

Local Large Deviations: A McMillian Theorem for Typed Random Graph Processes

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Abstract: For a finite typed graph on n nodes and with type law μ on finite alphabet Γ , we define the spectral potential $\rho_\lambda(\cdot, \mu)$, of the graph. From the $\rho_\lambda(\cdot, \mu)$ we define the Kullback action or the divergence function, $H_\lambda(\pi \parallel \nu)$, with respect to an empirical link measure, π , as the Legendre dual. For the finite typed random graph conditioned to have an empirical link measure π and empirical type measure μ , we prove a Local large Deviation Principle (LLDP), with rate function $H_\lambda(\pi \parallel \nu)$ and speed n . From this LLDP, we obtain a full conditional large deviation principle and a weak variant of the classical McMillian Theorem for the typed random graph models. Given the typical empirical link measure, $\lambda_{\mu \otimes \mu}$, we show that, the number of typed random graphs is equal to $e^{n \|\lambda_{\mu \otimes \mu}\| H(\lambda_{\mu \otimes \mu} \parallel \lambda_{\mu \otimes \mu})}$ approximately. Note that we do not require any topological restrictions on the space of finite graphs for these LLDPs.

Keywords: Local Large Deviation Principle, Variational Principle, Empirical Link Measure, Spectral Potential Theory, Relative Entropy, Typed Random Process

Introduction

We consider random graph models, where nodes are assigned types independently according to some type on a finite alphabet and between any two given nodes, a link is present with a probability type on a finite alphabet and between any two given nodes, a link is present with a probability that depends on the type of the nodes. This random graph model was first proposed and studied extensively by Penman (1998) as a generalization of the Erdos-Renyi graphs. Indeed, the random graph model which is known to model fairly well network structured data, has the Erdos-Renyi graph as a special case. See, Cannings and Penman (2003) for an exposition on this random graphs and their applications.

Now, some Large Deviation Principles (LDPs) and Coding Theorems exists for networked data structures modelled as the Typed Random Graph (TRG) models. See, O'Connell (1998), Biggins and Penman (2009), Doku-Amponsah and Mörters (2010), Bordenave and Caputo (2005), Doku-Amponsah (2012, 2017a) and the reference therein. O'Connell (1998) proved Large Deviation Principle (LDP) for the relative size of the largest connected component in the random graph with

small edge probability. Biggins and Penman (2009) have found LDPs for the number of edges of TRG models, where the link probabilities are independent of the number of nodes, using the Garten-Ellis Theorem, see Dembo and Zeitouni (1998). Doku-Amponsah and Mörters (2010) proved LDPs for the empirical measures of the TRG where the link probabilities are dependent on the number of nodes of the graph. In Bordenave and Caputo (2005), LDP for the empirical neighborhood distribution in sparse random graphs was proved using a technique that relies on the typical behavior within the framework of the local weak convergence of finite graph sequences. Asymptotic Equipartition Properties including the Lossy version have been found in Doku-Amponsah (2012; 2017a), by the techniques of exponential change of measure and random allocation, respectively.

We present in this article a conditional LLDP for the TRG models given the empirical type measure of the graph. Refer to Bakhtin (2015), Doku-Amponsah (2017) for similar results for the empirical measure of iid random variables and the empirical offspring measure of multitype Galton-Watson processes, respectively. This article shares similar features as Doku-Amponsah (2017c), but defers

from all the LDPs discussed above. i.e., O'Connell (1998) Biggins and Penman (2009), Doku-Amponsah and Mörters (2010) Bordenave and Caputo (2005), Doku-Amponsah (2012; 2017b). The main technique used to prove our main result is rooted in spectral potential theory. See, Doku-Amponsah (2017b) and the reference therein for similar idea for the LLDP for the multi-type Galton-Watson processes. To be precise about this technique, we define the spectral potential of the TRG and use it to calculate an extended version of the relative entropy and show that this relative entropy which has all the properties of the classical relative entropy, see Dembo and Zeitouni (1998), is the Legendre dual of our spectral potential of the TRG. From the LLDP for the TRG we deduce the weak variant of the classical McMillian Theorem and the full LDP for the TRG conditioned on the empirical link measure and under the conditional law of the TRG given the type law.

Coloured Random Process

Let $p_n: \Gamma \times \Gamma \rightarrow [0,1]$ be a symmetric function and μ on Γ be a probability measure. We can define the typed random graph Y with $[n] = \{1, 2, 3, \dots, n\}$ nodes as follows:

- Allocate to each node $v \in [n]$ type $Y(v)$ independently according to the *type law* μ
- For any pair of types $Y(u), Y(v)$, we connect two vertices $u, v \in [n]$, independently of everything else, with *link probability* $p_n(Y(u), Y(v))$

We always look at $Y = ((Y(v): v \in [n]), E)$ with respect to the combine law of the graph model and type and interpret Y as typed random graph model.

Denote by $\mathcal{G}([n], \Gamma)$ the set of all coloured graphs with colour set Γ and n vertices. We shall only study Y with link probabilities satisfying:

$$a_n^{-1} p_n(a, b) \rightarrow \lambda(a, b), \forall a, b \in \Gamma, \tag{1.1}$$

where $\lambda: \Gamma \times \Gamma \rightarrow [0, \infty]$ is a nonzero function.

If the sequence a_n in 1.1 above satisfies (i) $a_n \rightarrow 1$ (ii) $a_n \rightarrow 0$ and (iii) $a_n \rightarrow \infty$ we call X sparse, subcritical and supercritical respectively.

In this article we also assume that the sequence (a_n) converges to 0 as n approaches ∞ .

For any finite or countable set \mathcal{L} , we denote by $\mathcal{L}(\Gamma)$ the space of probability vectors by $\tilde{\mathcal{L}}(\Gamma)$ space of finite positive measures on Γ , by $\tilde{\mathcal{L}}(\Gamma)$ we denote the subspace of symmetric measures in $\tilde{\mathcal{L}}(\Gamma)$. By $\mathcal{P}(\Gamma)$ the space of all real-valued bounded measurable functions on Γ , by $\mathcal{P}_*(\Gamma)$ the space of continuous linear functionals

on $\mathcal{P}(\Gamma)$ and by $\mathcal{P}_+(\Gamma)$ the collection of all positive linear functional on $\mathcal{P}(\Gamma)$.

For every typed random graph Y , we define the empirical type distribution $L_Y^1 \in \mathcal{L}(\Gamma)$ by:

$$L_Y^1(a) = \frac{1}{n} \sum_{v \in [n]} \delta_{Y(v)}(a), \text{ for } a \in \Gamma. \tag{1.2}$$

and the empirical link distribution $L_Y^2 \in \tilde{\mathcal{L}}(\Gamma \times \Gamma)$ is defined by:

$$L_Y^2(a, b) = \frac{1}{a_n n^2} \sum_{(u, v) \in E} [\delta_{Y(v), Y(u)} + \delta_{Y(u), Y(v)}](a, b), \text{ for } a, b \in \Gamma. \tag{1.3}$$

Note that $a_n n^2$ is the maximum possible number of edges in the graph and we have that:

$$a_n n^2 L_X^2(a, b) = \begin{cases} \#\{\text{number of edges between} \\ \text{vertices of colours } a \text{ and } b\} & \text{if } a = b \\ 2 \times \#\{\text{number of edges between} \\ \text{vertices of colours } a\} & \text{if } a \neq b \end{cases} \tag{1.4}$$

The rest of the article is structured as follows: Section 2 contain the main results of the article (Theorem 2.1, Theorem 2.3, Corollary 2.2). Section 3 contain proofs of the main results of the article.

Statement of Main Results

We assume through out the remaining part of this article that the typed random graph process is near-critical or sparse. We shall write $\langle f, \sigma \rangle := \sum_{y \in \Gamma} \sigma(y) f(y)$ and

define the *spectral potential* $\rho_\lambda(g, \mu)$ of the near-critical typed random graph process Y by:

$$\rho_\lambda(g, \mu) = -\langle (1 - e^g), \lambda_{\mu \otimes \mu} \rangle / 2, \tag{2.1}$$

where $\lambda_{\mu \otimes \mu}(a, b) = \lambda(a, b) \mu(a) \mu(b), \forall a, b \in \Gamma \times \Gamma$

Notice, ρ_λ is (i) finite on $\left\{ g: \Gamma \times \Gamma \rightarrow R \mid e^{-\frac{1}{2} \langle (1 - e^g), \lambda_{\mu \otimes \mu} \rangle} < \infty \right\}$ (ii) monotone (iii)

additively homogeneous and convex in g . For $v \in \mathcal{P}(\Gamma \times \Gamma)$ we define the Kullback action by a nonlinear functional:

$$\mathcal{H}_\lambda(\pi \| \mu) := \left(\left\langle \pi, \log \frac{\pi}{\lambda_{\mu \otimes \mu}} \right\rangle + \|\lambda_{\mu \otimes \mu}\| - \|\pi\| \right) / 2 \tag{2.2}$$

and note that \mathcal{H}_λ above is nonlinear functional.

Let $P_\mu(y) = \mathbb{P}\{Y = y | L_y^1 = \mu\}$ be the probability law of the near-critical typed random graph process y on $[n]$ conditioned on the empirical type measure μ . In Theorem 2.1 below we present our main result, the LLDP for the typed random graph process.

Theorem 2.1.

Let $y = (y(v): v \in [n])$ be a typed random graph process with type law μ and link probabilities that satisfies $a_n^{-1} p_n(a, b) \rightarrow \lambda(a, b)$, for $a, b \in \Gamma$ and $na_n \rightarrow 1$. Then:

- for any $\omega \in \tilde{\mathcal{L}}_\ast(\Gamma \times \Gamma)$ and a number $\varepsilon > 0$, there exists a weak neighborhood B_ω such that:

$$P_\mu \{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in B_\omega\} \leq e^{-n\mathcal{H}_\lambda(\pi \| \mu) - n\varepsilon}.$$

- for any $\omega \in \tilde{\mathcal{L}}_\ast(\Gamma \times \Gamma)$, a number $\varepsilon > 0$ and a fine neighborhood B_ω we have the asymptotic estimate:

$$P_\mu \{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in B_\omega\} \geq e^{-n\mathcal{H}_\lambda(\pi \| \mu) - n\varepsilon}.$$

Next we state a corollary of Theorem 2.1, the McMillian-Breiman Theorem for the typed random graph process. We define an entropy by:

$$h^\lambda(\pi) := \left(\|\pi\| - \left\| \lambda_{\mu \otimes \mu} \right\| - \left\langle \pi, \log \frac{\pi}{\lambda_{\mu \otimes \mu}} \right\rangle \right) / 2. \tag{2.3}$$

Corollary 2.2.

Let $\mathcal{G}([n], \Gamma)$ be the space of all typed random graph process with type law μ and link probabilities that satisfies $a_n^{-1} p_n(a, b) \rightarrow \lambda(a, b)$, for $a, b \in \Gamma$ and $na_n \rightarrow 1$.

- For any empirical link measure ρ on $\Gamma \times \Gamma$ and $\varepsilon > 0$, there exists a neighborhood B_ρ such that:

$$\#\{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in B_\rho\} \geq e^{n(h^\lambda(\rho) + \varepsilon)}.$$

- for any neighborhood B_ρ and $\varepsilon > 0$, we have:

$$\#\{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in B_\rho\} \leq e^{n(h^\lambda(\rho) - \varepsilon)},$$

where, $\# \{A\}$ means the cardinality of A :

Remark 1

For $\rho = \lambda_{\mu \otimes \mu}$, equation 2.3 above reduces to $h^\lambda(\pi) := - \left\langle \lambda_{\mu \otimes \mu}, \log \frac{\lambda_{\mu \otimes \mu}}{\|\lambda_{\mu \otimes \mu}\|} \right\rangle$ and therefore, we have:

$$\#\{y \in \mathcal{G}([n], \Gamma)\} \approx e^{n\|\lambda_{\mu \otimes \mu}\| H(\lambda_{\mu \otimes \mu} / \|\lambda_{\mu \otimes \mu}\|)}.$$

We state in Theorem 2.3 the full LDP for the typed random graph process below:

Theorem 2.3.

Let $y = (y(v): v \in [n])$ be a typed random graph process with type law μ and link probabilities that satisfies $a_n^{-1} p_n(a, b) \rightarrow \lambda(b, a)$, for $b, a \in \Gamma$ and $na_n \rightarrow 1$.

- Let F be open subset of $\tilde{\mathcal{L}}_\ast(\Gamma \times \Gamma)$. Then:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_\mu \{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in F\} \geq - \inf_{\pi \in F} \mathcal{H}_\lambda(\pi \| \mu).$$

- Let G be closed subset of $\tilde{\mathcal{L}}_\ast(\Gamma \times \Gamma)$. Then:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_\mu \{y \in \mathcal{G}([n], \Gamma) | L_y^2 \in G\} \leq - \inf_{\pi \in G} \mathcal{H}_\lambda(\pi \| \mu).$$

Proof of Main Results

Properties of the Rate Function

In this subsection Lemma 3.1, which summaries the properties of 2.2 above. This will help us circumvent the topological problems faced by Doku-Amponsah (2017b), Dembo *et al.* (2005). Denote by C is the space of continuous functions $g: \Gamma \times \Gamma \rightarrow \mathbb{R}$.

Lemma 3.1.

The following holds for the Kullback action or divergence function $\mathcal{H}_\lambda(\pi \| \mu)$:

- $\mathcal{H}_\lambda(\pi \| \mu) = \frac{1}{2} \sup_{g \in C} \{ \langle g, \pi \rangle - \rho_\lambda(g, \mu) \}$
- The function $\mathcal{H}_\lambda(\pi \| \mu)$ is lower semi-continuous on the space $\mathcal{P}_\ast(\Gamma \times \Gamma)$
- For any real c , the set $\{v \in \mathcal{P}(\Gamma \times \Gamma) : \mathcal{H}_\lambda(\pi \| \mu) \leq c\}$ is weakly compact

Please we refer to (Bakhtin, 2015) for similar result and proof for the empirical measures on measurable spaces.

The proof presented below follows similar ideas as the proof of [(Bakhtin, 2015), Lemma 2.2] for the empirical measures on measurable spaces.

Proof

(i) Let $g \in C$ be such that $\langle g, \pi \rangle$ estimates the functional $\langle \phi, \pi \rangle$ and $\rho_\lambda(g, \mu)$ estimates $\rho_\lambda(\phi, \mu)$ where $\phi \in \mathcal{P}(\Gamma \times \Gamma)$. Suppose π is absolutely continuous with respect to $\lambda_{\mu \otimes \mu}$.

Define the function g by $g := \log \frac{\pi}{\lambda_{\mu \otimes \mu}}$. For $t > 0$, we define the estimate of g , $g_t \in \mathcal{P}(\Gamma \times \Gamma)$ as follows:

$$g_t(a, b) := \begin{cases} g(a, b), & \text{if } -t < g(a, b) < t, \\ e^t, & \text{if } g(a, b) > t \\ e^{-t}, & \text{if } g(a, b) < -t \end{cases} \quad (3.1)$$

for all $(a, c) \in \Gamma \times \Gamma^*$. Now for $t \rightarrow \infty$ we have that:

$$\begin{aligned} \langle (1 - e^{gt}), \lambda_{\mu \otimes \mu} \rangle &= \int (1 - e^{g(a,b)}) 1_{\{-t < g(a,b) < t\}} \lambda_{\mu \otimes \mu}(da, db) \\ &+ \int t 1_{\{g(a,b) > t\}} \lambda_{\mu \otimes \mu}(da, db) \\ &+ \int -t 1_{\{-t < g\}} \lambda_{\mu \otimes \mu}(da, db) \rightarrow \langle (1 - e^g), \lambda_{\mu \otimes \mu} \rangle = \|\lambda_{\mu \otimes \mu}\| - \|\pi\| \\ \langle g_t, \pi \rangle &= \int g 1_{\{-t < g < t\}} \pi(da, db) + \int t 1_{\{g > t\}} \pi(da, db) \\ &+ \int -t 1_{\{-t > g\}} \pi(da, db) \\ &\rightarrow \langle g, \pi \rangle = \left\langle \pi, \log \frac{\pi}{\lambda_{\mu \otimes \mu}} \right\rangle. \end{aligned}$$

Therefore we have $\lim_{t \rightarrow \infty} \frac{1}{2} (\langle g_t, \pi \rangle - \langle (1 - e^{gt}), \lambda_{\mu \otimes \mu} \rangle) \rightarrow \mathcal{H}_\lambda(\pi \| \mu)$ which proves Lemma 3.1 (i). Suppose π is not absolutely continuous with respect to $\lambda_{\mu \otimes \mu}$. i.e., there exists an $\varepsilon > 0$ such that for any $1 > \eta > 0$ there exists $B_\eta \subset \Gamma \times \Gamma^*$ with $\lambda_{\mu \otimes \mu}(B_\eta) \leq \eta / (1 - \eta)$ and at the same time we have $\pi(B_\eta) > \varepsilon$. For this η define the function:

$$g_\delta(a, b) := \begin{cases} -\log \eta & \text{if } (a, b) \in B_\eta, \\ 0, & \text{if } (a, b) \notin B_\eta. \end{cases} \quad (3.2)$$

Then we have $\lim_{\eta \downarrow 0} \frac{1}{2} (\langle g_\eta, \pi \rangle - \langle (1 - e^{g_\eta}), \lambda_{\mu \otimes \mu} \rangle) \geq \frac{\varepsilon}{2} \log \eta - \frac{1}{2} \langle (1 - e^{g_\eta}), \lambda_{\mu \otimes \mu} \rangle \geq -\frac{\varepsilon}{2} \log \eta + \frac{1}{2}$.

Taking limit as $\eta \downarrow 0$ we have $\mathcal{H}_\lambda(\pi \| \mu) = +\infty$, which ends the proof of Lemma 3.1 (i). (ii) & (iii). Observe from the properties of the relative entropy, see Dembo *et al.* (1998) and Lemma 3.1(i) that $\frac{1}{2} \sup_{g \in C} \{ \langle g, \pi \rangle - \rho_\lambda(g, \mu) \}$ reduces to equation 2.2 above. Now the relative entropy is lower semi-continuous and by Doku-Amponsah (2012) all its level sets are compact. Hence it holds $\mathcal{H}_\lambda(\pi \| \mu)$ is lower semi-continuous and all its level sets are weakly compact in the weak topology which ends the proof of the Lemma.

Note that Lemma 3.1 (i) above implies the so-called variational principle. See, example Komkov (1986).

Proof of Theorem 2.1.

By Lemma 3.1, for any $\varepsilon > 0$ there exists a function $g \in \mathcal{P}(\Gamma \times \Gamma)$ such that:

$$\mathcal{H}_\lambda(\pi \| \mu) - \frac{\varepsilon}{2} < \langle g, \pi \rangle - \rho_\lambda(g, \mu).$$

We define the probability distribution \tilde{P}_μ by:

$$\begin{aligned} \tilde{P}_\mu(Y) &= \prod_{(u,v) \in E} \tilde{p}_n(Y(u), Y(v)) \prod_{(u,v) \in E} 1 - \tilde{p}_n(Y(u), Y(v)) \\ &= \prod_{(u,v) \in E} \frac{\tilde{p}_n(Y(u), Y(v))}{n - n\tilde{p}_n(Y(u), Y(v))} \prod_{(u,v) \in E} (n - n\tilde{p}_n(Y(u), Y(v))) \\ &= \prod_{(u,v) \in E} e^{g(Y(u), Y(v))} \frac{p_n(Y(u), Y(v))}{n - np_n(Y(u), Y(v))} \prod_{(u,v) \in E} e^{n\tilde{h}_n(Y(u), Y(v))} (n - np_n(Y(u), Y(v))) \end{aligned} \quad (3.3)$$

Using 3.3 above, we have:

$$\begin{aligned} \frac{dP_\mu(Y)}{d\tilde{P}_\mu(Y)} &= \prod_{(u,v) \in E} e^{-g(Y(u), Y(v))} \\ &\prod_{(u,v) \in E} e^{-\frac{1}{n} \tilde{h}_n(Y(u), Y(v))} = e^{-n(\frac{1}{2} L^2, \tilde{g}) - n(\frac{1}{2} L^2 \otimes L, \tilde{h}_n) + \langle \frac{1}{2} L_\Delta, L, \tilde{h}_n \rangle}, \end{aligned} \quad (3.4)$$

While:

$$L'_\Delta = \delta_{(Y(u), Y(v))}.$$

Now we define a neighbourhood of the functional π as follows:

$$B_\pi = \{ \varpi \in \mathcal{P}(\Gamma \times \Gamma) : \langle g, \varpi \rangle > \langle g, \pi \rangle - \frac{\varepsilon}{2} \}.$$

Therefore, under the condition $L'_y \in B_\nu$ we have that:

$$\frac{dP_\mu(x)}{d\tilde{P}_\mu(x)} < e^{\frac{1}{2}(\rho_\lambda(g, \mu) - \langle g, \pi \rangle) + \frac{\varepsilon}{2}} < e^{-nH_\lambda(\pi \| \mu) + n\varepsilon}.$$

Hence, we have:

$$\begin{aligned} P_\mu \{ y \in \mathcal{G}([n], \Gamma) \mid L'_y \in B_\pi \} &\leq \int_{\mathcal{G}([n], \Gamma)} 1_{\{L'_y \in B_\pi\}} d\tilde{P}_\mu(x) \\ &\leq \int_{\mathcal{G}([n], \Gamma)} 1_{\{L'_y \in B_\pi\}} e^{-n\mathcal{H}_\lambda(\pi \| \mu) - n\varepsilon} d\tilde{P}_\mu(x) \\ &\leq e^{-n\mathcal{H}_\lambda(\pi \| \mu) - n\varepsilon}. \end{aligned}$$

Note that $\mathcal{H}_\lambda(\pi \| \mu) = \infty$ implies Theorem 2.1(ii) and so it suffice to prove that for a probability measure of the form $\pi = e^g \lambda_{\mu \otimes \mu}$, where the Kullback action $\mathcal{H}_\lambda(\pi \| \mu) = \langle g, \pi \rangle + \langle (1, e^g), \lambda_{\mu \otimes \mu} \rangle$ is finite. Fix any number $\varepsilon > 0$ and any neighbourhood $B_\pi \subset \mathcal{L}(\Gamma \times \Gamma)$. We define the sequence of sets:

$$\tilde{\mathcal{G}}([n], \Gamma) := \left\{ y \in \mathcal{G}([n], \Gamma) : L'_y \in B_\nu, \left| \langle g, L'_y \rangle - \langle g, \pi \rangle \right| \leq \frac{\varepsilon}{2} \right\}.$$

Observe that, for all $x \in \tilde{\mathcal{G}}([n], \Gamma)$ we have:

$$\frac{dP_\mu(x)}{d\tilde{P}_\mu(x)} = e^{-n\left(\frac{1}{2}L^2, \frac{\varepsilon}{2}\right) - n\left(\frac{1}{2}L^1 \otimes L^1, \tilde{h}_n\right) + \left(\frac{1}{2}L^1, \tilde{h}_n\right)} > e^{-n\left(\frac{1}{2}\pi, \log \frac{\pi}{\lambda_{\mu \otimes \mu}}\right) - n\frac{1}{2}\left\langle \lambda_{\mu \otimes \mu}, \left(1 - \frac{\pi}{\lambda_{\mu \otimes \mu}}\right)\right\rangle}$$

This gives:

$$\begin{aligned} P_\mu(\tilde{\mathcal{G}}([n], \Gamma)) &= \int_{\tilde{\mathcal{G}}([n], \Gamma)} dP_\mu(x) \\ &\geq \int_{\tilde{\mathcal{G}}([n], \Gamma)} e^{-\left(\frac{1}{2}\pi, \log \frac{\pi}{\lambda_{\mu \otimes \mu}}\right) - n\frac{1}{2}\left\langle \lambda_{\mu \otimes \mu}, \left(1 - \frac{\pi}{\lambda_{\mu \otimes \mu}}\right)\right\rangle + \frac{\varepsilon}{2}} d\tilde{P}_\mu(x) \\ &= e^{-n\mathcal{H}_\lambda(\pi \parallel \mu) + n\varepsilon} \tilde{P}_\mu(\tilde{\mathcal{G}}([n], \Gamma)). \end{aligned}$$

Using the law of large numbers we have $\lim_{n \rightarrow \infty} \tilde{P}_\mu(\tilde{\mathcal{G}}([n], \Gamma)) = 1$ which completes the proof.

Proof of Corollary 2.2 and Theorem 2.3.

Proof of Corollary 2.2.

The proof of Corollary 2.2 follows from the definition of the Kullback action and Theorem 2.1 if we set $\pi = \rho$ and $\lambda_{\mu \otimes \mu}(a, b) = \|\lambda_{\mu \otimes \mu}\|$, for all $(a, b) \in \Gamma \times \Gamma$.

The proof of Theorem 2.3 below, follows from Theorem 2.1 above using similar arguments as in [(Bakhtin, 2015), p. 544].

Proof of Theorem 2.3.

Proof

Note that the empirical link measure is a finite measure and so belongs to some ball in $\mathcal{B}_*(\Gamma \times \Gamma)$. Hence, without loss of generality we may assume that the set Γ in Theorem 2.3(ii) is relatively compact. See Lemma 3.1 (iii). Choose any $\varepsilon > 0$. Then for every functional $\pi \in \Gamma$ we can find a weak neighbourhood such that the estimate of Theorem 2.1(i) holds. We choose from all these neighbourhood a finite cover of $\mathcal{G}([n], \Gamma)$ and sum up over the estimate in Theorem 2.1(i) to obtain:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_n \left\{ y \in \mathcal{G}([n], \Gamma) \mid L_y^2 \in G \right\} \leq -\inf_{\pi \in \Gamma} \mathcal{H}_\lambda(\pi \parallel \mu) + \varepsilon.$$

As ε was arbitrarily chosen and the lower bound in Theorem 2.1(ii) implies the lower bound in Theorem 2.3(i) we have the desired results which ends the proof of the Theorem.

Conclusion

In this article, we have found a Local Large Deviation Principle (LLDP) for the typed random graph processes using techniques rooted in spectral potential theory. From this LLDP we have proved a weak variant of the famous McMillian Theorem and a full conditional LDP for the typed random graph model. These results like the ones in Bakhtin (2015), Doku-Amponsah

(2017c) differ from those in O'Connell (1998), Biggins and Penman (2009), Doku-Amponsah and Mörters, (2010), Bordenave and Caputo (2015), Doku-Amponsah (2012; 2017a) by the limited or no topology requirement for the space of finite graphs. This result has thrown more light on the basic information theorems developed for coloured random graphs in (Doku-Amponsah, 2012).

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Ethics

The author declares that there is no conflict of interest.

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