not \equiv 0\left(\bmod q_{n}\right)$ and $k_{1}+k_{2} p_{n} \equiv 0\left(\bmod q_{n}\right)$ we write $k_{1}+k_{2} p_{n}=l q_{n}$ with $l \in \mathbb{Z}$, and $k_{2}=m q_{n}+r$ for $m \in \mathbb{Z}$ and $r \in\left\{1, \ldots, q_{n}-1\right\}$. Then

$$
\begin{aligned}
\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0 \\
k_{1}+k_{2} p_{n} \equiv 0 \\
(\bmod ) \\
\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}} & =\sum_{\substack{k_{2} \in \mathbb{Z} \\
k_{2} \neq 0 \\
\left(\bmod q_{n}\right)}} \frac{1}{k_{2}^{2}} \sum_{\substack{l \in \mathbb{Z} \\
k_{1}=l q_{n}-k_{2} p_{n}}} \frac{1}{\left(l q_{n}-k_{2} p_{n}\right)^{2}} \\
& =\frac{1}{q_{n}^{2}} \sum_{\substack{k_{2} \in \mathbb{Z} \\
k_{2} \neq 0 \\
\left(\bmod q_{n}\right)}} \frac{1}{k_{2}^{2}} \sum_{l \in \mathbb{Z}} \frac{1}{\left(l-\frac{k_{2} p_{n}}{q_{n}}\right)^{2}} \\
& =\frac{1}{q_{n}^{4}} \sum_{r=1}^{q_{n}-1} \sum_{m \in \mathbb{Z}} \frac{1}{\left(m+\frac{r}{q_{n}}\right)^{2}} \frac{\pi^{2}}{\sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)} \\
& =\frac{\pi^{4}}{q_{n}^{4}} \sum_{r=1}^{q_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)} .
\end{aligned}
$$

Proof of Theorem 10. First we prove the result on the periodic $L_{2}$ discrepancy of $\mathcal{L}_{n}(\alpha)$. To this end we use the representation of the periodic $L_{2}$ discrepancy in terms of exponential sums as given in Proposition 3. Writing $\mathcal{L}_{n}(\alpha)=\left\{\boldsymbol{x}_{0}, \ldots, \boldsymbol{x}_{q_{n}-1}\right\}$, where $\boldsymbol{x}_{h}=\left(\frac{h}{q_{n}},\left\{\frac{h p_{n}}{q_{n}}\right\}\right)$ for $h=0,1, \ldots, q_{n}-1$, we have

$$
\begin{equation*}
\left(L_{2, q_{n}}^{\text {per }}\left(\mathcal{L}_{n}(\alpha)\right)\right)^{2}=\frac{1}{9} \sum_{\boldsymbol{k} \in \mathbb{Z}^{2} \backslash\{\mathbf{0}\}} \frac{1}{r(\boldsymbol{k})^{2}}\left|\sum_{h=0}^{q_{n}-1} \exp \left(2 \pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}_{h}\right)\right|^{2}, \tag{13}
\end{equation*}
$$

where the $r(\boldsymbol{k})$ are defined according to (3). Note that the following arguments are similar to those used in the proof of [4, Theorem 3]. In order to study the sum (13) we need to distinguish different instances for the vector $\boldsymbol{k}$.

- The case $\boldsymbol{k}=(k, 0), k \neq 0$. Then we have

$$
\begin{aligned}
& \sum_{\substack{k=1 \\
k=(k, 0)}}^{\infty} \frac{1}{r(\boldsymbol{k})^{2}}\left|\sum_{h=0}^{q_{n}-1} \exp \left(2 \pi \mathrm{i} k \frac{h}{q_{n}}\right)\right|^{2}+\sum_{\substack{k=1 \\
k=(-k, 0)}}^{\infty} \frac{1}{r(\boldsymbol{k})^{2}}\left|\sum_{h=0}^{q_{n}-1} \exp \left(-2 \pi \mathrm{i} k \frac{h}{q_{n}}\right)\right|^{2} \\
& \quad=2 \sum_{\substack{k=1 \\
q_{n} \mid k}}^{\infty} \frac{q_{n}^{2}}{r(k)^{2}}=2 \frac{6}{4 \pi^{2}} \sum_{l=1}^{\infty} \frac{q_{n}^{2}}{\left(l q_{n}\right)^{2}}=\frac{1}{2}
\end{aligned}
$$

where we used the the well known identity $\sum_{k=1}^{\infty} \frac{1}{k^{2}}=\frac{\pi^{2}}{6}$ and the fact that

$$
\sum_{h=0}^{q_{n}-1} \exp \left( \pm 2 \pi \mathrm{i} k \frac{h}{q_{n}}\right)= \begin{cases}q_{n} & \text { if } k \equiv 0 \quad\left(\bmod q_{n}\right) \\ 0 & \text { otherwise }\end{cases}
$$

- The case $\boldsymbol{k}=(0, k), k \neq 0$. This case can be treated analogously as the previous one and yields the same result. One has to use that $\operatorname{gcd}\left(p_{n}, q_{n}\right)=1$, which is a well known fact from the theory of continued fractions. Therefore

$$
\sum_{h=0}^{q_{n}-1} \exp \left( \pm 2 \pi \mathrm{i} k \frac{h p_{n}}{q_{n}}\right)= \begin{cases}q_{n} & \text { if } k \equiv 0 \quad\left(\bmod q_{n}\right) \\ 0 & \text { otherwise }\end{cases}
$$

- The case $\boldsymbol{k}=\left(k_{1}, k_{2}\right)$, where $k_{1}, k_{2} \neq 0$ and $k_{1} \equiv 0\left(\bmod q_{n}\right)$, but $k_{2} \not \equiv 0\left(\bmod q_{n}\right)$. In this case we find

$$
\sum_{\substack{\boldsymbol{k}=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2} \backslash\{\mathbf{0}\} \\ k_{1} \equiv 0 \\ k_{2} \neq 0\left(\bmod q_{n}\right)}}^{\infty} \frac{1}{r\left(\boldsymbol{\operatorname { m o d } ) ^ { 2 }} q_{n}\right)} \underbrace{\left|\sum_{h=0}^{q_{n}-1} \exp \left(2 \pi i k_{2} \frac{h p_{n}}{q_{n}}\right)\right|^{2}}_{=0}=0 .
$$

- The case $\boldsymbol{k}=\left(k_{1}, k_{2}\right)$, where $k_{1}, k_{2} \neq 0$ and $k_{2} \equiv 0\left(\bmod q_{n}\right)$, but $k_{1} \not \equiv 0\left(\bmod q_{n}\right)$ can be treated analogously as the previous one and yields the same result.
- The case $\boldsymbol{k}=\left(k_{1}, k_{2}\right)$, where $k_{1}, k_{2} \neq 0$ and $k_{1} \equiv 0\left(\bmod q_{n}\right)$ as well as $k_{2} \equiv 0\left(\bmod q_{n}\right)$. In this case we find

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{k}=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2} \backslash\{\mathbf{0}\} \\
k_{1} \equiv 0 \\
k_{2} \equiv 0 \\
\left(\bmod q_{n}\right) \\
\left(\bmod q_{n}\right)}}^{\infty} \frac{q_{n}^{2}}{r(\boldsymbol{k})^{2}} & =q_{n}^{2}\left(\frac{6}{4 \pi^{2}}\right)^{2} \sum_{\substack{l_{1}, l_{2} \in \mathbb{Z}^{*}}} \frac{1}{\left(q_{n} l_{1}\right)^{2}\left(q_{n} l_{2}\right)^{2}} \\
& =\frac{1}{q_{n}^{2}}\left(\frac{6}{4 \pi^{2}}\right)^{2}\left(2 \frac{\pi^{2}}{6}\right)^{2}=\frac{1}{4 q_{n}^{2}} .
\end{aligned}
$$

- The case $\boldsymbol{k}=\left(k_{1}, k_{2}\right)$, where $k_{1}, k_{2} \neq 0$ and $k_{1} \not \equiv 0\left(\bmod q_{n}\right)$ as well as $k_{2} \not \equiv 0\left(\bmod q_{n}\right)$. In this case we have to evaluate the sum

$$
q_{n}^{2} \sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \neq 0\left(\bmod q_{n}\right) \\ k_{1}+k_{2} p_{n} \equiv 0 \\\left(\bmod q_{n}\right)}} \frac{1}{r(\boldsymbol{k})^{2}},
$$

which equals

$$
q_{n}^{2}\left(\frac{6}{4 \pi^{2}}\right)^{2} \sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \neq 0 \\ k_{1}+k_{2} p_{n} \equiv 0 \\\left(\bmod q_{n}\right) \\\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}}=\frac{9}{4 q_{n}^{2}} \sum_{r=1}^{q_{n}-1} \frac{1}{\sin ^{2}\left(\frac{\pi r}{q_{n}}\right) \sin ^{2}\left(\frac{\pi r p_{n}}{q_{n}}\right)}
$$

by Lemma 18 .
The result on $\left(L_{2, q_{n}}^{\text {per }}\left(\mathcal{L}_{n}(\alpha)\right)\right)^{2}$ follows.
Finally it remains to prove the result for the extreme $L_{2}$ discrepancy of $\mathcal{L}_{n}(\alpha)$. Recall from Remark 14 that the extreme $L_{2}$ discrepancy of a point set $\mathcal{P}=\left\{\left(x_{h}, y_{h}\right): h=0,1, \ldots, N-1\right\}$ can be calculated via the formula

$$
\begin{equation*}
\left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2}=\frac{N^{2}}{144}-\frac{N}{2} \sum_{h=0}^{N-1} f\left(x_{h}\right) f\left(y_{h}\right)+\frac{1}{4} \sum_{h, l=0}^{N-1} g\left(x_{h}, x_{l}\right) g\left(y_{h}, y_{l}\right) \tag{14}
\end{equation*}
$$

where we define $f(x):=x(1-x)$ and $g(x, y)=x+y-2 x y-|x-y|$. We compute the Fourier series of these two functions. Let $\widehat{f}(k)$ and $\widehat{g}\left(k_{1}, k_{2}\right)$ for $k, k_{1}, k_{2} \in \mathbb{Z}$ be the Fourier coefficients of $f$ and $g$; i.e.

$$
\widehat{f}(k)=\int_{0}^{1} f(x) \exp (-2 \pi \mathrm{i} k x) \mathrm{d} x
$$

and

$$
\widehat{g}\left(k_{1}, k_{2}\right)=\int_{0}^{1} \int_{0}^{1} g(x, y) \exp \left(-2 \pi \mathrm{i}\left(k_{1} x+k_{2} y\right)\right) \mathrm{d} x \mathrm{~d} y
$$

It is not difficult to find that $\widehat{f}(0)=\frac{1}{6}$ and $\widehat{f}(k)=-\frac{1}{2 \pi^{2} k^{2}}$ for $k \in \mathbb{Z}^{*}$. Therefore

$$
f(x)=\frac{1}{6}-\sum_{k \in \mathbb{Z}^{*}} \frac{\exp (-2 \pi \mathrm{i} k x)}{2 \pi^{2} k^{2}}=\sum_{k \in \mathbb{Z}^{*}} \frac{1-\exp (-2 \pi \mathrm{i} k x)}{2 \pi^{2} k^{2}} .
$$

For the function $g$ we find

$$
\widehat{g}\left(k_{1}, k_{2}\right)= \begin{cases}\frac{1}{6} & \text { if } k_{1}=k_{2}=0 \\ -\frac{1}{2 \pi^{2} k_{1}^{2}} & \text { if } k_{1} \in \mathbb{Z}^{*} \text { and } k_{2}=0 \\ -\frac{1}{2 \pi^{2} k_{2}^{2}} & \text { if } k_{1}=0 \text { and } k_{2} \in \mathbb{Z}^{*} \\ \frac{1}{2 \pi^{2} k_{1}^{2}} & \text { if } k_{1} \in \mathbb{Z}^{*} \text { and } k_{2}=-k_{1} \\ 0 & \text { otherwise }\end{cases}
$$

Therefore

$$
\begin{aligned}
g(x, y)= & \frac{1}{6}-\sum_{k_{1} \in \mathbb{Z}^{*}} \frac{\exp \left(-2 \pi \mathrm{i} k_{1} x\right)}{2 \pi^{2} k_{1}^{2}}-\sum_{k_{2} \in \mathbb{Z}^{*}} \frac{\exp \left(-2 \pi \mathrm{i} k_{2} y\right)}{2 \pi^{2} k_{2}^{2}} \\
& +\sum_{k_{1} \in \mathbb{Z}^{*}} \frac{\exp \left(-2 \pi \mathrm{i} k_{1} x\right) \exp \left(2 \pi \mathrm{i} k_{1} y\right)}{2 \pi^{2} k_{1}^{2}} \\
= & \sum_{k \in \mathbb{Z}^{*}} \frac{1}{2 \pi^{2} k^{2}}-\sum_{k \in \mathbb{Z}^{*}} \frac{\exp (-2 \pi \mathrm{i} k x)}{2 \pi^{2} k^{2}}-\sum_{k \in \mathbb{Z}^{*}} \frac{\exp (2 \pi \mathrm{i} k y)}{2 \pi^{2} k^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{k \in \mathbb{Z}^{*}} \frac{\exp (-2 \pi \mathrm{i} k x) \exp (2 \pi \mathrm{i} k y)}{2 \pi^{2} k^{2}} \\
= & \sum_{k \in \mathbb{Z}^{*}} \frac{(1-\exp (-2 \pi \mathrm{i} k x))(1-\exp (2 \pi \mathrm{i} k y))}{2 \pi^{2} k^{2}}
\end{aligned}
$$

We insert the Fourier expansions of $f$ and $g$ into equation (14) and obtain after some simplifications

$$
\begin{aligned}
& \left(L_{2, N}^{\operatorname{extr}}(\mathcal{P})\right)^{2} \\
& =\frac{N^{2}}{144} \\
& \quad-\frac{N}{2} \sum_{k_{1}, k_{2} \in \mathbb{Z}^{*}} \frac{1}{4 \pi^{4} k_{1}^{2} k_{2}^{2}} \sum_{h=0}^{N-1}\left(1-\exp \left(-2 \pi \mathrm{i} k_{1} x_{h}\right)\right)\left(1-\exp \left(-2 \pi \mathrm{i} k_{2} y_{h}\right)\right) \\
& \quad+\frac{1}{4} \sum_{k_{1}, k_{2} \in \mathbb{Z}^{*}} \frac{1}{4 \pi^{4} k_{1}^{2} k_{2}^{2}}\left|\sum_{h=0}^{N-1}\left(1-\exp \left(-2 \pi \mathrm{i} k_{1} x_{h}\right)\right)\left(1-\exp \left(-2 \pi \mathrm{i} k_{2} y_{h}\right)\right)\right|^{2} .
\end{aligned}
$$

In order to find the exact formula for $L_{2, q_{n}}^{\mathrm{extr}}\left(\mathcal{L}_{n}(\alpha)\right)$, we need to investigate the expression

$$
\Sigma_{k_{1}, k_{2}}:=\sum_{h=0}^{q_{n}-1}\left(1-\exp \left(-2 \pi \mathrm{i} k_{1} \frac{h}{q_{n}}\right)\right)\left(1-\exp \left(-2 \pi \mathrm{i} k_{2} \frac{h p_{n}}{q_{n}}\right)\right)
$$

for non-zero integers $k_{1}$ and $k_{2}$. We observe that $\Sigma_{k_{1}, k_{2}}$ can have the following values:

$$
\Sigma_{k_{1}, k_{2}}=\left\{\begin{array}{lll}
q_{n} & \text { if } k_{1}, k_{2} \not \equiv 0 \quad\left(\bmod q_{n}\right) \text { and } k_{1}+k_{2} p_{n} \not \equiv 0 & \left(\bmod q_{n}\right) \\
2 q_{n} & \text { if } k_{1}, k_{2} \not \equiv 0 \quad\left(\bmod q_{n}\right) \text { and } k_{1}+k_{2} p_{n} \equiv 0 \quad\left(\bmod q_{n}\right) \\
0 & \text { otherwise } &
\end{array}\right.
$$

This leads to

$$
\begin{aligned}
& \left(L_{2, q_{n}}^{\text {extr }}\left(\mathcal{L}_{n}(\alpha)\right)\right)^{2} \\
& =\frac{q_{n}^{2}}{144} \\
& -\frac{q_{n}}{2}\left(\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0 \\
k_{1}+k_{2} p_{n} \neq 0 \\
\left(\bmod q_{n}\right) \\
\left(\bmod q_{n}\right)}} \frac{q_{n}}{4 \pi^{4} k_{1}^{2} k_{2}^{2}}+\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0\left(\bmod q_{n}\right) \\
k_{1}+k_{2} p_{n} \equiv 0 \\
\left(\bmod q_{n}\right)}} \frac{2 q_{n}}{4 \pi^{4} k_{1}^{2} k_{2}^{2}}\right) \\
& +\frac{1}{4}\left(\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0 \\
k_{1}+k_{2} p_{n} \neq 0 \\
\left(\bmod q_{n}\right) \\
\left(\bmod q_{n}\right)}} \frac{q_{n}^{2}}{4 \pi^{4} k_{1}^{2} k_{2}^{2}}+\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0 \\
k_{1}+k_{2} p_{n} \equiv 0}} \frac{4 q_{n}^{2}}{4 \pi^{4} k_{1}^{2} k_{2}^{2}}\right)
\end{aligned}
$$

$$
=\frac{q_{n}^{2}}{144}-\frac{q_{n}^{2}}{16 \pi^{4}} \sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \neq 0 \\ k_{1}+k_{2} p_{n} \neq 0}} \frac{1}{\left(\bmod q_{n}\right)}\left(\bmod q_{n}\right)<.
$$

We have

$$
\begin{equation*}
\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \neq 0 \\ k_{1}+k_{2} p_{n} \neq 0}} \frac{1}{\left.\bmod _{(1)}^{2} q_{n}\right)} \sum_{\substack{\left.\bmod q_{n}\right)}} \sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \not \equiv 0 \\\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}}-\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\ k_{1}, k_{2} \neq 0 \\ k_{1}+k_{2} p_{n} \equiv 0\left(\bmod q_{n}\right) \\\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}} \tag{15}
\end{equation*}
$$

For the first sum on the right hand side we find

$$
\begin{aligned}
\sum_{\substack{k_{1}, k_{2} \in \mathbb{Z}^{*} \\
k_{1}, k_{2} \neq 0 \\
\left(\bmod q_{n}\right)}} \frac{1}{k_{1}^{2} k_{2}^{2}} & =\left(\sum_{\substack{k \in \mathbb{Z}^{*} \\
k \neq 0 \\
\left(\bmod q_{n}\right)}} \frac{1}{k^{2}}\right)^{2} \\
& =\left(\sum_{k \in \mathbb{Z}^{*}} \frac{1}{k^{2}}-\sum_{k \in \mathbb{Z}^{*}} \frac{1}{\left(k q_{n}\right)^{2}}\right)^{2} \\
& =\frac{\pi^{4}}{9}\left(1-\frac{1}{q_{n}^{2}}\right)^{2}
\end{aligned}
$$

The value of the second sum in (15) is known by Lemma 18. Now the result follows.

Acknowledgements. The authors are supported by the Austrian Science Fund (FWF), Projects F5513-N26 (Hinrichs) and F5509-N26 (Kritzinger and Pillichshammer), which are parts of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

We are grateful to an anonymous referee for several comments.

## References

[1] D. Bilyk: Cyclic shifts of the van der Corput set. Proc. Amer. Math. Soc. 137: 2591-2600, 2009.
[2] D. Bilyk: On Roth's orthogonal function method in discrepancy theory. Unif. Distrib. Theory 6:143-184, 2011.
[3] D. Bilyk, V. N. Temlyakov, and R. Yu: Fibonacci sets and symmetrization in discrepancy theory. J. Complexity 28(1): 18-36, 2012.
[4] D. Bilyk, V. N. Temlyakov, and R. Yu: The $L_{2}$ discrepancy of twodimensional lattices. Recent Advances in Harmonic Analysis and Applications, pp. 63-77, Springer Proc. Math. Stat., 25, Springer, New York, 2013.
[5] B. Borda: On the theorem of Davenport and generalized Dedekind sums. J. Number Theory 172: 1-22, 2017.
[6] J. Dick, A. Hinrichs and F. Pillichshammer: Proof techniques in quasiMonte Carlo theory. J. Complexity 31(3): 327-371, 2015.
[7] J. Dick, A. Hinrichs, and F. Pillichshammer: A note on the periodic $L_{2}$-discrepancy of Korobov's $p$-sets. Arch. Math. 115(1): 67-78, 2020.
[8] J. Dick and F. Pillichshammer: On the mean square weighted $\mathcal{L}_{2}$ discrepancy of randomized digital $(t, m, s)$-nets over $\mathbb{Z}_{2}$. Acta Arith. 117(4): 371-403, 2005.
[9] J. Dick and F. Pillichshammer: Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.
[10] M. Drmota and R. F. Tichy: Sequences, Discrepancies and Applications. Lecture Notes in Mathematics, vol. 1651, Springer, Berlin, 1997.
[11] H. Faure, P. Kritzer, and F. Pillichshammer: From van der Corput to modern constructions of sequences for quasi-Monte Carlo rules. Indag. Math. 26(5): 760-822, 2015.
[12] M. Gnewuch: Bounds for the average $L_{p}$-extreme and the $L_{\infty}$-extreme discrepancy. Electron. J. Combin. 12, Research Paper 54, 11 pp., 2005.
[13] J.H. Halton and S.K. Zaremba: The extreme and $L^{2}$ discrepancies of some plane sets. Monatsh. Math. 73: 316-328, 1969.
[14] A. Hinrichs: Discrepancy, integration and tractability. Monte Carlo and Quasi-Monte Carlo Methods, pp. 123-163, Springer Proc. Math. Stat., 65, Springer, Berlin, 2013.
[15] A. Hinrichs, R. Kritzinger, and F. Pillichshammer: Optimal order of $L_{p}$ discrepancy of digit shifted Hammersley point sets in dimension 2. Unif. Distrib. Theory 10: 115-133, 2015.
[16] A. Hinrichs and J. Oettershagen: Optimal point sets for quasi-Monte Carlo integration of bivariate periodic functions with bounded mixed derivatives. Monte Carlo and Quasi-Monte Carlo Methods, pp. 385405, Springer Proc. Math. Stat., 163, Springer, [Cham], 2016.
[17] A. Hinrichs and H. Weyhausen: Asymptotic behavior of average $L_{p^{-}}$ discrepancies. J. Complexity 28: 425-439, 2012.
[18] J.F. Koksma: Some integrals in the theory of uniform distribution modulo 1. (Dutch) Mathematica, Zutphen. B. 11: 49-52, 1942.
[19] P. Kritzer and F. Pillichshammer: An exact formula for the $L_{2^{-}}$ discrepancy of the shifted Hammersley point set. Unif. Distrib. Theory 1: 1-13, 2006.
[20] R. Kritzinger and M. Passenbrunner: Extremal distributions of discrepancy functions. J. Complexity 54: 101409, 10 pp., 2019.
[21] L. Kuipers and H. Niederreiter: Uniform Distribution of Sequences. John Wiley, New York, 1974.
[22] V.F. Lev: On two versions of $L^{2}$-discrepancy and geometrical interpretation of diaphony. Acta Math. Hungar. 69(4): 281-300, 1995.
[23] W. J. Morokoff and R. E. Caflisch: Quasi-random sequences and their discrepancies. SIAM J. Sci.Comput. 15: 1251-1279, 1994.
[24] H. Niederreiter: Application of Diophantine approximations to numerical integration. Diophantine approximation and its applications (Proc. Conf., Washington, D.C., 1972), pp. 129-199. Academic Press, New York, 1973.
[25] H. Niederreiter: Random number generation and quasi-Monte Carlo methods. Number 63 in CBMS-NFS Series in Applied Mathematics, SIAM, Philadelphia, 1992.
[26] E. Novak and H. Woźniakowski: Tractability of Multivariate Problems, Volume II: Standard Information for Functionals. European Mathematical Society, Zürich, 2010.
[27] F. Pillichshammer: On the $L_{p}$ discrepancy of the Hammersley point set. Monatsh. Math. 136: 67-79, 2002.
[28] K.F. Roth: On irregularities of distribution. Mathematika 1: 73-79, 1954.
[29] K.F. Roth: On irregularities of distribution. IV. Acta Arith. 37: 67-75, 1980.
[30] T. T. Warnock: Computational investigations of low discrepancy point sets. Applications of Number Theory to Numerical Analysis. pp. 319343, Academic Press, New York, 1972.
[31] P. Zinterhof: Über einige Abschätzungen bei der Approximation von Funktionen mit Gleichverteilungsmethoden (German). Österr. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II 185: 121-132, 1976.

Institute of Analysis, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria

Email address: aicke.hinrichs@jku.at
Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria

Email address: ralph.kritzinger@jku.at
Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University Linz, Altenberger Strasse 69, 4040 Linz, Austria

Email address: friedrich.pillichshammer@jku.at


[^0]:    2020 Mathematics Subject Classification. Primary 11K38; Secondary 11K36.
    Key words and phrases. $L_{2}$ discrepancy, diaphony, Hammersley point set, rational lattice, lower bounds.

[^1]:    ${ }^{1}$ Set $m=\infty$ in [8, Lemma 3] and take care of the resulting consequences.

