STURM-LIOUVILLE PROBLEMS IN DOMAINS WITH NON-SMOOTH EDGES

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ABSTRACT. We consider a (generally, non-coercive) mixed boundary value problem in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$ for a second order elliptic differential operator $A(x, \partial)$. The differential operator is assumed to be of divergent form in $\mathcal{D}$ and the boundary operator $B(x, \partial)$ is of Robin type on $\partial\mathcal{D}$. The boundary of $\mathcal{D}$ is assumed to be a Lipschitz surface. Besides, we distinguish a closed subset $Y \subset \partial\mathcal{D}$ and control the growth of solutions near Y. We prove that the pair (A, B) induces a Fredholm operator L in suitable weighted spaces of Sobolev type, the weight function being a power of the distance to the singular set Y. Moreover, we prove the completeness of root functions related to L.

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Introduction

The Hilbert space methods take considerable part in the modern theory of partial differential equations. In particular, the spectral theorem for compact selfadjoint operators attributed to Hilbert and Schmidt allows one to look for solutions of boundary value problems for formally selfadjoint operators in the form of expansions over eigenfunctions of the operator.

Non-selfadjoint compact operators fail to have eigenvectors in general. Keldysh [Kel51] (see also [GK69, Ch. 5, §8] for more details) elaborated expansions over root functions for weak perturbations of compact selfadjoint operators. In particular, he applied successfully the theorem on the completeness of root functions to the Dirichlet problem for second order elliptic operators in divergent form.

The problem of completeness of the system of eigen- and associated functions of boundary value problems for elliptic operators in domains with smooth boundary was studied in many articles (see for instance [Bro53], [Bro59a], [Bro59b], [Agm62], [Kon99]). In a series of papers [Agr94a], [Agr94b], [Agr08], [Agr11b], [Agr11c], including two surveys [Agr02] and [Agr11a], Agranovich proved the completeness of root functions for a wide class of boundary value problems for second order elliptic equations with boundary conditions of the Dirichlet, Neumann and Zaremba type in standard Sobolev spaces over domains with Lipschitz boundary. In [ST12] this method was extended to a class of non-coercive mixed problems with Robin type boundary conditions over domains with Lipschitz boundary.

Root functions of general elliptic boundary value problems in weighted Sobolev spaces over domains with conic and edge type singularities on the boundary were studied in [EKS01] and [Tar06]. These papers used estimates of the resolvent of compact operators and the so-called rays of minimal growth. In order to realise fully to what extent the completeness criteria of [EKS01] and [Tar06] are efficient, we dwell on the concept of ellipticity on a compact manifold with smooth edges on the boundary. Such a singular space $\mathcal{X}$ has three smooth strata, more precisely, the interior part $\mathcal{X}_0$ of $\mathcal{X}$, the smooth part $\mathcal{X}_1$ of the boundary and the edge $\mathcal{X}_2$ which is assumed to be a compact closed manifold. Pseudodifferential operators on $\mathcal{X}$ are (3×3)-matrices $\mathcal{A}$ whose entries $A_{i,j}$ are operators mapping functions on $\mathcal{X}_j$ to functions on $\mathcal{X}_i$. To each operator $\mathcal{A}$ one assigns a principal symbol $\sigma(\mathcal{A}) := (\sigma_0(\mathcal{A}), \sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}))$ in such a way that $\sigma(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is compact, and $\sigma(\mathcal{B}\mathcal{A}) = \sigma(\mathcal{B})\, \sigma(\mathcal{A})$ for all operators $\mathcal{A}$ and $\mathcal{B}$ whose composition is well defined. The components $\sigma_i(\mathcal{A})$ of the principal symbol are functions on the cotangent bundles of $\mathcal{X}_i$ with values in operator spaces. They are smooth away from zero sections of the bundles and bear certain twisted homogeneity as operator families. An operator $\mathcal{A}$ is called elliptic if its principal symbol is invertible away from the zero sections of cotangent bundles. The invertibility of $\sigma_0(\mathcal{A})$ just amounts to the ellipticity of $\mathcal{A}$ in the interior of $\mathcal{X}$. The invertibility of $\sigma_1(\mathcal{A})$ is equivalent to the Shapiro-Lopatinskii condition on the smooth part of $\partial \mathcal{X}$. The invertibility of $\sigma_2(\mathcal{A})$ constitutes the most difficult problem, for this operator family is considered in weighted Sobolev spaces on an infinite cone. An operator $\mathcal{A}$ proves to be Fredholm if and only if it is elliptic. However, from what has been said it follows that there is no efficient criteria of ellipticity on compact manifolds with edges on the boundary. In general these techniques allow one to derive at most the following result. Consider a classical boundary value problem on $\mathcal{X}$ satisfying the Shapiro-Lopatinskii condition away from the edge $\mathcal{X}_2$. It is actually given by a column of operators $A_{i,0}$ with $i = 0, 1$, where $A_{0,0}$ is an elliptic differential operator in $\mathcal{X}_0$ and $A_{1,0}$ a differential operator near $\mathcal{X}_1$ followed by restriction to $\mathcal{X}_1$. We complete the column to a (2×2)-matrix A by setting $A_{0,1} = 0$ and $A_{1,1} = 0$. The Shapiro-Lopatinskii condition implies that $\sigma_2(A)(y, \eta)$ is a family of Fredholm operators on the unit sphere in $T^*\mathcal{X}_2$. Hence we can set $\sigma_2(A)(y, \eta)$ in the frame of a (3×3)-matrix $a(y, \eta)$ on the unit sphere of $T^*\mathcal{X}_2$ which is moreover invertible. A distinct quantisation procedure leads then immediately to a Fredholm operator of the type

$$\begin{pmatrix} A_{0,0} & A_{0,2} \\ A_{1,0} & A_{1,2} \\ A_{2,0} & A_{2,2} \end{pmatrix} : \quad \begin{array}{c} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_1}) \end{array} \quad \to \quad \begin{array}{c} C^\infty(\mathcal{X}) \\ \oplus \\ C^\infty(\partial\mathcal{X}, \mathbb{C}^m) \\ \oplus \\ C^\infty(\mathcal{X}_2, \mathbb{C}^{l_2}) \end{array}, \tag{0.1}$$

where l_1 and l_2 are non-negative integers. However, the Fredholm property of (0.1) elucidates by no means the original problem

$$\left\{ \begin{array}{rcll} A_{0,0}u & = & f & \text{in } \ \mathcal{X}_0, \\ A_{1,0}u & = & u_0 & \text{at } \ \mathcal{X}_1, \end{array} \right.$$

unless $\mathcal{X}_2$ is of dimension 0. Thus, operator-valued symbols make the condition of ellipticity ineffective.

In the present paper we study the completeness of root elements associated with a mixed boundary value problem (A, B) for a second order elliptic differential equation with Robin type boundary condition in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$. The differential operator $A(x, \partial)$ is assumed to be of divergent form and the boundary operator $B(x, \partial)$ includes an oblique derivative with discontinuous coefficients. The boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$ is assumed to be a Lipschitz surface. Besides, we distinguish a closed set $Y \subset \partial\mathcal{D}$ and control the behaviour of solutions to the problem near Y. To this end we consider the boundary value problem in weighted Sobolev spaces over $\mathcal{D}$, the weight being a power of the distance to Y. We allow Y to be empty, so the case of standard Sobolev spaces is not excluded. Within the framework of analysis on manifolds with singularities the set Y bears usually singularities of the boundary (cones, edges, etc.) or discontinuities of boundary operators.

The theory of [Tar06] applies in similar situation (with edge singularities) provided that one is able to establish the invertibility of the edge symbol. This latter is a family of Sturm-Liouville boundary value problems in an infinite plane cone parametrised by the points of the cotangent bundle of the edge. On reducing the family to the boundary of the cone one obtains two ordinary pseudodifferential equations on the rays constituting the boundary of the cone. The invertibility of the edge symbol just amounts to the unique solvability of these equations in certain weighted Sobolev spaces on the rays. This is a hard problem which deserves separate investigation.

Instead we exploit the classical approach of non-negative forms inducing inner products in spaces of smooth functions (cf. [Sch60], [LM72], [LU73], [Agr94a]), and the method of rays of minimal growth of the resolvent ([Agm62], [DS63]) to study the completeness of root elements of the boundary value problem in weighted Sobolev spaces. Note that usually one imposes Shapiro-Lopatinskii type conditions on the boundary value problem at the smooth part of $\partial\mathcal{D}$, cf. [AV64], [Tar06], etc. Our contribution consists in considering non-coercive forms, and hence the Shapiro-Lopatinskii condition can be violated. Indeed, a Hermitian form associated with a second order elliptic formally selfadjoint operator A is usually constructed through a factorization $A = C^*C$, where C is an overdetermined elliptic first order operator and C^* its formal adjoint. According to [SKK73], microlocally any first order operator C with complex-valued coefficients can be presented via the Lewy operator or the gradient operator or the multidimensional Cauchy-Riemann operator. The operators of the first type go beyond the elliptic theory, the second type operators correspond to coercive mixed problems related to A, and the operators of the third type inherit non-coercive boundary conditions. Thus, it is not fortuitous that non-coercive boundary value problems for elliptic differential operators attract considerable attention of mathematicians since the 1950 s, see for instance [ADN59], [KN65]. One of the typical problems of this type is the famous $\overline{\partial}$-Neumann problem in complex analysis whose boundary conditions involve precisely the multidimensional Cauchy-Riemann operator, see [Koh79]. The investigation of the problem resulted in the discovery of the subellipticity phenomenon which greatly influenced to the development of the theory of partial differential equations, cf. [Hör66]. To the best of our knowledge, there have been no advanced results on the completeness of root functions for non-coercive problems. However, the use of non-coercive forms enlarges essentially the class of those boundary conditions for which the root functions of the corresponding mixed problems are dense in weighted Lebesgue and Sobolev spaces. The enlargement allows one to perturb the boundary conditions by diverse tangential vector fields. In general, we lose on regularity of solutions, however, this gap is well motivated by the nature of problems.

Part 1. Weighted Sobolev-Slobodetskii spaces in Lipschitz domains

1. Sobolev-Slobodetskii spaces

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\mathcal{D}$, i.e. the surface $\partial\mathcal{D}$ is locally the graph of a Lipschitz function. More precisely, for each boundary point $p \in \partial\mathcal{D}$ there is a neighbourhood U of p in $\mathbb{R}^n$, such that, after a possible rotation, $\mathcal{D} \cap U = \{(x', x^n) \in U : x^n > f(x')\}$, where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, i.e. $|f(x') - f(y')| \leq L\,|x' - y'|$ for all $x', y' \in \mathbb{R}^{n-1}$. The smallest L for which the estimate holds is called the bound of the Lipschitz constants. By choosing finitely many balls $\{U_\nu\}$ covering $\partial\mathcal{D}$, the Lipschitz constant for a Lipschitz domain is the smallest L with the property that the Lipschitz constant is bounded by L for every ball U_ν.

Any bounded Lipschitz domain has actually a global Lipschitz defining function ϱ, i.e. $\varrho : \mathbb{R}^n \to \mathbb{R}$ satisfies $\varrho < 0$ in $\mathcal{D}$, $\varrho > 0$ outside $\overline{\mathcal{D}}$, and $c_1 < |\varrho'| < c_2$ almost everywhere at $\partial\mathcal{D}$, where c_1, c_2 are positive constants. The geometric interpretation of this description is that both $\mathcal{D}$ and $\mathbb{R}^n \setminus \overline{\mathcal{D}}$ are locally situated on exactly one side of the boundary $\partial\mathcal{D}$.

As a Lipschitz function is differentiable almost everywhere and the derivatives are bounded, the boundary $\partial\mathcal{D}$ possesses a tangent hyperplane and a normal vector almost everywhere.

We consider complex-valued functions defined in the domain $\mathcal{D}$. For $1 \le q < \infty$, we write $L^q(\mathcal{D})$ for the space of all (equivalence classes of) measurable functions u in $\mathcal{D}$, such that the Lebesgue integral of $|u|^q$ over $\mathcal{D}$ is finite. When endowed with the norm

$$\|u\|_{L^q(\mathcal{D})} = \Big(\int_{\mathcal{D}} |u|^q \, dx \Big)^{1/q},$$

the space $L^q(\mathcal{D})$ is Banach. As usual, this scale continues to include the case $q = \infty$, too. The C^∞ functions of compact support in $\mathcal{D}$ lie dense in $L^q(\mathcal{D})$ provided that $q < \infty$.

More generally, for $s = 1, 2, \ldots$, we denote by $H^s(\mathcal{D})$ the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\int_{\mathcal{D}} \sum_{|\alpha| \le s} |\partial^\alpha u|^2 \, dx \Big)^{1/2},$$

where the sum is over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of norm $|\alpha| := \alpha_1 + \ldots + \alpha_n$ not exceeding s, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x^j$. It is convenient to define $H^s(\mathcal{D}) := L^2(\mathcal{D})$ for $s = 0$. Obviously, every $H^s(\mathcal{D})$ with $s = 0, 1, \ldots$ specifies within $L^2(\mathcal{D})$. In this way we get a scale of Hilbert spaces $H^s(\mathcal{D})$ endowed with scalar product

$$(u, v)_{H^s(\mathcal{D})} = \int_{\mathcal{D}} \sum_{|\alpha| \le s} \partial^\alpha u \overline{\partial^\alpha v} \, dx,$$

for $u, v \in H^s(\mathcal{D})$.

In order to extend the scale $H^s(\mathcal{D})$ to the fractional values of $s > 0$, one can use an interpolation procedure. There is also a direct construction along more classical lines developed in [Slo58]. Given any non-integer $s > 0$, the so-called Sobolev-Slobodetskii space $H^s(\mathcal{D})$ is defined to be the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\mathcal{D})} = \Big(\|u\|^2_{H^{[s]}(\mathcal{D})} + \iint_{\mathcal{D} \times \mathcal{D}} \sum_{|\alpha| = [s]} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{n + 2(s - [s])}} \, dx dy \Big)^{1/2},$$

where $[s]$ is the integer part of s. The space $H^s(\mathcal{D})$ is endowed with obvious inner product under which it is a Hilbert space.

In the sequel, for a closed subset S of $\overline{\mathcal{D}}$, we denote by $H^s(\mathcal{D}, S)$ the closure of the subspace $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus S)$ in $H^s(\mathcal{D})$. When endowed with induced norm, $H^s(\mathcal{D}, S)$ is obviously a Hilbert space. If S is the whole boundary we get what is usually referred to as $\mathring{H}^s(\mathcal{D})$.

To define the spaces $H^s(\mathcal{D})$ for all negative $s \in \mathbb{R}$, too, we exploit an appropriate duality. More precisely, let H^+ and H^0 be complex Hilbert spaces with scalar products $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_0$, respectively. Suppose that H^+ is a subspace of H^0 and the natural inclusion

$$\iota : H^+ \to H^0 \tag{1.1}$$

is continuous. We also assume that there is a space $\Sigma \subset H^+$, such that Σ is dense in H^+ and $\iota(\Sigma)$ is dense in H^0. Write H^- for the completion of Σ with respect to the norm

$$\|u\|_- = \sup_{\substack{v\in\Sigma \\ v\neq 0}} \frac{|(v,u)_0|}{\|v\|_+}.$$

Remark 1.1. Since Σ is dense in H^+ and the norm $\|\cdot\|_+$ majorises $\|\cdot\|_0$, we conclude that

$$\|u\|_- = \sup_{\substack{v\in H^+ \\ v\neq 0}} \frac{|(v,u)_0|}{\|v\|_+}.$$

The following two lemmas are well known (see for instance [Sch60]).

Lemma 1.2. *The space H^0 is continuously embedded into H^-. If inclusion (1.1) is compact then the space H^0 is compactly embedded into H^-.*

Proof. By definition and the continuity of the map (1.1) we get

$$\|u\|_- \leq \sup_{\substack{v\in H^+ \\ v\neq 0}} \frac{\|u\|_{H^0}\|v\|_{H^0}}{\|v\|_+} \leq c\,\|u\|_{H^0}$$

for all $u \in H^0$, i.e. the space H^0 is continuously embedded into H^- indeed.

Suppose (1.1) is compact. Then the Hilbert space adjoint $\iota^* : H^0 \to H^+$ is compact, too. By Remark 1.1 we conclude that

$$\begin{aligned} \|u\|_- &= \sup_{\substack{v\in H^+ \\ v\neq 0}} \frac{|(\iota(v),u)_{H^0}|}{\|v\|_+} \\ &= \sup_{\substack{v\in H^+ \\ v\neq 0}} \frac{|(v,\iota^*(u))_+|}{\|v\|_+} \\ &= \|\iota^*(u)\|_+ \end{aligned} \tag{1.2}$$

for all $u \in H^0$. Therefore, any weakly convergent sequence in H^0 converges in H^-, which shows the second part of the lemma. □

Lemma 1.3. *The Banach space H^- is topologically isomorphic to the dual space $(H^+)'$ and the isomorphism is defined by the sesquilinear form*

$$\langle v,u\rangle = \lim_{\nu\to\infty} (v,u_\nu)_0 \tag{1.3}$$

for $u \in H^-$ and $v \in H^+$ where $\{u_\nu\}$ is any sequence in Σ converging to u.

That is, for every fixed $u \in H^-$, pairing (1.3) defines a continuous linear functional f_u on H^+ and, for each $f \in (H^+)'$, there is a unique $u \in H^-$ with $f(v) = f_u(v)$ for all $v \in H^+$. Moreover, the conjugate linear map $u \mapsto f_u$ is an isometry.

Proof. Cf. Lemma 3.3 of [Sch60] for Sobolev spaces. To show that the limit on the right-hand side of (1.3) exists for each fixed function $v \in H^+$, is suffices to show that $\{(v,u_\nu)_0\}$ is a Cauchy sequence. By definition,

$$|(v,u_\nu - u_\mu)_0| \leq \|v\|_+\|u_\nu - u_\mu\|_- \to 0$$

as $\nu,\mu \to \infty$, which is our claim. Clearly, this limit does not depend on the particular sequence $\{u_\nu\}$, for if $\|u_\nu\|_- \to 0$, then $|(v,u_\nu)_0| \to 0$ for all $v \in H^+$.

From the definition it follows that

$$|\langle v,u\rangle| \le \|u\|_- \, \|v\|_+$$

for all $u \in H^-$ and $v \in H^+$. Hence, for each fixed element $u \in H^+$, the formula $f_u(v) := (v,u)$ defines a continuous linear functional f_u on H^+, such that

$$\|f_u\|_{(H^+)'} \le \|u\|_- .$$

If $\{u_\nu\} \subset \Sigma$ approximates an element u in H^-, then equality (1.2) implies that the sequence $\{\iota^*\iota\, u_\nu\}$ converges to a function U in the space H^+ and

$$\begin{aligned} \|U\|_+ &= \lim_{\nu\to\infty} \|\iota^*\iota\, u_\nu\|_+ \\ &= \lim_{\nu\to\infty} \|u_\nu\|_- \\ &= \|u\|_- . \end{aligned}$$

Moreover,

$$\begin{aligned} f_u(v) &= \langle v,u\rangle \\ &= \lim_{\nu\to\infty} (\iota\, v, \iota\, u_\nu)_0 \\ &= \lim_{\nu\to\infty} (v, \iota^*\iota\, u_\nu)_+ \\ &= (v,U)_+ \end{aligned}$$

for all $v \in H^+$. Now, the Riesz theorem yields $\|U\|_+ = \|f_u\|_{(H^+)'}$ whence

$$\|f_u\|_{(H^+)'} = \|u\|_- .$$

It remains to show that any continuous linear functional f on H^+ has the form $f(v) = \langle v, u_f\rangle$ for some $u_f \in H^-$. By the Riesz theorem, for any $f \in (H^+)'$ there is a unique element $U_f \in H^+$, such that $f(v) = (v, U_f)_+$ for all $v \in H^+$. Besides, $\|U_f\|_+ = \|f\|_{(H^+)'}$. By definition, the operator ι is injective and its image is dense in H^0. Hence, the image of the operator $\iota^*\iota$ in H^+ is dense, too. Pick a sequence $\{u_\nu\} \subset H^+$ with the property that $\{\iota^*\iota\, u_\nu\}$ converges to U_f. Then, according to (1.2), $\{u_\nu\}$ is a Cauchy sequence in H^-, and so it converges to an element u_f in this space. It as easy to see that u_f is actually independent of the particular choice of the sequence $\{u_\nu\}$. Finally, we obtain

$$\begin{aligned} \langle v, u_f\rangle &= \lim_{\nu\to\infty} (\iota\, v, \iota\, u_\nu)_0 \\ &= \lim_{\nu\to\infty} (v, \iota^*\iota\, u_\nu)_+ \\ &= (v, U_f)_+ \\ &= f(v) \end{aligned}$$

for all $v \in H^+$, as desired. □

Remark 1.4. Note that H^+ is reflexive, since it is a Hilbert space. Hence it follows that $(H^+)' = H^+$, i.e., the spaces H^+ and H^- are dual to each other with respect to (1.3).

Now we define $H^{-s}(\mathcal{D})$ to be the dual to $H^s(\mathcal{D})$ with respect to the paring induced by $(\cdot,\cdot)_{L^2(\mathcal{D})}$. More precisely, by $H^{-s}(\mathcal{D})$ is meant the completion of $C^\infty(\overline{\mathcal{D}})$

with respect to the norm

$$\|u\|_{H^{-s}(\mathcal{D})} = \sup_{\substack{v\in H^s(\mathcal{D})\\ v\neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{H^s(\mathcal{D})}}.$$

Then, by Lemma 1.2 and Rellich Theorem, $H^s(\mathcal{D})$ is compactly embedded to $H^{s'}(\mathcal{D})$ for any $s, s' \in \mathbb{R}$ with $s > s'$.

It is also well known that any differential operator of order $m \geq 0$ with coefficients of class $C^{[s]-m,1}(\overline{\mathcal{D}})$ maps $H^s(\mathcal{D})$ continuously to $H^{s-m}(\mathcal{D})$, for $s \geq m$. To extend the proper action of differential operators to the whole scale of Sobolev spaces, one needs a slightly different definition of Sobolev spaces of negative smoothness. Namely, for $s > 0$, denote by $\tilde{H}^{-s}(\mathcal{D})$ the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{\tilde{H}^{-s}(\mathcal{D})} = \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\mathcal{D})\\ v\neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{H^s(\mathcal{D})}}.$$

Obviously, $H^{-s}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{-s}(\mathcal{D})$, if $s > 0$. According to Lemma 1.3, $\tilde{H}^{-s}(\mathcal{D})$ is the dual of $\overset{\circ}{H}{}^s(\mathcal{D})$ with respect to the pairing induced by $(\cdot,\cdot)_{L^2(\mathcal{D})}$.

Any differential operator of order $m \geq 0$ with coefficients of class $C^\infty(\overline{\mathcal{D}})$ proves to map $H^s(\mathcal{D})$ continuously to $\tilde{H}^{s-m}(\mathcal{D})$, if $0 \leq s < m - 1/2$, and $\tilde{H}^s(\mathcal{D})$ to $\tilde{H}^{s-m}(\mathcal{D})$, if $s < 0$.

Moreover, the following result holds, cf. Proposition 12.1 of [LM72, Ch. 1, § 12.8] for domains with smooth boundary.

Lemma 1.5. *Let $\partial\mathcal{D}$ be a Lipschitz surface. If $1/2 < s < 1$ then any first order differential operator with coefficients of class $C^\infty(\overline{\mathcal{D}})$ maps $H^s(\mathcal{D})$ continuously to $H^{s-1}(\mathcal{D})$.*

Proof. Indeed, the space $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $H^{s'}(\mathcal{D})$ for all $0 \leq s' < 1/2$ (see for instance Corollary 1.4.4.5 of [Gris85]). For $1/2 < s < 1$, we have $0 < 1 - s < 1/2$, and so

$$\|w\|_{H^{s-1}(\mathcal{D})} = \sup_{\substack{v\in H^{1-s}(\mathcal{D})\\ v\neq 0}} \frac{|(v,w)_{L^2(\mathcal{D})}|}{\|v\|_{H^{1-s}(\mathcal{D})}} = \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\mathcal{D})\\ v\neq 0}} \frac{|(v,w)_{L^2(\mathcal{D})}|}{\|v\|_{H^{1-s}(\mathcal{D})}}$$

whenever $w \in H^{s-1}(\mathcal{D})$. Hence, given any $u \in C^\infty(\overline{\mathcal{D}})$ and $1 \leq j \leq n$, we get

$$\begin{aligned}
\|\partial_j u\|_{H^{s-1}(\mathcal{D})} &= \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\mathcal{D})\\ v\neq 0}} \frac{|(v,\partial_j u)_{L^2(\mathcal{D})}|}{\|v\|_{H^{1-s}(\mathcal{D})}} \\
&= \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\mathcal{D})\\ v\neq 0}} \frac{|(\partial_j v,u)_{L^2(\mathcal{D})}|}{\|v\|_{H^{1-s}(\mathcal{D})}} \\
&\leq \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\mathcal{D})\\ v\neq 0}} \frac{\|\partial_j v\|_{\tilde{H}^{-s}(\mathcal{D})}\|u\|_{H^s(\mathcal{D})}}{\|v\|_{H^{1-s}(\mathcal{D})}} \\
&\leq c\,\|u\|_{H^s(\mathcal{D})},
\end{aligned}$$

for the operator $\partial_j : H^{1-s}(\mathcal{D}) \to \tilde{H}^{-s}(\mathcal{D})$ is bounded, if $s > 1/2$ (see for instance Corollary 1.4.4.6 of [Gris85]). $\square$

Also the traces of functions from $H^s(\mathcal{D})$, where $s > 1/2$, are well defined on the Lipschitz surface $\partial\mathcal{D}$. More precisely, for $0 < s < 1$, we define $H^s(\partial\mathcal{D})$ to be the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\partial\mathcal{D})} = \Big(\|u\|^2_{L^2(\partial\mathcal{D})} + \iint_{\partial\mathcal{D}\times\partial\mathcal{D}} \frac{|u(x)-u(y)|^2}{|x-y|^{n-1+2s}}\, ds_x ds_y \Big)^{1/2}.$$

If $s \geq 1$, then we define $H^s(\partial\mathcal{D})$ as the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{H^s(\partial\mathcal{D})} = \Big(\sum_{|\alpha|\leq[s]} \|\partial^\alpha u\|^2_{L^2(\partial\mathcal{D})} + \sum_{|\alpha|=[s]} \|\partial^\alpha u\|^2_{H^{s-[s]}(\partial\mathcal{D})} \Big)^{1/2}.$$

To justify the designation, we note that if $\partial\mathcal{D} \in C^{0,1}$, then, for $0 \leq s < 1$, one arrives at the same space $H^s(\partial\mathcal{D})$ when completing $C^{0,1}(\partial\mathcal{D})$ with respect to the (equivalent) norm

$$\Big(\|u\|^2_{L^2(\partial\mathcal{D})} + \|u\|^2_{H^s(\partial\mathcal{D})} \Big)^{1/2}.$$

If $s \geq 1$ and $\partial\mathcal{D} \in C^{[s],1}$, then using a proper partition of unity $\{\phi_\nu\}$ on $\partial\mathcal{D}$ one obtains the same space $H^s(\partial\mathcal{D})$ as the completion of $C^{[s],1}(\partial\mathcal{D})$ with respect to the (equivalent) norm

$$\Big(\sum_\nu \Big(\sum_{|\beta|\leq[s]} \|\partial^\beta(\phi_\nu u)\|^2_{L^2(\partial\mathcal{D})} + \sum_{|\beta|=[s]} \|\partial^\beta(\phi_\nu u)\|^2_{H^{s-[s]}(\partial\mathcal{D})} \Big) \Big)^{1/2}.$$

Here, ∂^β are (tangential) derivatives in appropriate local coordinates on the surface $\partial\mathcal{D}$.

Let $u \in H^s(\mathcal{D})$. There is a sequence $\{u_i\}$ in $C^\infty(\overline{\mathcal{D}})$ approximating u in the $H^s(\mathcal{D})$-norm. If $s > 1/2$ then $\{u_i\}$ is a Cauchy sequence in $L^2(\partial\mathcal{D})$. As usual, the limit $t_s(u)$ of $\{u_i\}$ in $L^2(\partial\mathcal{D})$ is called the trace of u on $\partial\mathcal{D}$. It is known that $t_s(u)$ does not depend on the approximating sequence $\{u_i\}$. For $1/2 < s < 3/2$, the trace operator t_s obtained in this way acts continuously from $H^s(\mathcal{D})$ to $H^{s-1/2}(\partial\mathcal{D})$. Moreover, it possesses a bounded right inverse, see for instance [McL00] and [LM72, Ch. 1, § 8] for domains with smooth boundary. If the surface $\partial\mathcal{D}$ is sufficiently smooth, then $t_s : H^s(\mathcal{D}) \to H^{s-1/2}(\partial\mathcal{D})$ is bounded and possesses a bounded right inverse for all $s > 1/2$.

2. Weighted Sobolev spaces of non-negative integer smoothness

Mixed problems for partial differential equations are often considered in weighted spaces of Sobolev type. One chooses a weight function to appropriately control the behavior of solutions near interface surface on the boundary where the boundary conditions change their character.

Another motivation to introduce weights consists in possible geometric singularities of the boundary of the manifold where the problem is posed. Indeed, local analysis of formal solutions to a partial differential equation immediately shows that there are solutions with typical behavior adequately described in weighted spaces only.

By the so-called Simonenko principle, the Fredholm property of a boundary value problem is equivalent to the local solvability of the problem. While the local solvability of elliptic problems is easily determined away from the set of singularities, localization at singular points requires a hard analysis. However, this approach does not work at all to study the spectrum of the problem, because the spectral problems are non-local by the very nature. So, we are going to consider singular sets rather globally.

To this end we fix a closed set $Y \subset \overline{\mathcal{D}}$ situated on an $(n-1)$-dimensional surface. Introduce special weighted Sobolev spaces associated with Y. Let ρ be a continuous non-negative function in $\overline{\mathcal{D}}$ and let ρ be smooth away from Y. We assume that $0 \leq \rho(x) \leq 1$ for all $x \in \overline{\mathcal{D}}$ and $\rho(x) = 0$ if and only if $x \in Y$. Moreover, we require

$$\rho^{|\alpha|-1} \partial^\alpha \rho \in L^\infty(\mathcal{D}) \tag{2.1}$$

for all multi-indices $\alpha \in \mathbb{Z}^n_{\geq 0}$. Estimates (2.1) guarantee various important properties of weighted Sobolev spaces with weight function ρ. One may think of $\rho(x)$ as the distance from x to Y locally near Y in $\overline{\mathcal{D}}$. If the set Y is empty, we choose $\rho \equiv 1$.

Remark 2.1. Our results apply also in the case where Y contains cuspidal points of $\partial\mathcal{D}$, except for statements on traces, e.g., Theorems 4.12, 4.13 and Corollaries 5.6, 5.7.

Let s be a non-negative integer and $\gamma \in \mathbb{R}$. On smooth functions with compact support in $\overline{\mathcal{D}} \setminus Y$ we introduce the scalar product

$$(u, v)_{\mathcal{H}^{s,\gamma}(\mathcal{D})} = \int_{\mathcal{D}} \rho^{-2\gamma} \sum_{|\alpha| \leq s} \rho^{2|\alpha|} \partial^\alpha u \overline{\partial^\alpha v}\, dx$$

and denote by $\mathcal{H}^{s,\gamma}(\mathcal{D})$ the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the corresponding norm. By the very construction, $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is a Hilbert space. Starting from the scalar product

$$(u, v)_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} = \int_{\mathcal{D}} \sum_{|\alpha| \leq s} \partial^\alpha (\rho^{s-\gamma} u) \overline{\partial^\alpha (\rho^{s-\gamma} v)}\, dx$$

in $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ we get similarly a Hilbert space $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$.

Remark 2.2. We emphasise that in order to define the spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ for a fixed $s \in \mathbb{Z}_{\geq 0}$, one needs $\rho \in C^s(\overline{\mathcal{D}} \setminus Y) \cap L^\infty(\mathcal{D})$ satisfying (2.1) for $|\alpha| \leq s$ only.

As we allow for the set Y to be empty, the standard Sobolev spaces are not excluded from consideration. Obviously,

$$\begin{array}{rcl} \mathcal{H}^{0,\gamma}(\mathcal{D}) & = & \tilde{\mathcal{H}}^{0,\gamma}(\mathcal{D}), \\ \tilde{\mathcal{H}}^{s,s}(\mathcal{D}) & = & H^s(\mathcal{D}, Y) \end{array}$$

for all $s \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \mathbb{R}$.

Lemma 2.3. *For a function u to belong to $\mathcal{H}^{0,\gamma}(\mathcal{D})$ it is necessary and sufficient that $\rho^{-\gamma} u$ be in $L^2(\mathcal{D})$.*

Proof. If $u \in \mathcal{H}^{0,\gamma}(\mathcal{D})$ then there is a sequence $\{u_i\}$ in $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$ which converges to u in the $\mathcal{H}^{0,\gamma}(\mathcal{D})$-norm. This implies that $\{\rho^{-\gamma}u_i\}$ is a Cauchy sequence in $L^2(\mathcal{D})$. As the space $L^2(\mathcal{D})$ is complete, the sequence $\{\rho^{-\gamma}u_i\}$ converges in it to a function $v \in L^2(\mathcal{D})$. Obviously, $v = \rho^{-\gamma}u$. Conversely, suppose that $\rho^{-\gamma}u$ belongs to $L^2(\mathcal{D})$. As $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$, there is a sequence $\{v_i\}$ in $C^\infty_{\mathrm{comp}}(\mathcal{D})$ converging to $\rho^{-\gamma}u$ in this space. It follows that the sequence $u_i = \rho^\gamma v_i$ lies in $C^\infty_{\mathrm{comp}}(\mathcal{D}) \subset C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$ and $\rho^{-\gamma}u_i$ converges to $\rho^{-\gamma}u$ in $L^2(\mathcal{D})$. Therefore, $\{u_i\}$ converges to u in $\mathcal{H}^{0,\gamma}(\mathcal{D})$. □

From the definition of $\mathcal{H}^{s,\gamma}(\mathcal{D})$ it follows readily that if $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$ then

$$\partial^\alpha u \in \mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})$$

for all multi-indices α with $|\alpha| \le s$.

Lemma 2.4. *The space $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $\mathcal{H}^{0,\gamma}(\mathcal{D})$.*

Proof. This follows immediately from the fact that $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$, cf. the proof of Lemma 2.3. □

Although $\mathcal{H}^{0,0}(\mathcal{D}) = H^0(\mathcal{D})$, the space $\mathcal{H}^{s,0}(\mathcal{D})$ does not coincide with $H^s(\mathcal{D}, Y)$ for integer $s > 0$. Besides, $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$ for all $\gamma \ge 0$.

For particular configurations of singularities Y, if we choose as ρ the distance from to Y in a suitable coordinate system, then the scale of Hilbert spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ coincides with that used in [BK06] for cone type singularities and [NP94] for edge type singularities, the only difference being in indexing the spaces.

Let us also introduce an important re-indexation. Namely, for each $s \in \mathbb{Z}_{\ge 0}$ and $\gamma \in \mathbb{R}$, we have $\mathcal{H}^{s,\gamma}(\mathcal{D}) = \mathcal{H}^{s,(\gamma-s)+s}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}) = \tilde{\mathcal{H}}^{s,(\gamma-s)+s}(\mathcal{D})$. Then we set

$$\begin{array}{rclcrcl} H^{0,\gamma}(\mathcal{D}) & =: & \mathcal{H}^{0,\gamma}(\mathcal{D}), & & H^{s,\gamma}(\mathcal{D}) & =: & \mathcal{H}^{s,s+\gamma}(\mathcal{D}), \\ \tilde{H}^{0,\gamma}(\mathcal{D}) & =: & \tilde{\mathcal{H}}^{0,\gamma}(\mathcal{D}), & & \tilde{H}^{s,\gamma}(\mathcal{D}) & =: & \tilde{\mathcal{H}}^{s,s+\gamma}(\mathcal{D}), \end{array} \tag{2.2}$$

for $s \in \mathbb{Z}_{\ge 0}$. The significance of this re-indexation will be clarified later. In any case, it allows one to distinguish important natural embeddings.

Our primary interest consists in the study of boundary value problems in the re-indexed scales $H^{s,\gamma}(\mathcal{D})$ and $\tilde{H}^{s,\gamma}(\mathcal{D})$. We first describe basic properties of these spaces. Notice that the scales under study possess embeddings similar to those for Sobolev spaces.

Lemma 2.5. *Suppose that s, s' are non-negative integers with $s \ge s'$ and the function $\rho \in C^s(\overline{\mathcal{D}}\setminus Y)$ satisfies (2.1) for all $\alpha \in \mathbb{Z}^n_{\ge 0}$ with $|\alpha| \le s$. If moreover $\gamma \ge \gamma'$, then*

1) the space $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s',\gamma'}(\mathcal{D})$;

2) the space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s',\gamma'}(\mathcal{D})$.

Proof. By definition,

$$\|u\|_{\mathcal{H}^{s-1,\gamma}(\mathcal{D})} \le \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$, i.e. $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s-1,\gamma}(\mathcal{D})$. Since $0 < \rho \le 1$ in $\overline{\mathcal{D}}\setminus Y$, we conclude that $\rho^{-\gamma} \ge \rho^{-\gamma'}$ in $\overline{\mathcal{D}}\setminus Y$, provided that $\gamma \ge \gamma'$. Hence, if $\gamma \ge \gamma'$, then

$$\|u\|_{\mathcal{H}^{0,\gamma'}(\mathcal{D})} \le \|u\|_{\mathcal{H}^{0,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, i.e. $\mathcal{H}^{0,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{0,\gamma'}(\mathcal{D})$. It follows that

$$\|\partial^\alpha u\|_{\mathcal{H}^{0,\gamma'-|\alpha|}(\mathcal{D})} \leq \|\partial^\alpha u\|_{\mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})}$$

whence

$$\|u\|_{\mathcal{H}^{s',\gamma'}(\mathcal{D})} \leq \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, i.e. $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s',\gamma'}(\mathcal{D})$ provided $s \geq s'$ and $\gamma \geq \gamma'$. In particular,

$$\|u\|_{\mathcal{H}^{s',s'+\gamma'}(\mathcal{D})} \leq \|u\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}$$

holds for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, which proves the continuity of the embedding $H^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{s',\gamma'}(\mathcal{D})$. □

The following theorem states that the spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ and their re-indexed versions are weighted indeed.

Theorem 2.6. *Let $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ and (2.1) hold for all multi-indices α with $|\alpha| \leq s$, where $s \in \mathbb{Z}_{\geq 0}$. Then, for any $\delta \in \mathbb{R}$, the correspondence*

$$\mathrm{Op}\,(\rho^\delta) : u \mapsto \rho^\delta u \tag{2.3}$$

induces bounded linear operators

$$\begin{array}{rclcrcl} \mathcal{H}^{s,\gamma}(\mathcal{D}) & \to & \mathcal{H}^{s,\gamma+\delta}(\mathcal{D}), & & H^{s,\gamma}(\mathcal{D}) & \to & H^{s,\gamma+\delta}(\mathcal{D}), \\ \tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}) & \to & \tilde{\mathcal{H}}^{s,\gamma+\delta}(\mathcal{D}), & & \tilde{H}^{s,\gamma}(\mathcal{D}) & \to & \tilde{H}^{s,\gamma+\delta}(\mathcal{D}). \end{array} \tag{2.4}$$

Moreover, the operators in the first line are topological isomorphisms, the operators in the second line are isometries.

Proof. As mentioned, all the spaces $\mathcal{H}^{0,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$, $\tilde{\mathcal{H}}^{0,\gamma}(\mathcal{D})$ and $\tilde{H}^{0,\gamma}(\mathcal{D})$ coincide and

$$\|\rho^\delta u\|_{\mathcal{H}^{0,\gamma+\delta}(\mathcal{D})} = \|u\|_{\mathcal{H}^{0,\gamma}(\mathcal{D})},$$

which establishes the theorem in the case $s = 0$. For arbitrary $s \in \mathbb{Z}_{\geq 0}$, we obtain immediately

$$\|\rho^\delta u\|_{\tilde{\mathcal{H}}^{s,\gamma+\delta}(\mathcal{D})} = \|\rho^{\delta+s-(\gamma+\delta)}\|_{H^s(\mathcal{D})} = \|\rho^{s-\gamma}\|_{H^s(\mathcal{D})} = \|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})}, \tag{2.5}$$

showing that the operators in the second line of (2.4) are isometries.

Note that (2.1) implies $\rho' \in L^\infty(\mathcal{D})$, where $u' = (u'_{x^1}, \ldots, u'_{x^n})$ is the gradient of u. As $(\rho^\delta u)' = \delta \rho^{\delta-1} \rho' u + \rho^\delta u'$, we obtain

$$\begin{array}{rcl} \|\rho^\delta u\|^2_{\mathcal{H}^{1,\gamma+\delta}(\mathcal{D})} & = & \|\rho^\delta u\|^2_{\mathcal{H}^{0,\gamma+\delta}(\mathcal{D})} + \|(\rho^\delta u)'\|^2_{\mathcal{H}^{0,\gamma-1+\delta}(\mathcal{D})} \\ & \leq & \|u\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + 2\delta^2\, \|\rho'\|^2_{L^\infty(\mathcal{D})} \|\rho^{\delta-1} u\|^2_{\mathcal{H}^{0,\gamma-1+\delta}(\mathcal{D})} + 2\, \|\rho^\delta u'\|^2_{\mathcal{H}^{0,\gamma-1+\delta}(\mathcal{D})} \\ & = & \|u\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + 2\delta^2\, \|\rho'\|^2_{L^\infty(\mathcal{D})} \|u\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + 2\|u'\|^2_{\mathcal{H}^{0,\gamma-1}(\mathcal{D})} \\ & \leq & \max\{2, 1+2\delta^2\, \|\rho'\|^2_{L^\infty(\mathcal{D})}\}\, \|u\|^2_{\mathcal{H}^{1,\gamma}(\mathcal{D})} \end{array}$$

for all $u \in \mathcal{H}^{1,\gamma}(\mathcal{D})$. This proves the continuity of the operators in the first line of (2.4) for $s = 1$.

To continue the proof we need an auxiliary assertion.

Lemma 2.7. *If $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ and (2.1) is fulfilled for all $|\alpha| \leq s$, where $s \in \mathbb{Z}_{\geq 0}$, then*

$$\rho^{|\alpha|-\delta} \partial^\alpha \rho^\delta \in L^\infty(\mathcal{D}) \tag{2.6}$$

whenever $\delta \in \mathbb{R}$ and $|\alpha| \leq s$.

Proof. If $|\alpha| = 1$, then $\alpha = e_k$ with $k = 1, \dots, n$, and so

$$\rho^{1-\delta}\partial_k \rho^{\delta} = \delta \rho^{1-\delta}\rho^{\delta-1}\partial_k \rho = \delta \partial_k \rho \in L^{\infty}(\mathcal{D})$$

for all $\delta \in \mathbb{R}$, which is due to (2.1). Analogously, if $|\alpha| = 2$, then $\alpha = e_j + e_k$ with $1 \leq j, k \leq n$, and so

$$\rho^{2-\delta}\partial_j \partial_k \rho^{\delta} = \delta(\delta-1)\,(\partial_j \rho)(\partial_k \rho) + \delta\, \rho(\partial_j \partial_k \rho) \in L^{\infty}(\mathcal{D})$$

for all $\delta \in \mathbb{R}$ because of (2.1).

We now proceed by induction. Assume that (2.6) holds for all $\delta \in \mathbb{R}$ and all multi-indices α with $|\alpha| \leq m$, where $m < s$. If α is a multi-index of length $|\alpha| = m+1$, then $\partial^{\alpha} = \partial^{\alpha'}\partial_k$ with some $1 \leq k \leq n$ and $\alpha' \in \mathbb{Z}^n_{\geq 0}$, where $|\alpha'| = m$. It follows that

$$\partial^{\alpha}\rho^{\delta} = \partial^{\alpha'}(\partial_k \rho^{\delta}) = \partial^{\alpha'}(\delta \rho^{\delta-1}\partial_k \rho) = \delta \sum_{\beta \leq \alpha'} \tbinom{\alpha'}{\beta}\, \partial^{\beta}(\rho^{\delta-1})\partial^{\alpha'-\beta}(\partial_k \rho)$$

whence

$$\rho^{|\alpha|-\delta}\partial^{\alpha}\rho^{\delta} = \delta \sum_{\beta \leq \alpha'} \tbinom{\alpha'}{\beta}\, (\rho^{|\beta|-(\delta-1)}\partial^{\beta}\rho^{\delta-1})\,(\rho^{|\alpha'-\beta|}\partial^{\alpha'-\beta+e_k}\rho)$$

which is in $L^{\infty}(\mathcal{D})$ by (2.1) and inductive assumption. Thus, (2.6) holds for all $\delta \in \mathbb{R}$ and for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying $|\alpha| \leq m+1$. □

Further, for any $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$ we have, by definition, $\rho^{|\alpha|-\gamma}\partial^{\alpha}u \in L^2(\mathcal{D})$, provided that $|\alpha| \leq s$. If $|\alpha| \leq s$, then

$$\begin{aligned} \rho^{|\alpha|-\gamma-\delta}\partial^{\alpha}(\rho^{\delta}u) &= \rho^{|\alpha|-\gamma-\delta} \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta}\, (\partial^{\beta}\rho^{\delta})\partial^{\alpha-\beta}u \\ &= \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta}\, \rho^{|\beta|-\delta}(\partial^{\beta}\rho^{\delta})\, \rho^{|\alpha-\beta|-\gamma}\partial^{\alpha-\beta}u. \end{aligned}$$

Hence, combining Lemmas 2.5 and 2.7 yields

$$\|\partial^{\alpha}(\rho^{\delta}u)\|_{\mathcal{H}^{0,\gamma+\delta-|\alpha|}(\mathcal{D})} \leq \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta}\, \|\rho^{|\beta|-\delta}(\partial^{\beta}\rho^{\delta})\|_{L^{\infty}(\mathcal{D})}\, \|\partial^{\alpha-\beta}u\|_{\mathcal{H}^{0,\gamma-|\alpha-\beta|}(\mathcal{D})}$$

for all $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, i.e. $\rho^{\delta}u \in \mathcal{H}^{s,\gamma+\delta}(\mathcal{D})$ and

$$\|\rho^{\delta}u\|_{\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})} \leq c\, \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$$

with c a constant independent of u. This establishes the continuity of the operator $\mathrm{Op}\,(\rho^{\delta})$ acting as $\mathcal{H}^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s,\gamma+\delta}(\mathcal{D})$, and hence its continuity as operator acting as $H^{s,\gamma}(\mathcal{D}) \to H^{s,\gamma+\delta}(\mathcal{D})$.

From what has already been proved it follows that for each $\delta \in \mathbb{R}$ there is a positive constant $c = c(s,\gamma,\delta)$, such that

$$1/c(s,\gamma{+}\delta,-\delta)\, \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} \leq \|\rho^{\delta}u\|_{\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})} \leq c(s,\gamma,\delta)\, \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} \tag{2.7}$$

for all $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$. Indeed, we only need to clarify the left-hand side estimate. On applying the argument to the multiplier $\rho^{-\gamma}$ we see that

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} = \|\rho^{-\delta}\,(\rho^{\delta}u)\|_{\mathcal{H}^{s,\gamma+\delta-\delta}(\mathcal{D})} \leq c(s,\gamma{+}\delta,-\delta)\, \|\rho^{\delta}u\|_{\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})}$$

for all $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, as desired. Substituting $s+\gamma$ for γ in (2.7) we arrive readily at the estimate

$$1/c(s,s{+}\gamma{+}\delta,-\delta)\, \|u\|_{H^{s,\gamma}(\mathcal{D})} \leq \|\rho^{\delta}u\|_{H^{s,\gamma+\delta}(\mathcal{D})} \leq c(s,s{+}\gamma,\delta)\, \|u\|_{H^{s,\gamma}(\mathcal{D})} \tag{2.8}$$

for all $u \in H^{s,\gamma}(\mathcal{D})$.

Thus, when acting as in (2.4), the map $\mathrm{Op}\,(\rho^{\gamma})$ is bounded and invertible with inverse $\mathrm{Op}\,(\rho^{-\gamma})$. $\square$

Lemma 2.8. *Assume that $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ and (2.1) holds true for $|\alpha| \le s$, where $s \in \mathbb{Z}_{\ge 0}$. If $\delta \ge 0$, then the space $\mathcal{H}^{s,s+\delta}(\mathcal{D}) = H^{s,\delta}(\mathcal{D})$ is continuously embedded into $H^s(\mathcal{D}, Y) = \tilde{H}^{s,0}(\mathcal{D})$.*

Proof. Indeed, if $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, then

$$\|u\|^2_{H^s(\mathcal{D})} = \sum_{|\alpha|\le s} \|\partial^{\alpha} u\|^2_{L^2(\mathcal{D})} \le \sum_{|\alpha|\le s} \|\partial^{\alpha} u\|^2_{\mathcal{H}^{0,s+\delta-|\alpha|}(\mathcal{D})} = \|u\|^2_{\mathcal{H}^{s,s+\delta}(\mathcal{D})},$$

because $\rho^{-2(s+\delta-|\alpha|)} \ge 1$ provided $|\alpha| \le s$ and $\delta \ge 0$. This establishes the continuous embedding $\mathcal{H}^{s,s+\delta}(\mathcal{D}) \hookrightarrow H^s(\mathcal{D}, Y)$. $\square$

Lemma 2.9. *Suppose that s, s' are non-negative integers with $s \ge s'$ and the function $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ satisfies (2.1) for all multi-indices α, such that $|\alpha| \le s$. Then*

1) the space $\tilde{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s',\gamma}(\mathcal{D})$;

2) the space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s',\gamma}(\mathcal{D})$.

Proof. Indeed,

$$\|u\|_{\tilde{H}^{s,\gamma}(\mathcal{D})} = \|\rho^{-\gamma} u\|_{H^s(\mathcal{D})} \ge c\, \|\rho^{-\gamma} u\|_{H^{s'}(\mathcal{D})} = c\, \|u\|_{\tilde{H}^{s',\gamma}(\mathcal{D})} \tag{2.9}$$

for all $u \in C^s(\overline{\mathcal{D}}, Y)$, with c a constant which depends only on s and s'. This proves the part 1).

By Lemma 2.8, the space $H^{s,0}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s,0}(\mathcal{D})$. Hence, there is a constant $c > 0$, such that

$$\|\rho^{-\gamma} u\|_{\tilde{H}^{s,0}(\mathcal{D})} \le c\, \|\rho^{-\gamma} u\|_{H^{s,0}(\mathcal{D})} \tag{2.10}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. Applying Theorem 2.6 we see that $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s,\gamma}(\mathcal{D})$. $\square$

Lemma 2.10. *Let $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ and (2.1) hold for $|\alpha| \le s$, where $s \in \mathbb{Z}_{\ge 0}$. Then any differential operator*

$$A = \sum_{|\alpha|\le m} a_{\alpha}(x)\, \rho^{|\alpha|-m}(x)\, \partial^{\alpha} \tag{2.11}$$

of order $m \le s$ with coefficients a_{α} of class $C^s(\overline{\mathcal{D}})$ maps $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously to $\mathcal{H}^{s-m,\gamma-m}(\mathcal{D})$ and $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-m,\gamma}(\mathcal{D})$.

Proof. Combining Theorem 2.6 and Lemma 2.5 we conclude that the operator $\mathrm{Op}\,(\rho^{-1})$ maps the space $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously to $\mathcal{H}^{s-1,\gamma-1}(\mathcal{D})$. On the other hand, if $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, then

$$\begin{aligned} \|\partial_j u\|^2_{\mathcal{H}^{s-1,\gamma-1}(\mathcal{D})} &= \int_{\mathcal{D}} \rho^{-2(\gamma-1)} \sum_{|\alpha|\le s-1} \rho^{2|\alpha|} |\partial^{\alpha} \partial_j u|^2\, dx \\ &\le \int_{\mathcal{D}} \rho^{-2\gamma} \sum_{1\le|\alpha|\le s} \rho^{2|\alpha|} |\partial^{\alpha} u|^2\, dx \\ &\le \|u\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{D})} \end{aligned}$$

for all $1 \leq j \leq n$. Hence, the derivative ∂_j maps $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously into $\mathcal{H}^{s-1,\gamma-1}(\mathcal{D})$. As multiplication by a function from $C^s(\overline{\mathcal{D}})$ is a bounded operator on $H^s(\mathcal{D})$, we see that the statement is true for all first order operators of type (2.11). For higher order operators one may argue by induction with the use of Theorem 2.6 and Lemmas 2.5, 2.7. The second statement follows from the definition of the re-indexed scale. $\square$

Lemma 2.11. *Let $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ and (2.1) hold true for $|\alpha| \leq s$, where $s \in \mathbb{Z}_{\geq 0}$. Then $u \in \tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ if and only if $\rho^{s-\gamma} u \in H^s(\mathcal{D})$. Moreover, if $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$ then $\rho^{s-\gamma} u \in H^s(\mathcal{D}, Y)$.*

Proof. This is a direct consequence of Theorem 2.6 and Lemma 2.8. $\square$

3. Weighted Sobolev spaces of fractional and negative smoothness

Lemma 2.11 enables us to introduce weighted Sobolev spaces of fractional smoothness $s > 0$. Namely, for functions $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, we consider two norms

$$\begin{array}{rcl} \|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} & = & \Big(\|u\|^2_{\mathcal{H}^{[s],\gamma}(\mathcal{D})} + \|\rho^{s-\gamma} u\|^2_{H^s(\mathcal{D})} \Big)^{1/2}, \\ \|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} & = & \|\rho^{s-\gamma} u\|_{H^s(\mathcal{D})}. \end{array}$$

Write $\mathcal{H}^{s,\gamma}(\mathcal{D})$ for the completion of the space $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$, and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ for the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the norm $\|\cdot\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})}$.

A scale similar to $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ was used in [Kon66] to study boundary value problems for parabolic equations. However, he restricted the study to resolved singularities of transversal intersections, cf. Section 5.

Remark 3.1. We emphasise that in order to define the spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ for a fixed $s \geq 0$ one needs merely that $\rho \in C^{[s]+1}(\overline{\mathcal{D}} \setminus Y)$ and that (2.1) be fulfilled for $|\alpha| \leq [s] + 1$.

From Lemma 2.8 it follows that, for $s \geq s' \geq 0$, the embeddings

$$\begin{array}{rcl} \mathcal{H}^{s,s}(\mathcal{D}) & \hookrightarrow & \tilde{\mathcal{H}}^{s,s}(\mathcal{D}) = H^s(\mathcal{D}, Y), \\ \mathcal{H}^{s,\gamma}(\mathcal{D}) & \hookrightarrow & \mathcal{H}^{[s],\gamma}(\mathcal{D}), \\ \mathcal{H}^{s,s}(\mathcal{D}) & \hookrightarrow & \mathcal{H}^{s',s'}(\mathcal{D}), \end{array} \tag{3.1}$$

are continuous. As before, we set

$$\begin{array}{rcl} H^{s,\gamma}(\mathcal{D}) & = & \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ \tilde{H}^{s,\gamma}(\mathcal{D}) & = & \tilde{\mathcal{H}}^{s,s+\gamma}(\mathcal{D}) \end{array}$$

for all fractional $s \geq 0$ and $\gamma \in \mathbb{R}$, thus extending the scales $H^{s,\gamma}(\mathcal{D})$ and $\tilde{H}^{s,\gamma}(\mathcal{D})$ from $s \in \mathbb{Z}_{\geq 0}$ to all real $s \geq 0$.

As the realization of the dual space depends essentially on the pairing (see Lemma 1.3), we should be motivated in the choice of paring to introduce weighted Sobolev spaces of negative smoothness. In the study of boundary value problems for second order differential operators in the scale $H^{s,\gamma}(\mathcal{D})$ one denotes by $H^{-s,\gamma}(\mathcal{D})$ the dual space of $H^{s,\gamma}(\mathcal{D})$, $s > 0$, with respect to the pairing induced by the scalar product in $H^{0,\gamma}(\mathcal{D})$. In other words, $H^{-s,\gamma}(\mathcal{D})$ is the completion of $C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ in the norm

$$\|u\|_{H^{-s,\gamma}(\mathcal{D})} = \sup_{\substack{v \in H^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma}(\mathcal{D})}}.$$

By $\tilde{H}^{-s,\gamma}(\mathcal{D})$ is meant the dual space of $\tilde{H}^{s,\gamma}(\mathcal{D})$, $s > 0$, with respect to the pairing induced by the scalar product in $H^{0,\gamma}(\mathcal{D})$. Thus, $\tilde{H}^{-s,\gamma}(\mathcal{D})$ is the completion of $C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ in the norm

$$\|u\|_{\tilde{H}^{-s,\gamma}(\mathcal{D})} = \sup_{\substack{v \in \tilde{H}^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{\tilde{H}^{s,\gamma}(\mathcal{D})}}.$$

This definition leads to a generalised setting of the mixed problem using the pairing in the space $H^{0,\gamma}(\mathcal{D})$. The advantage of the approach is that it allows one to argue as the classics did in the usual Sobolev spaces. In Lemma 3.5 we prove that the definition is actually equivalent to the standard one using the pairing in $H^{0,0}(\mathcal{D})$.

For the initial scales $\mathcal{H}^{s,\gamma}(\mathcal{D})$, $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ one ought to use another duality. Namely, $\mathcal{H}^{-s,\gamma}(\mathcal{D})$ is defined to be the dual of $\mathcal{H}^{s,-\gamma}(\mathcal{D})$ with respect to the pairing induced by the scalar product in $\mathcal{H}^{0,0}(\mathcal{D})$, and similarly for $\tilde{\mathcal{H}}^{-s,\gamma}(\mathcal{D})$, where $s > 0$. As already noted, the representation of the dual space depends essentially on the pairing. While all the realizations coincide as topological spaces, the routine calculations might be different. Thus the definition $\mathcal{H}^{s,-\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,-\gamma}(\mathcal{D})$, $s > 0$, needs further clarification (see below).

The scales of function spaces just introduced are still scales of proper weighted spaces.

Corollary 3.2. *Let $s \in \mathbb{R}$. Assume that $\rho \in C^{[|s|]+1}(\overline{\mathcal{D}} \setminus Y) \cap L^{\infty}(\mathcal{D})$ satisfies (2.1) for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq [|s|] + 1$. Then, for any $\delta \in \mathbb{R}$, the operator* $\mathrm{Op}\,(\rho^{\delta})$ *induces isometries*

$$\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}) \;\to\; \tilde{\mathcal{H}}^{s,\gamma+\delta}(\mathcal{D}), \qquad \tilde{H}^{s,\gamma}(\mathcal{D}) \;\to\; \tilde{H}^{s,\gamma+\delta}(\mathcal{D}).$$

Moreover,

$$\begin{array}{rcl} \|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} & = & \|\rho^{s-\gamma} u\|_{H^{s}(\mathcal{D})}, \\ \|u\|_{\tilde{H}^{s,\gamma}(\mathcal{D})} & = & \|\rho^{-\gamma} u\|_{H^{s}(\mathcal{D})} \end{array} \qquad (3.2)$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$.

Proof. Note that for $s \in \mathbb{Z}_{\geq 0}$ the statement has already been proved, see Theorem 2.6. The mappings in question are actually isometries for all real $s \geq 0$, for the equality (2.5) still holds for any $s \geq 0$. Moreover, the first equality of (3.2) follows directly from the definition of the space for all real $s \geq 0$. If $s > 0$, then, for each $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, we get

$$\begin{array}{rcl} \|u\|_{\tilde{\mathcal{H}}^{-s,\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in \tilde{\mathcal{H}}^{s,-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\tilde{\mathcal{H}}^{s,-\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \tilde{\mathcal{H}}^{s,-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{s+\gamma} v, \rho^{-s-\gamma} u)_{L^2(\mathcal{D})}|}{\|\rho^{s+\gamma} v\|_{H^{s}(\mathcal{D})}} \\ & = & \|\rho^{-s-\gamma} u\|_{H^{-s}(\mathcal{D})}, \end{array}$$

i.e. the first equality of (3.2) is valid for all $s \in \mathbb{R}$. Hence it follows that identity (2.5) holds true for all real $s < 0$, and so $\operatorname{Op}(\rho^\delta) : \tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}) \to \tilde{\mathcal{H}}^{s,\gamma+\delta}(\mathcal{D})$ is an isometry for $s < 0$, too.

Similarly, for $s \geq 0$, the second equality of (3.2) is an immediate consequence of the definition of the space. Using the first equality of (3.2) we see that

$$\begin{aligned} \|u\|_{\tilde{H}^{-s,\gamma}(\mathcal{D})} &= \sup_{\substack{v \in \tilde{H}^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{\tilde{H}^{s,\gamma}(\mathcal{D})}} \\ &= \sup_{\substack{v \in \tilde{H}^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{-\gamma} v, \rho^{-\gamma} u)_{L^2(\mathcal{D})}|}{\|\rho^{-\gamma} v\|_{H^s(\mathcal{D})}} \\ &= \|\rho^{-\gamma} u\|_{H^{-s}(\mathcal{D})} \end{aligned}$$

holds for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. Therefore, the second equality of (3.2) is still true and the operator $\operatorname{Op}(\rho^\delta) : \tilde{H}^{s,\gamma}(\mathcal{D}) \to \tilde{H}^{s,\gamma+\delta}(\mathcal{D})$ is an isometry for all real $s < 0$, as desired. □

Corollary 3.3. *Let $s \in \mathbb{R}$ and let $\rho \in C^{[|s|]+1}(\overline{\mathcal{D}} \setminus Y) \cap L^\infty(\mathcal{D})$ satisfy (2.1) for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq [|s|] + 1$. Then, for any $\delta \in \mathbb{R}$, the operator $\operatorname{Op}(\rho^\delta)$ induces topological isomorphisms*

$$\mathcal{H}^{s,\gamma}(\mathcal{D}) \;\to\; \mathcal{H}^{s,\gamma+\delta}(\mathcal{D}), \qquad H^{s,\gamma}(\mathcal{D}) \;\to\; H^{s,\gamma+\delta}(\mathcal{D}).$$

Proof. We first notice that, for $s \in \mathbb{Z}_{\geq 0}$, the statement is contained in Theorem 2.6.

Further, for $0 < s < 1$ and $\delta \in \mathbb{R}$, we get

$$\begin{aligned} \|\rho^\delta u\|^2_{\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})} &= \|\rho^{-(\gamma+\delta)} \rho^\delta u\|^2_{L^2(\mathcal{D})} + \|\rho^{s-(\gamma+\delta)} \rho^\delta u\|^2_{H^s(\mathcal{D})} \\ &= \|u\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + \|\rho^{s-\gamma} u\|^2_{H^s(\mathcal{D})} \\ &= \|u\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{D})}, \end{aligned}$$

and, in general, for non-integral $s \geq 0$ and $\delta \in \mathbb{R}$, we obtain

$$\begin{aligned} \|\rho^\delta u\|^2_{\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})} &= \|\rho^\delta u\|^2_{\mathcal{H}^{[s],\gamma}(\mathcal{D})} + \|\rho^{s-(\gamma+\delta)} \rho^\delta u\|^2_{H^s(\mathcal{D})} \\ &\leq c \, \|u\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{D})} \end{aligned}$$

for all $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, where c is a constant independent of u. Therefore, the operator $\operatorname{Op}(\rho^\delta)$ maps $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously to $\mathcal{H}^{s,\gamma+\delta}(\mathcal{D})$ for all $s \geq 0$. In particular, it maps $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s,\gamma+\delta}(\mathcal{D})$ for all $s \geq 0$. Our next objective is to prove this for $s < 0$.

Pick $s > 0$. Since the operator $\mathrm{Op}\,(\rho^{-\delta}) : H^{s,\gamma+\delta}(\mathcal{D}) \to H^{s,\gamma}(\mathcal{D})$ is bounded, it follows that

$$\begin{aligned}
\|\rho^\delta u\|^2_{H^{-s,\gamma+\delta}(\mathcal{D})} &= \sup_{\substack{v\in H^{s,\gamma+\delta}(\mathcal{D})\\ v\neq 0}} \frac{|(v,\rho^\delta u)_{H^{0,\gamma+\delta}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma+\delta}(\mathcal{D})}}\\
&= \sup_{\substack{v\in H^{s,\gamma+\delta}(\mathcal{D})\\ v\neq 0}} \frac{|(\rho^{-\delta}v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|\rho^{-\delta}v\|_{H^{s,\gamma}(\mathcal{D})}}\frac{\|\rho^{-\delta}v\|_{H^{s,\gamma}(\mathcal{D})}}{\|v\|_{H^{s,\gamma+\delta}(\mathcal{D})}}\\
&\leq c \sup_{\substack{w\in H^{s,\gamma}(\mathcal{D})\\ w\neq 0}} \frac{|(w,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|w\|_{H^{s,\gamma}(\mathcal{D})}}\\
&= c\,\|u\|^2_{H^{-s,\gamma}(\mathcal{D})}
\end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, i.e. the operator $\mathrm{Op}\,(\rho^\delta) : H^{-s,\gamma}(\mathcal{D}) \to H^{-s,\gamma+\delta}(\mathcal{D})$ is bounded, too.

Arguing as in Theorem 2.6 we easily obtain estimates (2.7) for all $s \geq 0$ and estimates (2.8) for all real s. This means that the maps $\mathcal{H}^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s,\gamma+\delta}(\mathcal{D})$, for $s \geq 0$, and $H^{s,\gamma}(\mathcal{D}) \to H^{s,\gamma+\delta}(\mathcal{D})$, for $s \in \mathbb{R}$, induced by the multiplication operator $\mathrm{Op}\,(\rho^\delta)$, are bounded and invertible and their bounded inverse maps are induced by $\mathrm{Op}\,(\rho^{-\delta})$.

To settle the question with the maps $\mathrm{Op}\,(\rho^\delta) : \mathcal{H}^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s,\gamma+\delta}(\mathcal{D})$ we need to clarify the pairing defining the space $\mathcal{H}^{s,\gamma}(\mathcal{D})$ for negative indices s. If $s \geq 0$ and $\gamma \geq 0$ then $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$ and we may realise the scheme above with $H^0 = \mathcal{H}^{0,0}(\mathcal{D}) = L^2(\mathcal{D})$. According to it, $H^+ = \mathcal{H}^{s,\gamma}(\mathcal{D})$ produces the dual space $H^- = \mathcal{H}^{-s,-\gamma}(\mathcal{D})$ as the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{-s,-\gamma}(\mathcal{D})} = \sup_{\substack{v\in\mathcal{H}^{s,\gamma}(\mathcal{D})\\ v\neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}}.$$

Then, as the operator $\mathrm{Op}\,(\rho^\delta) : \mathcal{H}^{s,\gamma-\delta}(\mathcal{D}) \to \mathcal{H}^{s,\gamma}(\mathcal{D})$ is bounded for all $s > 0$, we have

$$\begin{aligned}
\|\rho^\delta u\|^2_{\mathcal{H}^{-s,-\gamma+\delta}(\mathcal{D})} &= \sup_{\substack{v\in\mathcal{H}^{s,\gamma-\delta}(\mathcal{D})\\ v\neq 0}} \frac{|(v,\rho^\delta u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,\gamma-\delta}(\mathcal{D})}}\\
&= \sup_{\substack{v\in\mathcal{H}^{s,\gamma-\delta}(\mathcal{D})\\ v\neq 0}} \frac{|(\rho^\delta v,u)_{L^2(\mathcal{D})}|}{\|\rho^\delta v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}}\frac{\|\rho^\delta v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,\gamma-\delta}(\mathcal{D})}}\\
&\leq c \sup_{\substack{w\in\mathcal{H}^{s,\gamma}(\mathcal{D})\\ w\neq 0}} \frac{|(w,u)_{L^2(\mathcal{D})}|}{\|w\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}}\\
&= c\,\|u\|^2_{\mathcal{H}^{-s,-\gamma}(\mathcal{D})}
\end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, provided that $\gamma \geq 0$ and $\gamma - \delta \geq 0$. Hence, the operator $\mathrm{Op}\,(\rho^\delta) : \mathcal{H}^{-s,-\gamma}(\mathcal{D}) \to \mathcal{H}^{-s,-\gamma+\delta}(\mathcal{D})$ is bounded. For $\gamma < 0$ we use the following trick.

Lemma 3.4. *If $s \geq 0$, then the norm in the space $\mathcal{H}^{-s,-\gamma}(\mathcal{D})$ can be equivalently described as*

$$\sup_{\substack{v \in \mathcal{H}^{s,0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, \rho^{\gamma} u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}}.$$

Proof. This follows from the fact that $\mathrm{Op}\,(\rho^{\gamma}) : \mathcal{H}^{s,0}(\mathcal{D}) \to \mathcal{H}^{s,\gamma}(\mathcal{D})$ is a topological isomorphism for all $s \geq 0$ and $\gamma \in \mathbb{R}$. This has already been proved above, as desired. □

Let us continue the proof of the corollary. By Lemma 3.4,

$$\begin{array}{rcl} \|\rho^{\delta} u\|^2_{\mathcal{H}^{-s,-\gamma+\delta}(\mathcal{D})} & \sim & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s,0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, \rho^{\gamma-\delta} \rho^{\delta} u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s,0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, \rho^{\gamma} u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \\ & \sim & \|u\|^2_{\mathcal{H}^{-s,-\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, i.e. the operator $\mathrm{Op}\,(\rho^{\delta}) : \mathcal{H}^{-s,-\gamma}(\mathcal{D}) \to \mathcal{H}^{-s,-\gamma+\delta}(\mathcal{D})$ is bounded.

Finally, the assertion about the isomorphisms follows by the same arguments as those in the proof of Theorem 2.6. □

The re-indexing relation between the scales $H^{s,\gamma}(\mathcal{D})$ and $\mathcal{H}^{s,s+\gamma}(\mathcal{D})$ still holds for negative s.

Lemma 3.5. *For each $s > 0$, the norms of the spaces $H^{-s,\gamma}(\mathcal{D})$ and $\mathcal{H}^{-s,-s+\gamma}(\mathcal{D})$ are equivalent on $C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. In particular, the spaces are isomorphic as Banach spaces.*

Proof. Let first $s > 0$ and $s - \gamma > 0$. Then, using Corollary 3.3, we get

$$\begin{array}{rcl} \|u\|_{H^{-s,\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in H^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{-2\gamma} v, u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{-2\gamma} v, u)_{L^2(\mathcal{D})}|}{\|\rho^{-2\gamma} v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}} \, \frac{\|\rho^{-2\gamma} v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \\ & \leq & \displaystyle\sup_{\substack{w \in \mathcal{H}^{s,s-\gamma}(\mathcal{D}) \\ w \neq 0}} \frac{|(w, u)_{L^2(\mathcal{D})}|}{\|w\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}} \sup_{\substack{v \in \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{\|\rho^{-2\gamma} v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \\ & \leq & c\, \|u\|_{\mathcal{H}^{-s,-s+\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a positive constant independent of u. On the other hand, in this case we have

$$\begin{aligned}
\|u\|_{\mathcal{H}^{-s,-s+\gamma}(\mathcal{D})} &= \sup_{\substack{v \in \mathcal{H}^{s,s-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}} \\
&= \sup_{\substack{v \in \mathcal{H}^{s,s-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{2\gamma} v,u)_{\mathcal{H}^{0,\gamma}(\mathcal{D})}|}{\|\rho^{2\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \frac{\|\rho^{2\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}} \\
&\leq \sup_{\substack{w \in H^{s,\gamma}(\mathcal{D}) \\ w \neq 0}} \frac{|(w,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|w\|_{H^{s,\gamma}(\mathcal{D})}} \sup_{\substack{v \in \mathcal{H}^{s,s-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{\|\rho^{2\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s-\gamma}(\mathcal{D})}} \\
&\leq c\, \|u\|_{H^{-s,\gamma}(\mathcal{D})}
\end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, which is due to Corollary 3.3.

If $s - \gamma < 0$ then

$$\begin{aligned}
\|u\|_{\mathcal{H}^{-s,\gamma-s}(\mathcal{D})} &\sim \sup_{\substack{v \in \mathcal{H}^{s,0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,\rho^{s-\gamma} u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \\
&= \sup_{\substack{v \in \mathcal{H}^{s,0}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{s+\gamma} v,u)_{\mathcal{H}^{0,\gamma}(\mathcal{D})}|}{\|\rho^{s+\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \frac{\|\rho^{s+\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \\
&\leq \sup_{\substack{w \in H^{s,\gamma}(\mathcal{D}) \\ w \neq 0}} \frac{|(w,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|w\|_{H^{s,\gamma}(\mathcal{D})}} \sup_{\substack{v \in \mathcal{H}^{s,s-\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{\|\rho^{s+\gamma} v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \\
&\leq c\, \|u\|_{H^{-s,\gamma}(\mathcal{D})}
\end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, which is due to Corollary 3.3. Similarly,

$$\begin{aligned}
\|u\|_{H^{-s,\gamma}(\mathcal{D})} &= \sup_{\substack{v \in H^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma}(\mathcal{D})}} \\
&= \sup_{\substack{v \in \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{-(s+\gamma)} v,\rho^{s-\gamma} u)_{L^2(\mathcal{D})}|}{\|\rho^{-(s+\gamma)} v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \frac{\|\rho^{-(s+\gamma)} v\|_{\mathcal{H}^{s,0}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \\
&\leq \sup_{\substack{w \in \mathcal{H}^{s,0}(\mathcal{D}) \\ w \neq 0}} \frac{|(w,\rho^{s-\gamma} u)_{L^2(\mathcal{D})}|}{\|w\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \sup_{\substack{v \in \mathcal{H}^{s,s+\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{\|\rho^{-(s+\gamma)} v\|_{\mathcal{H}^{s,0}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s,s+\gamma}(\mathcal{D})}} \\
&\leq c\, \|u\|_{\mathcal{H}^{-s,-s+\gamma}(\mathcal{D})}
\end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ where c is a constant independent of u and different in various applications.

Hence it follows that the spaces $H^{-s,\gamma}(\mathcal{D})$ and $\mathcal{H}^{-s,-s+\gamma}(\mathcal{D})$ coincide as Banach spaces for all $s > 0$. $\square$

It is well known that the Sobolev spaces of fractional smoothness can be defined with the aid of appropriate interpolation procedure. When the field of scalars is the real numbers, one uses the so-called (real) trace method [Tri78, 4.3, 4.4] and (real) K-method [Tri78, 1.3, 1.18]. The two real interpolation methods are often equivalent, see Theorem 1.8.2 of [Tri78].

We now wish to test interpolation properties of spaces $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$. For $0 < \theta < 1$, we denote by $H_\theta = [H_0, H_1]_\theta$ the result of interpolation between Banach spaces H_0 and H_1.

Lemma 3.6. *Let $s \in \mathbb{R}_{\geq 0}$ be non-integral. Then*

$$\begin{array}{rcl} \tilde{\mathcal{H}}^{s,s}(\mathcal{D}) & = & [\tilde{\mathcal{H}}^{[s],[s]}(\mathcal{D}), \tilde{\mathcal{H}}^{[s]+1,[s]+1}(\mathcal{D})]_{s-[s]} \;=\; [H^{[s]}(\mathcal{D},Y), H^{[s]+1}(\mathcal{D},Y)]_{s-[s]}, \\ \tilde{H}^{s,\gamma}(\mathcal{D}) & = & [\tilde{H}^{[s],\gamma}(\mathcal{D}), \tilde{H}^{[s]+1,\gamma}(\mathcal{D})]_{s-[s]}. \end{array}$$

Proof. For non-integral $s \geq 0$, the Sobolev space $H^s(\mathcal{D})$ can be obtained as a result of interpolation between the spaces $H^{[s]}(\mathcal{D})$ and $H^{[s]+1}(\mathcal{D})$ by the (real) trace method. Since the norm in $\tilde{\mathcal{H}}^{s,s}(\mathcal{D})$ coincides with that in $H^s(\mathcal{D})$ for all $s \in \mathbb{Z}_{\geq 0}$, it follows that the space $\tilde{\mathcal{H}}^{s,s}(\mathcal{D}) = H^s(\mathcal{D}, Y)$ with fractional $s > 0$ may be alternatively described as the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the norm obtained as a result of interpolation between $\tilde{\mathcal{H}}^{[s],[s]}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{[s]+1,[s]+1}(\mathcal{D})$. Finally, Corollary 3.2 implies that $\|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} = \|\rho^{s-\gamma} u\|_{\tilde{\mathcal{H}}^{s,s}(\mathcal{D})}$ for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, whence we get

$$\|u\|_{\tilde{H}^{s,\gamma}(\mathcal{D})} = \|\rho^{-\gamma} u\|_{H^s(\mathcal{D})}.$$

On arguing as above we establish the second interpolation formula of the lemma, as desired. $\square$

For the scale $\mathcal{H}^{s,\gamma}(\mathcal{D})$ the arguments are much subtler.

Theorem 3.7. *For any non-integral $s \geq 0$, we get*

$$\begin{array}{rcl} \mathcal{H}^{s,s}(\mathcal{D}) & = & [\mathcal{H}^{[s],[s]}(\mathcal{D}), \mathcal{H}^{[s]+1,[s]+1}(\mathcal{D})]_{s-[s]}, \\ H^{s,\gamma}(\mathcal{D}) & = & [H^{[s],\gamma}(\mathcal{D}), H^{[s]+1,\gamma}(\mathcal{D})]_{s-[s]}. \end{array}$$

Proof. We begin with an auxiliary result.

Lemma 3.8. *Let $\gamma \in \mathbb{R}$ and $0 < \theta < 1$. Then*

$$\mathcal{H}^{0,\gamma+\theta}(\mathcal{D}) = [\mathcal{H}^{0,\gamma}(\mathcal{D}), \mathcal{H}^{0,\gamma+1}(\mathcal{D})]_\theta.$$

Proof. We exploit the (real) K-method mentioned above. According to it, the norm in the space $H_\theta = [\mathcal{H}^{0,\gamma}(\mathcal{D}), \mathcal{H}^{0,\gamma+1}(\mathcal{D})]_\theta$ is given by the integral

$$\|u\|_{H_\theta} = \Big(\int_0^\infty t^{-2\theta-1} K^2(t,u)\, dt \Big)^{1/2},$$

where

$$K(t,u) = \inf_{u=u_0+u_1} \big(\|u_0\|_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + t \|u_1\|_{\mathcal{H}^{0,\gamma+1}(\mathcal{D})} \big)$$

with $u_0 \in \mathcal{H}^{0,\gamma}(\mathcal{D})$ and $u_1 \in \mathcal{H}^{0,\gamma+1}(\mathcal{D})$. It is easily seen that

$$\begin{array}{rcl} K(t,u) & \sim & \displaystyle\inf_{u=u_0+u_1} \Big(\|u_0\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} + t^2 \|u_1\|^2_{\mathcal{H}^{0,\gamma+1}(\mathcal{D})} \Big)^{1/2} \\ & = & \displaystyle\inf_{u=u_0+u_1} \Big(\int_{\mathcal{D}} \rho^{-2\gamma} \left(|u_0|^2 + (t\rho^{-1})^2 |u_1|^2 \right) dx \Big)^{1/2}, \end{array}$$

and so

$$K(t,u) \sim \Big(\int_{\mathcal{D}} \rho^{-2\gamma} |u \min\{1, t\rho^{-1}\}|^2 dx \Big)^{1/2}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, cf. [Tri78, 1.18]. Obviously,

$$\min\{1, t\rho^{-1}\} = \begin{cases} t\,\rho(x), & \text{if} \quad t < \rho(x), \\ 1, & \text{if} \quad t > \rho(x). \end{cases}$$

Now, by direct calculation with the use of Fubini's theorem, we obtain

$$\begin{aligned} \|u\|^2_{H_\theta} &= \int_0^\infty t^{-2\theta-1} K^2(t,u)\, dt \\ &= \int_{\mathcal{D}} \rho^{-2\gamma} \Big(\rho^{-2} \int_0^{\rho(x)} t^{1-2\theta} dt + \int_{\rho(x)}^\infty t^{-1-2\theta} dt \Big) |u|^2\, dx \\ &= \frac{1}{2\theta(1-\theta)} \int_{\mathcal{D}} \rho^{-2(\gamma+\theta)} |u|^2\, dx \\ &\sim \|u\|^2_{\mathcal{H}^{0,\gamma+\theta}(\mathcal{D})} \end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. Thus, the norms in H_θ and $\mathcal{H}^{0,\gamma+\theta}(\mathcal{D})$ are equivalent and these spaces coincide as Banach spaces. $\square$

Let us continue the proof of Theorem 3.7. To this end we note that

$$\begin{aligned} \|u\|^2_{\mathcal{H}^{0,0}(\mathcal{D})} &\sim \|u\|^2_{\mathcal{H}^{0,0}(\mathcal{D})} + \|u\|^2_{L^2(\mathcal{D})}, \\ \|u\|^2_{\mathcal{H}^{1,1}(\mathcal{D})} &= \|u\|^2_{\mathcal{H}^{0,1}(\mathcal{D})} + \|u\|^2_{H^1(\mathcal{D})} \end{aligned}$$

whenever $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. Using Lemmas 3.6 and 3.8, we readily deduce that, given any $0 < s < 1$,

$$\|u\|^2_{\mathcal{H}^{s,s}(\mathcal{D})} \sim \|u\|^2_{\mathcal{H}^{0,s}(\mathcal{D})} + \|u\|^2_{H^s(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$.

For arbitrary fractional $s > 0$ we may argue in much the same way. Indeed, by Lemma 2.8, we get

$$\begin{aligned} \|u\|^2_{\mathcal{H}^{[s],[s]}(\mathcal{D})} &= \sum_{|\alpha| \le [s]} \|\partial^\alpha u\|^2_{\mathcal{H}^{0,[s]-|\alpha|}(\mathcal{D})} + \|u\|^2_{H^{[s]}(\mathcal{D})}, \\ \|u\|^2_{\mathcal{H}^{[s]+1,[s]+1}(\mathcal{D})} &= \sum_{|\alpha| \le [s]+1} \|\partial^\alpha u\|^2_{\mathcal{H}^{0,[s]+1-|\alpha|}(\mathcal{D})} + \|u\|^2_{H^{[s]+1}(\mathcal{D})} \end{aligned}$$

on functions $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. Again, applying Lemmas 3.6 and 3.8 we see that

$$\begin{aligned} \|u\|_{[\mathcal{H}^{[s],[s]}(\mathcal{D}), \mathcal{H}^{[s]+1,[s]+1}(\mathcal{D})]_{s-[s]}} &= \sum_{|\alpha| \le [s]} \|\partial^\alpha u\|^2_{\mathcal{H}^{0,s-|\alpha|}(\mathcal{D})} + \|u\|^2_{H^s(\mathcal{D})} \\ &= \|u\|^2_{\mathcal{H}^{s,s}(\mathcal{D})} \end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$.

Finally, Corollary 3.3 implies $\|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})} = \|\rho^{s-\gamma} u\|_{\mathcal{H}^{s,s}(\mathcal{D})}$ on functions u of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. In particular, $\|u\|_{H^{s,\gamma}(\mathcal{D})} = \|\rho^{-\gamma} u\|_{H^{s,s}(\mathcal{D})}$ and so we apply what has already been proved to establish the second interpolation formula of Theorem 3.7. $\square$

Part 2. Embedding theorems

4. Embedding theorems for weighted Sobolev spaces

The proof of embeddings below is especially simple if one applies interpolation arguments. To begin with, we mention an obvious fact.

Lemma 4.1. *Let s be an integral number and let $\rho \in C^{|s|}(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for all $|\alpha| \leq |s|$. If $\gamma \geq \gamma'$, then the space $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s,\gamma'}(\mathcal{D})$.*

Proof. For integer $s \geq 0$, the assertion is contained in Lemma 2.5. As the embedding $\mathcal{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s,\gamma'}(\mathcal{D})$, $\gamma \geq \gamma'$, is continuous and has dense range for $s \in \mathbb{Z}_{\geq 0}$, we see that the embedding

$$\mathcal{H}^{-s,-\gamma'}(\mathcal{D}) = (\mathcal{H}^{s,\gamma'}(\mathcal{D}))' \hookrightarrow (\mathcal{H}^{s,\gamma}(\mathcal{D}))' = \mathcal{H}^{-s,-\gamma}(\mathcal{D})$$

is bounded, too, with $-\gamma' \geq -\gamma$. □

Theorem 4.2. *Let s and s' be real numbers with $s \geq s'$, and $\rho \in C^{[s]+1}(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for all multi-indices α with $|\alpha| \leq [s]+1$. If $s \geq s'$, then the space $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s',\gamma}(\mathcal{D})$.*

Proof. Suppose $s \geq s' \geq 0$. Using Corollary 3.3, Lemma 2.8 (for integer s') and the definition of the norm in $\mathcal{H}^{s',s'}(\mathcal{D})$ (for non-integral s'), we obtain

$$\begin{array}{rcl} \|\rho^{s'-\gamma}u\|_{H^{s'}(\mathcal{D})} & \leq & \|\rho^{s'-\gamma}u\|_{\mathcal{H}^{s',s'}(\mathcal{D})} \\ & = & \|\rho^{s-\gamma}\rho^{s'-s}u\|_{\mathcal{H}^{s',s'}(\mathcal{D})} \\ & \leq & c\,\|\rho^{s-\gamma}u\|_{\mathcal{H}^{s',s'-(s'-s)}(\mathcal{D})} \\ & = & c\,\|\rho^{s-\gamma}u\|_{\mathcal{H}^{s',s}(\mathcal{D})} \end{array} \tag{4.1}$$

for all $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent of u. On the other hand, as $0 \leq [s'] \leq [s]$, from Lemma 2.5, Theorem 2.6 and the Sobolev embedding theorem it follows that

$$\begin{array}{rcl} \|\rho^{s-\gamma}u\|^2_{\mathcal{H}^{s',s}(\mathcal{D})} & = & \|\rho^{s-\gamma}u\|^2_{\mathcal{H}^{[s'],s}(\mathcal{D})} + \|\rho^{s-\gamma}u\|^2_{H^{s'}(\mathcal{D})} \\ & \leq & c\,(\|u\|^2_{\mathcal{H}^{[s'],\gamma}(\mathcal{D})} + \|\rho^{s-\gamma}u\|^2_{H^{s}(\mathcal{D})}) \\ & \leq & c'\,(\|u\|^2_{\mathcal{H}^{[s],\gamma}(\mathcal{D})} + \|\rho^{s-\gamma}u\|^2_{H^{s}(\mathcal{D})}) \end{array} \tag{4.2}$$

for all $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, where c and c' are constants independent on u. Combining (4.1), (4.2) and the definition of the space $\mathcal{H}^{s',\gamma}(\mathcal{D})$ yields

$$\|u\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})} \leq c\,\|u\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}$$

for all $C^\infty(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent on u. This establishes the continuous embedding $\mathcal{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s',\gamma}(\mathcal{D})$ for $0 \leq s' \leq s$.

Assume that $-s' < 0 \le s$ and $\gamma \ge 0$. Then

$$\begin{array}{rcl} \|u\|_{\mathcal{H}^{-s',-\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(\rho^{-\gamma}v,\rho^{\gamma}u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{\|v\|_{\mathcal{H}^{0,\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})}} \|u\|_{\mathcal{H}^{0,-\gamma}(\mathcal{D})} \\ & \le & c\,\|u\|_{\mathcal{H}^{s,-\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$, the last estimate is a consequence of continuous embeddings $\mathcal{H}^{s,-\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{0,-\gamma}(\mathcal{D})$ and $\mathcal{H}^{s',\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{0,\gamma}(\mathcal{D})$. If still $-s' < 0 \le s$ but $\gamma < 0$, then we obtain

$$\begin{array}{rcl} \|u\|_{\mathcal{H}^{-s',-\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,\rho^{\gamma}u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s',0}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',0}(\mathcal{D}) \\ v \neq 0}} \frac{\|v\|_{L^2(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s',0}(\mathcal{D})}} \|\rho^{\gamma}u\|_{L^2(\mathcal{D})} \\ & \le & c\,\|u\|_{\mathcal{H}^{s,-\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$, the first equality being due to Lemma 3.4 and the last estimate to the continuous embedding $\mathcal{H}^{s',0}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$.

Finally, let $-s' \le -s < 0$, i.e. $s' \ge s$. If $\gamma \ge 0$, then

$$\begin{array}{rcl} \|u\|_{\mathcal{H}^{-s',-\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}} \frac{\|v\|_{\mathcal{H}^{s,\gamma}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s',\gamma}(\mathcal{D})}} \\ & \le & c\,\|u\|_{\mathcal{H}^{-s,-\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$, since $\mathcal{H}^{s',\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s,\gamma}(\mathcal{D})$. If $\gamma < 0$, then

$$\begin{array}{rcl} \|u\|_{\mathcal{H}^{-s',-\gamma}(\mathcal{D})} & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,\rho^{\gamma}u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s',0}(\mathcal{D})}} \\ & = & \displaystyle\sup_{\substack{v \in \mathcal{H}^{s',0}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,\rho^{\gamma}u)_{L^2(\mathcal{D})}|}{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}} \frac{\|v\|_{\mathcal{H}^{s,0}(\mathcal{D})}}{\|v\|_{\mathcal{H}^{s',0}(\mathcal{D})}} \\ & \le & c\,\|u\|_{\tilde{\mathcal{H}}^{-s,-\gamma}(\mathcal{D})} \end{array}$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)$, where the first equality is due to Lemma 3.4 and the last estimate due to the continuous embedding $\mathcal{H}^{s',0}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s,0}(\mathcal{D})$.

We have thus established the continuous embedding $\mathcal{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s',\gamma}(\mathcal{D})$ for all $s \geq s'$, as desired. $\square$

We are in a position to say more about the re-indexed scales.

Lemma 4.3. *If $s \geq 0$, then the space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s,\gamma}(\mathcal{D})$. On the contrary, if $s \leq 0$, then the space $\tilde{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s,\gamma}(\mathcal{D})$.*

Proof. For $s \in \mathbb{Z}_{\geq 0}$ the statement is contained in Lemma 2.9. For non-integral $s \geq 0$, the lemma follows from inequality (2.10) which is still true in this case by the definition of spaces in question. If $s > 0$, then

$$\|u\|_{H^{-s,\gamma}(\mathcal{D})} = \sup_{\substack{v \in H^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{s,\gamma}(\mathcal{D})}} \leq c \sup_{\substack{v \in \tilde{H}^{s,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{\tilde{H}^{s,\gamma}(\mathcal{D})}} = \|u\|_{\tilde{H}^{-s,\gamma}(\mathcal{D})}$$

with $c > 0$ a constant independent on u, for $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s,\gamma}(\mathcal{D})$ provided $s > 0$. It follows that the space $\tilde{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s,\gamma}(\mathcal{D})$ for $s \leq 0$. For $s = 0$ the spaces coincide. $\square$

Theorem 4.4. *Suppose that s, s' are real numbers, such that $s \geq s'$, and $\rho \in C^{[s]+1}(\overline{\mathcal{D}} \setminus Y)$ satisfies (2.1) for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq [s]+1$. Then the space $\tilde{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s',\gamma}(\mathcal{D})$. If $s > s'$, then the embedding is actually compact.*

Proof. To establish the continuous embedding $\tilde{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow \tilde{H}^{s',\gamma}(\mathcal{D})$, one can exploit the same arguments as those in the proof of Lemma 2.9 (cf. inequality (2.9) which is still true for all $s \geq s'$ because of Corollary 3.2). Now we conclude, by Corollary 3.2, that a sequence $\{u_\nu\}$ is bounded in $\tilde{H}^{s,\gamma}(\mathcal{D})$ if and only if the sequence $\{\rho^{-\gamma}u_\nu\}$ is bounded in $H^s(\mathcal{D})$. Then, by Rellich's theorem, the sequence $\{\rho^{-\gamma}u_\nu\}$ is precompact in $H^{s'}(\mathcal{D})$, and so the sequence $\{u_\nu\}$ is precompact in $\tilde{H}^{s',\gamma}(\mathcal{D})$, provided that $s > s'$. This proves that the embedding $\tilde{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow \tilde{H}^{s',\gamma}(\mathcal{D})$ is compact for $s > s'$. $\square$

Theorem 4.5. *Let $s, s' \in \mathbb{R}$ be such that $s \geq s'$, and let $\rho \in C^{[s]+1}(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for all multi-indices α with $|\alpha| \leq [s]+1$. Then the space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s',\gamma}(\mathcal{D})$. If moreover $s > s'$, then the embedding is compact.*

Proof. We will divide the proof into several steps.

Lemma 4.6. *If $s \geq s'$, then the space $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s',\gamma}(\mathcal{D})$.*

Proof. It follows immediately from Lemma 3.4 and Theorem 4.2. $\square$

Lemma 4.7. *Suppose $s > s'$. Then the space $H^{s,\gamma}(\mathcal{D})$ is compactly embedded into $H^{s',\gamma}(\mathcal{D})$.*

Proof. First let $[s] = [s'] \geq 0$ or $0 \leq s = [s'] + 1$. If $s' < s$ and Σ is a bounded subset of $H^{s,\gamma}(\mathcal{D})$, then Σ is bounded in $\mathcal{H}^{[s],s+\gamma}(\mathcal{D})$ and $\mathrm{Op}\,(\rho^{-\gamma})(\Sigma)$ is bounded in $H^s(\mathcal{D})$, which is due to Theorem 2.6 and Lemma 2.8. By Rellich's theorem, $\mathrm{Op}\,(\rho^{-\gamma})(\Sigma)$ is precompact in $H^{s'}(\mathcal{D})$.

By the weak compactness principle, there is a sequence $\{u_\nu\}$ in Σ weakly convergent in $H^{s,\gamma}(\mathcal{D})$ to an element $u_0 \in H^{s,\gamma}(\mathcal{D})$. Without loss of generality we may certainly assume that $u_0 = 0$. Then the sequence $\{\rho^{-\gamma} u_\nu\}$ contains a subsequence which converges to zero in $H^{s'}(\mathcal{D})$. Passing to a subsequence, if necessary, we may actually assume that $\{\rho^{-\gamma} u_\nu\}$ converges to zero in $H^{s'}(\mathcal{D})$ (and, in particular, in $H^{[s']}(\mathcal{D})$).

Hence it follows that $\{u_\nu\}$ converges to zero in $\mathcal{H}^{[s'],\gamma}(\mathcal{D})$. Indeed, by construction, $\{\rho^{-\gamma} u_\nu\}$ converges to zero in $L^2(\mathcal{D})$, i.e. $\{u_\nu\}$ converges to zero in $\mathcal{H}^{0,\gamma}(\mathcal{D})$. This is enough, if $[s'] = 0$.

Suppose $[s'] = 1$. As the sequence $(\rho^{-\gamma} u_\nu)' = -\gamma \rho^{-\gamma-1} \rho' u_\nu + \rho^{-\gamma} u_\nu'$ converges to zero in $L^2(\mathcal{D})$, we see that the sequence

$$\rho (\rho^{-\gamma} u_\nu)' = -\gamma \rho^{-\gamma} \rho' u_\nu + \rho^{1-\gamma} u_\nu'$$

converges to zero in $L^2(\mathcal{D})$, too. This means that $\{u_\nu'\}$ converges to zero in $\mathcal{H}^{0,\gamma-1}(\mathcal{D})$.

In general, for $[s'] \geq 1$, let $\partial^\alpha u_\nu$ converge to zero in $\mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})$ for all multi-indices α satisfying $0 \leq |\alpha| \leq m$, where $m \leq [s'] - 1$. If α is a multi-index of $|\alpha| = m+1$, then the sequence

$$\partial^\alpha (\rho^{-\gamma} u_\nu) = \rho^{-\gamma} \partial^\alpha u_\nu + \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \rho^{-\gamma}\, \partial^\beta u_\nu$$

converges to zero in $L^2(\mathcal{D})$. From Lemma 2.7 it follows that $\rho^{|\alpha-\beta|+\gamma} \partial^{\alpha-\beta} \rho^{-\gamma}$ is uniformly bounded in $\mathcal{D}$. Hence the sequence

$$\rho^{|\alpha|} \partial^\alpha (\rho^{-\gamma} u_\nu) = \rho^{|\alpha|-\gamma} \partial^\alpha u_\nu + \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \binom{\alpha}{\beta} \rho^{|\alpha-\beta|+\gamma} (\partial^{\alpha-\beta} \rho^{-\gamma})\, \rho^{|\beta|-\gamma} \partial^\beta u_\nu$$

converges to zero in $L^2(\mathcal{D})$, too. This means precisely that $\{\partial^\alpha u_\nu\}$ converges to zero in $\mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})$. Summarizing we conclude by induction that $\{\partial^\alpha u_\nu\}$ converges to zero in $\mathcal{H}^{0,\gamma-|\alpha|}(\mathcal{D})$ for all α with $|\alpha| \leq [s']$, and hence $\{u_\nu\}$ converges to zero in $\mathcal{H}^{[s'],\gamma}(\mathcal{D})$, as desired.

On the other hand, $\{u_\nu\}$ is bounded in $\mathcal{H}^{[s],s+\gamma}(\mathcal{D})$, and so an easy computation shows that

$$\begin{aligned}
\|u_\nu\|^2_{\mathcal{H}^{[s'],\delta}(\mathcal{D})} &= \int_{\mathcal{D}} \rho^{-2\delta} \sum_{|\alpha| \leq [s']} \rho^{-s-\gamma+|\alpha|} \partial^\alpha u_\nu\, \rho^{s+\gamma+|\alpha|} \overline{\partial^\alpha u_\nu} dx \\
&\leq \sum_{|\alpha| \leq [s']} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,s+\gamma-|\alpha|}(\mathcal{D})} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,2\delta-s-\gamma-|\alpha|}(\mathcal{D})} \\
&\leq c\, \|u_\nu\|_{\mathcal{H}^{[s'],2\delta-s-\gamma}(\mathcal{D})},
\end{aligned}$$

which tends to zero, as $\nu \to \infty$, if $2\delta - s - \gamma \leq \gamma$ or, equivalently, $\delta \leq (1/2)s + \gamma$. Now we see that

$$\begin{aligned}
\|u_\nu\|^2_{\mathcal{H}^{[s'],\delta}(\mathcal{D})} &= \int_{\mathcal{D}} \rho^{-2\delta} \sum_{|\alpha| \leq [s']} \rho^{(1/2)s+\gamma+|\alpha|} \partial^\alpha u_\nu\, \rho^{-(1/2)s-\gamma+|\alpha|} \overline{\partial^\alpha u_\nu} dx \\
&\leq \sum_{|\alpha| \leq [s']} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,2\delta-(1/2)s-\gamma-|\alpha|}(\mathcal{D})} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,(1/2)s+\gamma-|\alpha|}(\mathcal{D})} \\
&\leq c\, \|u_\nu\|_{\mathcal{H}^{[s'],(1/2)s+\gamma}(\mathcal{D})}
\end{aligned}$$

with c a constant independent of u_ν, provided that $2\delta - (1/2)s - \gamma \le s + \gamma$ or, equivalently, if $\delta \le (3/4)s + \gamma$. From what has been proved above it follows that the right-hand side tends to zero, as $\nu \to \infty$. On repeating the same arguments once again we obtain

$$\begin{aligned} \|u_\nu\|^2_{\mathcal{H}^{[s'],\delta}(\mathcal{D})} &= \int_{\mathcal{D}} \rho^{-2\delta} \sum_{|\alpha|\le[s']} \rho^{(3/4)s+\gamma+|\alpha|} \partial^\alpha u_\nu \rho^{-(3/4)s-\gamma+|\alpha|} \overline{\partial^\alpha u_\nu} dx \\ &\le \sum_{|\alpha|\le[s']} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,2\delta-(3/4)s-\gamma-|\alpha|}(\mathcal{D})} \|\partial^\alpha u_\nu\|_{\mathcal{H}^{0,(3/4)s+\gamma-|\alpha|}(\mathcal{D})} \\ &\le c\,\|u_\nu\|_{\mathcal{H}^{[s'],(3/4)s+\gamma}(\mathcal{D})}, \end{aligned}$$

where c is a constant independent of u_ν, provided that $2\delta - (3/4)s - \gamma \le s + \gamma$ or, equivalently, if $\delta \le (7/8)s + \gamma$. By the above, the right-hand tends to zero, as $\nu \to \infty$.

Now we may argue by the induction. Set

$$\begin{aligned} q_0 &= 1/2, \\ q_j &= (1 + q_{j-1})/2, \end{aligned} \tag{4.3}$$

for $j = 1, 2, \ldots$. Assume that $\|u_\nu\|^2_{\mathcal{H}^{[s'],\delta}(\mathcal{D})} \to 0$ for all δ satisfying $\delta \le q_j s + \gamma$. Then

$$\begin{aligned} \|u_\nu\|^2_{\mathcal{H}^{[s'],\delta}(\mathcal{D})} &= \int_{\mathcal{D}} \rho^{-2\delta} \sum_{|\alpha|\le[s']} \rho^{q_j s+\gamma+|\alpha|} \partial^\alpha u_\nu\, \rho^{-q_j s-\gamma+|\alpha|} \overline{\partial^\alpha u_\nu} dx \\ &\le \sum_{|\alpha|\le[s']} \|\partial^\alpha u_\nu\|_{H^{0,2\delta-q_j s-\gamma}(\mathcal{D})} \|\partial^\alpha u_\nu\|_{H^{0,(q_j s+\gamma}(\mathcal{D})} \\ &\le c\,\|u_\nu\|_{\mathcal{H}^{[s'],q_\mu s+\gamma}(\mathcal{D})}, \end{aligned}$$

the constant c being independent of u_ν, if $2\delta - q_j s - \gamma \le s + \gamma$ or, equivalently, if $\delta \le q_{j+1} s + \gamma$. The right-hand side converges to zero, as $\nu \to \infty$.

It is easily seen that $\{q_j\}$ is a decreasing sequence of positive numbers. Moreover, it has the limit $q = 1$, as it follows from recurrent formula (4.3). Hence, $\{u_\nu\}$ converges to zero in $\mathcal{H}^{[s'],s+\gamma}(\mathcal{D})$, and so in the space $\mathcal{H}^{s',s'+\gamma}(\mathcal{D})$, if $s > s' \ge 0$. This just amounts to saying that $H^{s,\gamma}(\mathcal{D})$ is compactly embedded into $H^{s',\gamma}(\mathcal{D})$, if $s > s' \ge 0$.

Furthermore, if $s > [s'] + 1 \ge 0$, then we have the following line of continuous embeddings

$$H^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{[s']+1,\gamma}(\mathcal{D}) \hookrightarrow H^{s',\gamma}(\mathcal{D}),$$

the first one being compact. Hence we conclude that the theorem is true for all $s \ge 0$ and $s' \ge -1$ satisfying $s > s'$.

If $s' < 0 \le s$, then Lemmas 1.2 and 2.5 yield immediately the continuous embeddings

$$H^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{[s],\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D}) \hookrightarrow H^{s',\gamma}(\mathcal{D})$$

The last of these embeddings is compact because of Lemma 1.2 and the discussion above. Hence the embedding $H^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{s',\gamma}(\mathcal{D})$ is compact, too.

Finally, for $s' < s < 0$ we may argue by duality. Namely, we have already proved that the embedding $e_{-s',-s} : H^{-s',\gamma}(\mathcal{D}) \to H^{-s,\gamma}(\mathcal{D})$ is compact (provided that $-s' > -s > 0$). As the spaces $H^{-s,\gamma}(\mathcal{D})$ and $H^{-s',\gamma}(\mathcal{D})$ are reflexive (see Remark 1.4), we see that the adjoint $e'_{-s',-s} : H^{s,\gamma}(\mathcal{D}) \to H^{s',\gamma}(\mathcal{D})$ is compact, too. If Σ

is a bounded set in $H^{-s,\gamma}(\mathcal{D})$ then, by reflexivity, it contains a weakly convergent sequence, say, $\{u_\nu\}$. We get

$$\begin{aligned}
\|u_\nu - u_\mu\|_{H^{s',\gamma}(\mathcal{D})} &= \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)\\ v\neq 0}} \frac{|(e_{-s',-s}v, u_\nu - u_\mu)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{-s',\gamma}(\mathcal{D})}}\\
&= \sup_{\substack{v\in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}}\setminus Y)\\ v\neq 0}} \frac{|(v, e'_{-s',-s}(u_\nu - u_\mu))_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{H^{-s',\gamma}(\mathcal{D})}}\\
&= \|e'_{-s',-s}(u_\nu - u_\mu)\|_{H^{s',\gamma}(\mathcal{D})}
\end{aligned}$$

for all μ and ν. Since $e'_{-s',-s}$ is compact, it follows that $\{e'_{-s',-s}u_\nu\}$ is a Cauchy sequence in $H^{s',\gamma}(\mathcal{D})$, and so $\{u_\nu\}$ is a Cauchy sequence in $H^{s',\gamma}(\mathcal{D})$ Hence, Σ is precompact in $H^{s',\gamma}(\mathcal{D})$. $\square$

Theorem 4.5 is proved. $\square$

We are now in a position to establish embedding theorems for the initial scales $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$.

Corollary 4.8. *Let $s, s' \in \mathbb{Z}$, $k = \max\{|s|, |s'|\}$ and let $\rho \in C^k(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for all multi-indices α with $|\alpha| \le k$. If $s > s'$ and $\gamma > \gamma'$, then $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is compactly embedded into $\mathcal{H}^{s',\gamma'}(\mathcal{D})$.*

Proof. By definition, $\mathcal{H}^{s,\gamma}(\mathcal{D}) = H^{s,\gamma-s}(\mathcal{D})$. Under the hypothesis of the corollary we have $s \ge s' + 1$. But then Theorem 4.5 yields the compact embedding

$$\mathcal{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{s-\Delta s,\gamma-s}(\mathcal{D}) = \mathcal{H}^{s-\Delta s,\gamma-\Delta s}(\mathcal{D})$$

with any $\Delta s > 0$. Choose $\Delta s \in (0, 1)$ in such a way that $0 < \Delta s < \gamma - \gamma'$. Then Theorem 4.2 and Lemma 4.1 yield continuous embeddings

$$\mathcal{H}^{s-\Delta s,\gamma-\Delta s}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s-1,\gamma-\Delta s}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s-1,\gamma'}(\mathcal{D}) \hookrightarrow \mathcal{H}^{s',\gamma'}(\mathcal{D}),$$

showing the corollary. $\square$

Corollary 4.9. *Let $s \in \mathbb{R}$ be non-integral and let $\rho \in C^{[|s|]+1}(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) with $|\alpha| \le [|s|] + 1$. If $\gamma > \gamma'$ then $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is compactly embedded into $\mathcal{H}^{[s],\gamma'}(\mathcal{D})$.*

Proof. As we already mentioned, $\mathcal{H}^{s,\gamma}(\mathcal{D}) = H^{s,\gamma-s}(\mathcal{D})$. Under the hypothesis of the lemma we have $[s] + 1 > s > [s]$. But then Theorem 4.5 yields the compact embedding

$$\mathcal{H}^{s,\gamma}(\mathcal{D}) \hookrightarrow H^{s-\Delta s,\gamma-s}(\mathcal{D}) = \mathcal{H}^{s-\Delta s,\gamma-\Delta s}(\mathcal{D})$$

for all $0 < \Delta s < s$. Choose now Δs in such a way that $0 < \Delta s < s - [s]$ and $0 < \Delta s < \gamma - \gamma'$. Combining Theorem 4.5, Lemma 2.5 and Lemma 4.1 we get continuous embeddings

$$\mathcal{H}^{s-\Delta s,\gamma-\Delta s}(\mathcal{D}) \hookrightarrow \mathcal{H}^{[s],\gamma-\Delta s}(\mathcal{D}) = \mathcal{H}^{[s],\gamma'+(\gamma-\gamma'-\Delta s)}(\mathcal{D}) \hookrightarrow \mathcal{H}^{[s],\gamma'}(\mathcal{D}),$$

establishing the corollary. $\square$

Lemma 4.10. *Suppose $s \in \mathbb{R}_{\ge 0}$ and $\gamma \ge \gamma'$. Then the following conditions are equivalent:*

1) $\mathcal{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\mathcal{H}^{s,\gamma'}(\mathcal{D})$;

2) $H^{s,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s,\gamma'}(\mathcal{D})$;

3) There is a constant $c > 0$, such that $\|\rho^{\gamma-\gamma'}u\|_{H^s(\mathcal{D})} \leq c\,\|u\|_{\mathcal{H}^{s,s}(\mathcal{D})}$ for all $u \in \mathcal{H}^{s,s}(\mathcal{D})$.

4) There is a constant $c > 0$, such that $\|\rho^{\gamma-\gamma'}u\|_{H^s(\mathcal{D})} \leq c\,\|u\|_{H^{s,0}(\mathcal{D})}$ for all $u \in H^{s,0}(\mathcal{D})$.

Proof. The equivalence of 1) and 3) follows immediately from the definition of the spaces and Lemma 2.5, for

$$\begin{aligned}\|u\|^2_{\mathcal{H}^{s,\gamma'}(\mathcal{D})} &= \|u\|^2_{\mathcal{H}^{[s],\gamma'}(\mathcal{D})} + \|\rho^{s-\gamma'}u\|^2_{H^s(\mathcal{D})} = \|u\|^2_{\mathcal{H}^{[s],\gamma'}(\mathcal{D})} + \|\rho^{\gamma-\gamma'}\rho^{s-\gamma}u\|^2_{H^s(\mathcal{D})},\\ \|u\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{D})} &= \|u\|^2_{\mathcal{H}^{[s],\gamma}(\mathcal{D})} + \|\rho^{s-\gamma}u\|^2_{H^s(\mathcal{D})}\end{aligned}$$

because $\rho^{s-\gamma}u \in \mathcal{H}^{s,s}(\mathcal{D})$ for all $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$ (see Corollary 3.3).

Similarly, the equivalence of 2) and 4) follows immediately from the definition of the spaces and Lemma 2.5. Indeed,

$$\begin{aligned}\|u\|^2_{H^{s,\gamma'}(\mathcal{D})} &= \|u\|^2_{\mathcal{H}^{[s],s+\gamma'}(\mathcal{D})} + \|\rho^{-\gamma'}u\|^2_{H^s(\mathcal{D})} = \|u\|^2_{\mathcal{H}^{[s],s+\gamma'}(\mathcal{D})} + \|\rho^{\gamma-\gamma'}\rho^{-\gamma}u\|^2_{H^s(\mathcal{D})},\\ \|u\|_{H^{s,\gamma}(\mathcal{D})} &= \|u\|^2_{\mathcal{H}^{[s],s+\gamma}(\mathcal{D})} + \|\rho^{-\gamma}u\|^2_{H^s(\mathcal{D})}\end{aligned}$$

because $\rho^{-\gamma}u \in H^{s,0}(\mathcal{D})$ for all $u \in H^{s,\gamma}(\mathcal{D})$ (see Corollary 3.3).

Finally, the assertions 3) and 4) are equivalent for the spaces $H^{s,0}(\mathcal{D})$ and $\mathcal{H}^{s,s}(\mathcal{D})$ coincide. □

Thus, embeddings (3.1) allow one to establish natural embedding theorems for the scales $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $H^{s,\gamma}(\mathcal{D})$ with respect to the smoothness index $s \in \mathbb{R}$. In order to get analogous embedding theorems for the scales $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $H^{s,\gamma}(\mathcal{D})$ with respect to the weight index γ one needs to ensure an estimate

$$\|\rho^{\delta}u\|_{H^s(\mathcal{D})} \leq c\,\|u\|_{\mathcal{H}^{s,s}(\mathcal{D})}$$

for all $u \in \mathcal{H}^{s,s}(\mathcal{D})$ with $s \in \mathbb{R}_{\geq 0}$ and $\delta > 0$, where c is a constant depending on s and δ but not on u. For integer s such an estimate follows from Lemma 2.5 and Theorem 2.6 immediately. Our task is to derive the estimate for all non-integral s. However, it may cause the function ρ to satisfy additional restrictions, cf. Section 5 below.

Lemma 4.11. *Assume $s \in \mathbb{R}_{\geq 0}$ and $\gamma \geq \gamma'$. Then $\tilde{H}^{s,\gamma}(\mathcal{D})$ is continuously embedded into $\tilde{H}^{s,\gamma'}(\mathcal{D})$ if and only if there is a constant $c > 0$, such that*

$$\|\rho^{\gamma-\gamma'}u\|_{H^s(\mathcal{D})} \leq c\,\|u\|_{H^s(\mathcal{D})}$$

for all $u \in C^\infty(\overline{\mathcal{D}} \setminus Y)$.

Proof. The lemma follows from the definition of the spaces involved immediately, for

$$\begin{aligned}\|u\|_{\tilde{H}^{s,\gamma'}(\mathcal{D})} &= \|\rho^{-\gamma'}u\|^2_{H^s(\mathcal{D})} = \|\rho^{\gamma-\gamma'}\rho^{-\gamma}u\|^2_{H^s(\mathcal{D})},\\ \|u\|^2_{\tilde{H}^{s,\gamma}(\mathcal{D})} &= \|\rho^{-\gamma}u\|^2_{H^s(\mathcal{D})}\end{aligned}$$

because $\rho^{-\gamma}u \in \tilde{H}^{s,0}(\mathcal{D})$ for all $u \in \tilde{H}^{s,\gamma}(\mathcal{D})$ (see Corollary 3.2). □

To finish the preliminary discussion, let us describe the boundary properties of functions of the weighted spaces.

For this purpose we assume that $Y \cap \partial \mathcal{D}$ is situated on an $(n-2)$-dimensional surface in $\partial \mathcal{D}$. Define the trace on $\partial \mathcal{D}$ for functions of $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$, where $1/2 < s < 3/2$. As before, we require

$$\begin{array}{rcl} \rho & \in & C^1(\overline{\mathcal{D}} \setminus Y), \\ \rho' & \in & L^\infty(\mathcal{D}), \end{array} \tag{4.4}$$

if $1/2 < s \leq 1$, and

$$\begin{array}{rcl} \rho & \in & C^2(\overline{\mathcal{D}} \setminus Y), \\ \rho', \rho\rho'' & \in & L^\infty(\mathcal{D}), \end{array} \tag{4.5}$$

if $1 < s < 3/2$. These assumptions enable us to keep the definitions of spaces $\mathcal{H}^{s,\gamma}(\partial\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\partial\mathcal{D})$ with $0 \leq s < 1$ and all the statements proved above for $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$, where $0 \leq s \leq 3/2$.

More precisely, let ds stand for the area form on $\partial\mathcal{D}$ induced by the Lebesgue measure in $\mathbb{R}^n$. We introduce the scalar product

$$(u, v)_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} = \int_{\partial\mathcal{D}} \rho^{-2\gamma} u \overline{v}\, ds \tag{4.6}$$

for $u, v \in C_{\mathrm{comp}}(\partial\mathcal{D} \setminus Y)$. Denote by $\mathcal{H}^{0,\gamma}(\partial\mathcal{D})$ the completion of $C_{\mathrm{comp}}(\partial\mathcal{D} \setminus Y)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} = \Big(\int_{\partial\mathcal{D}} \rho^{-2\gamma} |u|^2\, ds \Big)^{1/2}$$

induced by (4.6).

For $0 < s < 1$, we write $\mathcal{H}^{s,\gamma}(\partial\mathcal{D})$ for the completion of $C^{0,1}_{\mathrm{comp}}(\partial\mathcal{D} \setminus Y)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\partial\mathcal{D})} = \Big(\|u\|^2_{\mathcal{H}^{0,\gamma}(\partial\mathcal{D})} + \|\rho^{s-\gamma} u\|^2_{H^s(\partial\mathcal{D})} \Big)^{1/2}.$$

Similarly, let $\tilde{\mathcal{H}}^{s,\gamma}(\partial\mathcal{D})$ denote the completion of $C^{0,1}_{\mathrm{comp}}(\partial\mathcal{D} \setminus Y)$ with respect to the norm

$$\|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\partial\mathcal{D})} = \|\rho^{s-\gamma} u\|^2_{H^s(\partial\mathcal{D})}.$$

Suppose $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$, where $s \geq 0$. From Corollary 3.3 it follows that the function $v = \rho^{s-\gamma} u$ belongs to $\mathcal{H}^{s,s}(\mathcal{D})$. By the above, $\mathcal{H}^{s,s}(\mathcal{D})$ is continuously embedded into $H^s(\mathcal{D})$, which is by the very definition for non-integral $s \geq 0$ and due to Lemma 2.8 for $s \in \mathbb{Z}_{\geq 0}$. Since for the Sobolev spaces $H^s(\mathcal{D})$ with $s > 1/2$ there is well-defined bounded linear trace operator $t_s : H^s(\mathcal{D}) \to L^2(\partial\mathcal{D})$ (see for instance [Tri78, § 4.7]), we define

$$t_s u := \rho^{\gamma - s} t_s(\rho^{s-\gamma} u) \tag{4.7}$$

to be the trace of a function $u \in \mathcal{H}^{s,\gamma}(\mathcal{D})$ on the boundary. For the re-indexed scale $H^{s,\gamma}(\mathcal{D})$ we introduce the trace operator by

$$t_s u := \rho^{\gamma} t_s(\rho^{-\gamma} u), \tag{4.8}$$

if $u \in H^{s,\gamma}(\mathcal{D})$ with $s > 1/2$.

On arguing in much the same way we define the traces of functions of $\tilde{\mathcal{H}}^{s,\gamma}(\partial\mathcal{D})$ and $\tilde{H}^{s,\gamma}(\partial\mathcal{D})$ at the surface $\partial\mathcal{D}$.

Theorem 4.12. *Let $Y \cap \partial\mathcal{D}$ be situated on an $(n-2)$-dimensional surface in $\partial\mathcal{D}$, $1/2 < s < 3/2$, and (4.4) or (4.5) be fulfilled. Then formulas (4.7) and (4.8) induce bounded trace operators*

$$\begin{array}{rcl} \tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}) & \to & \tilde{\mathcal{H}}^{s-1/2,\gamma-1/2}(\partial\mathcal{D}), \\ \tilde{H}^{s,\gamma}(\mathcal{D}) & \to & \tilde{H}^{s-1/2,\gamma}(\partial\mathcal{D}) \end{array}$$

provided $s > 1/2$. Moreover, these trace operators possess bounded right inverses.

Proof. From the trace theorem for Sobolev spaces it follows that the operator t_s maps $H^s(\mathcal{D})$ continuously to $H^{s-1/2}(\partial\mathcal{D})$, if $1/2 < s < 3/2$ (see for instance [McL00]). And Corollary 3.2 yields the desired continuity of the trace operator, showing the first part of the theorem.

Pick $u_0 \in \tilde{\mathcal{H}}^{s-1/2,\gamma-1/2}(\partial\mathcal{D})$. By definition, the function $v_0 = \rho^{s-\gamma}u_0$ belongs to $\tilde{\mathcal{H}}^{s-1/2,s-1/2}(\partial\mathcal{D}) = H^{s-1/2}(\partial\mathcal{D})$. By the trace theorem for Sobolev spaces, there is a function $v \in H^s(\mathcal{D})$, such that $t_s(v) = v_0$ on $\partial\mathcal{D}$ and

$$\begin{array}{rcl} \|v\|_{H^s(\mathcal{D})} & \leq & c\,\|v_0\|_{H^{s-1/2}(\partial\mathcal{D})} \\ & = & c\,\|v_0\|_{\tilde{\mathcal{H}}^{s-1/2,s-1/2}(\partial\mathcal{D})} \\ & = & c\,\|\rho^{s-\gamma}u_0\|_{\tilde{\mathcal{H}}^{s-1/2,s-1/2}(\partial\mathcal{D})}, \end{array} \tag{4.9}$$

where the constant c is independent of u_0. Moreover, Corollary 3.2 implies readily that

$$\|\rho^{s-\gamma}u_0\|_{\tilde{\mathcal{H}}^{s-1/2,s-1/2}(\partial\mathcal{D})} = \|u_0\|_{\tilde{\mathcal{H}}^{s-1/2,\gamma-1/2}(\partial\mathcal{D})}. \tag{4.10}$$

As the function u_0 is the limit of a sequence $\{u_\nu\} \subset C^{0,1}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ in the $\tilde{\mathcal{H}}^{s-1/2,\gamma-1/2}(\partial\mathcal{D})$-norm, we see that v_0 is the limit of $v_\nu = \rho^{s-\gamma}u_\nu \in C^{0,1}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ in the $H^{s-1/2}(\partial\mathcal{D})$-norm. By (4.9), the sequence $\{v_\nu\}$ converges to v in $H^s(\mathcal{D})$. In particular, $v \in H^s(\mathcal{D}, Y)$.

Now we set $u = \rho^{\gamma-s}v$. As $\tilde{\mathcal{H}}^{s,s}(\mathcal{D}) = H^s(\mathcal{D}, Y)$, the function v actually belongs to $\mathcal{H}^{s,s}(\mathcal{D})$ and

$$\|u\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} = \|\rho^{\gamma-s}v\|_{\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})} = \|v\|_{H^s(\mathcal{D})},$$

which is due to Corollary 3.2. Combining this equality with (4.9) and (4.10) yields the second part of the theorem. □

Theorem 4.13. *Let $Y \cap \partial\mathcal{D}$ be situated in an $(n-2)$-dimensional surface in $\partial\mathcal{D}$, $1/2 < s < 3/2$ and (4.4) or (4.5) be fulfilled. Then formulas (4.7) and (4.8) induce bounded trace operators*

$$\begin{array}{rcl} \mathcal{H}^{s,\gamma}(\mathcal{D}) & \to & \mathcal{H}^{s-1/2,\gamma-1/2}(\partial\mathcal{D}), \\ H^{s,\gamma}(\mathcal{D}) & \to & H^{s-1/2,\gamma}(\partial\mathcal{D}). \end{array}$$

Proof. From the trace theorem for Sobolev spaces we deduce that the operator t_s maps $H^s(\mathcal{D})$ continuously to $H^{s-1/2}(\partial\mathcal{D})$, if $1/2 < s < 3/2$ (see for instance [McL00]).

Using Theorem 2.6 and Lemma 2.8 we see that $\rho^{-\gamma}u \in H^{1,0}(\mathcal{D}) \subset H^1(\mathcal{D})$, and so $\rho^{-\gamma}t_s(u) \in H^{1/2}(\partial\mathcal{D})$ for all $u \in H^{1,\gamma}(\mathcal{D})$. According to Lemma 2.8 and Theorem

2.6, we obtain

$$
\begin{array}{rcl}
\|\rho^{-\gamma} t_s(u)\|_{H^{1/2}(\partial\mathcal{D})} & = & \|t_s(\rho^{-\gamma}u)\|_{H^{1/2}(\partial\mathcal{D})} \\
& \leq & c\,\|\rho^{-\gamma}u\|_{H^1(\mathcal{D})} \\
& \leq & c\,\|\rho^{-\gamma}u\|_{H^{1,0}(\mathcal{D})} \\
& \leq & c\,\|u\|_{H^{1,\gamma}(\mathcal{D})}
\end{array}
\tag{4.11}
$$

for all $u \in C^{0,1}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, where c stands for a constant independent of u and different in diverse applications.

If $1/2 < s < 3/2$ is non-integral, i.e. different from 1, then, by definition, $\rho^{-\gamma}u \in H^s(\mathcal{D})$ for all $u \in H^{s,\gamma}(\mathcal{D})$. Hence it follows that $\rho^{-\gamma}\tau_s(u) \in H^{s-1/2}(\partial\mathcal{D})$. Furthermore,

$$
\begin{array}{rcl}
\|\rho^{-\gamma} t_s(u)\|_{H^{s-1/2}(\partial\mathcal{D})} & = & \|t_s(\rho^{-\gamma}u)\|_{H^{s-1/2}(\partial\mathcal{D})} \\
& \leq & c\,\|\rho^{-\gamma}u\|_{H^s(\mathcal{D})} \\
& \leq & c\,\|u\|_{H^{s,\gamma}(\mathcal{D})}
\end{array}
\tag{4.12}
$$

for all $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent of u and different in diverse applications.

Let $1 \leq j \leq n$ and ϕ a smooth function in $\overline{\mathcal{D}}$. For any $u \in C^{\infty}_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, the Gauß-Ostrogradskii formula yields

$$
\begin{array}{rl}
\displaystyle\int_{\partial\mathcal{D}} \rho^{-(2s+2\gamma-1)} |u|^2 \phi\, \nu_j ds \;=\; \int_{\mathcal{D}} \partial_j \Big(\rho^{-(2s+2\gamma-1)}|u|^2\phi\Big) dx & \\
= \displaystyle\int_{\mathcal{D}} (-2s-2\gamma+1)|\rho^{-(s+\gamma)}u|^2\,(\partial_j\rho)\,\phi dx + \int_{\mathcal{D}} (\rho^{-(s+\gamma-1)}\partial_j u)(\rho^{-(s+\gamma)}\overline{u})\phi dx & \\
+ \displaystyle\int_{\mathcal{D}} (\rho^{-(s+\gamma)}u)(\rho^{-(s+\gamma-1)}\partial_j\overline{u})\phi dx + \int_{\mathcal{D}} |\rho^{-(s+\gamma)}u|^2 \rho(\partial_j\phi) dx. &
\end{array}
\tag{4.13}
$$

To continue the proof we need several lemmas.

Lemma 4.14. *If $1 \leq s < 3/2$, then formula (4.8) induces a bounded trace operator $t_s : H^{s,\gamma}(\mathcal{D}) \to H^{s-1/2,\gamma}(\partial\mathcal{D})$.*

Proof. Choose a coordinate neighborhood U_ν of a boundary point, such that $U_\nu \cap \mathcal{D}$ is given by $x^n > f_\nu(x')$, where f_ν is a Lipschitz function on $\mathbb{R}^{n-1}$. Then the part of the surface $\partial\mathcal{D}$ in U_ν is given as the graph of the function $x^n = f_\nu(x')$, cf. the beginning of Section 1. A finite number of such U_ν cover all of $\partial\mathcal{D}$. Choose a partition $\phi_\nu \in C^{\infty}_{\mathrm{comp}}(U_\nu)$ of unity on $\partial\mathcal{D}$ which is subordinate to the covering $\{U_\nu\}$.

By the Rademacher theorem, each Lipschitz function is differentiable almost everywhere and its derivatives are bounded. Hence, using (4.13) we arrive in a familiar manner at an estimate

$$
\begin{array}{rcl}
\displaystyle\int_{\partial\mathcal{D}} \rho^{-(2s+2\gamma-1)}|u|^2 ds & = & \displaystyle\sum_\nu \int_{\partial\mathcal{D}\cap U_\nu} \rho^{-(2s+2\gamma-1)}|u|^2 \phi_\nu ds \\
& \leq & c\,\|u\|^2_{\mathcal{H}^{1,s+\gamma}(\mathcal{D})} \\
& = & c\,\|u\|^2_{\mathcal{H}^{[s],s+\gamma}(\mathcal{D})}
\end{array}
\tag{4.14}
$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent of u.

Combining (4.11), (4.12) and (4.14) we see that

$$\begin{aligned} \|t_s(u)\|^2_{H^{s-1/2,\gamma}(\partial\mathcal{D})} &= \|t_s(u)\|^2_{\mathcal{H}^{s-1/2,s-1/2+\gamma}(\partial\mathcal{D})} \\ &= \|t_s(u)\|^2_{\mathcal{H}^{0,s-1/2+\gamma}(\partial\mathcal{D})} + \|t_s(u)\|^2_{H^{s-1/2}(\partial\mathcal{D})} \\ &\le c\,\|u\|^2_{H^{s,\gamma}(\mathcal{D})} \end{aligned}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. It follows that (4.8) induces a bounded trace operator $H^{s,\gamma}(\mathcal{D}) \to H^{s-1/2,\gamma}(\partial\mathcal{D})$ for $1 \le s < 3/2$. □

In order to proceed with $1/2 < s < 1$ we need the following lemma similar to Lemma 1.5.

Lemma 4.15. *If $1/2 < s < 1$ then the total derivative operator maps $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-1,\gamma}(\mathcal{D}, \mathbb{C}^n)$.*

Proof. By definition, $H^{s-1,\gamma}(\mathcal{D})$ is the dual space of $H^{1-s,\gamma}(\mathcal{D}) = \mathcal{H}^{1-s,1-s+\gamma}(\mathcal{D})$ with respect to the pairing in $H^{0,\gamma}(\mathcal{D})$. Therefore, given any $u \in C^\infty(\overline{\mathcal{D}} \setminus Y)$, we get

$$\|u'\|_{H^{s-1,\gamma}(\mathcal{D},\mathbb{C}^n)} = \sup_{\substack{v \in H^{1-s,\gamma}(\mathcal{D},\mathbb{C}^n) \\ v \neq 0}} \frac{|(v,u')_{H^{0,\gamma}(\mathcal{D},\mathbb{C}^n)}|}{\|v\|_{H^{1-s,\gamma}(\mathcal{D},\mathbb{C}^n)}}.$$

On the other hand,

$$(v,u')_{H^{0,\gamma}(\mathcal{D},\mathbb{C}^n)} = (\rho^{-\gamma}v, \rho^{-\gamma}u')_{L^2(\mathcal{D},\mathbb{C}^n)} = (\rho^{-\gamma}v, (\rho^{-\gamma}u)' + \gamma\rho^{-\gamma-1}\rho' u)_{L^2(\mathcal{D},\mathbb{C}^n)}.$$

By definition, $(\rho^{-\gamma}u) \in H^s(\mathcal{D})$ for any $u \in H^{s,\gamma}(\mathcal{D})$. Hence, using Lemma 1.5 we see that

$$\begin{aligned} |(\rho^{-\gamma}v, (\rho^{-\gamma}u)')_{L^2(\mathcal{D},\mathbb{C}^n)}| &\le \|\rho^{-\gamma}v\|_{H^{1-s}(\mathcal{D},\mathbb{C}^n)} \|(\rho^{-\gamma}u)'\|_{H^{s-1}(\mathcal{D},\mathbb{C}^n)} \\ &\le c\,\|\rho^{-\gamma}v\|_{H^{1-s}(\mathcal{D},\mathbb{C}^n)} \|\rho^{-\gamma}u\|_{H^s(\mathcal{D})}. \end{aligned} \tag{4.15}$$

Moreover, since $u \in \mathcal{H}^{0,s+\gamma}(\mathcal{D})$ for all $u \in H^{s,\gamma}(\mathcal{D})$, and $\rho' \in L^\infty(\mathcal{D})$, we conclude that

$$\begin{aligned} |(\rho^{-\gamma}v, \rho^{-\gamma-1}\rho' u)_{L^2(\mathcal{D},\mathbb{C}^n)}| &= |(\rho^{s-1-\gamma}v, \rho^{-s-\gamma}\rho' u)_{L^2(\mathcal{D},\mathbb{C}^n)}| \\ &\le \|\rho'\|_{L^\infty(\mathcal{D},\mathbb{C}^n)} \|\rho^{s-1-\gamma}v\|_{L^2(\mathcal{D})} \|\rho^{-s-\gamma}u\|_{L^2(\mathcal{D})} \\ &= \|\rho'\|_{L^\infty(\mathcal{D},\mathbb{C}^n)} \|v\|_{\mathcal{H}^{0,1-s+\gamma}(\mathcal{D})} \|u\|_{\mathcal{H}^{0,s+\gamma}(\mathcal{D})} \end{aligned} \tag{4.16}$$

provided that $v \in H^{1-s,\gamma}(\mathcal{D}, \mathbb{C}^n) = \mathcal{H}^{1-s,1-s+\gamma}(\mathcal{D}, \mathbb{C}^n)$.

Finally, (4.15) and (4.16) along with the Schwarz type inequality for the pairing in $H^{0,\gamma}(\mathcal{D})$ imply that

$$|(v,u')_{H^{0,\gamma}(\mathcal{D},\mathbb{C}^n)}| \le c\,\|v\|_{H^{1-s,\gamma}(\mathcal{D},\mathbb{C}^n)} \|u\|_{H^{s,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ and $v \in H^{s-1,\gamma}(\mathcal{D}, \mathbb{C}^n)$, with c a constant independent on u and v. Hence

$$\|u'\|_{H^{s-1,\gamma}(\mathcal{D},\mathbb{C}^n)} \le c\,\|u\|_{H^{s,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, as desired. □

Lemma 4.16. *If $1/2 < s < 1$, then (4.8) induces a bounded trace operator $H^{s,\gamma}(\mathcal{D}) \to H^{s-1/2,\gamma}(\partial\mathcal{D})$.*

Proof. It is clear that the first and the last integrals on the right-hand side of formula (4.13) are dominated by

$$c\,\|u\|^2_{\mathcal{H}^{0,s+\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent of u.

On the other hand, using Corollary 3.3 we conclude that the multiplication by ρ^{1-2s} maps $H^{s,\gamma}(\mathcal{D})$ continuously to the space $H^{s,1-2s+\gamma}(\mathcal{D}) = \mathcal{H}^{s,1-s+\gamma}(\mathcal{D})$. As $0 < 1-s < 1/2 < s$, Theorem 4.2 yields a continuous embedding

$$\mathcal{H}^{s,1-s+\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}^{1-s,1-s+\gamma}(\mathcal{D}) = H^{1-s,\gamma}(\mathcal{D}).$$

Hence it follows that the multiplication by ρ^{1-2s} maps $H^{s,\gamma}(\mathcal{D})$ continuously to the space $H^{1-s,\gamma}(\mathcal{D})$.

It is known that multiplication by a smooth function is a bounded operator in $H^s(\mathcal{D})$ for $|s| \leq 1/2$ (see for instance [Agr02, p. 865]). Hence,

$$\begin{array}{rcl}
\|\phi v\|^2_{H^{1-s,\gamma}(\mathcal{D})} & = & \|\phi v\|^2_{\mathcal{H}^{0,1-s+\gamma}(\mathcal{D})} + \|\phi(\rho^{-\gamma} v)\|^2_{H^{1-s}(\mathcal{D})} \\
& \leq & c\left(\|v\|^2_{\mathcal{H}^{0,1-s+\gamma}(\mathcal{D})} + \|\rho^{-\gamma} v\|^2_{H^{1-s}(\mathcal{D})}\right) \\
& = & c\,\|v\|^2_{H^{1-s,\gamma}(\mathcal{D})}
\end{array}$$

for all $v \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, where c is a constant independent of v. In particular, since $0 < 1-s < s$, we get

$$\|\phi(\rho^{1-2s}u)\|_{H^{1-s,\gamma}(\mathcal{D})} \leq c\,\|\rho^{1-2s}u\|_{H^{1-s,\gamma}(\mathcal{D})} \leq c\,\|u\|_{H^{s,\gamma}(\mathcal{D})}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, the constant c being independent on u and different in diverse applications. On combining this estimate with Lemma 4.15 we readily deduce that

$$\begin{array}{rcl}
\Big|\displaystyle\int_{\mathcal{D}} (\rho^{-(s+\gamma-1)}\partial_j u)(\rho^{-(s+\gamma)}\overline{u})\phi dx\Big| & = & |(\partial_j u, \phi(\rho^{1-2s}u))_{\mathcal{H}^{0,\gamma}(\mathcal{D})}| \\
& \leq & c\,\|\partial_j u\|_{H^{s-1,\gamma}(\mathcal{D})}\|\phi(\rho^{1-2s}u)\|_{H^{1-s,\gamma}(\mathcal{D})} \\
& \leq & c\,\|u\|^2_{H^{s,\gamma}(\mathcal{D})}
\end{array} \tag{4.17}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{D} \setminus Y)$.

Arguing as in the proof of Lemma 4.14 and using equality (4.13) and estimate (4.17), we obtain

$$\int_{\partial\mathcal{D}} \rho^{-(2s+2\gamma-1)}|u|^2 ds \leq c\,\|u\|^2_{\mathcal{H}^{s,\gamma}(\mathcal{D})} \tag{4.18}$$

for all $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, with c a constant independent of u. From (4.12) and (4.18) it follows that (4.8) induces a bounded trace operator

$$t_s : H^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s-1/2,\gamma-1/2}(\partial\mathcal{D}) = H^{s-1/2,\gamma}(\partial\mathcal{D}),$$

provided that $1/2 < s < 1$. □

Finally, as $\mathcal{H}^{s,\gamma}(\mathcal{D}) = H^{s,\gamma-s}(\mathcal{D})$, we derive immediately a bounded trace operator $\mathcal{H}^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s-1/2,\gamma-1/2}(\partial\mathcal{D})$, thus completing the proof of Theorem 4.13, as desired. □

Remark 4.17. For $s \geq 3/2$, one needs to increase the smoothness of $\partial\mathcal{D} \setminus Y$ in order to obtain adequate trace theorems, for, in general, there is no way to define a bounded trace operator $H^{3/2}(\mathcal{D}) \to H^1(\partial\mathcal{D})$ for domains with Lipschitz boundary (see for instance [McL00]).

5. Regular singularities at the boundary

Clearly, the properties of weighted spaces essentially depend on the asymptotic behavior of the weight function ρ near Y. Hence, in order to discuss further properties of the scales $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$, we need to impose additional conditions on the function ρ. Actually this will lead us to typical situations of analysis on manifolds with singularities.

Assume that there is a neighbourhood U of the set Y in $\overline{\mathcal{D}}$ and smooth functions $\rho_1, \ldots, \rho_{n-1}$ in U, such that

$$|d\rho_1(x) \wedge \ldots \wedge d\rho_{n-1}(x) \wedge d\rho(x)| \geq c \tag{5.1}$$

for all $x \in U \setminus Y$, where $c > 0$ is a constant independent of x. Note that the differential form in (5.1) has the form $(\det J(x))dx$, where $J(x)$ is the Jacobi matrix of the functional system $\rho_1, \ldots, \rho_{n-1}, \rho$. Hence, condition (5.1) means that the modulus of $\det J$ is bounded away from zero in $U \setminus Y$. Thus, ρ can be completed to a coordinate system in U.

Theorem 5.1. *Let s be a natural number and $\rho \in C^s(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for all multi-indices α with $|\alpha| \leq s$. If (5.1) holds then the normed spaces $H^{s,0}(\mathcal{D})$ and $H^s(\mathcal{D}, Y)$ are isomorphic.*

Proof. Using a suitable partition of unity near the boundary, we can restrict ourselves to those functions $u \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ whose supports are contained in a small neighborhood U of a point $x_0 \in Y$ in $\overline{\mathcal{D}}$. Shrinking U, if necessary, we introduce the new coordinates

$$\begin{array}{rcl} y^1 & = & \rho_1(x), \\ & \ldots & \\ y^{n-1} & = & \rho_{n-1}(x), \\ r & = & \rho(x) \end{array}$$

in U. By assumption, there is a constant $c > 0$, such that

$$|\det J| = \Big| \det \frac{\partial(y, r)}{\partial x} \Big| \geq c \tag{5.2}$$

in $U \setminus Y$

The summands involving the derivatives of order s in the norms $\|u\|_{\mathcal{H}^{s,s}(\mathcal{D})}$ and $\|u\|_{H^s(\mathcal{D})}$ coincide. To handle lower order terms, we fix a multi-index $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| \leq s - 1$ and set $u_\alpha(r) = \partial^\alpha u\,(y, r)$.

Since u vanishes in a neighbourhood of Y, it follows that the support of u is contained in a set of the form $U' \times (0, R)$ in the new coordinates (y, r), where U' is an open set in the hyperplane $\mathbb{R}^{n-1}$ of variables y. Then, by Fubini theorem, we obtain

$$\begin{array}{rcl} c \displaystyle\int_U \rho^{2(|\alpha|-s)} |\partial^\alpha u|^2 dx & \leq & \displaystyle\int_U \rho^{2(|\alpha|-s)} |\partial^\alpha u|^2 |\det J(x)| dx \\ & = & \displaystyle\int_{U'} \int_0^R |r^{|\alpha|-s} u_\alpha(r)|^2 dr\, dy. \end{array} \tag{5.3}$$

We next make use of the Hardy-Littlewood inequality for measurable functions on the half-axis with values in a normed space. Namely,

$$\|r^{p-1}\int_0^r f(\varrho)d\varrho\|_{L^q(\mathbb{R}_{\geq 0})} \leq \Big(\frac{1}{q'}-p\Big)^{-1}\|r^p f(r)\|_{L^q(\mathbb{R}_{\geq 0})}, \tag{5.4}$$

where $1 \leq q \leq \infty$, $1/q+1/q' = 1$, and $p < 1/q'$. Take $f(r) = (\partial/\partial r)u_\alpha(r)$ and $q = 2$, $p = |\alpha| - s + 1$, and observe that

$$|f(r)| = \Big|\sum_{j=1}^n \partial_j\,(\partial^\alpha u)\,\frac{\partial x^j}{\partial r}\Big| \leq c\sum_{j=1}^n |\partial_j \partial^\alpha u|, \tag{5.5}$$

which is due to (5.1). It follows from Hardy-Littlewood inequality (5.4) and (5.2), (5.5) that

$$\int_U \rho^{2(|\alpha|-s)}|\partial^\alpha u|^2 dx \leq c\sum_{|\beta|=|\alpha|+1}\|\partial^\beta u\|^2_{\mathcal{H}^{0,s-|\beta|}(\mathcal{D})},$$

for all u under consideration, with c a constant independent of u.

Repeated application of Hardy-Littlewood inequality (5.4) and (5.2), (5.5) therefore yields

$$\sum_{|\alpha|\leq s-1}\int_U \rho^{2(|\alpha|-s)}|\partial^\alpha u|^2 dx \leq c\sum_{|\beta|=s}\|\partial^\beta u\|^2_{L^2(U)}$$

for all u supported in U and vanishing in a neighborhood of Y, where c is a constant independent of u.

Summarizing we conclude that the $\mathcal{H}^{s,s}(\mathcal{D})$-norm (or, equivalently, $H^{s,0}(\mathcal{D})$-norm) is majorised by the $H^s(\mathcal{D})$-norm on functions vanishing near Y. This completes the proof. □

To clarify the geometric nature of regularity condition (5.1), we consider several examples.

Example 5.2. Let $\mathcal{D}$ be the cube $(0,1)^n$ in $\mathbb{R}^n$ and Y be the q-dimensional edge of the cube given by

$$Y = \{x \in [0,1]^n : x^{q+1} = \ldots = x^n = 0\},$$

where $0 \leq q \leq n-1$. By construction, the set Y contains the origin. Obviously, the function

$$\rho(x) = \Big(\sum_{j=q+1}^n (x^j)^2\Big)^{1/2}$$

is continuous in the closure of $\mathcal{D}$ and satisfies $0 \leq \rho(x) \leq \sqrt{n-q}$ in $\mathcal{D}$.

Furthermore, it is easy to see that

$$\partial^\alpha \rho(x) = \sum_{k=0}^{|\alpha|}\frac{p_{\alpha,k}(x)}{\rho^{|\alpha|+k-1}(x)}, \tag{5.6}$$

where $p_{\alpha,k}(x)$ are homogeneous polynomials of degree k. Indeed, we argue by induction with respect to $|\alpha|$. For $|\alpha| = 1$ the statement follows from the obvious formulas $\partial_j\rho(x) = 0$, if $1 \leq j \leq q$, and $\partial_j\rho(x) = x_j/\rho(x)$, if $q+1 \leq j \leq n$. Suppose

(5.6) holds for $|\alpha| = l$. If now $|\alpha| = l+1$, then one can write $\partial^\alpha = \partial_j \partial^\beta$, where $|\beta| = l$ and $1 \le j \le n$. Hence

$$\begin{aligned} \partial^\alpha \rho(x) &= \partial_j \sum_{k=0}^{|\beta|} \frac{p_{\beta,k}(x)}{\rho^{|\beta|+k-1}(x)} \\ &= \sum_{k=0}^{|\beta|} \frac{\partial_j p_{\beta,k}(x)}{\rho^{|\beta|+k-1}(x)} + \sum_{k=0}^{|\beta|} \frac{(1-k-|\beta|) x_j \, p_{\beta,k}(x)}{\rho^{|\beta|+k+1}(x)} \\ &= \sum_{k=0}^{|\alpha|} \frac{p_{\alpha,k}(x)}{\rho^{|\alpha|+k-1}(x)}, \end{aligned}$$

for $\partial_j p_{\beta,k}(x)$ and $x_j p_{\beta,k}(x)$ are homogeneous polynomials of degrees $k-1$ and $k+1$, respectively.

We have thus proved (5.6) for all multi-indices α. From (5.6) it follows immediately that (2.1) is fulfilled for all $\alpha \in \mathbb{Z}^n_{\ge 0}$.

Obviously, $|\rho'| = 1$ holds in $\overline{\mathcal{D}} \setminus Y$. Our next objective is to show that there are smooth functions $\rho_1, \ldots, \rho_{n-1}$ in $\overline{\mathcal{D}}$, such that

$$|d\rho_1 \wedge \ldots \wedge d\rho_{n-1} \wedge d\rho| \ge c$$

in $\overline{\mathcal{D}} \setminus Y$, where c is a positive constant, cf. (5.1). We argue by induction in q. For $q = n-1$, we get $\rho(x) = x^n$. On choosing $\rho_j(x) = x^j$, for $j = 1, \ldots, n-1$, one obtains

$$|d\rho_1 \wedge \ldots \wedge d\rho_{n-1} \wedge d\rho| \equiv 1$$

in $\overline{\mathcal{D}}$, as desired. If $q = n-2$, then we have $d\rho = (x^{n-1} dx^{n-1} + x^n dx^n)/\rho$. On choosing

$$\begin{aligned} \rho_j(x) &= x^j, \quad \text{for } j = 1, \ldots, n-2, \\ \rho_{n-1}(x) &= x^{n-1} - x^n, \end{aligned}$$

we get

$$|d\rho_1 \wedge \ldots \wedge d\rho_{n-1} \wedge d\rho| = (x^{n-1} + x^n)/\rho \ge 1$$

in $\overline{\mathcal{D}} \setminus Y$. Now, for arbitrary $1 \le q \le n-3$, one verifies that

$$d\rho = \Big(\sum_{j=q+1}^{n} x^j dx^j \Big) / \rho.$$

On choosing

$$\begin{aligned} \rho_j(x) &= x^j, \quad \text{for } j = 1, \ldots, q, \\ \rho_{q+1}(x) &= x^{q+1} - x^n, \\ &\ldots \\ \rho_{n-1}(x) &= x^{n-1} - x^n, \end{aligned}$$

we get

$$|d\rho_1 \wedge \ldots \wedge d\rho_{n-1} \wedge d\rho| = (x^{q+1} + \ldots + x^n)/\rho \ge 1$$

in $\overline{\mathcal{D}} \setminus Y$.

Thus, (5.1) holds for the domain $\mathcal{D}$ and the function $\rho(x)$ under consideration, i.e., Y is a regular singular set in $\overline{\mathcal{D}}$.

Example 5.3. Let $\mathcal{D}$ be the cube $(0,1)^3$ in $\mathbb{R}^3$ and Y the boundary of the face $\{x \in [0,1]^3 : x^3 = 0\}$ on $\partial\mathcal{D}$. In other words, Y is the boundary of the square $[0,1]^2$ in the plane of variables $x' = (x^1, x^2)$ The function

$$\rho(x) = \left\{ \begin{array}{lll} ((x^2)^2 + (x^3)^2)^{1/2}, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \leq 1, \\ ((1-x^1)^2 + (x^3)^2)^{1/2}, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \geq 1, \\ ((1-x^2)^2 + (x^3)^2)^{1/2}, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \geq 1, \\ ((x^1)^2 + (x^3)^2)^{1/2}, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \leq 1, \end{array} \right.$$

just amounts to the distance from a point $x \in \mathbb{R}^3$ to Y. This function is continuous in $\overline{\mathcal{D}}$, takes on its values in the interval $[0, \sqrt{5}/2]$ and vanishes on Y. Moreover, ρ is C^∞ in all of $\overline{\mathcal{D}}$ except for Y and the diagonal hyperplanes $\{x^1 - x^2 = 0\}$ and $\{x^1 + x^2 = 1\}$. The singularities of ρ at the hyperplanes are caused by the corner points of Y.

It is easy to verify that

$$\rho'(x) = \left\{ \begin{array}{lll} (0, x^2, x^3)/\rho(x), & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \leq 1, \\ (x^1 - 1, 0, x^3)/\rho(x), & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \geq 1, \\ (0, x^2 - 1, x^3)/\rho(x), & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \geq 1, \\ (x^1, 0, x^3)/\rho(x), & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \leq 1, \end{array} \right.$$

whence $|\rho'(x)| = 1$ for all $x \in \overline{\mathcal{D}}$ at which ρ is differentiable.

Moreover, let $\alpha \in \mathbb{Z}^3_{\geq 0}$ be an arbitrary multi-index. In the domain $x^1 - x^2 > 0$, $x^1 + x^2 < 1$ the power $\rho^{|\alpha|-1}$ is a homogeneous function of degree $|\alpha| - 1$ in x^2 and x^3, while the derivative $\partial^\alpha \rho$ is homogeneous of degree $1 - |\alpha|$. It follows that $\rho^{|\alpha|-1}\partial^\alpha\rho$ is a homogeneous function of degree zero in x^2 and x^3, and so it is bounded. The same reasoning shows that $\rho^{|\alpha|-1}\partial^\alpha\rho$ is bounded in the other three domains of function ρ. Hence, inequalities (2.1) are fulfilled for all multi-indices $\alpha \in \mathbb{Z}^3_{\geq 0}$.

Our next objective is to show that there are smooth functions ρ_1, ρ_2 in $\overline{\mathcal{D}}$, such that

$$|d\rho_1 \wedge d\rho_2 \wedge d\rho| \geq c$$

in $\overline{\mathcal{D}} \setminus Y$, where c is a positive constant. To this end, we consider two piecewise affine functions

$$\rho_1(x) = \left\{ \begin{array}{lll} x^1 - x^3, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \leq 1, \\ 1 - x^2 - x^3, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \geq 1, \\ 1 - x^1 - x^3, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \geq 1, \\ x^2 - x^3, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \leq 1, \end{array} \right.$$

and

$$\rho_2(x) = \left\{ \begin{array}{lll} x^2 - x^3, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \leq 1, \\ 1 - x^1 - x^3, & \text{if} & x^1 - x^2 \geq 0, \quad x^1 + x^2 \geq 1, \\ 1 - x^2 - x^3, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \geq 1, \\ x^1 - x^3, & \text{if} & x^1 - x^2 \leq 0, \quad x^1 + x^2 \leq 1, \end{array} \right.$$

for $x \in \overline{\mathcal{D}}$. Obviously, they are continuous in the closure of $\mathcal{D}$ and smooth away from the diagonal hyperplanes $\{x^1 - x^2 = 0\}$ and $\{x^1 + x^2 = 1\}$. A straightforward calculation shows that

$$d\rho_1 \wedge d\rho_2 \wedge d\rho = \begin{cases} ((x^2+x^3)/\rho(x))dx, & \text{if} \quad x^1-x^2 \geq 0, \quad x^1+x^2 \leq 1, \\ -(((1-x^1)+x^3)/\rho(x))dx, & \text{if} \quad x^1-x^2 \geq 0, \quad x^1+x^2 \geq 1, \\ (((1-x^2)+x^3)/\rho(x))dx, & \text{if} \quad x^1-x^2 \leq 0, \quad x^1+x^2 \geq 1, \\ -((x^1+x^3)/\rho(x))dx, & \text{if} \quad x^1-x^2 \leq 0, \quad x^1+x^2 \leq 1, \end{cases}$$

whence $|d\rho_1 \wedge d\rho_2 \wedge d\rho| \geq 1$ for all $x \in \overline{\mathcal{D}}$ except for those in the diagonal hyperplanes of $\overline{\mathcal{D}}$.

We thus conclude that the function ρ possesses the desired properties except for the differentiability in the complement of Y in $\overline{\mathcal{D}}$. This function can be certainly smoothen away from Y in $\overline{\mathcal{D}}$, however, the smoothing might lead to the violation of uniform estimate $|d\rho_1 \wedge d\rho_2 \wedge d\rho| \geq c$ with $c > 0$ at the corner points of Y. This actually reflects the fact that the corner points are of higher order than the smooth edges approaching them in the hierarchy of singularities of the stratified manifold $\overline{\mathcal{D}}$.

One may ask what singularities of smooth structure of the closure of $\mathcal{D}$ survive under the analytical condition (5.1). By the very nature they seem to be a kind of singularities of transversal intersection like conic points or edges. The function $\rho(x)$ enters itself into the new smooth structure of $\overline{\mathcal{D}}$ as a new singular coordinate. The hypersurface $\{\rho(x) = 0\}$ intersects the boundary of $\mathcal{D}$ along the closed set Y. Depending on $\mathcal{D}$ the intersection might be a manifold with boundary which bears singularities itself, cf. Example 5.2. We restrict our discussion to those Y which are closed manifolds of dimension $0 \leq q < n-1$. For $q = 0$, we think of Y as a conic point of the surface $\partial\mathcal{D}$. For $q \geq 1$, we think of Y as an edge which is locally a cone bundle over $\mathbb{R}^q$. Then, near Y, the function $\rho(x)$ can be thought of as the distance from x to Y.

In the case where Y is a smooth edge of dimension $0 \leq q \leq n-1$ on the boundary of $\mathcal{D}$ Theorem 5.1 is well known, see for instance Theorem 2.1 of [SST03].

Corollary 5.4. *Let the hypotheses of Theorem 5.1 hold. Then the spaces* $\mathcal{H}^{s,\gamma}(\mathcal{D})$ *and* $\tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D})$ *coincide for all* $s \in \mathbb{R}$.

Proof. If s is a non-negative integer, then

$$\begin{aligned} \mathcal{H}^{s,\gamma}(\mathcal{D}) &= \mathrm{Op}\,(\rho^{s-\gamma})\mathcal{H}^{s,s}(\mathcal{D}) \\ &= \mathrm{Op}\,(\rho^{s-\gamma})H^{s,0}(\mathcal{D}) \\ &= \mathrm{Op}\,(\rho^{s-\gamma})H^{s}(\mathcal{D}, Y) \\ &= \mathrm{Op}\,(\rho^{s-\gamma})\tilde{\mathcal{H}}^{s,s}(\mathcal{D}) \\ &= \tilde{\mathcal{H}}^{s,\gamma}(\mathcal{D}), \end{aligned}$$

where the first and the fifth equalities are consequences of Theorem 2.6, the second and the fourth equalities hold by definition, and the third equality is due to Theorem 5.1. For negative integral s the equality follows from what has already been proved by duality.

For all real $s \geq 0$, the assertion follows from Lemma 3.6, Theorem 3.7 and what has been shown for integral $s \geq 0$. To complete the proof for real $s < 0$, it suffices to exploit duality. □

Under reasonable assumptions on coefficients, partial differential operators act properly in weighted Sobolev spaces $H^{s,\gamma}(\mathcal{D})$ of fractional smoothness, too.

Theorem 5.5. *Let $s \in \mathbb{R}_{\geq 0}$ and $\rho \in C^{[s]+1}(\overline{\mathcal{D}} \setminus Y)$ satisfy (2.1) for $|\alpha| \leq [s]+1$. If the norms of the spaces $\mathcal{H}^{k,k}(\mathcal{D})$ and $H^k(\mathcal{D}, Y)$ are equivalent for all integer k with $[s]-m \leq k \leq [s]+1$, then any differential operator of order $m \leq s$ of type (2.11) maps*

1) $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-m,\gamma}(\mathcal{D})$;

2) $\tilde{H}^{s,\gamma}(\mathcal{D})$ continuously to $\tilde{H}^{s-m,\gamma}(\mathcal{D})$.

Proof. By Corollary 3.3 and Theorem 4.2, the operator ρ^{-1} maps $\mathcal{H}^{s,\gamma}(\mathcal{D})$ continuously to $\mathcal{H}^{s-1,\gamma-1}(\mathcal{D})$. Hence, it maps $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-1,\gamma}(\mathcal{D})$ because of (2.2). Of course, a similar statement holds for the scale $\tilde{H}^{s,\gamma}(\mathcal{D})$, for we may apply Corollary 3.2 and Theorem 4.4.

Under the hypotheses of the theorem the norms of the spaces $H^{k,\gamma}(\mathcal{D})$ and $\tilde{H}^{k,\gamma}(\mathcal{D})$ are equivalent for $[s]-m \leq k \leq [s]+1$, which is due to Corollaries 3.2 and 3.3. Then, for $s \in \mathbb{Z}_{\geq 0}$, the statement follows from Corollary 5.4 and Lemma 2.10. For fractional s, on applying Lemma 2.10 we conclude immediately that the operators

$$\begin{array}{rrcl} \partial_j: & \tilde{H}^{[s],\gamma}(\mathcal{D}) & \to & \tilde{H}^{[s]-1,\gamma}(\mathcal{D}), \\ \partial_j: & \tilde{H}^{[s]+1,\gamma}(\mathcal{D}) & \to & \tilde{H}^{[s],\gamma}(\mathcal{D}) \end{array}$$

are bounded. By Lemma 3.6, the space $\tilde{H}^{s,\gamma}(\mathcal{D})$ can be obtained as the result of interpolation between $\tilde{H}^{[s+1],\gamma}(\mathcal{D})$ and $\tilde{H}^{[s],\gamma}(\mathcal{D})$. Then, familiar interpolation arguments show (see for instance [Tri78]) that the operator $\partial_j : \tilde{H}^{s,\gamma}(\mathcal{D}) \to \tilde{H}^{s-1,\gamma}(\mathcal{D})$ is bounded.

Since Lemma 2.10 implies the boundedness of the operator ∂_j acting from $H^{[s],\gamma}(\mathcal{D})$ to $H^{[s]-1,\gamma}(\mathcal{D})$, we see that ∂_j maps $H^{s,\gamma}(\mathcal{D})$ continuously to $H^{s-1,\gamma}(\mathcal{D})$, too, by the definition of the norms included. This establishes the theorem for the first order differential operators, for multiplication by a function from $C^{[s],s-[s]}(\overline{\mathcal{D}})$ induces a bounded operator on the scale $H^s(\mathcal{D})$ of Sobolev space of smoothness $s \in \mathbb{R}_{\geq 0}$.

For higher order partial differential operators one may argue by induction, completing the proof. □

Corollary 5.6. *Suppose that $\partial\mathcal{D}$ be a Lipschitz surface, and $\rho \in C^1(\overline{\mathcal{D}} \setminus Y)$ and $\rho' \in L^\infty(\mathcal{D})$. If the norms of the spaces $\mathcal{H}^{1,1}(\mathcal{D})$ and $H^1(\mathcal{D}, Y)$ are equivalent then the bounded trace operator $t_1 : \mathcal{H}^{1,\gamma}(\mathcal{D}) \to \mathcal{H}^{1/2,\gamma-1/2}(\partial\mathcal{D})$ has a bounded right inverse.*

Proof. Pick $u_0 \in \mathcal{H}^{1/2,\gamma-1/2}(\partial\mathcal{D})$. By definition, the function $v_0 = \rho^{1-\gamma} u_0$ belongs to $\mathcal{H}^{1/2,1/2}(\partial\mathcal{D}) \subset H^{1/2}(\partial\mathcal{D})$. By the trace theorem for Sobolev spaces, there is a function $v \in H^1(\mathcal{D})$, such that $t_1(v) = v_0$ on $\partial\mathcal{D}$ and

$$\|v\|_{H^1(\mathcal{D})} \leq c\,\|v_0\|_{H^{1/2}(\partial\mathcal{D})} \leq c\,\|v_0\|_{\mathcal{H}^{1/2,1/2}(\partial\mathcal{D})} \tag{5.7}$$

where the constant c and $\tilde{c}$ is independent of v_0 and can be different in diverse applications. The right-hand side of (5.7) is majorised by

$$\|\rho^{1-\gamma} u_0\|_{\mathcal{H}^{1/2,1/2}(\partial\mathcal{D})} \leq c\,\|u_0\|_{\mathcal{H}^{1/2,\gamma-1/2}(\partial\mathcal{D})}, \tag{5.8}$$

which is due to Corollary 3.3.

As u_0 is the $\mathcal{H}^{1/2,\gamma-1/2}(\partial\mathcal{D})$-limit of a sequence $u_\nu \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$, it follows that v_0 is the $H^{1/2}(\partial\mathcal{D})$-limit of the sequence $v_\nu = \rho^{1-\gamma} u_\nu \in C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$. By (5.7), the sequence $\{v_\nu\}$ converges to v in the space $H^1(\mathcal{D})$. Therefore, we get $v \in H^1(\mathcal{D}, Y)$.

Set now $u = \rho^{\gamma-1} v$. As the norms of the spaces $\mathcal{H}^{1,1}(\mathcal{D})$ and $H^1(\mathcal{D}, Y)$ are equivalent, v belongs to $\mathcal{H}^{1,1}(\mathcal{D})$ whence, by Corollary 3.3,

$$\|u\|_{\mathcal{H}^{1,\gamma}(\mathcal{D})} \leq c\, \|v\|_{\mathcal{H}^{1,1}(\mathcal{D})} \leq c\, \|v\|_{H^1(\mathcal{D})} \tag{5.9}$$

with c a constant independent of v.

Combining inequalities (5.7), (5.8) and (5.9) we arrive readily at the desired assertion. □

Corollary 5.7. *Let $\partial\mathcal{D}$ be a Lipschitz surface and (4.4), (4.5) be fulfilled. If the norms of the spaces $\mathcal{H}^{1,1}(\mathcal{D})$ and $H^1(\mathcal{D}, Y)$ are equivalent, then the trace operator $\mathcal{H}^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s-1/2,\gamma-1/2}(\partial\mathcal{D})$ has a bounded right inverse for $1/2 < s \leq 1$. If, in addition, the norms of $\mathcal{H}^{2,2}(\mathcal{D})$ and $H^2(\mathcal{D}, Y)$ are equivalent then the statement holds for $1/2 < s < 3/2$.*

Proof. This follows from Corollary 5.4 and Theorem 4.12. □

Remark 5.8. If $s \geq 3/2$, then one needs to increase the smoothness of $\partial\mathcal{D} \setminus Y$ (see Remark 4.17).

We finish this section by shortly discussing singularities of non-transversal intersections. Choosing $\rho(x)$ as a singular coordinate, we regularise the singularity at the boundary, still making the coefficients of differential operators under considerations singular with respect to the new smooth structure. This motivates hard analysis of singularities.

Example 5.9. Consider the planar domain $\mathcal{D} \subset \mathbb{R}^2$ with a cuspidal point at the origin given by

$$\mathcal{D} = \{(x^1, x^2) :\, -(x^2)^2 < x^1 < (x^2)^2,\, 0 < x^2 < 1\}.$$

Thus, $Y = \{0\}$ and we take $\rho(x) = (x^2)^2$, so that $|\rho'(x)| = 2x^2$ and ρ fails to satisfy regularity condition (5.1). The change of variables

$$\begin{array}{rcl} y^1 & = & x^1, \\ y^2 & = & (x^2)^2 \end{array}$$

transforms $\mathcal{D}$ into the cone $\tilde{\mathcal{D}} = \{(y^1, y^2) :\, |y^1| < y^2,\, 0 < y^2 < 1\}$ with Lipschitz boundary. The new function $\tilde{\rho}(y) = y^2$ satisfies $|\tilde{\rho}'(y)| = 1$, i.e. regularity condition (5.1) is fulfilled. From

$$\det J(y) = \det \frac{\partial x}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{\rho(y)}}$$

we see immediately that the space $\mathcal{H}^{0,\gamma}(\mathcal{D}) = \tilde{\mathcal{H}}^{0,\gamma}(\mathcal{D})$ is pulled back under the change $x = x(y)$ to the space $\mathcal{H}^{0,\gamma+1/4}(\tilde{\mathcal{D}}) = \tilde{\mathcal{H}}^{0,\gamma+1/4}(\tilde{\mathcal{D}})$. On the other hand, we have

$$\begin{array}{rcl} \partial_{x^1} & = & \partial_{y^1}, \\ \partial_{x^2} & = & 2\sqrt{y^2}\, \partial_{y^2} \end{array}$$

under the change of variables. It is easily verified that the degenerate elliptic operator

$$\partial_{x^1}^2 + \frac{1}{4} \frac{1}{\rho(x)}\, \partial_{x^2}^2$$

in $\mathcal{D}$ transforms to the degenerate elliptic operator

$$\partial_{y^1}^2 + \partial_{y^2}^2 + \frac{1}{2} \frac{1}{\tilde{\rho}(y)}\, \partial_{y^2} = \frac{1}{(y^2)^2} \Big((y^2 \partial_{y^1})^2 + (y^2 \partial_{y^2})^2 - \frac{1}{2} (y^2 \partial_{y^2}) \Big)$$

in $\tilde{\mathcal{D}}$. This latter is specified within the framework of Fuchs-type operators in $\tilde{\mathcal{D}}$ developed in Section 8 below.

Part 3. **Meromorphic families of compact operators**

We begin with the discussion of main tools in the study of spectral properties of compact operators.

6. Weak perturbations of compact selfadjoint operators

Let H be a separable (complex) Hilbert space and $A : H \to H$ a linear operator. As usual, $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A if there is a non-zero element $u \in H$, such that $(A - \lambda I)u = 0$, where I is the identity operator in H. The element u is called an eigenvector of A corresponding to the eigenvalue λ. When supplemented with the zero element, all eigenvectors corresponding to an eigenvalue λ form a vector subspace $E(\lambda)$ in H. It is called an eigenspace of A corresponding to λ, and the dimension of $E(\lambda)$ is called the (geometric) multiplicity of λ. The famous spectral theorem of Hilbert and Schmidt asserts that the system of eigenvectors of a compact selfadjoint operator in H is complete.

Theorem 6.1. *Let $A : H \to H$ be compact and selfadjoint. Then all eigenvalues of A are real, each non-zero eigenvalue has finite multiplicity, and the system of all eigenvalues counted with their multiplicities is countable and has the only accumulation point $\lambda = 0$. Moreover, there is an orthonormal basis in H consisting of eigenvectors of A.*

As already mentioned, a non-selfadjoint compact operator might have no eigenvalues. However, each non-zero eigenvalue (if exists) is of finite multiplicity, see for instance [DS63]. Similarly to the Jordan normal form of a linear operator on a finite-dimensional vector space one uses the more general concept of root functions of operators.

More precisely, an element $u \in H$ is called a root vector of A corresponding to an eigenvalue $\lambda \in \mathbb{C}$ if $(A - \lambda I)^m u = 0$ for some natural number m. The set of all root vectors corresponding to an eigenvalue λ form a vector subspace in H whose dimension is called the (algebraic) multiplicity of λ.

If the linear span of the set of all root elements is dense in H one says that the root elements of A are complete in H. Aside from selfadjoint operators, the question arises under what conditions on a compact operator A the system of its root elements is complete.

If the dimension of H is finite then the completeness is equivalent to the possibility of reducing the matrix A to the Jordan normal form. Of course, this is always the case for linear operators in complex vector spaces, see, for instance, [VdW67, § 88].

In order to formulate the simplest completeness result for Hilbert spaces we need the definition of the order of a compact operator A. Since $A : H \to H$ is compact, the operator A^*A is compact, selfadjoint and non-negative. Hence it follows that A^*A possesses a unique non-negative selfadjoint compact square root $(A^*A)^{1/2}$ often denoted by $|A|$. By Theorem 6.1 the operator $|A|$ has countable system of non-negative eigenvalues $s_\nu(A)$ which are called the s-numbers of A. It is clear that if A is selfadjoint then $s_\nu = |\lambda_\nu|$, where $\{\lambda_\nu\}$ is the system of eigenvalues of A.

Definition 6.2. The operator A is said to belong to the Schatten class $\mathfrak{S}_p$, with $0 < p < \infty$, if

$$\sum_{\nu} |s_\nu(A)|^p < \infty.$$

Note that $\mathfrak{S}_2$ is the set of all Hilbert-Schmidt operators while $\mathfrak{S}_1$ is the ideal of all trace class operators.

The following lemma will be very useful in the sequel; it is taken from [DS63] (see also [GK69, Ch. 2, § 2]).

Lemma 6.3. *Let A be a compact operator of class $\mathfrak{S}_p$, with $0 < p < \infty$, in a Hilbert space H, and B be a bounded operator in H. Then the compositions BA and AB belong to $\mathfrak{S}_p$.*

After M.V. Keldysh a compact operator A is said to be of finite order if it belongs to a Schatten class $\mathfrak{S}_p$. The infimum of such numbers p is called the order of A. The following result is usually referred to as theorem on weak perturbations of compact selfadjoint operators. It was first proved in [Kel51], see also [Kel71]. Here we present its formulation from [GK69, Ch. 5, § 8].

Theorem 6.4. *Let A_0 be a compact selfadjoint operator of finite order in H. If δA is a compact operator and the operator $A_0(I + \delta A)$ is injective, then the system of root elements of $A_0(I + \delta A)$ is complete in H and, for any $\varepsilon > 0$, all eigenvalues of $A_0(I + \delta A)$ (except for a finite number) belong to the angles $|\arg \lambda| < \varepsilon$ and $|\arg \lambda - \pi| < \varepsilon$. Moreover,*

1) If A_0 has only a finite number of negative eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda - \pi| < \varepsilon$.

2) If A_0 has only a finite number of positive eigenvalues, then $A_0(I + \delta A)$ has at most a finite number of eigenvalues in the angle $|\arg \lambda| < \varepsilon$.

As is easy to see, both operators $A_0(I + \delta A)$ and A_0 are in fact injective under the hypothesis of Theorem 6.4.

However there is a more general concept than the notion of a root element of a linear operator. It is the concept of a characteristic function for a meromorphic family of linear operators.

7. Characteristic values of meromorphic families

Now let B be a Banach space and $\mathcal{L}(B)$ the algebra of all bounded linear operators acting in B.

Suppose $\lambda_0 \in \mathbb{C}$ and $F(\lambda)$ is a holomorphic function in a punctured neighborhood of λ_0 which takes on its values in $\mathcal{L}(B)$.

The point λ_0 is called a characteristic point of $F(\lambda)$ if there exists a holomorphic function $u(\lambda)$ in a neighborhood of λ_0 with values in B, such that $u(\lambda_0) \neq 0$ but $F(\lambda)u(\lambda)$ extends to a holomorphic function near λ_0 and vanishes at this point. We call $u(\lambda)$ a root function of $F(\lambda)$ at λ_0.

Assume that λ_0 is a characteristic point of $F(\lambda)$ and $u(\lambda)$ a root function at λ_0. The order of λ_0 as a zero of $F(\lambda)u(\lambda)$ is called the multiplicity of $u(\lambda)$, and the vector $u_0 = u(\lambda_0)$ an eigenvector of $F(\lambda)$ at λ_0. If supplemented by the zero vector, the eigenvectors of $F(\lambda)$ at λ_0 form a vector space. The closure of the set of all eigenvectors of $F(\lambda)$ at λ_0 is called the kernel of $F(\lambda)$ at λ_0, and it is denoted by $\ker F(\lambda_0)$.

By the rank of an eigenvector $u_0 \in B$ is meant the maximum of the multiplicities of all root functions $u(\lambda)$ such that $u(\lambda_0) = u_0$, if the set of multiplicities of these functions is bounded. If this set is unbounded, the rank of u_0 is taken to be infinity.

Suppose that $\ker F(\lambda_0)$ is of finite dimension I and that the ranks of all eigenvectors $u_0 \in \ker F(\lambda_0)$ are finite. By a canonical system of eigenvectors of $F(\lambda)$ at λ_0 we mean any system of eigenvectors $u_{0,1}, \ldots, u_{0,I}$ with the property that the rank of $u_{0,1}$ is maximal among the ranks of all eigenvectors of $F(\lambda)$ at λ_0 and the rank of $u_{0,i}$ is maximal among the ranks of all eigenvectors of $F(\lambda)$ at λ_0 in any direct complement in $\ker F(\lambda_0)$ of the linear span of the vectors $u_{0,1}, \ldots, u_{0,i-1}$, for $i = 2, \ldots, I$.

Let r_i be the rank of $u_{0,i}$. It is easy to see that the rank of any eigenvector u_0 corresponding to the characteristic point λ_0 is equal to one of the r_i. Consequently, the numbers $r_1, \ldots, r_I$ are uniquely determined by the function $F(\lambda)$. Note that a canonical system of eigenvectors is not uniquely determined in general. The numbers r_i are said to be partial null multiplicities of the characteristic point λ_0 of $F(\lambda)$. Following [GS71], we call $n(F(\lambda_0)) = r_1 + \ldots + r_I$ the null multiplicity of the characteristic point λ_0 of $F(\lambda)$. If $F(\lambda)$ has no root functions at λ_0, we set $n(F(\lambda_0)) = 0$.

We now apply these arguments again, with $F(\lambda)$ replaced by the inverse family $F^{-1}(\lambda)$. Suppose that $\lambda_0 \in \mathbb{C}$ is a characteristic point of $F^{-1}(\lambda)$ and the kernel of $F^{-1}(\lambda)$ at λ_0 is of finite dimension J. If $\varrho_1, \ldots, \varrho_J$ are the partial null multiplicities of this characteristic point of $F^{-1}(\lambda)$, then we call $\varrho_1, \ldots, \varrho_J$ the partial polar multiplicities of the characteristic point λ_0 of $F(\lambda)$. Moreover, we call the number $n(F^{-1}(\lambda_0)) = \varrho_1 + \ldots + \varrho_J$ the polar multiplicity of the characteristic point λ_0 of $F(\lambda)$ and denote it by $p(F(\lambda_0))$. If $F^{-1}(\lambda)$ has no root functions at λ_0, we set $p(F(\lambda_0)) = 0$.

The quantity $m(F(\lambda_0)) = n(F(\lambda_0)) - p(F(\lambda_0))$ is called the multiplicity of the characteristic point λ_0 of $F(\lambda)$.

If $F(\lambda)$ is holomorphic at the point λ_0 and the operator $F(\lambda_0)$ is invertible, then λ_0 is called a regular point of $F(\lambda)$. Note that the multiplicity of any regular point of $F(\lambda)$ is equal to zero.

In the scalar case it is evident that the multiplicity of a characteristic point λ_0 of a function $F(\lambda)$ is equal to the multiplicity of the zero if λ_0 is a zero of $F(\lambda)$, and is equal to the order of the pole if λ_0 is a pole.

Assume that λ_0 is a pole of the operator-valued function $F(\lambda)$. In some neighborhood of λ_0 we get an expansion

$$F(\lambda) = \sum_{j=-m}^{\infty} F_j (\lambda - \lambda_0)^j, \tag{7.1}$$

where $F_j \in \mathcal{L}(B)$.

If the operators $F_{-1}, \ldots, F_{-m}$ in (7.1) are of finite rank, then $F(\lambda)$ is called finitely meromorphic at λ_0.

The operator-valued function $F(\lambda)$ is said to be of Fredholm type at the point λ_0 if the operator F_0 in the expansion (7.1) is Fredholm. This is equivalent to saying that the value of F at λ_0 is a Fredholm operator.

A point λ_0 is called a normal point of $F(\lambda)$ if $F(\lambda)$ is finitely meromorphic and of Fredholm type at λ_0 and if all points of some punctured neighborhood of λ_0 are regular for $F(\lambda)$.

By [GS71], each normal point λ_0 of $F(\lambda)$ is a normal point of $F^{-1}(\lambda)$. If, in addition, λ_0 is a pole of either $F(\lambda)$ or $F^{-1}(\lambda)$, then it is a characteristic point of finite multiplicity of the other.

Expanding $F(\lambda)$ and $u(\lambda)$ as Laurent series (7.1) and

$$u(\lambda) = \sum_{k=0}^{\infty} u_k \, (\lambda - \lambda_0)^k,$$

respectively, we get

$$F(\lambda)u(\lambda) = \sum_{n=-m}^{r-1} \Big(\sum_{j+k=n} F_j u_k \Big) (\lambda - \lambda_0)^n + O\Big(|\lambda - \lambda_0|^r \Big)$$

for λ close to λ_0. It follows that for $u(z)$ to be a root function of $F(\lambda)$ at λ_0 of multiplicity $r \geq 1$ it is necessary and sufficient that

$$\sum_{k=0}^{n+m} F_{n-k} u_k = 0$$

for all $n = -m, \dots, r-1$.

The derivatives

$$u_k = \frac{1}{k!} \, u^{(k)}(\lambda_0),$$

$k = 1, \dots, r-1$, are said to be associated vectors for the eigenvector $u_0 = u(\lambda_0)$ of $F(\lambda)$ at λ_0. Any subsystem $u_0, u_1, \dots, u_s$ with $s \leq r-1$ is called a Jordan chain of length $s+1$ of $F(\lambda)$ at $\lambda = \lambda_0$.

Suppose $u_{0,1}, \dots, u_{0,I}$ is a canonical system of eigenvectors of $F(\lambda)$ at λ_0, I being the dimension of $\ker F(\lambda_0)$. Denote by r_i the rank of $u_{0,i}$. If, for every $i = 1, \dots, I$, the vectors $u_{0,i}, \dots, u_{r_i-1,i}$ form a Jordan chain consisting of an eigenvectors and associated vectors of $F(\lambda)$ at λ_0, then the system

$$\Big(u_{0,i}, u_{1,i}, \dots, u_{r_i-1,i} \Big)_{i=1,\dots,I}$$

is called a canonical system of Jordan chains corresponding to the characteristic point λ_0 of $F(\lambda)$.

Let $F(\lambda)$ be a holomorphic function in a punctured neighborhood of λ_0 with values in $\mathcal{L}(B)$. Then we define the transposed family $F'(\lambda)$ with values in $\mathcal{L}(B')$, where B' is the dual of B, by the equality $\langle F'g, u \rangle = \langle g, Fu \rangle$ for all $g \in B'$ and $u \in B$.

The following result is proved by Gokhberg and Sigal [GS71] for meromorphic operator-valued functions as a consequence of their normal factorisation theorem. They refer to Keldysh [Kel51] for the case of polynomials with values in operators on a Hilbert space.

Theorem 7.1. *Let λ_0 be a characteristic point of the operator-valued function $F(\lambda)$, which is a normal point of $F(\lambda)$. Then there are biorthonormal canonical systems*

$$\begin{aligned} &(u_{0,i}, u_{1,i}, \dots, u_{r_i-1,i})_{i=1,\dots,I} \, , \\ &(g_{0,i}, g_{1,i}, \dots, g_{r_i-1,i})_{i=1,\dots,I} \end{aligned}$$

of eigenvectors and associated vectors of $F(\lambda)$ and $F'(\lambda)$ at λ_0, respectively, such that

$$\text{p.p.}\, F^{-1}(\lambda) = \sum_{i=1}^{I} \sum_{j=-r_i}^{-1} (\lambda - \lambda_0)^j \sum_{k=0}^{r_i+j} \langle g_{k,i}, \cdot \rangle\, u_{r_i+j-k,i}.$$

Here, the abbreviation p.p. indicates the principal part of the Laurent expansion around λ_0.

Part 4. Spectral properties of Sturm-Liouville problems

By a Sturm-Liouville problem in $\mathbb{R}^n$ we mean any boundary value problem for solutions of second order elliptic partial differential equation with Robin-type boundary condition. The coefficients of the Robin boundary condition are allowed to have discontinuities of the first kind on the boundary of a connected open subset of $\partial\mathcal{D}$. The boundary of this domain is assumed to be a subset of Y. Thus, mixed boundary conditions are included as well. We are interested in studying the spectrum of such problems in weighted Sobolev spaces. For this purpose, we fix a function $\rho \in C^1(\overline{\mathcal{D}} \setminus Y) \cap C(\overline{\mathcal{D}})$, such that $\rho' \in L^\infty(\mathcal{D})$. As before, we define $\rho \equiv 1$ if the set Y is empty.

8. The Sturm-Liouville problem

Using Lemma 2.10 and Theorem 5.5, we consider a second order partial differential operator A of divergence form

$$A(x,\partial)u = -\sum_{i,j=1}^{n} \partial_i (a_{i,j}(x)\partial_j u) + \sum_{j=1}^{n} a_j(x)\partial_j u + a_0(x)u$$

in the domain $\mathcal{D}$. The coefficients $a_{i,j}$ are assumed to be complex-valued functions of class $L^\infty(\mathcal{D})$, and $\rho\, a_j \in L^\infty(\mathcal{D})$, $\rho^2 a_0 \in L^\infty(\mathcal{D})$.

Let $v(x) = (v_1(x), \ldots, v_n(x))$ be a vector field in $\mathbb{R}^n$ defined at the surface $\partial\mathcal{D}$. The coordinates $v_1(x), \ldots, v_n(x)$ are assumed to be bounded measurable functions on $\partial\mathcal{D}$.

Denote by ∂_v the oblique derivative

$$\partial_v = \sum_{j=1}^{n} v_j(x)\partial_j$$

and introduce a first order boundary operator $B = \partial_v + B_0$. We allow the vector $v(x)$ to vanish on a closed subset S of $\partial\mathcal{D}$. Our focus will be upon the case where S is the closure of an open connected subset of $\partial\mathcal{D}$ with piecewise smooth boundary and $Y = \partial S$. Similar considerations apply to the case where the boundary of S is a part of Y.

Concerning the summand B_0 we assume that it is a densely defined linear operator in $H^{0,\gamma}(\partial\mathcal{D})$ whose domain contains $C^\infty_{\mathrm{comp}}(\partial\mathcal{D} \setminus Y)$. Moreover, we require that

$$\ker B_0 \subset H^{0,\gamma}(\partial\mathcal{D}, S). \tag{8.1}$$

In the simplest case, the operator B_0 is given by multiplication $B_0 u := b_0 u$ with a function $b_0 \in L^\infty_{\mathrm{loc}}(\partial\mathcal{D} \setminus Y)$ which does not vanish on S.

Consider the following boundary value problem with Robin-type condition on the surface $\partial\mathcal{D}$. Given data f in $\mathcal{D}$ and u_0 on $\partial\mathcal{D}$, find a distribution u in $\mathcal{D}$ which satisfies

$$\left\{ \begin{array}{rclcl} A(x,\partial)u & = & f & \text{in} & \mathcal{D}, \\ B(x,\partial)u & = & u_0 & \text{at} & \partial\mathcal{D}. \end{array} \right. \tag{8.2}$$

In order to get substantial results, we put specific restrictions on the operators A and B.

Suppose that the matrix

$$(a_{i,j}(x))_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

is Hermitian and satisfies

$$\sum_{i,j=1}^{n} a_{i,j}(x)\overline{w}_i w_j \geq 0 \tag{8.3}$$

for all $(x,w) \in \overline{\mathcal{D}} \times \mathbb{C}^n$, and

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq m\,|\xi|^2 \tag{8.4}$$

for all $(x,\xi) \in \overline{\mathcal{D}} \times (\mathbb{R}^n \setminus \{0\})$, where m is a positive constant independent of x and ξ. Estimate (8.4) is nothing but the statement that the operator A is strongly elliptic. It should be noted that, since the coefficients of the operator and the functions under consideration are complex-valued, inequalities (8.3) and (8.4) are weaker than the (strong) coercivity of the Hermitian form, i.e. the existence of a constant m such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\,\overline{w}_i w_j \geq m\,|w|^2 \tag{8.5}$$

for all $(x,w) \in \overline{\mathcal{D}} \times (\mathbb{C}^n \setminus \{0\})$.

To specify the choice of the boundary operator we assume for a moment that $a_{i,j}$ are continuous up to the boundary of $\mathcal{D}$. Consider the complex vector field c at $\partial\mathcal{D}$, whose components are

$$c_j(x) = \sum_{i=1}^{n} a_{i,j}(x)\nu_i(x),$$

where $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the unit outward normal vector of $\partial\mathcal{D}$ at $x \in \partial\mathcal{D}$. From condition (8.4) it follows that there is a complex-valued function $b_1(x)$ on the boundary with the property that the difference $v(x) - b_1(x)c(x)$ is orthogonal to $\nu(x)$ for almost all $x \in \partial\mathcal{D}$. In fact, the pointwise equality $(v - b_1 c, \nu)_x = 0$ just amounts to

$$b_1(x) = \frac{(v(x),\nu(x))_x}{(c(x),\nu(x))_x}$$

for $x \in \partial\mathcal{D}$. Obviously, $b_1(x)$ is a bounded measurable function on $\partial\mathcal{D}$ and the vector field $t(x) = v(x) - b_1(x)c(x)$ takes on its values in the complexified tangent hyperplane $T_x(\partial\mathcal{D})$ of $\partial\mathcal{D}$ at x. Summarizing we conclude that if $\partial\mathcal{D}$ is a Lipschitz surface then

$$B(x,\partial) = b_1(x)\partial_c + \partial_t + B_0,$$

where $t(x)$ is a tangential vector field on $\partial\mathcal{D}$ whose components belong to $L^\infty(\partial\mathcal{D})$. By assumption, both b_1 and t vanish on S. Concerning the behavior of b_1 in $\partial\mathcal{D}\setminus S$

we require that $b_1(x) \neq 0$ for almost all $x \in \partial\mathcal{D} \setminus S$ and $1/b_1$ is integrable away from Y on $\partial\mathcal{D} \setminus S$.

From now on we drop the continuity assumption for the coefficients $a_{i,j}(x)$ and we keep the same choice for B. Note that in this case the Shapiro-Lopatinskii condition can be violated on the smooth part of $\partial\mathcal{D} \setminus S$ unless the coefficients $a_{i,j}(x)$ are real-valued.

As we wish to study the spectral properties of problem (8.2) we will mostly be concentrated on the case where $u_0 = 0$. Then, since on S the boundary operator reduces to $B = B_0$ satisfying (8.1), the functions of $H^{0,\gamma}(\overline{\mathcal{D}})$, satisfying $Bu = 0$ at $\partial\mathcal{D}$, actually vanish on S.

Since we want to apply standard perturbation arguments, we split the coefficient a_0 into two parts

$$a_0 = a_{0,0} + \Delta a_0,$$

where $a_{0,0}$ is a non-negative function satisfying $\rho^2 a_{0,0} \in L^\infty(\mathcal{D})$. In order to split the operator B_0 we denote by χ_S the characteristic function of the set S on ∂D. Set

$$B_0 = B_{0,0} + \Delta B_0,$$

where

$$B_{0,0}u = \chi_S u + b_1 \rho^\gamma\, \Psi^* \Psi(\rho^{-\gamma} u)$$

with a bounded linear operator $\Psi : H^r(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ and $-1/2 \leq r \leq 1/2$. The range of r is motivated by trace and duality arguments, cf. Theorem 4.13. It might be more natural to think of Ψ as a bounded operator $H^{r,r}(\partial\mathcal{D}) \to H^{0,0}(\partial\mathcal{D})$, but for coercive forms and regular singularities it is the same anyway. As but one example of Ψ we show a pseudodifferential operator of order r on $\partial\mathcal{D}$. As $b_1 \equiv 0$ on S we conclude that condition (8.1) is fulfilled for $B_{0,0}$. Note that the pair $\{1, \partial_c\}$ is the so-called Dirichlet system of order 1 on $\partial\mathcal{D}$, and so the pair $\{\chi_S, \partial_c\}$ is a Dirichlet system of order one on S. Furthermore, $\{\Psi^* \Psi, \partial_c\}$ inherits the surjectivity property of Dirichlet systems on $\partial\mathcal{D} \setminus S$, at least if the operator $\Psi^* \Psi$ has an inverse in $L^2(\mathcal{D})$ (cf. Theorem 8.4 and Example 8.8).

For $r = 0$, a typical operator Ψ is given by $\Psi u = \psi u$, where ψ is a function on $\partial\mathcal{D}$ locally bounded away from Y. Then $(\Psi^* \Psi u)(x) = |\psi(x)|^2 u(x)$ is invertible provided that $|\psi(x)| \geq c > 0$. If $\partial\mathcal{D}$ is C^2-smooth then a model operator Ψ is $\Psi = (1 + \Delta_{\partial\mathcal{D}})^{r/2}$ where $\Delta_{\partial\mathcal{D}}$ is the Laplace-Beltrami operator on the boundary (see Example 8.8).

If the functional

$$\|u\|_{+,\gamma} = \Big(\sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + \|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})} + \|\Psi(\rho^{-\gamma}u)\|^2_{L^2(\partial\mathcal{D})} \Big)^{1/2}$$

defines a norm on $H^{1,\gamma}(\mathcal{D}, S)$, we denote by $H^{+,\gamma}(\mathcal{D})$ the completion of $H^{1,\gamma}(\mathcal{D}, S)$ with respect to this norm. Obviously, $H^{+,\gamma}(\mathcal{D})$ is a Hilbert space with scalar product

$$(u, v)_{+,\gamma} = \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i v)_{H^{0,\gamma}(\mathcal{D})} + (a_{0,0}u, v)_{H^{0,\gamma}(\mathcal{D})} + (\Psi(\rho^{-\gamma}u), \Psi(\rho^{-\gamma}v))_{L^2(\partial\mathcal{D})}.$$

From now on we assume that the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into the space $H^{0,\gamma}(\mathcal{D})$, i.e.,

$$\|u\|_{H^{0,\gamma}(\mathcal{D})} \leq c\, \|u\|_{+,\gamma} \tag{8.6}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, where c is a constant independent of u. This condition is not particularly restrictive.

Lemma 8.1. *Let there be $\delta \geq 0$ and $c(\delta) > 0$, such that*

$$a_{0,0} \geq c(\delta)\, \rho^{-2\delta} \tag{8.7}$$

in $\mathcal{D}$. Then the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{0,\gamma+\delta}(\mathcal{D})$. In particular, (8.6) holds with $c = (c(\delta))^{-1}$.

Proof. From (8.7) it follows that the norm $\| \sqrt{a_{0,0}} \cdot \|_{H^{0,\gamma}(\mathcal{D})}$ is not weaker than the norm $\| \cdot \|_{H^{0,\gamma+\delta}(\mathcal{D})}$ on $H^{1,\gamma}(\mathcal{D}, S)$. This establishes the continuous embedding

$$H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma+\delta}(\mathcal{D}),$$

for the norm $\| \cdot \|_{+,\gamma}$ is not weaker than $\| \sqrt{a_{0,0}} \cdot \|_{H^{0,\gamma}(\mathcal{D})}$. Now, (8.6) follows from Lemma 2.5. $\square$

Write ι for the inclusion

$$H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D}), \tag{8.8}$$

which is continuous by (8.6).

The sesquilinear form $(\cdot, \cdot)_{+,\gamma}$ is said to be coercive if there is a constant $c > 0$, such that

$$\|u\|_{H^{1,\gamma}(\mathcal{D})} \leq c\, \|u\|_{+,\gamma} \tag{8.9}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, that is the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1,\gamma}(\mathcal{D}, S)$.

Our next concern will be adequate embedding theorems for the space $H^{+,\gamma}(\mathcal{D})$. To this end, denote by $H^{-,\gamma}(\mathcal{D})$ be the completion of $H^{1,\gamma}(\mathcal{D}, S)$ with respect to the norm

$$\|u\|_{-,\gamma} = \sup_{\substack{v \in H^{+,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{H^{0,\gamma}(\mathcal{D})}|}{\|v\|_{+,\gamma}}.$$

As explained in Lemma 1.3, the space $H^{-,\gamma}(\mathcal{D})$ can be specified as the dual of $H^{+,\gamma}(\mathcal{D})$ with respect to the pairing

$$(v, u)_\gamma = \lim_{\nu \to \infty} (v, u_\nu)_{H^{0,\gamma}(\mathcal{D})}.$$

According to Lemma 1.2, the space $H^{0,\gamma}(\mathcal{D})$ is continuously embedded into $H^{-,\gamma}(\mathcal{D})$; we write ι' for this embedding.

Since the norm $\| \cdot \|_{H^{0,\gamma}(\mathcal{D})}$ majorises the norm $\| \cdot \|_{-,\gamma}$, we deduce from Lemma 2.4 that $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $H^{-,\gamma}(\mathcal{D})$, too.

The following lemma is contained in [ST12, Lemma 6.1]. It corresponds to $Y = \emptyset$, i.e., $\rho \equiv 1$. In [ST12], we write $H_{SL}(\mathcal{D})$ for the space $H^{+,\gamma}(\mathcal{D}) =: H^+(\mathcal{D})$ and $H^-_{SL}(\mathcal{D})$ for $H^{-,\gamma}(\mathcal{D}) =: H^-(\mathcal{D})$.

Lemma 8.2. *Let $Y = \emptyset$. Suppose estimate (8.5) is fulfilled. Then there are continuous embeddings*

$$\begin{array}{rcl} H^+(\mathcal{D}) & \hookrightarrow & H^1(\mathcal{D}, S), \\ H^{-1}(\mathcal{D}) & \hookrightarrow & H^-(\mathcal{D}) \end{array}$$

if at least one of the following conditions holds:

1) S is not empty;

2) $\displaystyle\int_{\mathcal{D}} a_{0,0}(x) dx > 0$;

3) $\| \Psi(1) \|_{L^2(\partial\mathcal{D})} > 0$.

In particular, the form $(\cdot,\cdot)_+$ is coercive and in either of the cases inclusion (8.8) is compact.

Lemma 8.3. *Assume that $Y = \emptyset$ and $S = \partial\mathcal{D}$. Then there are continuous embeddings*

$$\begin{array}{rcl} H^+(\mathcal{D}) & \hookrightarrow & H^1(\mathcal{D},\partial\mathcal{D}), \\ (H^1(\mathcal{D},\partial\mathcal{D}))' & \hookrightarrow & H^-(\mathcal{D}). \end{array}$$

In particular, the sesquilinear form $(\cdot,\cdot)_+$ is coercive and inclusion (8.8) is compact.

Proof. The lemma follows from the Gårding inequality. The compactness of inclusion (8.8) is due to the Rellich theorem. □

The following theorem will be used to include into consideration the non-coercive forms, too.

Theorem 8.4. *Suppose $Y = \emptyset$, the coefficients $a_{i,j}$ are smooth in a neighborhood of $\overline{\mathcal{D}}$, and there is $r \in [-1/2, 1/2]$ and a constant $c > 0$, such that*

$$\| \Psi u \|_{L^2(\partial\mathcal{D})} \geq c \, \|u\|_{H^r(\partial\mathcal{D})} \tag{8.10}$$

for all $u \in H^1(\partial\mathcal{D}, S)$. If moreover $a_{0,0} \geq c_1 > 0$ in $\mathcal{D}$ or the operator A is strongly elliptic in a neighbourhood $\mathcal{X}$ of $\overline{\mathcal{D}}$ and

$$\int_{\mathcal{X}} \sum_{i,j=1}^{n} a_{i,j} \partial_j u \overline{\partial_i u} \, dx \geq m \, \|u\|^2_{L^2(\mathcal{X})} \tag{8.11}$$

for all $u \in C^\infty_{\mathrm{comp}}(\mathcal{X})$, with $m > 0$ a constant independent of u, then the space $H^+(\mathcal{D})$ is continuously embedded into $H^s(\mathcal{D})$, where s is given by

$$s = \begin{cases} 1/2 - \epsilon \ \mathit{with}\ \epsilon > 0, & \mathit{if} \quad r = 0, \\ 1/2, & \mathit{if} \quad r = 0 \ \ \mathit{and} \ \ \partial\mathcal{D} \in C^2, \\ 1/2 + r, & \mathit{if} \quad 0 < |r| \leq 1/2. \end{cases} \tag{8.12}$$

Proof. By shrinking $\mathcal{X}$, if necessary, we may assume that the coefficients $a_{i,j}$ are continuous in $\overline{\mathcal{X}}$. As the operator

$$A_0 = - \sum_{i,j=1}^{n} \partial_i (a_{i,j} \partial_j)$$

is strongly elliptic on $\mathcal{X}$, the classical Gårding inequality yields the existence of a Hodge parametrix $\mathcal{G}$ for the Dirichlet problem related to A_0 in $\mathcal{X}$ (see for instance [LU73] or [Sch60]). To formulate this more precisely, we define $\tilde{H}^{-1}(\mathcal{X})$ to be the dual space of $H^1(\mathcal{X},\partial\mathcal{X})$ with respect to the $L^2(\mathcal{X})$-pairing, as discussed above. Clearly, $H^{-1}(\mathcal{X})$ is continuously embedded into $\tilde{H}^{-1}(\mathcal{X})$. As usual, the operator A_0 is given the domain $H^1(\mathcal{X},\partial\mathcal{X})$ to map it to $\tilde{H}^{-1}(\mathcal{X})$. Then there are bounded linear operators

$$\begin{array}{rrcl} \mathcal{G}: & \tilde{H}^{-1}(\mathcal{X}) & \to & H^1(\mathcal{X},\partial\mathcal{X}), \\ \mathcal{H}: & \tilde{H}^{-1}(\mathcal{X}) & \to & \mathcal{H}(\mathcal{X}) \end{array}$$

satisfying

$$\begin{array}{rcl} \mathcal{G} A_0 & = & I - \mathcal{H}, \\ A_0 \mathcal{G} & = & I - \mathcal{H} \end{array} \tag{8.13}$$

on $H^1(\mathcal{X}, \partial\mathcal{X})$ and $\tilde{H}^{-1}(\mathcal{X})$, respectively, where $\mathcal{H}(\mathcal{X}) \subset H^1(\mathcal{X}, \partial\mathcal{X}) \cap C^\infty(\overline{\mathcal{X}})$ stands for the null space of the Dirichlet problem in $\mathcal{X}$. The dimension of $\mathcal{H}(\mathcal{X})$ is finite and $\mathcal{H}$ is actually the $L^2(\mathcal{X})$-orthogonal projection onto $\mathcal{H}(\mathcal{X})$. Moreover, $\mathcal{H}$ maps $H^s(\mathcal{X})$ continuously to $C^\infty(\overline{\mathcal{X}})$ for all $s \geq -1$.

On applying the trace theorem for Sobolev spaces we introduce the so-called Poisson operator $\mathcal{P} : H^{1/2}(\partial\mathcal{X}) \to H^1(\mathcal{X})$ to satisfy

$$\mathcal{P} \circ t_1 + \mathcal{G} A_0 = I - \mathcal{H},$$

where t_1 stands for the trace operator $H^1(\mathcal{X}) \to H^{1/2}(\partial\mathcal{X})$. In particular, the range of $\mathcal{P}$ is $L^2(\mathcal{X})$-orthogonal to $\mathcal{H}(\mathcal{X})$. If the boundary of $\mathcal{X}$ is a Lipschitz surface then the Green and Poisson operators bear adequate regularity properties. More precisely,

$$\begin{array}{rrcl} \mathcal{G}: & \tilde{H}^{s-1}(\mathcal{X}) & \to & H^{s+1}(\mathcal{X}), \\ \partial_j \mathcal{G}: & H^{s-1}(\mathcal{X}) & \to & H^{s}(\mathcal{X}), \\ \mathcal{P}: & H^{s+1/2}(\partial\mathcal{X}) & \to & H^{s+1}(\mathcal{X}) \end{array} \tag{8.14}$$

for all $0 \leq s < 1/2$ (see for instance [Agr11a, § 12]). If $\partial\mathcal{X}$ is C^2-smooth, then the mappings

$$\begin{array}{rrcl} \mathcal{G}: & L^2(\mathcal{X}) & \to & H^2(\mathcal{X}), \\ \mathcal{P}: & H^{3/2}(\partial\mathcal{X}) & \to & H^2(\mathcal{X}) \end{array}$$

are continuous, too. We need subtler properties of the operators $\mathcal{G}$ and $\mathcal{P}$.

Lemma 8.5. *If $0 < r \leq 1/2$ then the operator $\partial_c \mathcal{G}$ extends to map $H^{-r-1/2}(\mathcal{X})$ continuously to $H^{-r}(\partial\mathcal{X})$.*

It should be noted that invoking the space $H^{-r-1/2}(\mathcal{X})$ instead of $\tilde{H}^{-r-1/2}(\mathcal{X})$ is of crucial importance here.

Proof. If $f \in H^{-r-1/2}(\mathcal{X})$, then $\mathcal{G}f$ is known to be a strong solution of the Dirichlet problem, i.e. there is a sequence $\{u_\nu\}$ of functions in $H^2(\mathcal{X})$ vanishing at the boundary and satisfying

$$\begin{array}{rcl} \|u_\nu - \mathcal{G}f\|_{H^{-r+3/2}(\mathcal{X})} & \to & 0, \\ \|A_0 u_\nu - f\|_{H^{-r-1/2}(\mathcal{X})} & \to & 0 \end{array} \tag{8.15}$$

as $\nu \to \infty$, see [Sch60], [ST03]. In particular, using (8.14) we deduce that, for each $1 \leq j \leq n$, the operator $\partial_j \mathcal{G}$ maps the space $H^{-r-1/2}(\mathcal{X})$ continuously to $H^{-r+1/2}(\mathcal{X})$ and

$$\|\partial_j u_\nu - \partial_j \mathcal{G} f\|_{H^{-r+1/2}(\mathcal{X})} \to 0$$

when $\nu \to \infty$.

Let $u \in H^2(\mathcal{X})$ vanish at $\partial\mathcal{X}$. From the existence of a continuous right inverse $t_{r+1/2}^{-1}$ for the trace operator

$$t_{r+1/2} : H^{r+1/2}(\mathcal{X}) \to H^r(\partial\mathcal{X})$$

it follows that

$$\begin{aligned} \|\partial_c u\|_{H^{-r}(\partial\mathcal{X})} &= \sup_{\substack{v\in H^r(\partial\mathcal{X})\\ v\neq 0}} \frac{|(v,\partial_c u)_{L^2(\partial\mathcal{X})}|}{\|v\|_{H^r(\partial\mathcal{X})}} \\ &= \sup_{\substack{v\in H^1(\partial\mathcal{X})\\ v\neq 0}} \frac{|(t_{r+1/2}t^{-1}_{r+1/2}v,\partial_c u)_{L^2(\partial\mathcal{X})}|}{\|v\|_{H^r(\partial\mathcal{X})}} \\ &= \sup_{\substack{v\in H^1(\partial\mathcal{X})\\ v\neq 0}} \frac{\lim\limits_{\nu\to\infty}|(t_{r+1/2}g_\nu,\partial_c u)_{L^2(\partial\mathcal{X})}|}{\|v\|_{H^r(\partial\mathcal{X})}} \end{aligned}$$

where $\{g_\nu\}$ is a sequence of smooth functions on $\overline{\mathcal{X}}$ approximating $g := t^{-1}_{r+1/2}v$ in $H^{r+1/2}(\mathcal{X})$. Furthermore,

$$(t_{r+1/2}g_\nu,\partial_c u)_{L^2(\partial\mathcal{X})} = (\sum_{i,j=1}^{n} a_{i,j}\partial_j g_\nu,\partial_i u)_{L^2(\mathcal{X})} + (g_\nu, A_0 u)_{L^2(\mathcal{X})},$$

which is due to Stokes' formula and (8.13). Since $r+1/2 > 1/2$, for $r>0$, it follows from Lemma 1.5 that there is a constant c, such that

$$\|\partial_j g_\nu\|_{H^{r-1/2}(\mathcal{X})} \leq c\,\|g_\nu\|_{H^{r+1/2}(\mathcal{X})}$$

for all ν, and

$$\lim_{\nu\to\infty}\partial_j g_\nu = \partial_j g$$

in $H^{r-1/2}(\mathcal{X})$. As $a_{i,j}\in L^\infty(\mathcal{X})$, formula (8.14) implies that

$$\begin{aligned} |\sum_{i,j=1}^{n} a_{i,j}\partial_j g_\nu,\partial_i u)_{L^2(\mathcal{X})}| &\leq c\sum_{i,j=1}^{n}\|\partial_j g_\nu\|_{H^{r-1/2}(\mathcal{X})}\|\partial_i u\|_{H^{-r+1/2}(\mathcal{X})} \\ &\leq c\sum_{j=1}^{n}\|\partial_j g_\nu\|_{H^{r-1/2}(\mathcal{X})}\,\|u\|_{H^{-r+3/2}(\mathcal{X})} \end{aligned}$$

with c a constant independent of u and ν and different in diverse applications. On the other hand,

$$|(g_\nu, A_0 u)_{L^2(\mathcal{X})}| \leq \|g_\nu\|_{H^{r+1/2}(\mathcal{X})}\|A_0 u\|_{H^{-r-1/2}(\mathcal{X})}$$

and therefore

$$\|\partial_c u\|_{H^{-r}(\partial\mathcal{X})} \leq c\left(\|u\|_{H^{-r+3/2}(\mathcal{X})} + \|A_0 u\|_{H^{-r-1/2}(\mathcal{X})}\right), \tag{8.16}$$

where c is a constant independent of u.

In particular, we get

$$\|\partial_c(u_\mu - u_\nu)\|_{H^{-r}(\partial\mathcal{X})} \leq c\left(\|u_\mu - u_\nu\|_{H^{-r+3/2}(\mathcal{X})} + \|A_0(u_\mu - u_\nu)\|_{H^{-r-1/2}(\mathcal{X})}\right)$$

for all ν and μ. From (8.15) it follows readily that the sequence $\{\partial_c u_\nu\}$ converges in $H^{-r}(\partial\mathcal{X})$. We write $\partial_c\mathcal{G}f$ for the limit, which is thus well defined for any function $f\in H^{-r-1/2}(\mathcal{X})$. By construction and (8.16),

$$\|\partial_c\mathcal{G}f\|_{H^{-r}(\partial\mathcal{X})} \leq c\left(\|\mathcal{G}f\|_{H^{-r+3/2}(\mathcal{X})} + \|f\|_{H^{-r-1/2}(\mathcal{X})}\right)$$

for all $f \in H^{-r-1/2}(\mathcal{X})$. Finally, (8.14) with $s = -r + 1/2$ implies the continuity of the operator $\partial_c \mathcal{G} : H^{-r-1/2}(\mathcal{X}) \to H^{-r}(\partial\mathcal{X})$ constructed above, provided that $0 < r \leq 1/2$. □

Lemma 8.6. *If $0 < r \leq 1/2$, then the operator $\mathcal{P}$ maps $H^r(\partial\mathcal{X})$ continuously to $H^{r+1/2}(\mathcal{X})$. If, moreover, $\partial\mathcal{D} \in C^2$, then the operator $\mathcal{P}$ maps $L^2(\partial\mathcal{X})$ continuously to $H^{1/2}(\mathcal{X})$.*

Proof. Indeed, fix $0 < r \leq 1/2$. On arguing as in (8.15) we obtain

$$\begin{aligned}
\|\mathcal{P}u\|_{H^{r+1/2}(\mathcal{X})} &= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{X}) \\ v \neq 0}} \frac{|(v, \mathcal{P}u)_{L^2(\mathcal{X})}|}{\|v\|_{H^{-r-1/2}(\mathcal{X})}} \\
&= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{X}) \\ v \neq 0}} \frac{|(A_0 \mathcal{G}v + \mathcal{H}v, \mathcal{P}u)_{L^2(\mathcal{X})}|}{\|v\|_{H^{-r-1/2}(\mathcal{X})}} \\
&= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{X}) \\ v \neq 0}} \frac{|(A_0 \mathcal{G}v, \mathcal{P}u)_{L^2(\mathcal{X})}|}{\|v\|_{H^{-r-1/2}(\mathcal{X})}} \\
&= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{X}) \\ v \neq 0}} \frac{\lim_{\nu \to \infty} |(\partial_c g_\nu, u)_{L^2(\partial\mathcal{X})}|}{\|v\|_{H^{-r-1/2}(\mathcal{X})}}
\end{aligned}$$

for all $u \in H^r(\partial\mathcal{X})$, where $\{g_\nu\}$ is a sequence of functions in $H^2(\mathcal{X})$ vanishing at the boundary and satisfying

$$\begin{aligned}
\|g_\nu - \mathcal{G}v\|_{H^{-r+3/2}(\mathcal{X})} &\to 0, \\
\|A_0 g_\nu - v\|_{H^{-r-1/2}(\mathcal{X})} &\to 0
\end{aligned}$$

as $\nu \to \infty$. Applying Lemma 8.5 yields

$$\lim_{\nu \to \infty} |(\partial_c g_\nu, u)_{L^2(\partial\mathcal{X})}| \leq c\, \|v\|_{H^{-r-1/2}(\mathcal{X})}\, \|u\|_{H^r(\partial\mathcal{X})}$$

whence

$$\|\mathcal{P}u\|_{H^{r+1/2}(\mathcal{X})} \leq c\, \|u\|_{H^r(\partial\mathcal{X})}$$

with c a constant independent of u. This proves the continuity of the operator $\mathcal{P} : H^r(\partial\mathcal{X}) \to H^{r+1/2}(\mathcal{X})$, for $0 < r \leq 1/2$.

Finally, if $r = 0$ and $\partial\mathcal{X} \in C^2$, then we can exploit the familiar regularity theorem for the Dirichlet problem in $\mathcal{X}$. Our task is to show that the Poisson integral $\mathcal{P}$ maps $L^2(\partial\mathcal{X})$ continuously into $H^{1/2}(\mathcal{X})$. For this purpose, given any $u \in H^{-1/2}(\partial\mathcal{X})$, we choose a sequence $\{u_\nu\}$ in $H^{1/2}(\partial\mathcal{X})$ converging to u in $H^{-1/2}(\partial\mathcal{X})$. Integrating by parts we get

$$\begin{array}{rcl}
\|\mathcal{P}u_\nu\|_{L^2(\mathcal{X})} & = & \displaystyle\sup_{\substack{v\in L^2(\mathcal{X})\\ v\neq 0}} \frac{|(v,\mathcal{P}u_\nu)_{L^2(\mathcal{X})}|}{\|v\|_{L^2(\mathcal{X})}} \\
& = & \displaystyle\sup_{\substack{v\in L^2(\mathcal{X})\\ v\neq 0}} \frac{|(A_0\mathcal{G}v+\mathcal{H}v,\mathcal{P}u_\nu)_{L^2(\mathcal{X})}|}{\|v\|_{L^2(\mathcal{X})}} \\
& = & \displaystyle\sup_{\substack{v\in L^2(\mathcal{X})\\ v\neq 0}} \frac{|(A_0\mathcal{G}v,\mathcal{P}u_\nu)_{L^2(\mathcal{X})}|}{\|v\|_{L^2(\mathcal{X})}} \\
& \leq & \displaystyle\sup_{\substack{v\in L^2(\mathcal{X})\\ v\neq 0}} \frac{|(\partial_c\mathcal{G}v,u_\nu)_{L^2(\partial\mathcal{X})}|}{\|v\|_{L^2(\mathcal{X})}} \\
& \leq & \displaystyle\sup_{\substack{v\in L^2(\mathcal{X})\\ v\neq 0}} \frac{\|\partial_c\mathcal{G}v\|_{H^{1/2}(\partial\mathcal{X})}\|u_\nu\|_{H^{-1/2}(\partial D)}}{\|v\|_{L^2(\mathcal{X})}} \\
& \leq & c\,\|u_\nu\|_{H^{-1/2}(\partial\mathcal{X})}
\end{array}$$

for all ν, where the constant c does not depend on ν. It follows that the sequence $\{\mathcal{P}u_\nu\}$ converges in $L^2(\mathcal{X})$, and so the Poisson integral $\mathcal{P}$ induces a bounded linear operator $H^{-1/2}(\partial\mathcal{X}) \to L^2(\mathcal{X})$. We now use a familiar interpolations argument (see [LM72], [Tri78]). By interpolation, the Poisson integral $\mathcal{P}$ induces bounded linear operators

$$\mathcal{P}_\theta : [H^{-1/2}(\partial\mathcal{X}), H^{1/2}(\partial\mathcal{X})]_\theta \to [L^2(\mathcal{X}), H^1(\mathcal{X})]_\theta$$

for all $0<\theta<1$, where $[H_0,H_1]_\theta$ means the interpolation space for a pair $H_0 \hookrightarrow H_1$ of Hilbert spaces. As is known,

$$\begin{array}{rcl}
[L^2(\mathcal{X}), H^1(\mathcal{X})]_\theta & = & H^\theta(\mathcal{X}), \\
{}[H^{-1/2}(\partial\mathcal{X}), H^{1/2}(\partial\mathcal{X})]_\theta & = & H^{1/2-\theta}(\partial\mathcal{X}),
\end{array}$$

see for instance Theorems 9.6 and 12.5 of [LM72, Ch. I]. Therefore, choosing $\theta = 1/2$ we deduce that $\mathcal{P}$ induces a bounded linear operator $L^2(\partial\mathcal{X}) \to H^{1/2}(\mathcal{X})$, as desired. $\square$

Having disposed of this preliminary step we can now return to the proof of Theorem 8.4. Denote by e^+ the operator of extension by zero from $\mathcal{D}$ to $\mathcal{X}$, and by r^+ the restriction from $\mathcal{X}$ to the domain $\mathcal{D}$. Obviously, e^+ is a bounded linear operator from $L^2(\mathcal{D})$ to $L^2(\mathcal{X})$ and r^+ a bounded linear operator from $H^s(\mathcal{X})$ to $H^s(\mathcal{D})$, for any $s \in \mathbb{R}$.

Clearly, $\mathcal{H} \equiv 0$ if (8.11) is fulfilled. On the other hand, if condition (8.7) holds true then $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$. Hence the norm $\|\cdot\|_{+,\gamma}$ is not weaker than the norm $\|\cdot\|_a$ on $H^1(\mathcal{D},S)$ defined by

$$\|u\|_a = \Big(\int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j}\partial_j u\overline{\partial_i u}\,dx + \|u\|^2_{H^r(\partial\mathcal{D})} + \|\mathcal{H}e^+u\|^2_{L^2(\mathcal{X})}\Big)^{1/2}. \tag{8.17}$$

As the coefficients $a_{i,j}(x)$ are continuous up to the boundary of $\mathcal{D}$, the Stokes formula yields

$$\int_{\partial\mathcal{D}} \partial_c u \overline{v}\, ds = \int_{\mathcal{D}} \sum_{i,j=1}^{n} \big(a_{i,j}\partial_j u \overline{\partial_i v} + \partial_i (a_{i,j}\partial_j u)\overline{v} \big)\, dx. \tag{8.18}$$

for all $u \in H^2(\mathcal{D})$ and $v \in H^1(\mathcal{D})$.

Denote by

$$\begin{array}{rccl} \mathcal{G}_{\mathcal{D}}: & \tilde{H}^{-1}(\mathcal{D}) & \to & H^1(\mathcal{D}, \partial\mathcal{D}), \\ \mathcal{H}_{\mathcal{D}}: & \tilde{H}^{-1}(\mathcal{D}) & \to & \mathcal{H}(\mathcal{D}) \end{array}$$

the Green operator and projection onto the null space of the Dirichlet problem for A_0 in the $\mathcal{D}$. The properties of $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}}$ are similar to those of the operators $\mathcal{G}$ and $\mathcal{H}$ considered above for the domain $\mathcal{X}$, cf. (8.13). This enables us to introduce the Poisson operator $\mathcal{P}_{\mathcal{D}}$.

Note that $e^+ h \in \mathcal{H}(\mathcal{X})$ for each $h \in \mathcal{H}(\mathcal{D})$, i.e. the image of $\mathcal{H}(\mathcal{D})$ by e^+ can be thought of as a closed subspace of $\mathcal{H}(\mathcal{X})$. Choose an $L^2(\mathcal{D})$-orthonormal basis $\{e_k\}$ in $\mathcal{H}(\mathcal{D})$. Then there is an $L^2(\mathcal{X})$-orthonormal system $\{f_l\}$ in $\mathcal{H}(\mathcal{X})$, such that $\{e^+(e_k)\} \cup \{f_l\}$ is an $L^2(\mathcal{X})$-orthonormal basis in $\mathcal{H}(\mathcal{X})$. By the very construction we get

$$\begin{array}{rcl} \mathcal{H} e^+ u & = & \displaystyle\sum_k (u, e_k)_{L^2(\mathcal{D})} e^+(e_k) + \sum_l (e^+ u, f_l)_{L^2(\mathcal{X})} f_l \\ & = & \displaystyle e^+(\mathcal{H}_{\mathcal{D}} u) + \sum_l (e^+ u, f_l)_{L^2(\mathcal{X})} f_l \end{array}$$

whence

$$\| \mathcal{H} e^+ u \|^2_{L^2(\mathcal{X})} = \| \mathcal{H}_{\mathcal{D}} u \|^2_{L^2(\mathcal{D})} + \sum_l |(e^+ u, f_l)_{L^2(\mathcal{X})}|^2 \tag{8.19}$$

for all $u \in L^2(\mathcal{D})$.

On combining formulas (8.13), (8.18) and (8.19) one deduces by an easy computation that

$$\begin{array}{rcl} \|u\|_a^2 & \geq & \displaystyle\sum_{i,j=1}^{n} (a_{i,j}\partial_j \mathcal{G}_{\mathcal{D}} A_0 u, \partial_i \mathcal{G}_{\mathcal{D}} A_0 u)_{L^2(\mathcal{D})} + \sum_{i,j=1}^{n} (a_{i,j}\partial_j \mathcal{P}_{\mathcal{D}} u, \partial_i \mathcal{P}_{\mathcal{D}} u)_{L^2(\mathcal{D})} \\ & + & \| \mathcal{P}_{\mathcal{D}} u \|^2_{H^r(\partial D)} + \| \mathcal{H}_{\mathcal{D}} u \|^2_{L^2(\mathcal{D})} \end{array} \tag{8.20}$$

whenever $u \in H^1(\mathcal{D}, S)$. As $\mathcal{H}_{\mathcal{D}}$ is the projection onto the finite-dimensional subspace $\mathcal{H}(\mathcal{D})$, we conclude that

$$\| \mathcal{H}_{\mathcal{D}} u \|_{H^1(\mathcal{D})} \leq c\, \| \mathcal{H}_{\mathcal{D}} u \|_{L^2(\mathcal{D})} \tag{8.21}$$

for all $u \in H^1(\mathcal{D}, S)$, with c a constant independent of u.

On the other hand, the Gårding inequality yields

$$\| \mathcal{G}_{\mathcal{D}} A_0 u \|^2_{H^1(\mathcal{D})} \leq c \sum_{i,j=1}^{n} (a_{i,j}\partial_j \mathcal{G}_{\mathcal{D}} A_0 u, \partial_i \mathcal{G}_{\mathcal{D}} A_0 u)_{L^2(\mathcal{D})} \tag{8.22}$$

for all $u \in H^1(\mathcal{D}, S)$.

Using (8.13), (8.20), (8.21) and (8.22) we conclude readily that any sequence $\{u_\nu\} \subset H^1(\mathcal{D}, S)$ converging to a function u in the space $H^{+,\gamma}(\mathcal{D})$ can be presented as

$$u_\nu = \mathcal{H}_\mathcal{D} u_\nu + \mathcal{G}_\mathcal{D} A_0 u_\nu + \mathcal{P}_\mathcal{D} u_\nu,$$

where the sequences $\{\mathcal{H}_\mathcal{D} u_\nu\}$ and $\{\mathcal{G}_\mathcal{D} A_0 u_\nu\}$ converge in $H^1(\mathcal{D}, \partial\mathcal{D}) \subset H^1(\mathcal{D}, S)$ to elements u_H and u_G, respectively. Hence it follows that the sequence $\{\mathcal{P}_\mathcal{D} u_\nu\}$ converges to an element u_P in $H^{+,\gamma}(\mathcal{D})$, and so

$$u = u_H + u_G + u_P = \mathcal{H}_\mathcal{D} u + \mathcal{G}_\mathcal{D} A_0 u + \mathcal{P}_\mathcal{D} u, \tag{8.23}$$

where $\mathcal{P}_\mathcal{D} u$ is the Poisson integral of the "trace" $u \restriction_{\partial D} \in H^r(\partial\mathcal{D})$ of $u \in H^{+,\gamma}(\mathcal{D})$. Thus, the embedding theorem is completely determined by the behavior of the element $u_P = \mathcal{P}_\mathcal{D} u$ at the boundary.

Suppose $-1/2 \le r < 0$. As the coefficients $a_{i,j}$ are smooth in a neighbourhood of $\overline{\mathcal{D}}$, we can assume without loss of generality that $\mathcal{X}$ is a domain with smooth boundary. In this case any solution of the Dirichlet problem with $A_0 u \in L^2(\mathcal{X})$ and zero data on $\partial\mathcal{X}$ belongs actually to $H^2(\mathcal{X})$. Therefore, by a priori estimates, $\mathcal{G}$ and $\mathcal{H}$ give rise to the bounded operators

$$\begin{array}{rccl} r^+\mathcal{G}e^+ : & L^2(\mathcal{D}) & \to & H^2(\mathcal{D}), \\ r^+\mathcal{H}e^+ : & L^2(\mathcal{D}) & \to & H^2(\mathcal{D}), \end{array}$$

the operators r^+ and e^+ being defined above.

Let $s \ge 0$. It is clear that any element $u \in H^{-s}(\mathcal{D})$ extends to an element $U \in H^{-s}(\mathcal{X})$ by

$$\langle U, v\rangle_\mathcal{X} = \langle u, v\rangle_\mathcal{D}$$

for all $v \in H^s(\mathcal{X})$. Since U vanishes in $\mathcal{X} \setminus \overline{\mathcal{D}}$, it is natural to denote it by e^+u. The linear operator $e^+ : H^{-s}(\mathcal{D}) \to H^{-s}(\mathcal{X})$ obtained in this way is bounded, provided that $s \ge 0$.

The distribution e^+u is supported in $\overline{D}$. So, using the continuity properties of pseudodifferential operators on compact closed manifolds we deduce that both $r^+\mathcal{G}e^+$ and $r^+\mathcal{H}e^+$ extend to bounded linear operators

$$\begin{array}{rccl} r^+\mathcal{G}e^+ : & H^{-r-1/2}(\mathcal{D}) & \to & H^{-r+3/2}(\mathcal{D}), \\ r^+\mathcal{H}e^+ : & H^{-r-1/2}(\mathcal{D}) & \to & H^2(\mathcal{D}), \end{array}$$

with $-1/2 \le r \le 1/2$. Hence, the operators

$$\begin{array}{rccl} \partial_j\,(r^+\mathcal{G}e^+) : & H^{-r-1/2}(\mathcal{D}) & \to & H^{-r+1/2}(\mathcal{D}), \\ \partial_c\,(r^+\mathcal{G}e^+) : & H^{-r-1/2}(\mathcal{D}) & \to & H^{-r}(\partial\mathcal{D}) \end{array} \tag{8.24}$$

are bounded, too, if $-1/2 \le r < 0$, which is due to the trace theorem for Sobolev spaces in Lipschitz domains. Notice that for $r = 0$ the arguments fail, for the elements of $H^{1/2}(\mathcal{D})$ need not have traces on $\partial\mathcal{D}$.

Formula (8.18) and continuity properties (8.24) imply that

$$\begin{array}{rcl} (v, u)_{L^2(\mathcal{D})} & = & (\mathcal{H}(e^+v) + A_0\mathcal{G}(e^+v), u)_{L^2(\mathcal{D})} \\ & = & (\mathcal{H}(e^+v), u)_{L^2(\mathcal{D})} + \displaystyle\sum_{i,j=1}^n \int_\mathcal{D} a_{i,j}\partial_j\mathcal{G}(e^+v)\overline{\partial_i u}\,dx + (\partial_c(r^+\mathcal{G}e^+)v, u)_{L^2(\partial\mathcal{D})} \end{array} \tag{8.25}$$

for all $u \in H^1(\mathcal{D}, S)$ and $v \in L^2(\mathcal{D})$.

We next claim that the norm $\| \cdot \|_a$ is not weaker than the norm $\| \cdot \|_{H^{r+1/2}(\mathcal{D})}$ on $H^1(\mathcal{D}, S)$. Indeed,

$$\begin{aligned} \|u\|_{H^{r+1/2}(\mathcal{D})} &= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{D}) \\ v \neq 0}} \frac{|(v,u)_{L^2(\mathcal{D})}|}{\|v\|_{H^{-r-1/2}(\mathcal{D})}} \\ &= \sup_{\substack{v \in H^{-r-1/2}(\mathcal{D}) \\ v \neq 0}} \frac{\lim\limits_{\nu \to \infty} |(v_\nu, u)_{L^2(D)}|}{\|v\|_{H^{-r-1/2}(D)}} \end{aligned} \tag{8.26}$$

for all $u \in H^1(\mathcal{D}, S)$, where $\{v_\nu\}$ is a sequence of smooth functions on $\overline{\mathcal{D}}$ approximating v in the space $H^{-r-1/2}(\mathcal{D})$. Using formula (8.25) for u and $v = v_\nu$, we evaluate the nominator on the right-hand side of (8.26) to be

$$\Big| (\mathcal{H}(e^+ v), u)_{L^2(\mathcal{D})} + \sum_{i,j=1}^n \int_{\mathcal{D}} a_{i,j} \partial_j \mathcal{G}(e^+ v) \overline{\partial_i u}\, dx + (\partial_c (r^+ \mathcal{G} e^+) v, u)_{L^2(\partial \mathcal{D})} \Big|.$$

As $\mathcal{H}$ is an orthogonal projection in $L^2(\mathcal{X})$, we get

$$\begin{aligned} |(\mathcal{H}(e^+ v), u)_{L^2(\mathcal{D})} &= |(\mathcal{H}(e^+ v), e^+ u)_{L^2(\mathcal{X})}| \\ &= |(\mathcal{H}(e^+ v), (\mathcal{H}(e^+ u))_{L^2(\mathcal{X})}| \\ &\leq c\, \|v\|_{H^{-r-1/2}(\mathcal{D})} \, \|(\mathcal{H}(e^+ u)\|_{L^2(\mathcal{X})} \end{aligned} \tag{8.27}$$

and

$$\begin{aligned} |(\partial_c (r^+ \mathcal{G} e^+) v, u)_{L^2(\partial \mathcal{D})}| &\leq \|\partial_c (r^+ \mathcal{G} e^+) v\|_{H^{-r}(\mathcal{D})} \|u\|_{H^r(\partial \mathcal{D})} \\ &\leq c\, \|v\|_{H^{-r-1/2}(\mathcal{D})} \, \|u\|_{H^r(\partial \mathcal{D})} \end{aligned} \tag{8.28}$$

for all $u \in H^1(\mathcal{D}, S)$ and $v \in H^{-r-1/2}(\mathcal{D})$, the last inequality being a consequence of (8.24). Here, c stands for a constant independent of u and v and different in diverse applications.

As the matrix

$$(a_{i,j}(x))_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$$

is Hermitian and non-negative, we get a generalised Cauchy inequality

$$\Big| \sum_{i,j=1}^n a_{i,j}(x) \overline{z}_i \zeta_j \Big|^2 \leq \Big(\sum_{i,j=1}^n a_{i,j}(x) \overline{z}_i z_j \Big) \Big(\sum_{i,j=1}^n a_{i,j}(x) \overline{\zeta}_i \zeta_j \Big) \tag{8.29}$$

for all $z, \zeta \in \mathbb{C}^n$. On applying (8.29) we see that

$$\Big| \sum_{i,j=1}^n \int_{\mathcal{D}} a_{i,j} \partial_j \mathcal{G}(e^+ v) \overline{\partial_i u}\, dx \Big| \leq c \Big(\sum_{i,j=1}^n \int_{\mathcal{D}} a_{i,j} \partial_j u \overline{\partial_i u}\, dx \Big)^{1/2} \|v\|_{H^{-r-1/2}(\mathcal{D})} \tag{8.30}$$

with c a constant independent of u and v.

Combining (8.26), (8.27), (8.28) and (8.30) we deduce that there are positive constants c and C, such that

$$c\, \|u\|_{H^{r+1/2}(\mathcal{D})} \leq \|u\|_a \leq C\, \|u\|_{+,\gamma}$$

for all $u \in H^1(\mathcal{D}, S)$, where $-1/2 \le r < 0$. In particular, this establishes a continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{r+1/2}(\mathcal{D})$, provided that $-1/2 \le r < 0$, as desired.

Consider now the case of Lipschitz domain and $r = 0$. Then (8.10) is fulfilled for $r = -\epsilon$ with any $\epsilon > 0$, too. Hence, from what has already been proved it follows immediately that the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1/2-\epsilon}(\mathcal{D})$ for all $\epsilon > 0$.

Finally, let $0 < r \le 1/2$ or $r = 0$ and $\partial\mathcal{D} \in C^2$. Since the sequence $\{u_\nu\}$ behind (8.23) converges in $H^{+,\gamma}(\partial\mathcal{D})$ and the norm of this space is not weaker than the auxiliary norm $\| \cdot \|_a$, the sequence converges in $H^r(\partial\mathcal{D})$, too. From Lemma 8.6 it follows that the sequence $\{\mathcal{P}_{\mathcal{D}} u_\nu\}$ converges to an element u_P in $H^{r+1/2}(\mathcal{D})$. Hence, (8.23) gives a continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{r+1/2}(\mathcal{D})$ in this case. □

Remark 8.7. Denote by $\mathcal{S}_{A_0}(\mathcal{D})$ the space of all generalised solutions to the equation $A_0 u = 0$ in the sense of distributions in $\mathcal{D}$. As the sequence $\{\mathcal{P}_{\mathcal{D}} u_\nu\}$ of (8.23) converges to an element u_P in $H^s(\mathcal{D})$, with $s \ge 0$ given by (8.12), the Stieltjes-Vitali theorem implies that u_P satisfies $A_0 u_P = 0$ in the sense of distributions in $\mathcal{D}$. Thus, it follows from (8.23) that, under the hypothesis of Theorem 8.4, there is a continuous embedding

$$H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^1(\mathcal{D}, \partial\mathcal{D}) \oplus \Big(\mathcal{S}_{A_0}(\mathcal{D}) \cap H^s(\mathcal{D}) \Big).$$

Example 8.8. If $r = 0$ and $(\varPsi u)(x) = \psi(x) u(x)$ with a function $\psi \in L^\infty_{\mathrm{loc}}(\partial\mathcal{D} \setminus Y)$ then (8.10) is fulfilled if

$$|\psi| \ge c > 0 \tag{8.31}$$

in $\partial\mathcal{D} \setminus S$. For arbitrary r in the interval $-1/2 \le r \le 1/2$, an invertible operator $\varPsi : H^r(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ satisfying (8.10) is specified within the so-called order-reducing operators. Suppose $\partial\mathcal{D}$ is a smooth hypersurface in $\mathbb{R}^n$. Denote by $\varDelta_{\partial\mathcal{D}}$ the (non-negative) Laplace-Beltrami operator on $\partial\mathcal{D}$. Then the pseudodifferential operator $\varPsi = (1 + \varDelta_{\partial\mathcal{D}})^{r/2}$ maps $H^s(\partial\mathcal{D})$ continuously to $H^{s-r}(\partial\mathcal{D})$ for all $s \in \mathbb{R}$. Moreover, the operator $\varPsi$ is continuously invertible with inverse $(1 + \varDelta_{\partial\mathcal{D}})^{-r/2}$ and hence (8.10) holds true. It is clear that the operator $\varPsi^* \varPsi$ is continuously invertible, too. In particular, the operator $\varDelta B_0 := \varPsi$ maps $H^s(\partial\mathcal{D})$ compactly to $H^{-s}(\partial\mathcal{D})$ if $r < 2s$.

Our next goal is to describe the properties of the space $H^{+,\gamma}(\mathcal{D})$ in the case where Y is a non-empty subset of $\partial\mathcal{D}$. We first observe that $H^{+,\gamma}(\mathcal{D})$ is really a weighted space.

Lemma 8.9. *Let $Y \neq \emptyset$, $\rho' \in L^\infty(\mathcal{D})$ and (8.7) hold with $\delta = 1$. Then, for any $\delta \in \mathbb{R}$, the correspondence* $\mathrm{Op}\,(\rho^\delta) : u \mapsto \rho^\delta u$ *(cf. (2.3)) induces bounded linear operators*

$$\begin{array}{rcl} H^{+,\gamma}(\mathcal{D}) & \to & H^{+,\gamma+\delta}(\mathcal{D}), \\ H^{-,\gamma}(\mathcal{D}) & \to & H^{-,\gamma+\delta}(\mathcal{D}), \end{array}$$

which are actually topological isomorphisms.

Proof. By definition, we have

$$
\begin{aligned}
&\|\rho^{\delta} u\|^2_{+,\gamma+\delta} \\
&= \sum_{i,j=1}^{n} (a_{i,j}\partial_j(\rho^{\delta}u), \partial_i(\rho^{\delta}u))_{H^{0,\gamma+\delta}(\mathcal{D})} + \|\sqrt{a_{0,0}}\rho^{\delta}u\|^2_{H^{0,\gamma+\delta}(\mathcal{D})} + \|\,\Psi(\rho^{-\gamma)}u)\|^2_{L^2(\partial\mathcal{D})}
\end{aligned}
\tag{8.32}
$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$. Obviously,

$$
\|\sqrt{a_{0,0}}\rho^{\delta}u\|_{H^{0,\gamma+\delta}(\mathcal{D})} = \|\sqrt{a_{0,0}}u\|_{H^{0,\gamma}(\mathcal{D})}. \tag{8.33}
$$

On the other hand, we get

$$
\begin{aligned}
&(a_{i,j}\partial_j(\rho^{\delta}u), \partial_i(\rho^{\delta}u))_{H^{0,\gamma+\delta}(\mathcal{D})} \\
&= \quad (a_{i,j}\rho^{\delta}\partial_j u, \rho^{\delta}\partial_i u)_{H^{0,\gamma+\delta}(\mathcal{D})} + \delta^2\,(a_{i,j}\rho^{\delta-1}(\partial_j\rho)u, \rho^{\sigma-1}(\partial_i\rho)u)_{H^{0,\gamma+\delta}(\mathcal{D})} \\
&+ \quad \delta\,(a_{i,j}\rho^{\delta-1}(\partial_j\rho)u, \rho^{\delta}\partial_i u)_{H^{0,\gamma+\delta}(\mathcal{D})} + \delta\,(a_{i,j}\rho^{\delta}(\partial_j u, \rho^{\delta-1}(\partial_i\rho)u)_{H^{0,\gamma+\delta}(\mathcal{D})}.
\end{aligned}
\tag{8.34}
$$

Clearly,

$$
\begin{array}{rcl}
(a_{i,j}\rho^{\delta}\partial_j u, \rho^{\delta}\partial_i u)_{H^{0,\gamma+\delta}(\mathcal{D})} & = & (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})}, \\
|(a_{i,j}\rho^{\delta-1}(\partial_j\rho)u, \rho^{\delta-1}(\partial_i\rho)u)_{H^{0,\gamma+\delta}(\mathcal{D})}| & \leq & c\,\|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})}
\end{array}
\tag{8.35}
$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, with c a constant independent on u because $\rho' \in L^{\infty}(\mathcal{D})$ and $a_{i,j} \in L^{\infty}(\mathcal{D})$.

Hence, by (8.29),

$$
\begin{aligned}
&\Big|\sum_{i,j=1}^{n} (a_{i,j}\rho^{\delta-1}(\partial_j\rho)u, \rho^{\delta}\partial_i u)_{H^{0,\gamma+\delta}(\mathcal{D})}\Big|^2 \\
&\quad = \quad \Big|\sum_{i,j=1}^{n} (a_{i,j}\rho^{-1}(\partial_j\rho)u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})}\Big|^2 \\
&\quad \leq \quad \Big(\sum_{i,j=1}^{n} a_{i,j}\partial_j u, \partial_i u\Big)_{H^{0,\gamma}(\mathcal{D})} \Big(\sum_{i,j=1}^{n} a_{i,j}(\partial_j\rho)u, (\partial_i\rho)u\Big)_{H^{0,\gamma+1}(\mathcal{D})} \\
&\quad \leq \quad c\Big(\sum_{i,j=1}^{n} a_{i,j}\partial_j u, \partial_i u\Big)_{H^{0,\gamma}(\mathcal{D})} \|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\Big|\sum_{i,j=1}^{n} (a_{i,j}\rho^{\delta}\partial_j u, \rho^{\delta-1}(\partial_i\rho)u)_{H^{0,\gamma+\delta}(\mathcal{D})}\Big|^2 \\
&\quad \leq \quad c\Big(\sum_{i,j=1}^{n} a_{i,j}\partial_j u, \partial_i u\Big)_{H^{0,\gamma}(\mathcal{D})} \|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})},
\end{aligned}
$$

where c is a constant independent of u and different in diverse applications. Now, using (8.34) and (8.35), we see that

$$\begin{array}{rl} & \Big| \sum_{i,j=1}^{n} (a_{i,j} \partial_j (\rho^\delta u), \partial_i (\rho^\delta u))_{H^{0,\gamma+\delta}(\mathcal{D})} \Big| \\ \leq & c \Big(\Big(\sum_{i,j=1}^{n} (a_{i,j} \partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} \Big)^{1/2} + \| \sqrt{a_{0,0}} u \|_{H^{0,\gamma}(\mathcal{D})} \Big)^2 \end{array} \tag{8.36}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, with c a constant independent of u.

It follows from (8.32), (8.33) and (8.36) that

$$\|\rho^\delta u\|_{+,\gamma+\delta} \leq c\, \|u\|_{+,\gamma}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, where c is a constant independent of u. This shows that $\mathrm{Op}\,(\rho^\delta)$ maps $H^{+,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma+\delta}(\mathcal{D})$. Then, the bounded inverse operator is given by $\mathrm{Op}\,(\rho^{-\delta})$.

Finally, the assertion on the operator $\mathrm{Op}\,(\rho^\delta) : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma+\delta}(\mathcal{D})$ follows by duality because

$$\begin{array}{rcl} \|\rho^\delta u\|_{-,\gamma+\delta} & = & \sup\limits_{\substack{v \in H^{+,\gamma+\delta}(\mathcal{D}) \\ v \neq 0}} \dfrac{|(v, \rho^\delta u)_{\mathcal{H}^{0,\gamma+\delta}(\mathcal{D})}|}{\|v\|_{+,\gamma+\delta}} \\ & = & \sup\limits_{\substack{v \in H^{+,\gamma+\delta}(\mathcal{D}) \\ v \neq 0}} \dfrac{|(\rho^{-\delta} v, u)_{\mathcal{H}^{0,\gamma}(\mathcal{D})}|}{\|\rho^{-\delta} v\|_{+,\gamma}} \, \dfrac{\|\rho^{-\delta} v\|_{+,\gamma}}{\|v\|_{+,\gamma+\delta}} \\ & \leq & c\, \|u\|_{-,\gamma}\| \end{array}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$. Once again the inverse operator is given by $\mathrm{Op}\,(\rho^{-\delta})$ which is bounded by the above. $\square$

Let us indicate some important cases where there are reasonable embedding theorems for the spaces $H^{+,\gamma}(\mathcal{D})$ and $H^{-,\gamma}(\mathcal{D})$ with $Y \neq \emptyset$.

Lemma 8.10. *Suppose $Y \neq \emptyset$. If (8.5) is fulfilled and inequality (8.7) holds with $\delta = 1$, then there are continuous embeddings*

$$\begin{array}{rcl} H^{+,\gamma}(\mathcal{D}) & \hookrightarrow & H^{1,\gamma}(\mathcal{D}, S), \\ H^{-1,\gamma}(\mathcal{D}) & \hookrightarrow & H^{-,\gamma}(\mathcal{D}). \end{array}$$

Moreover the embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$ is compact and the form $(\cdot,\cdot)_{+,\gamma}$ is coercive.

Proof. Inequality (8.5) implies

$$\|u'\|^2_{H^{0,\gamma}(\mathcal{D})} \leq c \sum_{i,j=1}^{n} (a_{i,j}(x) \partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})}, \tag{8.37}$$

where by u' is meant the gradient of u. Now it follows from Lemma 8.1 that the coercive estimate (8.9) is fulfilled. This establishes the continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1,\gamma}(\mathcal{D}, S)$. Since $H^{1,\gamma}(\mathcal{D}, S) \hookrightarrow H^{1,\gamma}(\mathcal{D})$, the second embedding follows by duality. Finally, on applying Theorem 4.5 we get the compact embedding $H^{1,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$ and hence the compact embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$, as desired. $\square$

Lemma 8.11. *Let $Y \neq \emptyset$, $S = \partial\mathcal{D}$ and inequality (8.7) hold with $\delta = 1$. Then there is continuous embedding*

$$H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1,\gamma}(\mathcal{D}, \partial\mathcal{D}).$$

Moreover, the embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$ is compact and the form $(\cdot,\cdot)_{+,\gamma}$ is coercive.

Proof. The Gårding inequality for strongly elliptic systems gives

$$\|u'\|^2_{L^2(\mathcal{D})} \leq c \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{L^2(\mathcal{D})} + \|u\|^2_{L^2(\mathcal{D})} \tag{8.38}$$

for all $u \in H^1_{\mathrm{comp}}(\mathcal{D})$. Hence,

$$\|(\rho^{-\gamma}u)'\|^2_{L^2(\mathcal{D})} \leq c \sum_{i,j=1}^{n} (a_{i,j}\partial_j(\rho^{-\gamma}u), \partial_i(\rho^{-\gamma}u))_{L^2(\mathcal{D})} + \|\rho^{-\gamma}u\|^2_{L^2(\mathcal{D})} \tag{8.39}$$

for $u \in H^1_{\mathrm{comp}}(\mathcal{D})$, with c a constant independent on u.

It is easy to see that

$$\begin{aligned}
&\sum_{i,j=1}^{n} (a_{i,j}\partial_j(\rho^{-\gamma}u), \partial_i(\rho^{-\gamma}u))_{L^2(\mathcal{D})} \\
&= \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + \gamma^2 \sum_{i,j=1}^{n} (a_{i,j}(\partial_j\rho)u, (\partial_i\rho)u)_{H^{0,\gamma+1}(\mathcal{D})} \\
&- \gamma \sum_{i,j=1}^{n} \Big((a_{i,j}(\partial_j\rho)\rho^{-1}u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + (a_{i,j}\partial_j u, (\partial_i\rho)\rho^{-1}u)_{H^{0,\gamma}(\mathcal{D})} \Big)
\end{aligned}$$

for all $u \in H^1_{\mathrm{comp}}(\mathcal{D})$. Using the generalised Cauchy inequality we conclude that there is a constant $c > 0$, such that

$$\sum_{i,j=1}^{n} (a_{i,j}\partial_j(\rho^{-\gamma}u), \partial_i(\rho^{-\gamma}u))_{L^2(\mathcal{D})} \leq c \Big(\sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + \|u\|^2_{H^{0,\gamma+1}(\mathcal{D})} \Big) \tag{8.40}$$

for all $u \in H^1_{\mathrm{comp}}(\mathcal{D})$.

Furthermore,

$$\|\rho^{-\gamma}u'\|^2_{L^2(\mathcal{D})} \leq 2 \Big(\|(\rho^{-\gamma}u)'\|^2_{L^2(\mathcal{D})} + \gamma^2 \, \|\rho^{-(\gamma+1)}\rho' u\|^2_{L^2(\mathcal{D})} \Big) \tag{8.41}$$

for all $u \in H^1_{\mathrm{comp}}(\mathcal{D})$. Combining (8.39), (8.40), (8.41) and estimate (8.7) with $\delta = 1$, we conclude that

$$\begin{aligned}
\|\rho^{-\gamma}u'\|^2_{L^2(\mathcal{D})} &\leq c \Big(\sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + \|u\|^2_{H^{0,\gamma+1}(\mathcal{D})} \Big) \\
&\leq c \Big(\sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i u)_{H^{0,\gamma}(\mathcal{D})} + \|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})} \Big) \\
&\leq c \, \|u\|^2_{+,\gamma}
\end{aligned}$$

and

$$\|\rho^{-(\gamma+1)}u\|^2_{L^2(\mathcal{D})} \leq c \, \|\sqrt{a_{0,0}}u\|^2_{H^{0,\gamma}(\mathcal{D})}$$

for all $u \in H^1_{\mathrm{comp}}(\mathcal{D})$, the constant c does not depend on u and need not be the same in diverse applications. These two inequalities establish the continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1,\gamma}(\mathcal{D},S)$. In particular, the embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$ is compact due to Theorem 4.5. □

Corollary 8.12. *Assume that (8.7) is fulfilled with $\delta = 1$ and the coefficients $a_{i,j}$ are smooth in a neighbourhood of $\overline{\mathcal{D}}$. If $|r| \le 1/2$ and (8.10) holds, then the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{s,\gamma}(\mathcal{D})$, where s is given by (8.12).*

Proof. Consider the norm $u \mapsto \|\rho^{-\gamma}u\|_a$ on $H^{1,\gamma}(\mathcal{D},S)$, see (8.17). According to Theorem 8.4, there is $c > 0$ such that

$$\|\rho^{-\gamma}u\|_{H^s(\mathcal{D})} \le c\,\|u\|_a \tag{8.42}$$

for all $u \in H^{1,\gamma}(\mathcal{D},S)$, where s is given by (8.12). On the other hand, estimate (8.10) implies

$$\|\rho^{-\gamma}u\|_{H^r(\partial\mathcal{D}\setminus S)} \le c\,\|\Psi(\rho^{-\gamma})\|_{L^2(\partial\mathcal{D})}$$

for all $u \in H^{1,\gamma}(\mathcal{D},S)$ Besides, as $a_{0,0} \ge c(1)\,\rho^{-2}$, we get

$$\|\mathcal{H}e^+(\rho^{-\gamma}u)\|_{L^2(\mathcal{X})} \le c\,\|u\|_{H^{0,\gamma}(\mathcal{D})} \le c\,(c(1))^{-1/2}\,\|\sqrt{a_{0,0}}u\|_{H^{0,\gamma}(\mathcal{D})}$$

and

$$\begin{aligned} &\int_{\mathcal{D}} \sum_{i,j=1}^n a_{i,j}\partial_j(\rho^{-\gamma}u)\overline{\partial_i(\rho^{-\gamma}u)}\,dx \\ &\quad\le\quad c\Big(\Big(\sum_{i,j=1}^n (a_{i,j}\partial_j u,\partial_i u)_{H^{0,\gamma}(\mathcal{D})}\Big)^{1/2} + \|\sqrt{a_{0,0}}u\|_{H^{0,\gamma}(\mathcal{D})}\Big)^2 \end{aligned}$$

for all $u \in H^{1,\gamma}(\mathcal{D},S)$, the last inequality being the consequence of (8.36) with $\delta = -\gamma$. From these estimates it follows that there is a constant $c > 0$ with the property that

$$\|\rho^{-\gamma}u\|_a \le c\,\|u\|_{+,\gamma},$$

for $u \in H^{1,\gamma}(\mathcal{D},S)$. Now (8.42) implies that the embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow \tilde{H}^{s,\gamma}(\mathcal{D})$ is continuous.

Finally, for $0 \le s \le 1$, we have

$$\|u\|_{H^{0,s+\gamma}(\mathcal{D})} \le \|u\|_{H^{0,1+\gamma}(\mathcal{D})} \le (c(1))^{-1/2}\,\|\sqrt{a_{0,0}}u\|_{H^{0,\gamma}(\mathcal{D})} \le (c(1))^{-1/2}\,\|u\|_{+,\gamma},$$

for $u \in H^{1,\gamma}(\mathcal{D},S)$. This yields the continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{s,\gamma}(\mathcal{D})$ for all s satisfying $0 \le s \le 1$. For $s < 0$ corresponding to $r = 0$ and $\epsilon > 1/2$, the embedding follows by duality. □

From now on we assume that (8.7) is fulfilled with $\delta = 1$, provided that $Y \neq \emptyset$. If

$$\begin{aligned} \Delta A &= \sum_{j=1}^n a_j\partial_j + a_0, \\ \Delta B &= \partial_t + \Delta B_0 \end{aligned}$$

then integrating by parts with the use of (8.18) yields readily that

$$
\begin{aligned}
& (Au, v)_{H^{0,\gamma}(\mathcal{D})} \\
& = \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i v)_{H^{0,\gamma}(\mathcal{D})} - 2\gamma \int_{\mathcal{D}} \rho^{-(2\gamma+1)} \sum_{i,j=1}^{n} a_{i,j}\partial_j u (\partial_i \rho) \overline{v}\, dx \\
& + (\Delta A u, v)_{H^{0,\gamma}(\mathcal{D})} + (\Psi(\rho^{-\gamma}u), \Psi(\rho^{-\gamma}v))_{L^2(\partial\mathcal{D})} + (b_1^{-1}\Delta B u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}
\end{aligned}
$$

for all functions $u \in H^{1,\gamma}(\mathcal{D}, S)$ and $v \in H^{1,\gamma}(\mathcal{D}, S)$ satisfying the boundary condition of (8.2).

It follows from the generalised Cauchy inequality (8.29) (and Lemma 8.1 if Y is non-empty) that

$$
\begin{aligned}
\Big| \int_{\mathcal{D}} \rho^{-(2\gamma+1)} \sum_{i,j=1}^{n} a_{i,j}\partial_j u (\partial_i \rho) \overline{v}\, dx \Big|^2 & \leq c \sum_{i,j=1}^{n} \big(a_{i,j}(x)\partial_j u, \partial_i u\big)_{H^{0,\gamma}(\mathcal{D})} \|v\|^2_{H^{0,\gamma+1}(\mathcal{D})} \\
& = c\, \|u\|^2_{+,\gamma}\, \|v\|^2_{H^{0,\gamma+1}(\mathcal{D})} \\
& \leq c\, c(1)\, \|u\|^2_{+,\gamma} \|v\|^2_{+,\gamma}
\end{aligned}
\tag{8.43}
$$

for all $u, v \in H^{1,\gamma}(\mathcal{D}, S)$, where c is a positive constant independent of u and v.

Suppose

$$
\Big| (\Delta A u, v)_{H^{0,\gamma}(\mathcal{D})} + (b_1^{-1}\Delta B u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)} \Big| \leq c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma} \tag{8.44}
$$

for all $u, v \in H^{1,\gamma}(\mathcal{D}, S)$, with c a constant independent of u and v. Then, for each fixed $u \in H^{+,\gamma}(\mathcal{D})$, the sesquilinear form

$$
\begin{aligned}
& Q(u, v) \\
& := \sum_{i,j=1}^{n} (a_{i,j}\partial_j u, \partial_i v)_{H^{0,\gamma}(\mathcal{D})} - 2\gamma \int_{\mathcal{D}} \rho^{-(2\gamma+1)} \sum_{i,j=1}^{n} a_{i,j}\partial_j u (\partial_i \rho) \overline{v}\, dx \\
& + (\Delta A u, v)_{H^{0,\gamma}(\mathcal{D})} + (\Psi(\rho^{-\gamma}u), \Psi(\rho^{-\gamma}v))_{L^2(\partial\mathcal{D})} + (b_1^{-1}\Delta B u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}
\end{aligned}
$$

determines a continuous linear functional f on $H^{+,\gamma}(\mathcal{D})$ by $f(v) := \overline{Q(u,v)}$ for $v \in H^{+,\gamma}(\mathcal{D})$.

In the following lemma by c is meant a constant which is independent on u and v and may be different in diverse applications.

Lemma 8.13. *Suppose $Y = \emptyset$ or (8.7) holds with $\delta = 1$.*

1) If $\rho^2 a_0 \in L^\infty(\mathcal{D})$ then $|(a_0 u, v)_{H^{0,\gamma}(\mathcal{D})}| \leq c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}$ for all functions $u, v \in H^{1,\gamma}(\mathcal{D})$.

2) If $\rho a_j \in L^\infty(\mathcal{D})$ then $|(a_j \partial_j u, v)_{H^{0,\gamma}(\mathcal{D})}| \leq c\, \|\partial_j u\|_{H^{0,\gamma}(\mathcal{D})} \|v\|_{+,\gamma}$ is valid for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

3) If $r = 0$ and the operators Ψ and ΔB_0 are given by multiplication with functions ψ and Δb_0, respectively, satisfying

$$
|b_1^{-1}\Delta b_0| \leq c\, |\psi|^2 \tag{8.45}
$$

on $\partial\mathcal{D} \setminus S$, then $|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \leq c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}$ for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

4) If (8.10) is fulfilled and the operator $\rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^\gamma$ maps $H^r(\partial\mathcal{D}, S)$ continuously into $H^{-r}(\partial\mathcal{D})$, then $|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \leq c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}$ for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

Proof. For $\rho \equiv 1$ the lemma follows immediately from the Cauchy inequality. If $Y \neq \emptyset$, the Cauchy inequality should be combined with Lemma 8.1. For instance, we get

$$\begin{aligned}
|(b_1^{-1} \Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| &= |(\rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^{\gamma} \rho^{-\gamma} u, \rho^{-\gamma} v)_{L^2(\partial\mathcal{D}\setminus S)}| \\
&\leq \|\rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^{\gamma} (\rho^{-\gamma} u)\|_{H^{-r}(\partial\mathcal{D})} \|\rho^{-\gamma} v\|_{H^{r}(\partial\mathcal{D})} \\
&\leq c \,\| \Psi(\rho^{-\gamma} u)\|_{L^2(\partial\mathcal{D})} \,\| \Psi(\rho^{-\gamma} v)\|_{L^2(\partial\mathcal{D})} \\
&\leq c \,\|u\|_{+,\gamma} \,\|v\|_{+,\gamma}
\end{aligned}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D}, S)$, as desired. □

Lemma 8.13 and estimate (8.43) suggest that under condition (8.7) with $\delta = 1$ estimate (8.44) focuses upon the terms

$$\sum_{j=1}^{n} (a_j \partial_j u, v)_{H^{0,\gamma}(\mathcal{D})} \quad \text{and} \quad (b_1^{-1} \partial_t u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}$$

in the form $Q(u, v)$. In the general case no substantial results are possible. However, we can say much more under reasonable conditions discussed in Sections 11 and 12) below.

Thus, if estimate (8.44) holds true, then, by Lemma 1.3, for each $u \in H^{+,\gamma}(\mathcal{D})$ there is a unique element in $H^{-,\gamma}(\mathcal{D})$, which we denote by Lu, such that

$$f(v) = (v, Lu)_{\gamma}$$

for all $v \in H^{+,\gamma}(\mathcal{D})$. We have thus introduced a linear operator L acting as $H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$. From (8.43), (8.44) it follows that L is bounded.

The bounded linear operator $L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ defined in the same way via the sesquilinear form $(\cdot, \cdot)_{+,\gamma}$, i.e.

$$(v, u)_{+,\gamma} = (v, L_0 u)_{\gamma} \tag{8.46}$$

for all $u \in H^{+,\gamma}(\mathcal{D})$ and $v \in H^{+,\gamma}(\mathcal{D})$, corresponds to the case $A = A_0$, $B = B_0$, where

$$\begin{aligned}
A_0(x, \partial) &= -\sum_{i,j=1}^{n} \partial_i (a_{i,j} \partial_j \cdot) + 2\gamma\, \rho^{-1} \sum_{i,j=1}^{n} a_{i,j} (\partial_i \rho) \partial_j + a_{0,0}, \\
B_0(x, \partial) &= b_1\, \partial_c + \chi_S + b_1\, \rho^{\gamma}\, \Psi^* \, \Psi \rho^{-\gamma}.
\end{aligned}$$

We are thus lead to a weak formulation of problem (8.2). Given $f \in H^{-,\gamma}(\mathcal{D})$, find $u \in H^{+,\gamma}(\mathcal{D})$, such that

$$\overline{Q(u, v)} = (v, f)_{\gamma} \tag{8.47}$$

for all $v \in H^{+,\gamma}(\mathcal{D})$.

Now one can handle problem (8.47) by standard techniques of functional analysis, see for instance [LM72, Ch. 2, § 9], [LU73, Ch. 3, §§ 4-6]) for the coercive case.

Lemma 8.14. *Let $A = A_0$, $B = B_0$. Then for each $f \in H^{-,\gamma}(\mathcal{D})$ there is a unique solution $u \in H^{+,\gamma}(\mathcal{D})$ to (8.47), i.e. the operator $L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is continuously invertible. Moreover, the norms of both L_0 and its inverse L_0^{-1} are equal to* 1.

Proof. Under the hypotheses of the lemma, (8.47) is just a weak formulation of problem (8.2) with A and B replaced by A_0, B_0, respectively. The corresponding bounded operator in Hilbert spaces just amounts to $L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ defined by (8.46). Its norm equals 1, for, by Lemma 1.3, we get

$$\|L_0 u\|_{-,\gamma} = \sup_{\substack{v \in H^{+,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, L_0 u)_\gamma|}{\|v\|_{+,\gamma}} = \sup_{\substack{v \in H^{+,\gamma}(\mathcal{D}) \\ v \neq 0}} \frac{|(v, u)_{+,\gamma}|}{\|v\|_{+,\gamma}} = \|u\|_{+,\gamma} \tag{8.48}$$

whenever $u \in H^{+,\gamma}(\mathcal{D})$.

The existence and uniqueness of solutions to problem (8.47) follows immediately from the Riesz theorem on the general form of continuous linear functionals on Hilbert spaces. From (8.48) we conclude that L_0 is actually an isometry of $H^{-,\gamma}(\mathcal{D})$ onto $H^{+,\gamma}(\mathcal{D})$, as desired. □

Consider the sesquilinear form on $H^{-,\gamma}(\mathcal{D})$ given by

$$(u, v)_{-,\gamma} := (L_0^{-1} u, v)_\gamma$$

for $u, v \in H^{-,\gamma}(\mathcal{D})$. Since

$$(L_0^{-1} u, v)_\gamma = (L_0^{-1} u, L_0 L_0^{-1} v)_\gamma = (L_0^{-1} u, L_0^{-1} v)_{+,\gamma} \tag{8.49}$$

for all $u, v \in H^{-,\gamma}(\mathcal{D})$, the last equality being due to (8.46), this form is Hermitian. Combining (8.48) and (8.49) yields

$$\sqrt{(u, u)_{-,\gamma}} = \|u\|_{-,\gamma}$$

for all $u \in H^{-,\gamma}(\mathcal{D})$. From now on we endow the space $H^{-,\gamma}(\mathcal{D})$ with the scalar product $(\cdot, \cdot)_{-,\gamma}$.

Lemma 8.15. *Let estimate (8.44) be fulfilled with constant $c < 1$. Then, for each $f \in H^{-,\gamma}(\mathcal{D})$, there exists a unique solution $u \in H^{+,\gamma}(\mathcal{D})$ to problem (8.47), i.e. the operator $L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is continuously invertible.*

Proof. If (8.44) holds with $c < 1$ then the operator $L : H^{+,\gamma}(\mathcal{D})(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ corresponding to problem (8.47) is easily seen to differ from L_0 by a bounded operator $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ whose norm does not exceed $c < 1$. As L_0 is invertible and the inverse operator L_0^{-1} has norm 1, a familiar argument shows that L is invertible, too. □

Lemma 8.16. *Assume that $Y = \emptyset$ or estimate (8.7) holds with $\delta = 1$. Let the map $\iota : H^{+,\gamma}(\mathcal{D}) \to H^{0,\gamma}(\mathcal{D})$ be compact. If moreover*

$$\begin{array}{rcl} \rho a_0 & \in & L^\infty(\mathcal{D}), \\ a_j & = & 2\gamma\, \rho^{-1} \displaystyle\sum_{i=1}^n a_{i,j} \partial_i \rho \end{array}$$

for $1 \le j \le n$ and $t = 0$, $\Delta B_0 = 0$, then the operator $\Delta L = L - L_0$ acting as $H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is compact.

Proof. Indeed, in this case the multiplication operator by a_0 maps the space $H^{+,\gamma}(\mathcal{D})$ continuously to $H^{0,\gamma}(\mathcal{D})$, for

$$\begin{array}{rcl} \|a_0 u\|^2_{H^{0,\gamma}(\mathcal{D})} & = & \displaystyle\int_{\mathcal{D}} \rho^{-2(\gamma+1)} |\rho a_0|^2 |u|^2\, dx \\ & \leq & c\, \|u\|^2_{H^{0,\gamma+1}(\mathcal{D})} \\ & \leq & c\, \|u\|^2_{+,\gamma}, \end{array}$$

the last inequality being a consequence of Lemma 8.1, if $Y \neq \emptyset$. According to Lemma 1.2, if the map $\iota : H^{+,\gamma}(\mathcal{D}) \to H^{0,\gamma}(\mathcal{D})$ is compact then the embedding $\iota' : H^{0,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is compact, too. As $\Delta L = \iota' a_0$, the proof is complete. □

As mentioned, we can say much more on compact perturbation of the operator L_0 under additional restrictions of Sections 11 and 12.

Since $C^\infty_{\mathrm{comp}}(\mathcal{D}) \hookrightarrow H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{0,\gamma}(\mathcal{D})$, the elements of $H^{-,\gamma}(\mathcal{D})$ are distributions in $\mathcal{D}$ and any solution to problem (8.2) satisfies $Au = f$ in $\mathcal{D}$ in the sense of distributions. Though the boundary conditions are interpreted in a weak sense, they agree with those in terms of restrictions to $\partial\mathcal{D}$ if the solution is sufficiently smooth up to the boundary, e.g. belongs to $C^1(\overline{\mathcal{D}})$. Suppose for instance that the coefficients $a_{i,j}$ are smooth in $\overline{\mathcal{D}}$ and $f \in L^2_{\mathrm{loc}}(\mathcal{D})$. Since A is elliptic, we deduce readily that $u \in H^2_{\mathrm{loc}}(\mathcal{D})$ and the equality $Au = f$ is actually satisfied almost everywhere in $\mathcal{D}$. If, in addition, $u \in H^{2,\gamma}(\mathcal{D})$ then

$$((\partial_c + b_1^{-1}(\partial_t + B_0))u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)} = 0$$

for all $v \in H^{+,\gamma}(\mathcal{D})$. As any smooth function v in $\overline{\mathcal{D}}$ whose support does not meet S belongs to $H^{+,\gamma}(\mathcal{D})$, we conclude that $(b_1\partial_c + \partial_t + B_0)u = 0$ on $\partial\mathcal{D} \setminus S$. Hence, in this case $Bu = 0$ on $\partial\mathcal{D}$, for $u = 0$ and $b_1 = 0$ on S. If $Au = f$ is not sufficiently regular in the closure of $\mathcal{D}$, then f may behave wildly at the boundary which may cause $Bu = u_0$ in some very implicit sense at $\partial\mathcal{D}$, with $u_0 = u_0(f)$ different from zero.

9. Completeness of root functions for weak perturbations

We are now in a position to study the completeness of root functions related to problem (8.47). We begin with the selfadjoint operator L_0. To this end we write ι' for the continuous embedding of $H^{0,\gamma}(\mathcal{D})$ into $H^{-,\gamma}(\mathcal{D})$, as it is described by Lemma 1.2.

Lemma 9.1. *Suppose that estimate (8.6) is fulfilled and inclusion (8.8) is continuous. Then the inverse L_0^{-1} of the operator given by (8.46) induces positive selfadjoint operators*

$$\begin{array}{rclcrcl} Q_1 & = & \iota'\iota\, L_0^{-1} & : & H^{-,\gamma}(\mathcal{D}) & \to & H^{-,\gamma}(\mathcal{D}), \\ Q_2 & = & \iota\, L_0^{-1}\, \iota' & : & H^{0,\gamma}(\mathcal{D}) & \to & H^{0,\gamma}(\mathcal{D}), \\ Q_3 & = & L_0^{-1}\, \iota'\iota & : & H^{+,\gamma}(\mathcal{D}) & \to & H^{+,\gamma}(\mathcal{D}) \end{array}$$

which have the same systems of eigenvalues and eigenvectors. If the inclusion ι is compact then the operators are compact and there are orthonormal bases in $H^{+,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{-,\gamma}(\mathcal{D})$ consisting of the eigenvectors.

Proof. Easily, if ι is compact then, as ι', L_0^{-1} are bounded, all the operators Q_1, Q_2, Q_3 are compact.

Recall that we endow the space $H^{-,\gamma}(\mathcal{D})$ with the scalar product $(\cdot,\cdot)_{-,\gamma}$. Then, by (8.49),

$$\begin{array}{rcl} (Q_1 u, v)_{-,\gamma} & = & \overline{(v, \iota'\iota\, L_0^{-1} u)_{-,\gamma}} \\ & = & \overline{(L_0^{-1} v, \iota'\iota\, L_0^{-1} u)_{\gamma}} \\ & = & (\iota L_0^{-1} u, \iota L_0^{-1} v)_{H^{0,\gamma}(\mathcal{D})}, \end{array} \tag{9.1}$$

and

$$\begin{array}{rcl} (u, Q_1 v)_{-,\gamma} & = & \overline{(Q_1 v, u)_{-,\gamma}} \\ & = & (\iota L_0^{-1} u, \iota L_0^{-1} v)_{H^{0,\gamma}(\mathcal{D})} \end{array}$$

for all $u, v \in H^{-,\gamma}(\mathcal{D})$, i.e. the operator Q_1 is selfadjoint and non-negative.

Using (8.46) we get

$$\begin{array}{rcl} (Q_2 u, v)_{H^{0,\gamma}(\mathcal{D})} & = & (\iota(L_0^{-1}(\iota' u)), v)_{H^{0,\gamma}(\mathcal{D})} \\ & = & (L_0^{-1}(\iota' u), \iota' v)_{\gamma} \\ & = & (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_{+,\gamma} \end{array}$$

and

$$\begin{array}{rcl} (u, Q_2 v)_{H^{0,\gamma}(\mathcal{D})} & = & \overline{(Q_2 v, u)_{H^{0,\gamma}(\mathcal{D})}} \\ & = & (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_{+,\gamma} \end{array}$$

for all $u, v \in H^{0,\gamma}(\mathcal{D})$, i.e. the operator Q_2 is selfadjoint and non-negative.

On applying (8.46) once again we obtain

$$\begin{array}{rcl} (Q_3 u, v)_{+,\gamma} & = & (L_0^{-1}(\iota'\iota\, u), v)_{+,\gamma} \\ & = & (\iota'\iota\, u, v)_{\gamma} \\ & = & (\iota u, \iota v)_{H^{0,\gamma}(\mathcal{D})} \end{array} \tag{9.2}$$

and

$$\begin{array}{rcl} (u, Q_3 v)_{+,\gamma} & = & \overline{(Q_3 v, u)_{+,\gamma}} \\ & = & (\iota u, \iota v)_{H^{0,\gamma}(\mathcal{D})} \end{array}$$

for all $u, v \in H^{+,\gamma}(\mathcal{D})$, which shows that Q_3 is a non-negative selfadjoint operator.

Finally, as the operator L_0^{-1} is injective, so are the operators Q_1, Q_2 and Q_3. Hence, all these operators are actually positive. Moreover, all their eigenvectors $\{u_\nu\}$ belong to the space $H^{+,\gamma}(\mathcal{D})$, for $L_0^{-1} u_\nu$ lies in $H^{+,\gamma}(\mathcal{D})$. From the injectivity of ι and ι' we conclude immediately that the systems of eigenvalues and eigenvectors of Q_1, Q_2 and Q_3 coincide. The last part of the lemma follows from Theorem 6.1. □

Our next goal is to apply Theorem 6.4 to investigate the completeness of root functions of weak perturbations of Q_j. Lemmas 8.2, 8.3, 8.10, 8.11, Theorem 8.4 and Corollary 8.12 give sufficient conditions for the inclusion (8.8) to be compact. However, we need to describe typical situations where the operators Q_1, Q_2, Q_3 have finite order. With this purpose, we present a broad class of finite order compact operators acting in spaces of integrable functions. The following result goes back at least as far as [Agm62].

Theorem 9.2. *Let $s \in \mathbb{R}$ and $A : H^s(\mathcal{D}) \to H^s(\mathcal{D})$ be a compact operator. If there is $\Delta s > 0$ such that A maps $H^s(\mathcal{D})$ continuously to $H^{s+\Delta s}(\mathcal{D})$, then it belongs to Schatten class $\mathfrak{S}_{n/\Delta s+\varepsilon}$ for each $\varepsilon > 0$.*

Proof. For the case $s \in \mathbb{Z}_{\geq 0}$ see [Agm62]. For the case $s \in \mathbb{R}$ and Sobolev spaces on a compact closed manifold $\mathcal{D}$ see e.g. Proposition 5.4.1 in [Agr90]. For the general case we have not been able to find a proper reference and so we refer to our paper [ST12].

We indicate crucial steps of the proof for the completeness of exposition. Let Q be the cube

$$Q = \{x \in \mathbb{R}^n : |x_j| < \pi, \ j = 1, \ldots, n\}$$

in $\mathbb{R}^n$. Given a function $u \in L^2(Q)$, we consider the Fourier series expansion

$$u(x) \sim \sum_{k \in \mathbb{Z}^n} c_k(u) \, e^{\imath (\sum_{j=1}^n k_j x_j)}$$

and introduce the norm

$$\|u\|^2_{H^{(s)}} = |a_0(u)|^2 + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{2s} \, |c_k(u)|^2,$$

where s is a non-negative real number. The subspace of functions for which this norm is finite is denoted by $H^{(s)}$. Obviously, $H^{(s)}$ is a Hilbert space which, for non-negative integral s, can be regarded as a closed subspace of the Sobolev space $H^s(Q)$. We see readily that $H^s_{\mathrm{comp}}(Q) \hookrightarrow H^{(s)}$. For $s < 0$, we write $H^{(s)}$ for the dual of $H^{(-s)}$ with respect to the sesquilinear pairing $\langle \cdot, \cdot \rangle_{(0)}$ induced by the inner product $(\cdot, \cdot)_H$ (see Lemma 1.3). The norm in $H^{(s)}$ is still given by the same formula, as is easy to check.

Without loss of the generality we can assume that the closure of $\mathcal{D}$ is situated in the cube Q. For $s \geq 0$, we denote by $r_{s,\mathcal{D}}$ the restriction operator from $H^s(Q)$ to $H^s(\mathcal{D})$. By the above, $r_{s,\mathcal{D}}$ acts to the elements of $H^{(s)}$, too, mapping these continuously to $H^s(\mathcal{D})$. As the boundary of $\mathcal{D}$ is Lipschitz, for each $s \in \mathbb{Z}_{\geq 0}$ there is a bounded extension operator $e_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^s_{\mathrm{comp}}(Q)$ (see for instance [Bur98, Ch. 6]). We will think of $e_{s,\mathcal{D}}$ as bounded linear operator from $H^s(\mathcal{D})$ to $H^{(s)}$, provided that $s \in \mathbb{Z}_{\geq 0}$.

Given any non-negative integer s, an interpolation procedure applies to the pair $(H^s(\mathcal{D}), H^{s+1}(\mathcal{D}))$, thus giving a family of function spaces in $\mathcal{D}$ of fractional smoothness $(1-\theta)s + \theta(s+1) = s + \theta$ with $0 < \theta < 1$. The Banach spaces obtained in this way coincide with $H^{s+\theta}(\mathcal{D})$ up to equivalent norms. Thus, we can apply interpolation arguments to conclude that there is a bounded linear extension operator $e_{s,\mathcal{D}} : H^s(\mathcal{D}) \to H^{(s)}$ for all real $s \geq 0$. By construction,

$$r_{s,\mathcal{D}} \, e_{s,\mathcal{D}} u = u \tag{9.3}$$

holds for each $u \in H^s(\mathcal{D})$ with $s \geq 0$.

For $s < 0$ we introduce the mappings

$$\begin{array}{rcccl} r_{s,\mathcal{D}} & : & H^{(s)} & \to & H^s(\mathcal{D}), \\ e_{s,\mathcal{D}} & : & H^s(\mathcal{D}) & \to & H^{(s)}, \end{array}$$

using the duality between the spaces $H^{(s)}$ and $H^{(-s)}$. Namely, if $s < 0$ we set

$$\begin{array}{rcl} \langle r_{s,\mathcal{D}} u, v \rangle_{(0)} & := & \langle u, e_{-s,\mathcal{D}} v \rangle_{(0)}, \\ \langle e_{s,\mathcal{D}} u, v \rangle_{(0)} & := & \langle u, r_{-s,\mathcal{D}} v \rangle_{(0)} \end{array} \tag{9.4}$$

for all $u \in H^{(s)}$, $v \in H^{-s}(\mathcal{D})$ and for all $u \in H^{s}(\mathcal{D})$, $v \in H^{(-s)}$, respectively. As

$$|\langle u, e_{-s,\mathcal{D}} v\rangle_{(0)}| \leq \|u\|_{H^{(s)}} \, \|e_{-s,\mathcal{D}}\| \, \|v\|_{H^{-s}(\mathcal{D})}$$

for all $u \in H^{(s)}$ and $v \in H^{-s}(\mathcal{D})$, which is a consequence of duality between the spaces $H^{(s)}$ and $H^{(-s)}$, the first identity of (9.4) defines a bounded linear operator $r_{s,\mathcal{D}} : H^{(s)} \to H^{s}(\mathcal{D})$ indeed. Similarly, by the duality between $H^{s}(\mathcal{D})$ and $H^{-s}(\mathcal{D})$ (cf. Lemma 1.3), the second identity of (9.4) defines a bounded linear operator $e_{s,\mathcal{D}} : H^{s}(\mathcal{D}) \to H^{(s)}$.

On applying equality (9.3) we get $\langle r_{s,\mathcal{D}}\, e_{s,\mathcal{D}} u, v\rangle_{(0)} = \langle u, v\rangle_{(0)}$ for all $u \in H^{s}(\mathcal{D})$ and $v \in H^{-s}(\mathcal{D})$ with real $s < 0$. In other words, the operators $r_{s,\mathcal{D}}$ and $r_{s,\mathcal{D}}$ satisfy (9.3) for all $s \in \mathbb{R}$, i.e.,

$$r_{s,\mathcal{D}}\, e_{s,\mathcal{D}} = I_{H^{s}(\mathcal{D})}. \tag{9.5}$$

For $t > s$ we denote by

$$\begin{array}{rcccl} \iota_{t,s,\mathcal{D}} & : & H^{t}(\mathcal{D}) & \to & H^{s}(\mathcal{D}), \\ \iota_{t,s} & : & H^{(t)} & \to & H^{(s)} \end{array}$$

the natural inclusion mappings. If $t < 0$, by this is meant

$$\begin{array}{rcl} \langle \iota_{t,s,\mathcal{D}} u, v\rangle_{(0)} & = & \langle u, \iota_{-s,-t,\mathcal{D}} v\rangle_{(0)}, \\ \langle \iota_{t,s} u, v\rangle_{(0)} & = & \langle u, \iota_{-s,-t,\mathcal{D}} v\rangle_{(0)} \end{array} \tag{9.6}$$

for all $u \in H^{t}(\mathcal{D})$, $v \in H^{-s}(\mathcal{D})$ and $u \in H^{(t)}$, $v \in H^{(-s)}$, respectively. It is clear that

$$\begin{array}{rcl} r_{s,\mathcal{D}}\, \iota_{t,s} & = & \iota_{t,s,\mathcal{D}}\, r_{t,\mathcal{D}}, \\ \iota_{t,s}\, e_{t,\mathcal{D}} & = & e_{s,\mathcal{D}}\, \iota_{t,s,\mathcal{D}}, \\ r_{s,\mathcal{D}}\, \iota_{t,s}\, e_{t,\mathcal{D}} & = & \iota_{t,s,\mathcal{D}} \end{array} \tag{9.7}$$

provided $t \geq 0$. If $t < 0$ then combining (9.4), (9.6) and (9.7) yields

$$\begin{array}{rcl} \langle r_{s,\mathcal{D}}\, \iota_{t,s} u, v\rangle_{(0)} & = & \langle u, \iota_{-s,-t}\, e_{-s,\mathcal{D}} v\rangle_{(0)} \\ & = & \langle u, e_{-t,\mathcal{D}}\, \iota_{-s,-t,\mathcal{D}} v\rangle_{(0)} \\ & = & \langle \iota_{t,s,\mathcal{D}}\, r_{t,\mathcal{D}} u, v\rangle_{(0)} \end{array}$$

for all $u \in H^{(t)}$ and $v \in H^{-s}(\mathcal{D})$, and

$$\begin{array}{rcl} \langle \iota_{t,s}\, e_{t,\mathcal{D}} u, v\rangle_{(0)} & = & \langle u, r_{-t,\mathcal{D}}\, \iota_{-s,-t} v\rangle_{(0)} \\ & = & \langle u, \iota_{-s,-t}\, r_{-s,\mathcal{D}} v\rangle_{(0)} \\ & = & \langle e_{s,\mathcal{D}}\, \iota_{t,s,\mathcal{D}} u, v\rangle_{(0)} \end{array}$$

for all $u \in H^{t}(\mathcal{D})$ and $v \in H^{(-s)}$, whence $r_{s,\mathcal{D}}\, \iota_{t,s}\, e_{t,\mathcal{D}} = \iota_{t,s,\mathcal{D}}$. Therefore, equalities (9.7) are valid not only for real $t \geq 0$ but also for all $t \in \mathbb{R}$.

Lemma 9.3. *Let $s \in \mathbb{R}$ and $K : H^{(s)} \to H^{(s)}$ be a compact operator. If there is $\Delta s > 0$ such that K maps $H^{(s)}$ continuously to $H^{(s+\Delta s)}$, then K is of Schatten class $\mathfrak{S}_{n/\Delta s + \varepsilon}$ for each $\varepsilon > 0$.*

Proof. Put

$$\Lambda_r u\,(x) = \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^{r/2}\, c_k(u)\, e^{\imath (\sum_{j=1}^{n} k_j x_j)}.$$

Obviously, Λ_r maps $H^{(s)}$ continuously to $H^{(s-r)}$ for all $s \in \mathbb{R}$. For each fixed s, the operator $\Lambda_{-r}\, \iota_{s+r,s}$ is selfadjoint and compact in $H^{(s+r)}$. Its eigenvalues are $(1+|k|^2)^{-r/2}$ and the corresponding eigenfunctions are $e^{\imath(\sum_{j=1}^n k_j x_j)}$. The series

$$\sum_{k\in\mathbb{Z}^n} (1+|k|^2)^{-pr/2}$$

(counting the eigenvalues with their multiplicities) converges for all $p > n/r$, and so $\Lambda_{-r}\, \iota_{s+r,s}$ is of Schatten class $\mathfrak{S}_{n/r+\varepsilon}$ for any $\varepsilon > 0$.

Obviously, $\Lambda_{-r}\Lambda_r = I$ holds for all $r > 0$. By assumption, the operator K factors through the embedding $\iota_{s+\Delta s,s} : H^{(s+\Delta s)} \to H^{(s)}$, i.e., there is a bounded linear operator $K_0 : H^{(s)} \to H^{(s+\Delta s)}$ such that $K = \iota_{s+\Delta s,s} K_0$. Then

$$\begin{aligned} K &= \Lambda_{-\Delta s}\, \Lambda_{\Delta s} K \\ &= \Lambda_{-\Delta s}\, \Lambda_{\Delta s}\, \iota_{s+\Delta s,s} K_0 \\ &= \Lambda_{-\Delta s}\, \iota_{s+\Delta s,s}\, \Lambda_{\Delta s} K_0. \end{aligned}$$

Since the operator $\Lambda_{\Delta s} K_0 : H^{(s)} \to H^{(s)}$ is bounded, Lemma 6.3 implies that K belongs to the Schatten class $\mathfrak{S}_{n/\Delta s+\varepsilon}$ for any $\varepsilon > 0$. $\square$

We are now in a position to complete the proof of Theorem 9.2. Suppose that $A_0 : H^s(\mathcal{D}) \to H^{s+\Delta s}(\mathcal{D})$ is a bounded linear operator, such that $A = \iota_{s+\Delta s,s} A_0$. Set

$$K_0 = e_{s+\Delta s,\mathcal{D}}\, A_0\, r_{s,\mathcal{D}},$$

then K_0 maps $H^{(s)}$ continuously to $H^{(s+\Delta s)}$. By Lemma 9.3, the composition $K = \iota_{s+\Delta s,s} K_0$ is of Schatten class $\mathfrak{S}_{n/\Delta s+\varepsilon}$ for any $\varepsilon > 0$. Besides, we get

$$r_{s+\Delta s,\mathcal{D}}\, K_0 = A_0\, r_{s,\mathcal{D}} \tag{9.8}$$

because of (9.5).

Let λ be a non-zero eigenvalue of A and $u \in H^s(\mathcal{D})$ a root function corresponding to λ, i.e. $(A-\lambda I)^m u = 0$ for some natural number m. Then, using (9.7) and (9.8), we conclude that

$$\begin{aligned} (K-\lambda I)^m\, e_{s,\mathcal{D}} u &= e_{s,\mathcal{D}}\, (A-\lambda I)^m u \\ &= 0, \end{aligned}$$

that is each non-zero eigenvalue of A is actually an eigenvalue for K of the same multiplicity. Therefore, A belongs to the Schatten class $\mathfrak{S}_{n/\Delta s+\varepsilon}$ for any $\varepsilon > 0$, too. $\square$

Corollary 9.4. *Suppose that $\rho' \in L^\infty(\mathcal{D})$ and the coercive estimate (8.9) holds. Then any compact operator $R : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ which maps $H^{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$. In particular, its order is finite.*

Proof. We first observe that $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$ and the norm $\|\cdot\|_{L^2(\mathcal{D})}$ majorises the norm $\|\cdot\|_{H^{-1}(\mathcal{D})}$. Hence, $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $H^{-s}(\mathcal{D})$, too, i.e. each equivalence class of $H^{-s}(\mathcal{D})$ contains a Cauchy sequence consisting of smooth functions with compact support in $\mathcal{D}$.

As the coercive estimate (8.9) holds, the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1,\gamma}(\mathcal{D})$. On the other hand, it follows from Lemma 2.8 that the embedding $\iota_{(1,0),1,\mathcal{D}} : \mathcal{H}^{1,1}(\mathcal{D}) = H^{1,0}(\mathcal{D}) \to H^1(\mathcal{D})$ is continuous. Now using Theorem 2.6 we define the continuous map $\mathcal{S}^+ : H^{+,\gamma}(\mathcal{D}) \to H^1(\mathcal{D})$ via

$$\mathcal{S}^+ u = \iota_{(1,0),1,\mathcal{D}} \rho^{-\gamma} u$$

for any $u \in H^{+,\gamma}(\mathcal{D})$. For $v \in C^\infty_{\mathrm{comp}}(\mathcal{D})$, set

$$\mathcal{S}^- v = \rho^\gamma v.$$

Then Theorem 2.6 and Lemmas 2.8 and 8.10 yield

$$\begin{array}{rcl} |(\mathcal{S}^- v, u)_{H^{0,\gamma}(\mathcal{D})}| & = & |(v, \rho^{-\gamma} u)_{L^2(\mathcal{D})}| \\ & \leq & \|v\|_{H^{-1}(\mathcal{D})} \|\rho^{-\gamma} u\|_{H^1(\mathcal{D})} \\ & \leq & \|v\|_{H^{-1}(\mathcal{D})} \|\rho^{-\gamma} u\|_{H^{1,0}(\mathcal{D})} \\ & \leq & c\, \|v\|_{H^{-1}(\mathcal{D})} \|u\|_{H^{1,\gamma}(\mathcal{D})} \\ & \leq & c\, \|v\|_{H^{-1}(\mathcal{D})} \|u\|_{+,\gamma} \end{array}$$

for all $u \in H^{+,\gamma}(\mathcal{D})$, where the constant c does not depend on u and v. Therefore, we get

$$\|\mathcal{S}^- v\|_{-,\gamma} \leq c\, \|v\|_{H^{-1}(\mathcal{D})},$$

i.e. $\mathcal{S}^-$ maps $H^{-1}(\mathcal{D})$ continuously to $H^{-,\gamma}(\mathcal{D})$.

Denote by R_0 the operator R which is thought of as a bounded map of $H^{-,\gamma}(\mathcal{D})$ to $H^{+,\gamma}(\mathcal{D})$. Then the composition $\mathcal{S}^+ R_0 \mathcal{S}^-$ maps $H^{-1}(\mathcal{D})$ continuously to $H^1(\mathcal{D})$. Write $i : H^1(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ for the natural inclusion and $i' : L^2(\mathcal{D}) \hookrightarrow H^{-1}(\mathcal{D})$ for the corresponding operator induced by duality. It follows from Theorem 9.2 that the operator $i' i\, \mathcal{S}^+ R_0 \mathcal{S}^- : H^{-1}(\mathcal{D}) \to H^{-1}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$. By the very construction,

$$\mathcal{S}^- i'\, i \mathcal{S}^+ = \iota' \iota$$

whence $R = \iota' \iota\, R_0 = \mathcal{S}^- i' i \mathcal{S}^+ R_0$.

Let now λ be a non-zero eigenvalue of R and u be a root function of R corresponding to λ, i.e. $(R - \lambda I)^m u = 0$ for a natural number m. Then it follows from the binomial formula that u belongs to the image of the operator $\mathcal{S}^-$, i.e. $u = \mathcal{S}^- u_0$ for some $u_0 \in H^{-1}(\mathcal{D})$. Hence

$$\begin{array}{rcl} (R - \lambda I)^m u & = & (R - \lambda I)^m \mathcal{S}^- u_0 \\ & = & \mathcal{S}^- \, (i' i\, \mathcal{S}^+ \, R_0 \mathcal{S}^- - \lambda I)^m u_0 \\ & = & 0. \end{array}$$

As the operator $\mathcal{S}^-$ is obviously injective, each eigenvalue of the operator R is in fact an eigenvalue of $i' i\, \mathcal{S}^+ \, R_0 \mathcal{S}^-$ of the same multiplicity. Therefore, R lies in $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$, too. □

Corollary 9.5. *If for some* $0 < s < 1$ *there is a continuous embedding*

$$\iota_s : H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{s,\gamma}(\mathcal{D}, S), \tag{9.9}$$

then any compact operator $R : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ *which maps* $H^{-,\gamma}(\mathcal{D})$ *continuously to* $H^{+,\gamma}(\mathcal{D})$ *is of Schatten class* $\mathfrak{S}_{n/2s+\varepsilon}$ *for any* $\varepsilon > 0$*. In particular, its order is finite.*

Proof. The proof generalises that of Corollary 9.4 in obvious way.

We first observe that $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $L^2(\mathcal{D})$ and the norm $\|\cdot\|_{L^2(\mathcal{D})}$ majorises the norm $\|\cdot\|_{H^{-s}(\mathcal{D})}$. Hence, $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in $H^{-s}(\mathcal{D})$, too, i.e. each equivalence class of $H^{-s}(\mathcal{D})$ contains a Cauchy sequence consisting of smooth functions with compact support in $\mathcal{D}$. Thus, the operator ι_s induces via composition a bounded inclusion operator $\iota'_s : H^{-s,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$. This latter is actually the transpose of ι_s.

By definition there is a continuous embedding $\iota_{(s,0),s,\mathcal{D}} : H^{s,0}(\mathcal{D}) \to H^s(\mathcal{D})$. Using Corollary 3.3 we may define a continuous map $\mathcal{S}^+ : H^{s,\gamma} \to H^s(\mathcal{D})$ by means of

$$\mathcal{S}^+ u = \iota_{(s,0),s,\mathcal{D}} \rho^{-\gamma} u$$

for $u \in H^{s,\gamma}(\mathcal{D})$. Moreover, set

$$\mathcal{S}^- v = \rho^\gamma v$$

for $v \in C^\infty_{\mathrm{comp}}(\mathcal{D})$. Then Corollary 3.3 yields

$$\begin{array}{rcl} |(\mathcal{S}^- v, u)_{H^{0,\gamma}(\mathcal{D})}| & = & |(v, \rho^{-\gamma} u)_{L^2(\mathcal{D})}| \\ & \leq & \|v\|_{H^{-s}(\mathcal{D})} \|\rho^{-\gamma} u\|_{H^s(\mathcal{D})} \\ & \leq & c\,\|v\|_{H^{-s}(\mathcal{D})} \|\rho^{-\gamma} u\|_{H^{s,0}(\mathcal{D})} \\ & = & c\,\|v\|_{H^{-s}(\mathcal{D})} \|u\|_{H^{s,\gamma}(\mathcal{D})} \end{array}$$

for all $u \in H^{s,\gamma}(\mathcal{D})$. Therefore, we get

$$\|\mathcal{S}^- v\|_{H^{-s,\gamma}(\mathcal{D})} \leq c\,\|v\|_{H^{-s}(\mathcal{D})},$$

i.e. $\mathcal{S}^-$ extends to a continuous mapping of $H^{-s}(\mathcal{D})$ to $H^{-s,\gamma}(\mathcal{D})$.

Let R_0 stand for the operator R regarded as a bounded map of $H^{-,\gamma}(\mathcal{D})$ to $H^{+,\gamma}(\mathcal{D})$. Then the operator $S^+ \iota_s R_0 \iota'_s S^-$ maps the space $H^{-s}(\mathcal{D})$ continuously to $H^s(\mathcal{D})$.

Write $i : H^s(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ for the natural inclusion and $i' : L^2(\mathcal{D}) \hookrightarrow H^{-s}(\mathcal{D})$ for the corresponding map induced by duality. It follows from Theorem 9.2 that the operator $i' i\, \mathcal{S}^+ \iota_s R_0 \iota'_s \mathcal{S}^- : H^{-s}(\mathcal{D}) \to H^{-s}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$. By construction,

$$\iota'_s \mathcal{S}^- i'\, i \mathcal{S}^+ \iota_s = \iota'\, \iota,$$

and so $R = \iota' \iota\, R_0 = \iota'_s \mathcal{S}^- i'\, i \mathcal{S}^+ \iota_s R_0$.

Let now λ be a non-zero eigenvalue of R and u be a root function of R corresponding to λ, i.e. $(R - \lambda I)^m u = 0$ for a natural number m. Then it follows from the binomial formula that u belongs to the image of the operator $\iota'_s \mathcal{S}^-$, i.e. $u = \iota'_s \mathcal{S}^- u_0$ for some $u_0 \in H^{-s}(\mathcal{D})$. Hence

$$\begin{array}{rcl} (R - \lambda I)^m u & = & (R - \lambda I)^m \iota'_s \mathcal{S}^- u_0 \\ & = & \iota'_s \mathcal{S}^- \,(i'\, i\, \mathcal{S}^+ \iota_s\, R_0 \iota'_s \mathcal{S}^- - \lambda I)^m u_0 \\ & = & 0. \end{array}$$

As the operator $\iota' \mathcal{S}^-$ is injective, each eigenvalue of the operator R is in fact an eigenvalue of $i'\, i\, \mathcal{S}^+ \iota_s\, R_0 \iota'_s \mathcal{S}^-$ of the same multiplicity. Therefore, R lies in $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$, as desired. $\square$

Corollary 9.6. *If for some $0 < s < 1$ there is a continuous embedding*

$$\tilde{\iota}_s : H^{+,\gamma}(\mathcal{D}) \hookrightarrow \tilde{H}^{s,\gamma}(\mathcal{D}, S), \tag{9.10}$$

then any compact operator $R : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ which maps $H^{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$ is of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$. In particular, its order is finite.

Corollary 9.7. *Suppose $\rho' \in L^\infty(\mathcal{D})$ and the coercive estimate (8.9) holds. Then the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$ (and so they are of finite order).*

Proof. Since $Q_1 = \iota'\iota\, L_0^{-1}$ and L_0^{-1} maps $H^{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$, the operator Q_1 is of Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$, which is due to Corollary 9.4. On the other hand, Lemma 9.1 shows that the operators Q_1, Q_2 and Q_3 have the same eigenvalues. Hence, all these operators belong to the Schatten class $\mathfrak{S}_{n/2+\varepsilon}$ for any $\varepsilon > 0$, as desired. □

Corollary 9.8. *Suppose there is a continuous embedding (9.9) with some $s > 0$. Then the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$ (and so they are of finite order).*

Proof. Since $Q_1 = \iota'\iota\, L_0^{-1}$ and L_0^{-1} maps $H_{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$, the operator Q_1 is of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$, which is due to Corollary 9.5. On the other hand, Lemma 9.1 shows that the operators Q_1, Q_2 and Q_3 have the same eigenvalues. Hence, all these operators belong to the Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$, as desired. □

Corollary 9.9. *Suppose there is a continuous embedding (9.10) with some $s > 0$. Then the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$ (and so they are of finite order).*

Lemmas 8.2, 8.3, 8.10, 8.11, Theorem 8.4 and Corollary 8.12 provide sufficient conditions for a continuous embedding (9.9) to be true with $s = 1$ and $0 < s < 1/2$, respectively.

Theorem 9.10. *If the operator $Q_1 : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is of finite order then, for any invertible operator of the type $L_0 + \Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ with a compact operator $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$, the system of root functions of the compact operator*

$$P_1 = \iota'\iota\, (L_0 + \Delta L)^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$$

is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$.

Proof. By assumption, there is a bounded inverse

$$(L_0 + \Delta L)^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D}).$$

Since

$$I - L_0(L_0 + \Delta L)^{-1} = \Delta L\, (L_0 + \Delta L)^{-1},$$

we conclude that

$$\begin{array}{rcl} L_0^{-1} - (L_0 + \Delta L)^{-1} & = & L_0^{-1} \left(\Delta L\, (L_0 + \Delta L)^{-1} \right), \\ Q_1 - P_1 & = & Q_1 \left(\Delta L\, (L_0 + \Delta L)^{-1} \right). \end{array} \tag{9.11}$$

From the compactness of ΔL and boundedness of $(L_0 + \Delta L)^{-1}$ it follows that the operator

$$\Delta L\, (L_0 + \Delta L)^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$$

is compact.

Hence, P_1 is an injective weak perturbation of the compact selfadjoint operator Q_1. If in addition the order of Q_1 is finite then Theorem 6.4 implies that the countable system $\{u_\nu\}$ of root functions related to the operator P_1 is complete in the Hilbert space $H^{-,\gamma}(\mathcal{D})$.

Pick a root function u_ν of the operator P_1 corresponding to an eigenvalue λ_ν. Note that $\lambda_\nu \neq 0$, for the operator $(L_0 + \varDelta L)^{-1}$ is injective. By definition there is a natural number m, such that $(P_1 - \lambda_\nu I)^m u_\nu = 0$. Using the binomial formula yields

$$\sum_{j=0}^{m} \binom{m}{j} \lambda_\nu^{m-j} P_1^j u_\nu = 0.$$

In particular, since $\lambda_\nu \neq 0$, we get

$$u_\nu = \sum_{j=1}^{m} \binom{m}{j} \lambda_\nu^{-j} (\iota' \iota\, (L_0 + \varDelta L)^{-1})^j u_\nu.$$

Hence, $u_\nu \in H^{+,\gamma}(\mathcal{D})$ because the range of the operator $(L_0 + \varDelta L)^{-1}$ lies in the space $H^{+,\gamma}(\mathcal{D})$.

We have thus proved that $\{u_\nu\} \subset H^{+,\gamma}(\mathcal{D})$. Our next concern will be to show that the linear span $\mathcal{L}(\{u_\nu\})$ of the system $\{u_\nu\}$ is dense in $H^{+,\gamma}(\mathcal{D})$ (cf. Proposition 6.1 of [Agr11a] and [Agr11c, p. 12]). For this purpose, pick $u \in H^{+,\gamma}(\mathcal{D})$. As $L_0 + \varDelta L$ maps $H^{+,\gamma}(\mathcal{D})$ continuously onto $H^{-,\gamma}(\mathcal{D})$, we get $(L_0 + \varDelta L)u \in H^{-,\gamma}(\mathcal{D})$. Hence, there is a sequence $\{f_k\} \subset \mathcal{L}(\{u_\nu\})$ converging to $(L_0 + \varDelta L)u$ in $H^{-,\gamma}(\mathcal{D})$. On the other hand, the inverse $(L_0 + \varDelta L)^{-1}$ maps $H^{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$, and so the sequence

$$(L_0 + \varDelta L)^{-1} f_k = (L_0 + \varDelta L)^{-1} \iota' \iota\, f_k$$

converges to u in $H^{+,\gamma}(\mathcal{D})$.

If now $u_{\nu_0} \in \mathcal{L}(\{u_\nu\})$ corresponds to an eigenvalue λ_0 of multiplicity m_0, then the vector $v_{\nu_0} = P_1 u_{\nu_0}$ satisfies

$$(P_1 - \lambda_0 I)^{m_0} v_{\nu_0} = (P_1 - \lambda_0 I)^{m_0+1} u_{\nu_0} + \lambda_0 (P_1 - \lambda_0 I)^{m_0} u_{\nu_0} = 0.$$

Thus, the operator P_1 maps $\mathcal{L}(\{u_\nu\})$ to $\mathcal{L}(\{u_\nu\})$ itself. Therefore, the sequence $\{\iota' \iota\, (L_0 + \varDelta L)^{-1} f_k\}$ still belongs to $\mathcal{L}(\{u_\nu\})$ and we can think of $\{(L_0 + \varDelta L)^{-1} f_k\}$ as sequence of linear combinations of root functions of P_1 converging to u. These arguments show that the subsystem $(L_0 + \varDelta L)^{-1}\, \mathcal{L}(\{u_\nu\}) \subset \mathcal{L}(\{u_\nu\})$ is dense in $H^{+,\gamma}(\mathcal{D})$.

Finally, since $C^\infty_{\mathrm{comp}}(\mathcal{D}) \subset H^{+,\gamma}(\mathcal{D})$ and $C^\infty_{\mathrm{comp}}(\mathcal{D})$ is dense in the weighted Lebesgue space $H^{0,\gamma}(\mathcal{D})$, the space $H^{+,\gamma}(\mathcal{D})$ is dense in $H^{0,\gamma}(\mathcal{D})$ as well. This proves the completeness of the system of root functions in $H^{0,\gamma}(\mathcal{D})$. □

Similar assertions are also true for the weak perturbations of the operators Q_2 and Q_3.

The operator $L_0 + \varDelta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ with a compact operator $\varDelta L$ fails to be injective in general, and so Theorem 9.10 does not apply. However, as L_0 is continuously invertible, we conclude that $L = L_0 + \varDelta L$ is Fredholm. In particular, there is a constant c, such that

$$\|u\|_{+,\gamma} \leq c\, (\|Lu\|_{-,\gamma} + \|u\|_{-,\gamma}) \tag{9.12}$$

for all $u \in H^{+,\gamma}(\mathcal{D})$.

We next extend Theorem 9.10 to Fredholm operators. To this end denote by T the unbounded linear operator $H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ with domain $\mathcal{D}_T = H^{+,\gamma}(\mathcal{D})$ which maps an element $u \in \mathcal{D}_T$ to Lu. The operator T is clearly closed because of inequality (9.12). It is densely defined as $H^{1,\gamma}(\mathcal{D}, S) \subset H^{+,\gamma}(\mathcal{D})$ is dense in $H^{-,\gamma}(\mathcal{D})$. It is well known that the null space of T is finite dimensional in $H^{+,\gamma}(\mathcal{D})$ and its range is closed in $H^{-,\gamma}(\mathcal{D})$.

When speaking on eigen- and root functions u of the operator T we always assume that $u \in \mathcal{D}_T$ and $(T - \lambda I)^j u \in \mathcal{D}_T$ for all $j = 1, \ldots, m-1$.

Let $T_0 : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ correspond to the selfadjoint operator L_0. The operator T_0 is obviously continuously invertible and the inverse operator coincides with $\iota' \iota\, L_0^{-1} = Q_1$.

Lemma 9.11. *The spectrum of the operator T_0 consists of the points $\mu_\nu = \lambda_\nu^{-1}$ in $\mathbb{R}_{>0}$, where λ_ν are the eigenvalues of Q_1.*

Proof. Recall that all λ_ν are positive, which is due to Lemma 9.1, and so $\mu_\nu > 0$. If $\lambda \neq 0$ then

$$(T_0 - \lambda I)u = (I - \lambda\, \iota' \iota\, L_0^{-1})\, T_0 u = -\lambda \Big(Q_1 - \frac{1}{\lambda} I \Big)\, T_0 u$$

for all $u \in H^{+,\gamma}(\mathcal{D})$, showing the lemma. $\square$

If the spectrum of T is different from the whole complex plane, i.e., if the resolvent $\mathcal{R}(\lambda; T) = (T - \lambda I)^{-1}$ exists for some $\lambda = \lambda_0$, then it follows from the resolvent equation (since $\mathcal{R}(\lambda_0; T)$ is compact) that $\mathcal{R}(\lambda; T)$ exists for all $\lambda \in \mathbb{C}$ except for a discrete sequence of points $\{\lambda_\nu\}$ which are the eigenvalues of T (see [Kel71, p. 17]. In the general case, however, one cannot exclude the situation where the spectrum of T is all of $\mathbb{C}$.

Theorem 9.12. *Assume that $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is a compact operator and $Q_1 : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is of finite order. Then the spectrum of the closed operator $T : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ corresponding to $L = L_0 + \Delta L$, is different from $\mathbb{C}$ and the system of root functions of T is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$. Moreover, for any $\varepsilon > 0$, all eigenvalues of T (except for a finite number) belong to the corner $|\arg \lambda| < \varepsilon$.*

Proof. First we note that

$$T - \lambda I = L - \lambda\, \iota' \iota \tag{9.13}$$

on $H^{+,\gamma}(\mathcal{D}))$ for all $\lambda \in \mathbb{C}$. Let us prove that there is a natural number N, such that $\lambda_0 = -N$ is a resolvent point of T. For this purpose, using (9.13) and Lemma 9.11, we get

$$T + kI = (I + \Delta L (L_0 + k\, \iota' \iota)^{-1})(T_0 + kI) \tag{9.14}$$

for all $k \in \mathbb{N}$.

We will show that the operator $I + \Delta L (L_0 + k\, \iota' \iota)^{-1}$ is injective for some $k \in \mathbb{N}$. Indeed, we argue by contradiction. Suppose for any $k \in \mathbb{N}$ there is $f_k \in H^{-,\gamma}(\mathcal{D})$, such that $\|f_k\|_{-,\gamma} = 1$ and

$$(I + \Delta L (L_0 + k\, \iota' \iota)^{-1}) f_k = 0. \tag{9.15}$$

Given any $u \in H^{+,\gamma}(\mathcal{D})$ and $k \in \mathbb{N}$, an easy computation shows that

$$\begin{array}{rcl} \|(L_0 + k\,\iota'\iota)u\|^2_{-,\gamma} & = & \|u + k\,L_0^{-1}u\|^2_{+,\gamma} \\ & = & \|u\|^2_{+,\gamma} + 2k\,\|u\|^2_{H^{0,\gamma}(\mathcal{D})} + k^2\,\|L_0^{-1}u\|^2_{+,\gamma} \\ & \geq & \|u\|^2_{+,\gamma}. \end{array}$$

Hence, the sequence $u_k := (L_0 + k\,\iota'\iota)^{-1} f_k$ is bounded in $H^{+,\gamma}(\mathcal{D})$. Now the weak compactness principle for Hilbert spaces yields that there is a subsequence $\{f_{k_j}\}$ with the property that both $\{f_{k_j}\}$ and $\{u_{k_j}\}$ converge weakly in the spaces $H^{-,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$ to limits f and u, respectively. Since $\varDelta L$ is compact, it follows that the sequence $\{\varDelta L u_{k_j}\}$ converges to $\varDelta L u$ in $H^{-,\gamma}(\mathcal{D})$, and so $\{f_{k_j}\}$ converges to f because of (9.15). Obviously,

$$\|f\|_{-,\gamma} = 1.$$

In particular, we conclude that the sequence $\{\varDelta L(L_0 + k_j\,\iota'\iota)^{-1} f_{k_j}\}$ converges to $-f$ whence

$$f = -\varDelta L\,u. \tag{9.16}$$

Further, on passing to the weak limit in the equality $f_{k_j} = (L_0 + k_j\,\iota' \circ \iota)u_{k_j}$ we obtain

$$f = L_0 u - \lim_{k_j \to \infty} k_j\,\iota'\iota\,u_{k_j},$$

for the continuous operator $L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ maps weakly convergent sequences to weakly convergent sequences. As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\,u_{k_j}\}$ converges to $\iota'\iota\,u$ in the space $H^{-,\gamma}(\mathcal{D})$ and $\iota'\iota\,u \neq 0$ which is a consequence of (9.16) and the injectivity of $\iota'\iota$. This shows readily that the weak limit

$$\lim_{k_j \to \infty} k_j\,\iota'\iota\,u_{k_j} = L_0 u - f$$

does not exist, a contradiction.

We have proved more, namely that the operator $I + \varDelta L(L_0 + k\,\iota'\iota)^{-1}$ is injective for all but a finitely many natural numbers k. Since this is a Fredholm operator of index zero, it is continuously invertible. Hence, (9.14) and Lemma 9.11 imply that $(T - \lambda_0 I)^{-1}$ exists for some $\lambda_0 = -N$ with $N \in \mathbb{N}$.

As λ_0 is a resolvent point of T,

$$(T - \lambda_0 I)^{-1} = (L - \lambda_0\,\iota'\iota)^{-1}$$

on $H^{-,\gamma}(\mathcal{D})$. Since $L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is a Fredholm operator and the inclusion ι compact, the operator $L - \lambda_0\,\iota'\iota : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is Fredholm. So $(L - \lambda_0\,\iota'\iota)^{-1}$ maps $H^{-,\gamma}(\mathcal{D})$ continuously to $H^{+,\gamma}(\mathcal{D})$. Similarly to (9.11) we obtain

$$L_0^{-1} - (L - \lambda_0\,\iota'\iota)^{-1} = L_0^{-1}\left((\varDelta L - \lambda_0\,\iota'\iota)(L - \lambda_0\,\iota'\iota)^{-1}\right).$$

Then, Theorem 9.10 yields that the root functions $\{u_\nu\}$ of the operator $(L - \lambda_0\,\iota'\iota)^{-1}$ are complete in the spaces $H^{+,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{-,\gamma}(\mathcal{D})$.

From (9.13) it follows that the systems of root functions related to the operators $(L - \lambda_0\,\iota'\iota)^{-1}$ and $T - \lambda_0\,I$ coincide.

Finally, as the operators $T - \lambda_0 I$ and T have the same root functions, we conclude that $\mathcal{L}(\{u_\nu\})$ is dense in the spaces $H^{+,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{-,\gamma}(\mathcal{D})$. □

The equality $(T - \lambda I)u = 0$ for a function $u \in H^{+,\gamma}(\mathcal{D})$ may be equivalently reformulated by saying that u is a solution in a weak sense to the boundary value problem

$$\left\{ \begin{array}{rcll} Au & = & \lambda u & \text{in} \quad \mathcal{D}, \\ Bu & = & 0 & \text{at} \quad \partial\mathcal{D}, \end{array} \right. \tag{9.17}$$

where the pair (A, B) corresponds to the perturbation $L_0 + \Delta L$. For $n = 1$ such problems are known as Sturm-Liouville boundary problems for second order ordinary differential equations (see for instance [Har64, Ch. XI, § 4]). Thus, we may still refer to (9.17) as the Sturm-Liouville problem in many dimensions.

Now we want to study the completeness of root functions of "small" perturbations of compact selfadjoint operators instead of the weak ones. To this end we apply the so-called method of rays of minimal growth of resolvent which leads to more general results than Theorem 6.4. This idea seems to go back at least as far as [Agm62].

10. Rays of minimal growth

We first describe briefly the method of minimal growth rays following [DS63] and Theorem 6.1 of [GK69, p. 302].

Let $L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ be the bounded linear operator constructed in Section 8. We still assume that estimates (8.8) and (8.44) hold and that the operator L is Fredholm. In the sequel we confine ourselves to those Sturm-Liouville problems for which the spectrum of the corresponding unbounded closed operator $T : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is discrete, cf. [Agm62]. We denote by $\mathcal{R}(\lambda; T)$ the resolvent of the operator T.

Definition 10.1. A ray $\arg \lambda = \vartheta$ in the complex plane $\mathbb{C}$ is called a ray of minimal growth of the resolvent $\mathcal{R}(\lambda; T) : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ if the resolvent exists for all λ of sufficiently large modulus on this ray, and if, moreover, for all such λ an estimate

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \leq c\, |\lambda|^{-1} \tag{10.1}$$

holds with a constant $C > 0$.

Theorem 10.2. *Let $H^{+,\gamma}(\mathcal{D})$ be continuously embedded into $H^{s,\gamma}(\mathcal{D})$ or $\tilde{H}^{s,\gamma}(\mathcal{D})$, for some $0 < s \leq 1$. Suppose there are rays of minimal growth of the resolvent $\arg \lambda = \vartheta_j$, where $j = 1, \ldots, J$, in the complex plane, such that the angles between any two neighbouring rays are less than $2\pi s/n$. Then the spectrum of the operator T is discrete and the root functions form a complete system in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$.*

Proof. The proof actually follows by the same method as that in Theorem 3.2 of [Agm62].

Since the spectrum of the operator T is different from the whole complex plane it is actually discrete. It remains to show that if $g \in H^{-,\gamma}(\mathcal{D})$ is orthogonal to all eigen- and associated functions of the operator T then g is identically zero. By the Hahn-Banach theorem, this implies that the root functions of T are complete in $H^{-,\gamma}(\mathcal{D})$.

Since the operators T and $T - \lambda_0 I$ have the same root functions, we may assume without loss of generality that the origin is not in the spectrum of T. Choosing $\lambda_0 = 0$ in $R = (T - \lambda_0 I)^{-1}$, we set $R = T^{-1}$.

Consider now the function

$$f(\lambda) = (\mathcal{R}(1/\lambda; R)u, g)_{-,\gamma}, \tag{10.2}$$

where $u \in H^{-,\gamma}(\mathcal{D})$ and $(\cdot,\cdot)_{-,\gamma}$ stands for the scalar product in $H^{-,\gamma}(\mathcal{D})$. Since the resolvent of R is a meromorphic function with poles at the points of the spectrum of R, the function f is analytic for $\lambda \neq \lambda_\nu$, where $\{\lambda_\nu\}$ is the sequence of eigenvalues of $R^{-1} = T$. We shall use a familiar relation between the resolvents of the operators T and T^{-1}, namely

$$\mathcal{R}(1/\lambda; T^{-1}) = -\lambda I - \lambda^2 \mathcal{R}(\lambda; T). \tag{10.3}$$

Consider the expansion

$$\mathcal{R}(\lambda; T)u = \frac{f_{-N}}{(\lambda-\lambda_\nu)^N} + \frac{f_{-N+1}}{(\lambda-\lambda_\nu)^{N-1}} + \ldots + \frac{f_{-1}}{\lambda-\lambda_\nu} + \sum_{k=0}^{\infty} f_k\,(\lambda-\lambda_\nu)^k,$$

in a neighborhood of the point $\lambda = \lambda_\nu$, where λ_ν is a pole of $\mathcal{R}(\lambda; T)$. Here $N \geq 1$ and $f_{-N} \neq 0$, the functions $f_{-N}, \ldots, f_{-1} \in H^{-,\gamma}(\mathcal{D})$ form a chain of associated functions of T, and $f_k \in H^{-,\gamma}(\mathcal{D})$ for $k \geq 0$. This expansion implies that λ_ν is a regular point of $f(\lambda)$, for g is orthogonal to all $f_{-N}, \ldots, f_{-1}$. Therefore, $f(\lambda)$ is an entire function.

Relations (10.1), (10.2) and (10.3) imply that f is of exponential type, i.e., there is a constant $c > 0$, such that

$$|f(\lambda)| \leq c \exp|\lambda| \tag{10.4}$$

for $|\lambda| \to \infty$, provided that $\arg\lambda = \vartheta_j$ for some $j = 1, \ldots, J$. We use the following lemma taken from [DS63].

Lemma 10.3. *Assume that R is a compact linear operator of Schatten class $\mathfrak{S}_p$, with $0 < p < \infty$, in a Hilbert space H. Then there exists a sequence ρ_j satisfying $\rho_j \to 0$, such that*

$$\|\mathcal{R}(\lambda; R)\|_{\mathcal{L}(H)} \leq \text{const}\ \exp(c\,|\lambda|^{-p})$$

for $|\lambda| = \rho_j$.

According to Corollaries 9.4, 9.5, 9.6, the operator R belongs to $\mathfrak{S}_{n/2s+\varepsilon}$ for any $\varepsilon > 0$. Then it follows from Lemma 10.3 that for any $\varepsilon > 0$ there exists a sequence $\rho_j \to 0$, such that

$$|f(\lambda)| \leq \exp\left(|\lambda|^{-\frac{n}{2s}-\varepsilon}\right) \tag{10.5}$$

for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1/\rho_j$.

Consider $f(\lambda)$ in the closed corner between the rays $\arg\lambda = \vartheta_j$ and $\arg\lambda = \vartheta_{j+1}$. Its angle is less than $2\pi s/n$. Since

$$\mathcal{R}(1/\lambda; R) = -\lambda I - \lambda^2 \mathcal{R}(\lambda; T)$$

and each ray $\arg\lambda = \vartheta_j$ is a ray of minimal growth, inequality (10.4) is fulfilled on the sides of the corner and inequality (10.5) on a sequence of arcs which tends to infinity.

Choosing $\varepsilon > 0$ in (10.5) sufficiently small and applying the Fragmen-Lindelöf theorem we conclude that $|f(\lambda)| = O(|\lambda|)$ as $|\lambda| \to \infty$ in the whole complex plane. Therefore, $f(\lambda)$ is an affine function, i.e., $f(\lambda) = a_0 + c_1\lambda$. On the other hand, we have

$$\mathcal{R}(1/\lambda; R) = -\lambda\, I - \lambda^2\, R + \ldots,$$

and so

$$f(\lambda) = -\lambda\,(u,g)_{-,\gamma} - \lambda^2\,(Ru,g)_{-,\gamma} + \ldots.$$

Since $f(\lambda)$ is affine, we get

$$(Ru,g)_{-,\gamma} = 0$$

for all $u \in H^{-,\gamma}(\mathcal{D})$. Hence it follows that $g = 0$, for the range of the operator R is dense in $H^{-,\gamma}(\mathcal{D})$. Thus, the system of root functions of the operator T is complete in $H^{-,\gamma}(\mathcal{D})$.

As the operators T, $(T-\lambda_0 I)$ and $(T-\lambda_0 I)^{-1}$ have the same root functions, it suffices to repeat the arguments of the proof of Theorem 9.10 to see the completeness in the spaces $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$. □

This theorem raises the question under what conditions neighboring rays of minimal growth are close enough. We now indicate some conditions for a ray $\arg\lambda = \vartheta$ in the complex plane to be a ray of minimal growth for the resolvent of T.

Lemma 10.4. *Each ray $\arg\lambda = \vartheta$ with $\vartheta \neq 0$ is a ray of minimal growth for $\mathcal{R}(\lambda; T_0)$ and*

$$\|(T_0-\lambda I)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \leq \begin{cases} (|\lambda|\,|\sin(\arg\lambda)|)^{-1}, & \text{if} \quad |\arg\lambda| \in (0,\pi/2), \\ |\lambda|^{-1}, & \text{if} \quad |\arg\lambda| \in [\pi/2,\pi]. \end{cases} \tag{10.6}$$

Moreover, the operator $L_0 - \lambda\,\iota'\iota : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is continuously invertible and

$$\|(L_0-\lambda\,\iota'\iota)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}),H^{+,\gamma}(\mathcal{D}))} \leq \begin{cases} |\sin(\arg\lambda)|, & \text{if} \quad |\arg\lambda| \in (0,\pi/2), \\ 1, & \text{if} \quad |\arg\lambda| \in [\pi/2,\pi]. \end{cases} \tag{10.7}$$

Proof. According to Lemma 9.11 the resolvent

$$(T_0-\lambda I)^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$$

exists for all $\lambda \in \mathbb{C}$ away from the positive real axis. As the operator $Q_3 = L_0^{-1}\,\iota'\iota$ is selfadjoint, the operator T_0 is symmetric, i.e.,

$$\begin{array}{rcl} (T_0 u, g)_{-,\gamma} & = & (L_0 u, g)_{-,\gamma} \\ & = & (u, Q_3^{-1} g)_{+,\gamma} \\ & = & (Q_3^{-1} u, g)_{+,\gamma} \\ & = & (u, L_0 g)_{-,\gamma} \\ & = & (u, T_0 g)_{-,\gamma} \end{array}$$

for all $u, v \in H^{+,\gamma}(\mathcal{D})$. If $|\arg(\lambda)| \in (0,\pi/2)$, then

$$\begin{array}{rcl} \|(T_0-\lambda I)u\|^2_{-,\gamma} & = & \|(T_0-\Re\lambda\,I)u\|^2_{-,\gamma} + |\Im\lambda|^2\,\|u\|^2_{-,\gamma} \\ & \geq & |\lambda|^2|\sin(\arg\lambda)|^2\,\|u\|^2_{-,\gamma} \end{array}$$

for all functions $u \in H^{+,\gamma}(\mathcal{D})$, which establishes the first estimate of (10.6). If $|\arg\lambda| \in [\pi/2,\pi]$, then $\Re\lambda \leq 0$ whence

$$\|(T_0-\lambda I)u\|^2_{-,\gamma} \geq |\lambda|^2\,\|u\|^2_{-,\gamma}$$

and so the second estimate of (10.6) holds.

Now it follows from (9.13) that the operator $L_0 - \lambda\,\iota'\iota$ is injective for $\lambda \in \mathbb{C}$ away from the positive real axis. As this operator is Fredholm and its index is zero, it is continuously invertible. Finally, as the operator $Q_3 = L_0^{-1}\,\iota'\iota$ is positive, we deduce readily that

$$\begin{array}{rcl} \|(L_0 - \lambda\,\iota'\iota)u\|_{-,\gamma} & = & \|(I - \lambda L_0^{-1}\,\iota'\iota)u\|_{+,\gamma} \\ & \geq & |\lambda|\,|\Im\lambda^{-1}|\,\|u\|_{+,\gamma} \\ & = & |\sin(\arg\lambda)|\,\|u\|_{+,\gamma}, \end{array}$$

if $|\arg\lambda| \in (0,\pi/2)$, i.e. the second estimate of (10.7) is fulfilled. Similar arguments lead to the second estimate of (10.7). □

Theorem 10.5. *Let $H^{+,\gamma}(\mathcal{D})$ be continuously embedded into $H^{s,\gamma}(\mathcal{D})$ or $\tilde{H}^{s,\gamma}(\mathcal{D})$ for some $s > 0$ and estimate (8.44) be fulfilled with a constant $c < |\sin(\pi s/n)|$. Then all eigenvalues of the closed operator $T : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ belong to the corner $|\arg\lambda| \leq \arcsin c$, each ray $\arg\lambda = \vartheta$ with $|\vartheta| > \arcsin c$ is a ray of minimal growth for $\mathcal{R}(\lambda;T)$ and the system of root functions is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$.*

Proof. First we note that, by Lemma 8.15, the operator $L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is invertible. Indeed, $L = L_0 + \mathit{\Delta} L$ where $\mathit{\Delta} L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is a bounded operator with the norm

$$\|\mathit{\Delta} L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))} < 1 = \|L_0^{-1}\|^{-1}.$$

In particular, by (9.13), the spectrum of the corresponding operator T does not coincide with the whole complex plane.

Fix $\vartheta \neq 0$ and set $m_\vartheta = |\sin\vartheta|$, if $|\vartheta| \in (0,\pi/2)$, and $m_\vartheta = 1$, if $|\vartheta| \in [\pi/2,\pi]$. If $m_\vartheta > c$ then

$$\|\mathit{\Delta} L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))} \leq c < m_\vartheta \leq \|(L_0 - \lambda\,\iota'\iota)^{-1}\|^{-1}_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))}.$$

Hence it follows that the operator $L - \lambda\,\iota'\iota : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is continuously invertible and

$$\|(L - \lambda\,\iota'\iota)^{-1}\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))} \leq (m_\vartheta - c)^{-1}. \tag{10.8}$$

In order to establish estimate (10.1) we have to show that there is a constant $C > 0$, such that

$$C\,|\lambda|^{-1}\,\|(T - \lambda I)u\|_{-,\gamma} \geq \|u\|_{-,\gamma}$$

for all $u \in H^{+,\gamma}(\mathcal{D})$.

If $\arg\lambda = \vartheta$ with $m_\vartheta > c$, then, by (9.13), we get

$$\begin{array}{rcl} \|(T - \lambda I)u\|_{-,\gamma} & = & \|(L - \lambda\,\iota'\iota)u\|_{-,\gamma} \\ & \geq & (m_\vartheta - c)\,\|u\|_{+,\gamma} \\ & \geq & (m_\vartheta - c)\,\|u\|_{-,\gamma} \end{array}$$

for all $u \in H^{+,\gamma}(\mathcal{D})$. Therefore, given any λ on the ray $\arg\lambda = \vartheta$ with $m_\vartheta > c$, it follows that

1) The range of the operator $T - \lambda I : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is a closed subspace of $H^{-,\gamma}(\mathcal{D})$.

2) The null space of the operator $T - \lambda I : H^{-,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is trivial.

By (9.13), the range of $T - \lambda I$ coincides with the range of $L - \lambda\,\iota'\iota$ which is the whole space $H^{-,\gamma}(\mathcal{D})$. Hence, the resolvent $(T - \lambda I)^{-1}$ exists for all λ away from the corner $|\arg\lambda| \leq \arcsin c$ in the complex plane. On applying (9.13) and Lemma 10.4 we obtain

$$T - \lambda I = L_0 + \Delta L - \lambda\, \iota' \iota = (I + \Delta L (L_0 - \lambda\, \iota' \iota)^{-1})(T_0 - \lambda I) \tag{10.9}$$

on $H^{+,\gamma}(\mathcal{D})$ and

$$\begin{aligned} & \|(I + \Delta L (L_0 - \lambda\, \iota' \iota)^{-1}) u\|_{-,\gamma} \\ \geq\ & \|u\|_{-,\gamma} - \|\Delta L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}), H^{-,\gamma}(\mathcal{D}))} \|(L_0 - \lambda\, \iota' \iota)^{-1} u\|_{+,\gamma} \\ \geq\ & (1 - c/m_\vartheta)\, \|u\|_{-,\gamma}. \end{aligned}$$

Therefore the operator $I + \Delta L (L_0 - \lambda\, \iota' \iota)^{-1}$ is continuously invertible as Fredholm operator of zero index and trivial null space. Moreover,

$$\|(I + \Delta L (L_0 - \lambda\, \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \leq (1 - c/m_\vartheta)^{-1}.$$

Now (10.9) implies

$$\begin{aligned} & \|(T - \lambda I)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \\ \leq\ & \|(I + \Delta L (L_0 - \lambda\, \iota' \iota)^{-1})^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \|(T_0 - \lambda\, I)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \\ \leq\ & (1 - c/m_\vartheta)^{-1} m_\vartheta^{-1} |\lambda|^{-1} \end{aligned} \tag{10.10}$$

for all λ satisfying $\arg\lambda = \vartheta$ with $m_\vartheta > c$.

Thus, all rays outside of the corner $|\arg\lambda| \leq \arcsin c$ are rays of minimal growth. By the hypothesis of the theorem, the angles between the pairs of neighboring rays $\arg\lambda = \vartheta$ are less than $2\pi s/n$, and so the completeness of root functions follows from Theorem 10.2. □

We are now in a position to prove the main result of this section. When compared with [Agr11c] our contribution consists in developing dual function spaces which fit the problem.

Theorem 10.6. *Let the space $H^{+,\gamma}(\mathcal{D})$ be continuously embedded into $H^{s,\gamma}(\mathcal{D})$ or into $\tilde{H}^{s,\gamma}(\mathcal{D})$ for some $s > 0$, the operator $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ be bounded with norm less then $|\sin(\pi s/n)|$, and $C : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ be compact. Then the following is true:*

1) The spectrum of the operator T in $H^{-,\gamma}(\mathcal{D})$ corresponding to $L_0 + \Delta L + C$ is discrete.

2) For any $\varepsilon > 0$, all eigenvalues of the operator T (except for a finite number) belong to the corner $|\arg\lambda)| < \arcsin \|\Delta L\| + \varepsilon$.

3) Each ray $\arg\lambda = \vartheta$ with

$$|\vartheta| > \arcsin \|\Delta L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}), H^{-,\gamma}(\mathcal{D}))} \tag{10.11}$$

is a ray of minimal growth for $\mathcal{R}(\lambda; T)$.

4) The system of root functions is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$.

Proof. First we note that the operator $L_0 + \Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is continuously invertible and hence the operator $L_0 + \Delta L + C : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is actually Fredholm.

Theorem 10.5 implies that all rays satisfying (10.11) are rays of minimal growth for $\mathcal{R}(\lambda; T_0 + \Delta T)$ with the closed operator $T_0 + \Delta T$ in $H^{-,\gamma}(\mathcal{D})$ corresponding to $L_0 + \Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$.

Fix an arbitrary $\varepsilon > 0$. Then estimates (10.8) and (10.10) imply that there are constants c_1 and c_2 depending on ε, such that

$$\begin{aligned} \|(L_0 + \Delta L - \lambda\, \iota'\iota)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}),H^{+,\gamma}(\mathcal{D}))} &\leq c_1, & (10.12)\\ \|(T_0 + \Delta T - \lambda I)^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} &\leq c_2\, |\lambda|^{-1} & (10.13)\end{aligned}$$

for all λ satisfying

$$|\arg \lambda| \geq \arcsin \|\Delta L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))} + \varepsilon. \qquad (10.14)$$

Then, using (9.13), (10.12) and Theorem 10.5 we obtain

$$T - \lambda I = \left(I + C(L_0 + \Delta L - \lambda\, \iota'\iota)^{-1}\right)(T_0 + \Delta T - \lambda I) \qquad (10.15)$$

on $H^{+,\gamma}(\mathcal{D})$ for all rays satisfying (10.14).

We now prove that there is a constant $M_\varepsilon > 0$ depending on ε, such that the operator $I + C(L_0 + \Delta L - \lambda\, \iota'\iota)^{-1}$ is injective for all λ satisfying both (10.14) and $|\lambda| \geq M_\varepsilon$. To do this, we argue by contradiction in the same way as in the proof of Theorem 9.12. Suppose for each natural number k there are $f_k \in H^{-,\gamma}(\mathcal{D})$, satisfying $\|f_k\|_{-,\gamma} = 1$, and λ_k, satisfying (10.14) and $|\lambda_k| \geq k$, such that

$$\left(I + C(L_0 + \Delta L - \lambda_k\, \iota'\iota)^{-1}\right) f_k = 0. \qquad (10.16)$$

It follows from (10.12) that the sequence $u_k = (L_0 + \Delta L - \lambda_k \iota'\iota)^{-1} f_k$ is bounded in $H^{+,\gamma}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces one can assume without restriction of generality that the sequences $\{f_k\}$ and $\{u_k\}$ converge weakly in the spaces $H^{-,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$ to functions f and u, respectively. Since C is compact, it follows that the sequence $\{Cu_k\}$ converges to Cu in $H^{-,\gamma}(\mathcal{D})$ and so $\{f_k\}$ converges to f, which is due to (10.16). Obviously, the $H^{-,\gamma}(\mathcal{D})$-norm of f just amounts to 1. In particular, we conclude that $\{C(L_0 + \Delta L - \lambda_k\, \iota'\iota)^{-1}) f_k\}$ converges to $-f$ whence

$$f = -Cu. \qquad (10.17)$$

Further, as $f_k = (L_0 + \Delta L - \lambda_k\, \iota'\iota)\, u_k$, letting $k \to \infty$ in this formula yields readily

$$f = (L_0 + \Delta L)u - \lim_{k\to\infty} \lambda_k\, \iota'\iota\, u_k.$$

As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\, u_k\}$ converges to $\iota'\iota\, u$ in the space $H^{-,\gamma}(\mathcal{D})$, and $\iota'\iota u \neq 0$ because of (10.17) and the injectivity of $\iota'\iota$. Therefore, the weak limit

$$\lim_{k\to\infty} \lambda_k\, \iota'\iota\, u_k = (L_0 + \Delta L)u - f$$

fails to exist, for $\{\lambda_k\}$ is unbounded. A contradiction.

As the operator $I + C(L_0 + \Delta L - \lambda\, \iota'\iota)^{-1}$ is Fredholm and it has index zero, this operator is continuously invertible for all $\lambda \in \mathbb{C}$ satisfying both (10.14) and $|\lambda| \geq M_\varepsilon$. Set

$$N_\varepsilon = \inf\, \|(I + C(L_0 + \Delta L - \lambda\, \iota'\iota)^{-1}) f\|_{-,\gamma} \geq 0,$$

the infimum being over all $f \in H^{-,\gamma}(\mathcal{D})$ of norm 1 and all $\lambda \in \mathbb{C}$ satisfying (10.14) and $|\lambda| \geq M_\varepsilon$. We claim that $N_\varepsilon > 0$. To show this, we argue by contradiction. If $N_\varepsilon = 0$ then there are sequences $\{f_k\}$ in $H^{-,\gamma}(\mathcal{D})$, each f_k being of norm 1, and $\{\lambda_k\}$ satisfying (10.14) and $|\lambda| \geq M_\varepsilon$, such that

$$\lim_{k\to\infty} \|(I + C(L_0 + \Delta L - \lambda_k\, \iota'\iota)^{-1}) f_k\|_{-,\gamma} = 0. \qquad (10.18)$$

Again, by (10.12), the sequence $u_k = (L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1} f_k$ is bounded in $H^{+,\gamma}(\mathcal{D})$. By the weak compactness principle for Hilbert spaces we may assume that the sequences $\{f_k\}$ and $\{u_k\}$ are weakly convergent in the spaces $H^{-,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D})$ to functions f and u, respectively. Since C is compact, the sequence $\{Cu_k\}$ converges to Cu in $H^{-,\gamma}(\mathcal{D})$ and so $\{f_k\}$ converges to f because of (10.18); obviously, $\|f\|_{-,\gamma} = 1$. In particular, we deduce that the sequence $C(L_0 + \varDelta L - \lambda_k, \iota'\iota)^{-1})\, f_k$ converges to $-f$ whence

$$f = -Cu \tag{10.19}$$

with $u \neq 0$.

If the sequence $\{\lambda_k\}$ is bounded in $\mathbb{C}$, then using the weak compactness principle and passing to a subsequence, if necessary, we may assume that $\{\lambda_k\}$ converges to $\lambda_0 \in \mathbb{C}$ which satisfies (10.14) and $|\lambda| \geq M_\varepsilon$. Since

$$\begin{aligned} &(L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1} f_k - (L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1} f \\ &= \ (L_0 + \varDelta L - \lambda_j\, \iota'\iota)^{-1})(f_k - f) + \big((L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1} - (L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1}\big)\, f \end{aligned}$$

and

$$\begin{aligned} &\| \big((L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1} - (L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1}\big)\, f\|_{-,\gamma} \\ &\leq \ |\lambda_k - \lambda_0|\, \|(L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1}\|\, \|(L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1}\|\, \|f\|_{-,\gamma}, \end{aligned}$$

estimate (10.12) implies that in this case the sequence $\{(L_0 + \varDelta L - \lambda_k\, \iota'\iota)^{-1} f_k\}$ converges to $(L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1} f$, and so

$$\big(I + C(L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1}\big)\, f = 0$$

because of (10.18). But λ_0 satisfies (10.14) and $|\lambda| \geq M_\varepsilon$, and hence the injectivity of the operator $I + C(L_0 + \varDelta L - \lambda_0\, \iota'\iota)^{-1}$ established above yields $f = 0$. This contradicts $\|f\| = 1$.

If $\{\lambda_k\}$ is unbounded in $\mathbb{C}$ we can repeat the arguments above. Indeed, then $f_k = (L_0 + \varDelta L - \lambda_k\, \iota'\iota) u_k$ and on passing to the weak limit with respect to $k \to \infty$ we get

$$f = (L_0 + \varDelta L) u - \lim_{k \to \infty} \lambda_k\, \iota'\iota\, u_k.$$

As the operator $\iota'\iota$ is compact, the sequence $\{\iota'\iota\, u_k\}$ converges to $\iota'\iota\, u$ in the space $H^{-,\gamma}(\mathcal{D})$. Moreover, $\iota'\iota\, u \neq 0$ because of (10.19) and the injectivity of $\iota'\iota$. This shows that the weak limit

$$\lim_{n \to \infty} \lambda_k\, \iota'\iota\, u_k = (L_0 + \varDelta L) u - f$$

fails to exist if $\{\lambda_k\}$ is unbounded in $\mathbb{C}$, a contradiction. Therefore, $N_\varepsilon > 0$ and for all $\lambda \in \mathbb{C}$ satisfying (10.14) and $|\lambda| \geq M_\varepsilon$ we obtain

$$\| \big(I + C(L_0 + \varDelta L - \lambda\, \iota'\iota)^{-1}\big)^{-1} \|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \leq 1/N_\varepsilon. \tag{10.20}$$

From estimates (10.12), (10.20) and formula (10.15) it follows that, given any $\lambda \in \mathbb{C}$ satisfying (10.14) and $|\lambda| \geq M_\varepsilon$, the resolvent $\mathcal{R}(\lambda; T)$ exists and

$$\|\mathcal{R}(\lambda; T)\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}))} \leq \mathrm{const}\,(\varepsilon)\, |\lambda|^{-1}.$$

As C is compact, there are only finitely many $\lambda \in \mathbb{C}$ with $|\lambda| < N_\varepsilon$, such that the operator $(I + C(L_0 - \lambda\, \iota'\iota))$ is not injective. Therefore, it follows from formula (10.15) that all eigenvalues of the operator T corresponding to $L_0 + \varDelta L + C$ (except for a finite number) belong to the corner $|\arg \lambda| < \arcsin \|\varDelta L\| + \varepsilon$. Finally, since $\varepsilon > 0$ is arbitrary, all rays (10.11) are rays of minimal growth. By the hypothesis of the theorem, the angles between the pairs of neighboring rays $\arg \lambda = \vartheta$ satisfying (10.11) are less than $2\pi s/n$, and so the statement of the theorem follows from Theorem 10.2. □

Part 5. Non-coercive problems

11. The coercive case

We first consider coercive boundary value problems, i.e. we assume that estimate (8.9) is fulfilled. The sufficient conditions for (8.9) to be true are indicated in Lemmas 8.2, 8.3, 8.10 and 8.11.

Lemma 11.1. *Assume that estimate (8.9) is fulfilled.*

1) If $r = 0$ and the operator Ψ is given by multiplication with a function ψ satisfying $\rho|\psi|^2 \in L^\infty(\partial\mathcal{D}\setminus S)$, then the norms $\|\cdot\|_{+,\gamma}$ and $\|\cdot\|_{H^{1,\gamma}(\mathcal{D})}$ are equivalent, and so the Banach spaces $H^{+,\gamma}(\mathcal{D})$ and $H^{1,\gamma}(\mathcal{D}, S)$ are isomorphic.

2) The conclusion is the same if the operator Ψ maps $H^{1/2,0}(\partial\mathcal{D}, S)$ continuously to $L^2(\partial\mathcal{D})$.

Note that there are continuous embeddings

$$H^{1/2,0}(\partial\mathcal{D}) \hookrightarrow H^{1/2}(\partial\mathcal{D}) \hookrightarrow H^r(\partial\mathcal{D}),$$

if $|r| \le 1/2$. Hence, the hypothesis of the continuity of Ψ is natural, for the domain of Ψ is intended to belong to $H^r(\partial\mathcal{D})$. In the transversal case we get $H^{1/2,0}(\partial\mathcal{D}) = H^{1/2}(\partial\mathcal{D})$.

Proof. From estimate (8.9) it follows that the norm $\|\cdot\|_{+,\gamma}$ is not weaker than the norm $\|\cdot\|_{H^{1,\gamma}(\mathcal{D})}$ on $H^{1,\gamma}(\mathcal{D}, S)$. Furthermore, since the coefficients $a_{i,j}$ are bounded in $\mathcal{D}$, we obtain

$$\int_{\mathcal{D}} \sum_{i,j=1}^{n} \rho^{-2\gamma} a_{i,j} \partial_j u \overline{\partial_i u}\, dx \le c \sum_{j=1}^{n} \|\partial_j u\|^2_{\mathcal{H}^{0,\gamma}(\mathcal{D})} \le c\, \|u\|^2_{H^{1,\gamma}(\mathcal{D})}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$.

Obviously

$$\|a_{0,0} u\|_{H^{0,\gamma}(\mathcal{D})} \le c\, \|u\|_{H^{1,\gamma}(\mathcal{D})}$$

because $\rho^2 a_{0,0} \in L^\infty(\mathcal{D})$.

As $\rho|\psi|^2 \in L^\infty(\partial\mathcal{D} \setminus S)$,

$$\begin{array}{rcl} \displaystyle\int_{\partial\mathcal{D}\setminus S} \rho^{-2\gamma}\, |\psi(x)|^2 |u(x)|^2\, ds & = & \displaystyle\int_{\partial\mathcal{D}\setminus S} \rho|\psi(x)|^2\, \rho^{-2(\gamma+1/2)} |u(x)|^2\, ds \\ & \le & c\, \|u\|^2_{\mathcal{H}^{0,\gamma+1/2}(\partial\mathcal{D})} \\ & \le & c\, \|u\|^2_{H^{1/2,\gamma}(\partial\mathcal{D})} \\ & \le & c\, \|u\|^2_{H^{1,\gamma}(\mathcal{D})} \end{array} \tag{11.1}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, the last inequality being a consequence of Theorem 4.13. Here, by c is meant a constant independent of u, which can be diverse in different applications.

On combining the above estimates we deduce immediately that there is a constant c with the property that

$$\|u\|_{+,\gamma} \le c\,\|u\|_{H^{1,\gamma}(\mathcal{D})}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, as desired.

If $u \in H^{1,\gamma}(\mathcal{D}, S)$, then $\rho^{-\gamma}u \in H^{1,0}(\mathcal{D}, S)$ and so the restriction of $\rho^{-\gamma}u$ to the boundary belongs to $H^{1/2,0}(\partial\mathcal{D}, S)$. Under the coercive estimate (8.9), the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1,\gamma}(\mathcal{D})$. Now, if Ψ maps $H^{1/2,0}(\partial\mathcal{D}, S)$ continuously to $L^2(\partial\mathcal{D})$, then, by Corollary 3.3 and Theorem 4.13, we get

$$\begin{aligned} \|\,\Psi(\rho^{-\gamma}u)\|_{L^2(\partial\mathcal{D})} &\le c\,\|\rho^{-\gamma}u\|_{H^{1/2,0}(\partial\mathcal{D})} \\ &\le c\,\|u\|_{H^{1/2,\gamma}(\partial\mathcal{D})} \\ &\le c\,\|u\|_{H^{1,\gamma}(\mathcal{D})} \end{aligned} \tag{11.2}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$, with c a constant independent of u and different in diverse applications.

Thus, using (11.2) instead of (11.1) in the above arguments we again obtain the desired statement. □

Let us discuss the estimate (8.44).

Lemma 11.2. *Let (8.9) hold. In each case of 1)-3) there is a constant c with the property that*

1) If $\rho a_j \in L^\infty(\mathcal{D})$ for all $1 \le j \le n$, then

$$|(a_j\partial_j u, v)_{H^{0,\gamma}(\mathcal{D})}| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma} \tag{11.3}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

2) If the operator ΔB_0 is given by multiplication with a function Δb_0 satisfying $\rho\Delta b_0/b_1 \in L^\infty(\partial D \setminus S)$, then

$$|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma} \tag{11.4}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

3) If the operator $b_1^{-1}\Delta B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D}, S)$ continuously to $H^{-1/2,\gamma}(\partial\mathcal{D})$ then

$$|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

In the transversal case 3) just amounts to saying that the operator $\rho^\gamma b_1^{-1}\Delta B_0\rho^{-\gamma}$ maps $H^{1/2}(\partial\mathcal{D}, S)$ continuously to $H^{-1/2}(\partial\mathcal{D})$ (cf. Lemma 8.13 with $r = 1/2$).

Proof. Inequality (11.3) follows from estimates (8.9) and Lemma 8.13 in an obvious way.

Furthermore, (8.9) implies the continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1,\gamma}(\mathcal{D})$. For $u, v \in H^{1,\gamma}(\mathcal{D})$, the traces on the surface $\partial\mathcal{D}$ belong to $H^{1/2,\gamma}(\partial\mathcal{D})$, which is due to Lemma 4.14. Hence, using Lemmas 8.10 and 4.14 we obtain

$$\begin{aligned} |(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| &= |(\rho b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma+1/2}(\partial\mathcal{D}\setminus S)}| \\ &\le c\,\|u\|_{H^{1/2,\gamma}(\partial\mathcal{D})}\|v\|_{H^{1/2,\gamma}(\partial\mathcal{D})} \\ &\le c\,\|u\|_{H^{1,\gamma}(\mathcal{D})}\|v\|_{H^{1,\gamma}(\mathcal{D})} \end{aligned}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$, i.e. (11.4) holds true.

Finally, if $b_1^{-1}\Delta B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D},S)$ continuously to $H^{-1/2,\gamma}(\partial\mathcal{D})$, then, by duality and Theorem 4.13,

$$\begin{array}{rcl} |(b_1^{-1}\Delta B_0 u,v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| & \leq & \|b_1^{-1}\Delta B_0 u\|_{H^{-1/2,\gamma}(\partial\mathcal{D})}\|v\|_{H^{1/2,\gamma}(\partial\mathcal{D}\setminus S)} \\ & \leq & c\,\|u\|_{H^{1/2,\gamma}(\partial\mathcal{D})}\|v\|_{H^{1/2,\gamma}(\partial\mathcal{D}\setminus S)} \\ & \leq & c\,\|u\|_{H^{1,\gamma}(\mathcal{D})}\|v\|_{H^{1,\gamma}(\mathcal{D})} \end{array}$$

for all $u,v\in H^{1,\gamma}(\mathcal{D},S)$. □

Lemma 11.2 and estimate (8.43) show readily that in the coercive case estimate (8.44) concerns for the most part the mere summand

$$(b_1^{-1}\partial_t u,v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}$$

in the sesquilinear form $Q(u,v)$.

Let $t_1(x),\dots,t_{n-1}(x)$ be a basis of tangential vectors of the boundary surface at a point $x\in\partial\mathcal{D}$. Then we can write

$$\partial_t=\sum_{j=1}^{n-1}k_j(x)\,\partial_{t_j}$$

where $k_1,\dots,k_{n-1}$ are bounded functions on the boundary vanishing at S.

Lemma 11.3. *Suppose (8.9) is fulfilled. If k_j/b_1 is of Hölder class $C^{0,\lambda}$ in the closure of $\partial\mathcal{D}\setminus S$ for all $1\leq j\leq n-1$, with $\lambda>1/2$, then*

$$|(b_1^{-1}\partial_t u,v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}|\leq c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma} \tag{11.5}$$

for all $u,v\in H^{1,\gamma}(\mathcal{D},S)$.

Proof. By assumption, $\partial\mathcal{D}$ is a compact closed Lipschitz manifold and ∂_{t_j} first order differential operators with bounded coefficients on $\partial\mathcal{D}$. Such operators map $H^{1/2,\gamma}(\partial\mathcal{D})=\mathcal{H}^{1/2,1/2+\gamma}(\partial\mathcal{D})$ continuously to $H^{-1/2,\gamma}(\partial\mathcal{D})=\mathcal{H}^{-1/2,-1/2+\gamma}(\partial\mathcal{D})$, the dual of $\mathcal{H}^{1/2,1/2-\gamma}(\partial\mathcal{D})$.

Recall that for each Hölder continuous function $f\in C^{0,\lambda}(K)$ on a compact set $K\subset\mathbb{R}^n$, with $0<\lambda\leq 1$, there is an explicit extension F to all of $\mathbb{R}^n$, which is given by

$$F(x)=\inf_{y\in K}(f(y)+\|f\|_{C^{0,\lambda}(K)}|x-y|^{\lambda}) \tag{11.6}$$

and satisfies $\|F\|_{C^{0,\lambda}(\tilde{K})}\leq\|f\|_{C^{0,\lambda}(K)}$ for any larger compact $\tilde{K}$.

Applying this result to each function k_j/b_1 on the compact set $\overline{\partial\mathcal{D}\setminus S}$ with $1/2<\lambda\leq 1$, we find functions $F_j\in C^{0,\lambda}(\partial\mathcal{D})$ satisfying $F_j=k_j/b_1$ in $\partial\mathcal{D}\setminus S$ and such that

$$\|F_j\|_{C^{0,\lambda}(\partial\mathcal{D})}\leq\|k_j/b_1\|_{C^{0,\lambda}(\overline{\partial\mathcal{D}\setminus S})}.$$

It follows that

$$\begin{array}{rcl} |(b_1^{-1}\partial_t u,v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| & \leq & \displaystyle\sum_{j=1}^{n-1}|(\partial_{t_j}u,\overline{b_1^{-1}k_j}v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \\ & \leq & \displaystyle\sum_{j=1}^{n-1}\|\partial_{t_j}u\|_{H^{-1/2,\gamma}(\partial\mathcal{D})}\|\overline{F_j}v\|_{H^{1/2,\gamma}(\partial\mathcal{D})} \end{array} \tag{11.7}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D}, S)$, the last estimate being due to the generalised Schwarz inequality.

It is well known that multiplication by functions of Hölder class $C^{0,\lambda}(\partial\mathcal{D})$ with $1/2 < \lambda \le 1$ is a bounded linear operator in $H^{1/2}(\partial\mathcal{D})$ (see for instance [Slo58, § 3], [Pal96, Lemma 2.1] and elsewhere). Hence, there is a constant c with the property that

$$\|\overline{F_j} v\|_{H^{1/2}(\partial\mathcal{D})} \le c\, \|v\|_{H^{1/2}(\partial\mathcal{D})}$$

for all $v \in H^{1/2}(\partial\mathcal{D})$. If now $v \in H^{1/2,\gamma}(\partial\mathcal{D})$, then $\rho^{-\gamma} v \in H^{1/2,0}(\partial\mathcal{D}) = H^{1/2}(\partial\mathcal{D})$ whence

$$\begin{array}{rcl} \|\overline{F_j} v\|_{H^{1/2,\gamma}(\partial\mathcal{D})} & \le & \|\overline{F_j}(\rho^{-\gamma} v)\|_{H^{1/2}(\partial\mathcal{D})} \\ & \le & c\, \|\rho^{-\gamma} v\|_{H^{1/2}(\partial\mathcal{D})} \\ & \le & c\, \|v\|_{H^{1/2,\gamma}(\partial\mathcal{D})}. \end{array}$$

We have thus proved that multiplication by a function of class $C^{0,\lambda}(\overline{\partial\mathcal{D} \setminus S})$ is a bounded linear operator on $\mathcal{H}^{1/2,\gamma}(\partial\mathcal{D}, S)$. Summarising we estimate the right-hand side of (11.7) by

$$\begin{array}{rcl} c\, \|u\|_{H^{1/2,\gamma}(\partial\mathcal{D})} \|v\|_{H^{1/2,\gamma}(\partial\mathcal{D})} & \le & c\, \|u\|_{H^{1,\gamma}(\mathcal{D})} \|v\|_{H^{1,\gamma}(\mathcal{D})} \\ & \le & c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}, \end{array}$$

where the constant c does not depend on u and v and may be different in diverse applications. $\square$

Remark 11.4. Lemma 11.3 is actually true under weaker assumptions. We need not require than $k_j/b_1 \in C^{0,\lambda}(\overline{\partial\mathcal{D} \setminus S})$. These quotients must be just multipliers for the space $H^{1/2}(\partial\mathcal{D}, S)$. For example, this is the case for a measurable function m on $\partial\mathcal{D} \setminus S$ if

$$\sup_{x \in \partial\mathcal{D} \setminus S} |m(x)|^2 + \sup_{x \in \partial\mathcal{D} \setminus S} \int_{\partial\mathcal{D} \setminus S} \frac{|m(x) - m(y)|^2}{|x - y|^n}\, ds$$

is finite, see [Slo58, § 3], [Pal96, Lemma 2.1]) and elsewhere.

Our next concern will be to describe those perturbations of a_j and $\Delta b_0/b_1$ and t which preserve the completeness property of root functions of the operator L_0^{-1} under condition (8.9).

Lemma 11.5. *Let estimate (8.9) hold and $t = 0$. Suppose there is a number $\varepsilon > 0$ such that*

$$\begin{array}{rcl} \rho^{1-\varepsilon} \Big(a_j - 2\gamma\, \rho^{-1} \displaystyle\sum_{i=1}^{n} a_{i,j} (\partial_i \rho) \Big) & \in & L^{\infty}(\mathcal{D}), \\ \rho^{2-\varepsilon} a_0 & \in & L^{\infty}(\mathcal{D}) \end{array}$$

and either the operator ΔB_0 is given by multiplication with a function Δb_0 satisfying $\rho^{1-\varepsilon} \Delta b_0/b_1 \in L^{\infty}(\partial\mathcal{D} \setminus S)$ or $b_1^{-1} \Delta B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D})$ compactly to $H^{-1/2,\gamma}(\partial\mathcal{D})$. Then the operator

$$\Delta L = L - L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$$

is compact.

In the case of regular singularities the last condition means that the operator $\rho^\gamma b_1^{-1} \Delta B_0 \rho^{-\gamma}$ maps $H^{1/2}(\partial\mathcal{D})$ compactly to $H^{-1/2}(\partial\mathcal{D})$. It is the case, e.g., if $\rho^\gamma b_1^{-1} \Delta B_0 \rho^{-\gamma}$ maps $H^{1/2}(\partial\mathcal{D})$ continuously to $H^{-1/2+\varepsilon,\gamma}(\partial\mathcal{D})$ with some $\varepsilon > 0$.

Proof. Fix $v \in H^{1,\gamma}(\mathcal{D})$ and consider the function $w = \rho^{\varepsilon-1} v$. By Corollary 3.3, we get $w \in H^{1,\gamma+\varepsilon-1}(\mathcal{D}) = \mathcal{H}^{1,\gamma+\varepsilon}(\mathcal{D})$. The embedding $e : H^{1,\gamma+\varepsilon-1}(\mathcal{D}) \to \mathcal{H}^{0,\gamma}(\mathcal{D})$ is compact because of Corollary 4.8.

Fix a bounded sequence $\{u_\nu\}$ in $H^{+,\gamma}(\mathcal{D})$. Estimate (8.9) implies that $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1,\gamma}(\mathcal{D})$. Hence, the sequence

$$F_\nu = \rho^{1-\varepsilon} \sum_{j=1}^{n} \Big(a_j - 2\gamma\, \rho^{-1} \sum_{i=1}^{n} a_{i,j} (\partial_i \rho) \Big) \partial_j u_\nu + \rho^{2-\varepsilon} a_0 (\rho^{-1} u_\nu)$$

is bounded in $H^{0,\gamma}(\mathcal{D})$. According to the weak compactness principle we may assume without restriction of generality that the sequence converges weakly to zero in $H^{0,\gamma}(\mathcal{D})$.

On the other side, if $\Delta B_0 = 0$ then

$$(\Delta L u_\nu, v)_\gamma = (F_\nu, e\mathrm{Op}\,(\rho^{1-\varepsilon}) v)_{\mathcal{H}^{0,\gamma}(\mathcal{D})} = ((e\mathrm{Op}\,(\rho^{1-\varepsilon}))^* F_\nu, v)_{H^{1,\gamma}(\mathcal{D})}$$

whence

$$\begin{array}{rcl} |(\Delta L u_\nu, v)_\gamma| & \leq & \|(e\mathrm{Op}\,(\rho^{1-\varepsilon}))^* F_\nu\|_{H^{1,\gamma}(\mathcal{D})} \|v\|_{H^{1,\gamma}(\mathcal{D})} \\ & \leq & c\, \|(e\mathrm{Op}\,(\rho^{1-\varepsilon}))^* F_\nu\|_{H^{1,\gamma}(\mathcal{D})} \|v\|_{H^{+,\gamma}(\mathcal{D})}, \end{array}$$

where

$$(e\mathrm{Op}\,(\rho^{1-\varepsilon}))^* : \mathcal{H}^{0,\gamma}(\mathcal{D}) \to \mathcal{H}^{1,1+\gamma}(\mathcal{D})$$

is the Hilbert space adjoint for $e\mathrm{Op}\,(\rho^{1-\varepsilon})$ and the constant c does not depend on u and v.

The operator $e\mathrm{Op}\,(\rho^{1-\varepsilon})$ is compact, for $\mathrm{Op}\,(\rho^{1-\varepsilon})$ is bounded and e is compact. Hence

$$\|\Delta L u_\nu\|_{-,\gamma} \leq c\, \|e^* F_\nu\|_{\mathcal{H}^{1,\gamma+\varepsilon}(\mathcal{D})} \to 0$$

as $\nu \to \infty$, i.e. $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is compact, too.

Let

$$a_j = 2\gamma\, \rho^{-1} \sum_{i=1}^{n} a_{i,j} (\partial_i \rho),$$

for $0 \leq j \leq n$, and let the operator ΔB_0 be given by multiplication with a function Δb_0 satisfying $\rho^{1-\varepsilon} \Delta b_0 / b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$. According to Lemma 4.14 the trace operator $t_1 : H^{1,\gamma}(\mathcal{D}) \to H^{1/2,\gamma}(\partial\mathcal{D})$ is bounded. Hence it follows that the sequence

$$U_\nu = \begin{cases} \rho^{1-\varepsilon} b_1^{-1} \Delta b_0\, t_1 u_\nu & \text{on} \quad \partial\mathcal{D} \setminus S, \\ 0 & \text{on} \quad S \end{cases}$$

is bounded in $\mathcal{H}^{0,\gamma+1/2}(\partial\mathcal{D})$. Again we may assume that it converges weakly to zero in $\mathcal{H}^{0,\gamma+1/2}(\partial\mathcal{D})$. Then

$$\begin{array}{rcl} (\Delta L u_\nu, v)_\gamma & = & (b_1^{-1} \Delta b_0\, t_1 u_\nu, t_1 v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)} \\ & = & (\rho^{1-\varepsilon} b_1^{-1} \Delta b_0\, t_1 u_\nu, \rho^\varepsilon t_1 v)_{H^{0,\gamma+1/2}(\partial\mathcal{D}\setminus S)}. \end{array}$$

By Corollary 3.3, the operator $\mathrm{Op}\,(\rho^\varepsilon)$ maps the space $H^{1/2,\gamma}(\partial\mathcal{D})$ continuously to $H^{1/2,\gamma+\varepsilon}(\partial\mathcal{D}) = \mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})$. Furthermore, Corollary 4.9 implies that the embedding $e : \mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D}) \hookrightarrow \mathcal{H}^{0,1/2+\gamma}(\partial\mathcal{D})$ is compact. Therefore,

$$\begin{array}{rcl} (\Delta L u_\nu, v)_\gamma & = & (U_\nu, e\mathrm{Op}\,(\rho^\varepsilon)\, t_1 v)_{\mathcal{H}^{1/2,1/2+\gamma}(\partial\mathcal{D})} \\ & = & (e^* U_\nu, \mathrm{Op}\,(\rho^\varepsilon)\, t_1 v)_{\mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})} \end{array}$$

where $e^* : \mathcal{H}^{0,1/2+\gamma}(\partial\mathcal{D}) \to \mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})$ is the Hilbert space adjoint for e. As the operator

$$\mathrm{Op}\,(\rho^\varepsilon) t_1 : H^{1,\gamma}(\mathcal{D}) \to \mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})$$

is bounded (see Lemma 4.14 and Corollary 3.3) we see that

$$\begin{array}{rcl} |(\Delta L u_\nu, v)_\gamma| & \le & c\, \|e^* U_\nu\|_{\mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})} \|v\|_{H^{1,\gamma}(\mathcal{D})} \\ & \le & c\, \|e^* U_\nu\|_{\mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})} \|v\|_{H^{+,\gamma}(\mathcal{D})} \end{array}$$

with c a constant independent of u and v. It follows that

$$\|\Delta L u_\nu\|_{-,\gamma} \le \|e^* U_\nu\|_{\mathcal{H}^{1/2,1/2+\gamma+\varepsilon}(\partial\mathcal{D})} \to 0,$$

as $\nu \to \infty$, i.e. ΔL is compact in this case, too.

Finally, suppose that

$$a_j = 2\gamma\, \rho^{-1} \sum_{i=1}^{n} a_{i,j} (\partial_i \rho),$$

for $0 \le j \le n$, and $b_1^{-1} \Delta B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D})$ compactly to $H^{-1/2,\gamma}(\partial\mathcal{D})$. Then the sequence

$$U_\nu = b_1^{-1} \Delta B_0\, u_\nu$$

is precompact in $H^{-1/2,\gamma}(\partial\mathcal{D})$. We may assume that $\{u_\nu\}$ converges weakly to zero in $H^{1/2,\gamma}(\partial\mathcal{D})$, and then $\{U_\nu\}$ converges to zero in $H^{-1/2,\gamma}(\partial\mathcal{D})$. By duality and Theorem 4.13,

$$\begin{array}{rcl} |(\Delta L u_\nu, v)_\gamma| & = & |(b_1^{-1} \Delta B_0\, u_\nu, v)_{\gamma,\partial\mathcal{D}}| \\ & \le & \|U_\nu\|_{H^{-1/2,\gamma}(\partial\mathcal{D})} \|v\|_{H^{1/2,\gamma}(\partial\mathcal{D})} \\ & \le & c\, \|U_\nu\|_{H^{-1/2,\gamma}(\partial\mathcal{D})} \|v\|_{H^{1,\gamma}(\mathcal{D})} \\ & \le & c\, \|U_\nu\|_{H^{-1/2,\gamma}(\partial\mathcal{D})} \|v\|_{+,\gamma}, \end{array}$$

where c is a constant independent of v and ν. Therefore,

$$\|\Delta L u_\nu\|_{-,\gamma} \le c\, \|U_\nu\|_{H^{-1/2,\gamma}(\partial\mathcal{D})} \to 0,$$

as $\nu \to \infty$, i.e. ΔL is compact in this case, as desired. □

Typical compact mappings from $H^{1/2}(\partial\mathcal{D})$ to $H^{-1/2}(\partial\mathcal{D})$ are discussed in Example 8.8.

We now split

$$a_j = 2\gamma \rho^{-1} \sum_{i=1}^{n} a_{i,j} \partial_i \rho + \Delta_c a_j + \Delta_s a_j$$

and

$$\begin{array}{rcl} \Delta a_0 & = & \Delta_c a_0 + \Delta_s a_0, \\ \Delta B_0 & = & \Delta_c B_0 + \Delta_s B_0, \end{array}$$

where the terms $\Delta_c a_j$, $\Delta_c a_0$ and $\Delta_c B_0$ induce compact operators and the terms $\Delta_s a_j$, $\Delta_s a_0$ and $\Delta_s B_0$ "small" operators,

$$| \sum_{j=1}^{n} (\Delta_s a_j \partial_j u + \Delta_s a_0 u, v)_{H^{0,\gamma}(\mathcal{D})} + ((\partial_t + \Delta_s B_0) u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)} \, | \leq M \, \|u\|_{+,\gamma} \|v\|_{+,\gamma} \tag{11.8}$$

for all $u, v \in H^{+,\gamma}(\mathcal{D})$, with $M > 0$ a constant independent of u and v.

Corollary 11.6. *Suppose that estimate (8.9) is fulfilled, $k_j/b_1 \in C^{0,\lambda}(\overline{\partial\mathcal{D} \setminus S})$ for all $1 \leq j \leq n-1$, with $\lambda > 1/2$, $\Delta_s B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D}, S)$ continuously to $H^{-1/2,\gamma}(\partial\mathcal{D})$, and the constant M in (11.8) is less than one. If there is a number $\varepsilon > 0$, such that $\rho^{1-\varepsilon} \Delta_c a_j \in L^\infty(\mathcal{D})$, for $1 \leq j \leq n$, $\rho^{2-\varepsilon} \Delta_c a_0 \in L^\infty(\mathcal{D})$, and either the operator $\Delta_c B_0$ is given by multiplication with a function $\Delta_c b_0$ satisfying $\rho^{1-\varepsilon} \Delta_c b_0 / b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$ or $b_1^{-1} \Delta_c B_0$ maps $H^{1/2,\gamma}(\partial\mathcal{D})$ compactly to $H^{-1/2,\gamma}(\partial\mathcal{D})$, then problem (8.47) is Fredholm of index zero. Besides, if $\Delta_c a_j = 0$ for all $1 \leq j \leq n$, $\Delta_c a_0 = 0$, and $\Delta_c B_0 = 0$, then, in fact, problem (8.47) is uniquely solvable and the inverse operator $L^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D}) \subset H^{1,\gamma}(\mathcal{D}, S)$ is bounded.*

Proof. It follows from Lemmas 8.13, 11.2, 11.3 and 11.5 that (8.44) is fulfilled under the hypothesis of the corollary.

Denote by $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ the operator defined by the terms $\Delta_s a_j$, $\Delta_s a_0$, ∂_t, and $\Delta_s B_0$, as described in Section 8. As the constant M in (11.8) is less than one, we deduce that

$$\|\Delta L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}), H^{-,\gamma}(\mathcal{D}))} < 1 = \|L_0^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}), H^{+,\gamma}(\mathcal{D}))},$$

and so a familiar argument shows that the operator $L_0 + \Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is invertible.

According to Lemma 11.5, the operator $C = L - L_0 - \Delta L$ is compact. Therefore, problem (8.47) is equivalent to the Fredholm-type operator equation

$$(I + (L_0 + \Delta L)^{-1} C) u = (L_0 + \Delta L)^{-1} f$$

in $H^{+,\gamma}(\mathcal{D})$ with compact operator $(L_0 + \Delta L)^{-1} C : H^{+,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D})$. This establishes the corollary. □

Corollary 11.7. *Under the hypotheses of Corollary 11.6, if $M < \sin \pi/n$, with M being the constant from (11.8), then the system of root functions of the corresponding closed operator T in $H^{-,\gamma}(\mathcal{D})$ is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{1,\gamma}(\mathcal{D}, S)$, and, for any $\delta > 0$, all eigenvalues of T (except for a finite number) lie in the corner $|\arg \lambda| < \delta + \arcsin M$ in $\mathbb{C}$. If moreover $\Delta_c a_j = 0$ for all $1 \leq j \leq n$, $\Delta_c a_0 = 0$, and $\Delta_c B_0 = 0$, then all eigenvalues of T belong to the corner $|\arg \lambda| \leq \arcsin M$ in $\mathbb{C}$.*

Proof. This is a straightforward consequence of Theorem 10.6 and Corollary 11.6. As the constant M in (11.8) is less than one, it suffices to apply Theorem 9.10 combined with Corollary 11.6. The last statement follows from Theorem 10.5 and Corollary 11.6. □

We now discuss several examples. The most illustrative of them is of perhaps the Dirichlet problem.

Example 11.8. Let $S = \partial\mathcal{D}$ and (8.7) hold with $\delta = 1$. By Lemmas 8.11 and 11.1, we get $H^{+,\gamma}(\mathcal{D}) = H^{1,\gamma}(\mathcal{D}, \partial\mathcal{D})$. We thus arrive at the Dirichlet problem

$$\left\{ \begin{array}{rcll} Au & = & f & \text{in} \ \ \mathcal{D}, \\ u & = & 0 & \text{at} \ \ \partial\mathcal{D} \end{array} \right. \tag{11.9}$$

for a complex-valued function $u \in H^{+,\gamma}(\mathcal{D})$, with a given distribution $f \in H^{-,\gamma}(\mathcal{D})$. By Lemma 11.5, the problem is Fredholm provided there is $\varepsilon > 0$ such that

$$\begin{array}{rcl} \rho^{1-\varepsilon}\Big(a_j - 2\gamma\,\rho^{-1}\displaystyle\sum_{i=1}^{n} a_{i,j}(\partial_i\rho)\Big) & \in & L^\infty(\mathcal{D}), \\ \rho^{2-\varepsilon} a_0 & \in & L^\infty(\mathcal{D}) \end{array}$$

for all $1 \le j \le n$. The spectral properties of such a problem in weighted Sobolev spaces are similar to those in the usual Sobolev spaces (corresponding to the case $\rho = 1$), see for instance [Kel51], [Mik76]. However, we note that if ρ vanishes on $\partial\mathcal{D}$ then the summand $\varDelta a_0$ induces a bounded operator from $H^{+,\gamma}(\mathcal{D})$ to $H^{-,\gamma}(\mathcal{D})$ only in the case $\rho^2 \varDelta a_0 \in L^\infty(\mathcal{D})$. This means that in weighted spaces the Dirichlet problem for the Laplace equation might be non-Fredholm for certain weight indices γ. In the case of regular singularities our techniques allows one to we consider the Dirichlet problem also with non-zero boundary data. Indeed, let the norms of the spaces $H^{1,0}(\mathcal{D})$ and $H^1(\mathcal{D})$ be equivalent. Consider the inhomogeneous Dirichlet problem

$$\left\{ \begin{array}{rcll} Au & = & f & \text{in} \ \ \mathcal{D}, \\ u & = & u_0 & \text{at} \ \ \partial\mathcal{D} \end{array} \right. \tag{11.10}$$

for a function $u \in H^{1,\gamma}(\mathcal{D})$, where $f \in H^{-,\gamma}(\mathcal{D})$ and $u_0 \in H^{1/2,\gamma}(\partial\mathcal{D})$ are given data. Using spectral synthesis in Sobolev spaces (see [HW83]) we verify that the case $u_0 = 0$ corresponds to the Dirichlet problem (11.9). By Corollary 5.6, there is a bounded inverse $t_1^{-1} : H^{1/2,\gamma}(\partial\mathcal{D}) \to H^{1,\gamma}(\mathcal{D})$. Therefore, (11.10) is a Fredholm problem, for it is solvable if and only if problem (11.9) is solvable with f replaced by $f - Lt_1^{-1}u_0$, and its solution is given by $u = u(f) + t_1^{-1}u_0$, where $u(f)$ is a solution to the corresponding problem (11.9). Here, by $L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is meant the operator induced by (11.9), as explained above.

After Zaremba [Zar10], the following mixed boundary value problem was suggested to him by W. Wirtinger.

Example 11.9. Consider the mixed problem

$$\left\{ \begin{array}{rcll} -\varDelta_n u + \rho^{-1}\displaystyle\sum_{j=1}^{n}(\rho^\varepsilon a_j + 2\gamma\partial_j\rho)\partial_j u + \rho^{-2}(\rho^\varepsilon a_0 + 1)u & = & f & \text{in} \ \ \mathcal{D}, \\ u & = & 0 & \text{at} \ \ S, \\ \partial_v u & = & u_1 & \text{at} \ \ \partial\mathcal{D}\setminus S \end{array} \right. \tag{11.11}$$

for a real-valued function u, where $\varDelta_n$ is the Laplace operator in $\mathbb{R}^n$, the coefficients $a_1, \ldots, a_n$ and a_0 are assumed to be bounded functions in $\mathcal{D}$, and $\partial_v = \partial_\nu + \partial_t$ with a tangential vector field $t(x)$ on $\partial\mathcal{D}$ whose coefficients are functions of class $C^{0,\lambda}(\overline{\partial\mathcal{D}\setminus S})$ vanishing on S, and $\varepsilon > 0$. In this case $a_{i,j} = \delta_{i,j}$, $b_0 = \chi_S$ is the characteristic function of the boundary set S, and $b_1 = \chi_{\partial\mathcal{D}\setminus S}$ is that of $\partial\mathcal{D}\setminus S$. From the results of the previous section it follows that the root functions related to problem (11.11) in the space $H^{+,\gamma}(\mathcal{D}) = H^{1,\gamma}(\mathcal{D}, S)$ are complete in $H^{-,\gamma}(\mathcal{D})$,

$H^{0,\gamma}(\mathcal{D})$ and $H^{+,\gamma}(\mathcal{D},S)$ for all $t(x)$ of sufficiently small length. If $\rho \equiv 1$ then $H^{+,\gamma}(\mathcal{D}) = H^1(\mathcal{D},S)$ and $H^{0,\gamma}(\mathcal{D}) = L^2(\mathcal{D})$, i.e. the results are applicable to usual Sobolev spaces over Lipschitz domains. In this case (11.11) is a mixed problem of Zaremba type for the Helmholtz equation in Sobolev spaces. It becomes the classical problem of Zaremba for the Laplace operator provided $a_0 = -1$ and $a_j = 0$ for $1 \leq j \leq n$ (cf. [Zar10]).

Weight functions ρ enable to enlarge the class of function spaces which are used to find adequate function-theoretic setting of the boundary value problem. Although such functions ρ may be quite whimsical, merely property (5.1) is of crucial importance.

We finish the section by showing a second order elliptic differential operator in the plane for which no Zaremba-type problem is Fredholm (see [ST12]). The idea is traced back to a familiar example of A. V. Bitsadze (1948).

Example 11.10. Let $A = \bar{\partial}^2$ be the square of the Cauchy-Riemann operator in the plane of complex variable z. We choose $\mathcal{D}$ to be the upper half-disk of radius 1, i.e. the set of all $z \in \mathbb{C}$ satisfying $|z| < 1$ and $\Im z > 0$. As S we take the upper half-circle, i.e. the part of $\partial\mathcal{D}$ lying in the upper half-plane. Consider the function sequence

$$u_\nu(z) = (|z|^2 - 1)\, \frac{\sin(\nu z)}{\nu^s},$$

for $\nu = 1, 2, \ldots$, where s is a fixed positive number. Each function u_ν satisfies $Au_\nu = 0$ in the plane and vanishes on S. Moreover, for any differential operator B of order $< s$ with bounded coefficients, the sequence $\{Bu_\nu\}$ converges to zero uniformly on $\partial\mathcal{D} \setminus S = [-1, 1]$. Since $|u_\nu(z)| \to \infty$ for all $z \in \mathcal{D}$, we deduce that no reasonable setting of Zaremba-type problem is possible.

12. The non-coercive case

To the best of our knowledge the completeness of root functions has been studied for elliptic boundary value problems, i.e. for those satisfying the Shapiro-Lopatinskii condition. If the boundary is non-smooth, by the Shapiro-Lopatinskii condition is meant any generalisation of this condition in the context of operator algebras with symbolic structure. In this section we consider an example where the Shapiro-Lopatinskii condition is violated (cf. [ST12] for the case of usual Sobolev spaces).

Our focus will be upon the case where embedding (9.9) is fulfilled for some $0 < s < 1$.

Lemma 11.3 shows that there is no hope for the tangential operator ∂_t to be bounded in the non-coercive situation. But as in the non-coercive case the conormal derivative ∂_c contains already an essential tangential part, it is natural to assume in this section that $t = 0$.

It should be also noted that in the absence of coercivity the cases $1/2 < s < 1$ and $0 < s \leq 1/2$ may differ drastically because the functions from $H^{s,\gamma}(\mathcal{D})$ no longer need possess traces on $\partial\mathcal{D}$, if $0 < s \leq 1/2$. In particular, the significance of boundary terms in the norm $\| \cdot \|_{+,\gamma}$ increases for $0 < s \leq 1/2$.

Lemma 12.1. *Suppose that there is a continuous embedding (9.9) for some index s satisfying $1/2 < s < 1$.*

1) If $\rho^{2s-1}\Delta b_0/b_1 \in L^\infty(\partial D \setminus S)$ then there is a constant $c > 0$ with the property that

$$|(b_1^{-1}\Delta b_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

2) If the operator $b_1^{-1}\Delta B_0$ maps $H^{s-1/2,\gamma}(\partial\mathcal{D}, S)$ continuously to $H^{1/2-s,\gamma}(\partial\mathcal{D})$ then

$$|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$.

In the case of regular singularities the condition 2) just amounts to saying that the operator $\rho^\gamma b_1^{-1}\Delta B_0 \rho^{-\gamma}$ maps $H^{s-1/2}(\partial\mathcal{D}, S)$ continuously to $H^{1/2-s}(\partial\mathcal{D})$.

Proof. In this case, for $u, v \in H^{s,\gamma}(\mathcal{D})$, the traces $t_s u$ and $t_s v$ belong to $H^{s-1/2,\gamma}(\partial\mathcal{D})$ (see Lemma 4.16). Hence, using Lemma 4.16 we get

$$\begin{aligned}
|(b_1^{-1}\Delta b_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| &= |(\rho^{2s-1}b_1^{-1}\Delta b_0 u, v)_{H^{0,s-1/2+\gamma}(\partial\mathcal{D}\setminus S)}| \\
&\le c\,\|t_s u\|_{H^{0,s-1/2+\gamma}(\partial\mathcal{D})}\|t_s v\|_{H^{0,s-1/2+\gamma}(\partial\mathcal{D})} \\
&\le c\,\|u\|_{H^{s,\gamma}(\mathcal{D})}\|v\|_{H^{s,\gamma}(\mathcal{D})} \\
&\le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma}
\end{aligned}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$, where c is a constant independent on u and v and different in diverse applications. This proves part 1) of the lemma.

Finally, if $b_1^{-1}\Delta B_0$ maps $H^{s-1/2,\gamma}(\partial\mathcal{D}, S)$ continuously to $H^{1/2-s,\gamma}(\partial\mathcal{D})$, then, by duality and Theorem 4.13, we obtain

$$\begin{aligned}
|(b_1^{-1}\Delta B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| &\le \|b_1^{-1}\Delta B_0 u\|_{H^{1/2-s,\gamma}(\partial\mathcal{D})}\|v\|_{H^{s-1/2,\gamma}(\partial\mathcal{D}\setminus S)} \\
&\le c\,\|u\|_{H^{s-1/2,\gamma}(\partial\mathcal{D}\setminus S)}\|v\|_{H^{s-1/2,\gamma}(\partial\mathcal{D}\setminus S)} \\
&\le c\,\|u\|_{+,\gamma}\|v\|_{+\gamma}
\end{aligned}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D}, S)$, showing part 2). □

For $0 < s \le 1/2$, we no longer can exploit Lemma 12.1. In this case we have to use Lemma 8.13 in order to guarantee inequality (8.44).

Moreover, in the absence of coercivity we ought to assume additionally that the estimate

$$\Big|\Big(\sum_{j=1}^n (a_j - 2\gamma\,\rho^{-1}\sum_{i=1}^n a_{i,j}(\partial_i\rho))\partial_j u, v\Big)_{H^{0,\gamma}(\mathcal{D})}\Big| \le c\,\|u\|_{+,\gamma}\|v\|_{+,\gamma} \qquad (12.1)$$

holds for all $u, v \in H^{+,\gamma}(\mathcal{D})$ with a constant $c > 0$ independent on u and v.

To cope with this condition we need the following lemma.

Lemma 12.2. *The matrix*

$$A(x) = (a_{i,j}(x))_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

admits a factorisation, i.e. there is an $(m \times n)$-matrix $X(x)$ of bounded functions in $\mathcal{D}$, such that

$$(X(x))^* X(x) = A(x) \qquad (12.2)$$

for almost all $x \in \mathcal{D}$.

Proof. By (8.3), the matrix $A(x)$ induces a non-negative map of $\mathbb{C}^n$, for each fixed $x \in \mathcal{D}$. As is well known, this map possesses a unique non-negative square root in $\mathcal{L}(\mathbb{C}^n)$ presented by an $(n \times n)$-matrix $X(x) = \sqrt{A(x)}$ whose entries are complex-valued functions in $\mathcal{D}$. Write

$$X(x) = (X_{i,j}(x))_{\substack{i=1,\dots,n\\ j=1,\dots,n}}.$$

Since $X^* = X$ and $X^2 = A$, it follows that

$$a_{i,i}(x) = \sum_{j=1}^{n} |X_{i,j}(x)|^2$$

for almost all $x \in \mathcal{D}$, where $1 \le i \le n$. This shows, in particular, that $X_{i,j} \in L^\infty(\mathcal{D})$ for all $i, j = 1, \dots, n$, as desired. $\square$

Let

$$X(x) = (X_{i,j}(x))_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$$

be an arbitrary factorisation of $A(x)$, where $X_{i,j} \in L^\infty(\mathcal{D})$. Then

$$\begin{aligned} \sum_{i,j=1}^{n} a_{i,j}(x)\partial_j u\, \overline{\partial_i v} &= (\nabla v)^* A(x)\, \nabla u \\ &= (X(x)\nabla v)^*(X(x)\nabla u) \\ &= \sum_{k=1}^{m} \overline{X_k v}\, X_k u, \end{aligned} \tag{12.3}$$

for all smooth functions u and v in $\mathcal{D}$, where ∇u is thought of as n-column with entries $\partial_1 u, \dots, \partial_n u$, and

$$X_k u := \sum_{l=1}^{n} X_{k,l}(x)\partial_l u,$$

$k = 1, \dots, m$. If v has compact support in $\mathcal{D}$, then, by (12.3), we get

$$\begin{aligned} \sum_{k=1}^{m} (X_k^* X_k u, v)_{L^2(\mathcal{D})} &= \sum_{k=1}^{m} (X_k u, X_k v)_{L^2(\mathcal{D})} \\ &= \int_{\mathcal{D}} \sum_{k=1}^{m} \overline{X_k v}\, X_k u\, dx \\ &= \int_{\mathcal{D}} \sum_{i,j=1}^{n} a_{i,j}(x)\, \partial_j u\, \overline{\partial_i v}\, dx \\ &= -\,(\sum_{i,j=1}^{n} \partial_i(a_{i,j}(x)\, \partial_j u), v)_{L^2(\mathcal{D})} \end{aligned}$$

whence

$$-\sum_{i,j=1}^{n} \partial_i(a_{i,j}(x)\, \partial_j \cdot) = \sum_{k=1}^{m} X_k^* X_k. \tag{12.4}$$

Remark 12.3. The matrix $X(x)$ need not possess any left inverse with entries in $L^\infty(\mathcal{D})$, i.e. no decomposition

$$\partial_l = \sum_{k=1}^{m} Y_{l,k}(x) X_k, \tag{12.5}$$

$l = 1, \ldots, n$, with $L^\infty(\mathcal{D})$-coefficients is available in general. In fact, (12.3) and (12.5) yield (strong) coercive estimate (8.5).

Second order differential operators of the form (12.4) were considered in [Hör67]. This paper gave rise to a property of vector fields $X_1, \ldots, X_m$ that, if satisfied, has many useful consequences in the theory of partial and stochastic differential equations. In many interesting cases the matrix $X(x)$ is surjective, see a model example in Section 13. This is precisely a reason for non-coercive effects. The standard factorisation $A = \sqrt{A}\,\sqrt{A}$ leads to a boundary condition under which the form is coercive.

Lemma 12.2 and Remark 12.3 suggest to confine ourselves with first order perturbations of the form

$$\sum_{k=1}^{m} \tilde{a}_k(x) X_k, \tag{12.6}$$

where $\rho\, \tilde{a}_k \in L^\infty(\mathcal{D})$, instead of $\displaystyle\sum_{j=1}^{n} a_j(x) \partial_j$, where $\rho\, a_j \in L^\infty(\mathcal{D})$.

Integrating by parts as in (8.18) yields

$$\begin{aligned} & (Au, v)_{H^{0,\gamma}(\mathcal{D})} \\ = \quad & (u, v)_{+,\gamma} - 2\gamma \sum_{k=1}^{m} (X_k u, \rho^{-1} (X_k \rho) v)_{H^{0,\gamma}(\mathcal{D})} + (\sum_{k=1}^{m} \tilde{a}_k X_k u + \varDelta a_0 u, v)_{H^{0,\gamma}(\mathcal{D})} \\ + \quad & (b_1^{-1}\, \varDelta B u, v)_{H^{0,\gamma}(\partial\mathcal{D} \setminus S)} \end{aligned}$$

for all $u \in H^{1,\gamma}(\mathcal{D}, S)$ and $v \in H^{1,\gamma}(\mathcal{D}, S)$ satisfying the boundary condition of (8.2). In much the same way as in formulas (8.43) and (8.46) we deduce that the term

$$\sum_{k=1}^{m} (X_k u, \rho^{-1} (X_k \rho) v)_{H^{0,\gamma}(\mathcal{D})}$$

induces a bounded perturbation of the operator A_0.

Lemma 12.4. *If $\rho \tilde{a}_k \in L^\infty(\mathcal{D})$ for all $1 \le k \le m$, then there is a consant $c > 0$, such that*

$$|\sum_{k=1}^{m} (\tilde{a}_k X_k u, v)_{H^{0,\gamma}(\mathcal{D})}| \le c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$. Moreover, if there exists an $\varepsilon > 0$ with the property that $\rho^{1-\varepsilon}(\tilde{a}_k - 2\gamma \rho^{-1} \overline{X_k \rho}) \in L^\infty(\mathcal{D})$ for $1 \le k \le m$, then the corresponding perturbation $\varDelta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is compact.

Proof. The proof is similar to those of Lemmas 11.2 and 11.5. $\square$

From now on we will consider in the non-coercive case first order perturbations of the form (12.6) only. In this way one can also describe the class of tangential vectors admissible for small perturbations.

Our next concern will be to specify other compact perturbations of the problem if the form fails to be coercive.

Lemma 12.5. *Let for some $0 < s < 1$ there be a continuous embedding (9.9). If*

$$\tilde{a}_k = 2\gamma\, \rho^{-1} \overline{X_k \rho}$$

for $1 \leq k \leq m$, $t = 0$, $\Delta B_0 = 0$ and there is a number $\varepsilon > 0$ with the property that $\rho^{2-\varepsilon} \Delta a_0 \in L^\infty(\mathcal{D})$, then the operator $\Delta L = L - L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is compact.

Proof. Fix an arbitrary $v \in H^{+,\gamma}(\mathcal{D})$. Then $\iota_s v \in H^{s,\gamma}(\mathcal{D})$. Consider the function $w = \rho^{\varepsilon - s} \iota_s v$. It follows from Corollary 3.3 that $w \in H^{s,\gamma+\varepsilon-s}(\mathcal{D}) = \mathcal{H}^{s,\gamma+\varepsilon}(\mathcal{D})$. But then Corollary 4.9 yields compact embedding $e : H^{s,\gamma+\varepsilon-s}(\mathcal{D}) \to \mathcal{H}^{0,\gamma}(\mathcal{D})$.

Let $\{u_\nu\}$ be a bounded sequence in $H^{+,\gamma}(\mathcal{D})$. The continuous embedding (9.9) guarantees that it is bounded in $H^{s,\gamma}(\mathcal{D})$, too. Then the sequence

$$F_\nu = \rho^{2-\varepsilon} \Delta a_0 (\rho^{-s} u_\nu)$$

is bounded in $\mathcal{H}^{s,\gamma}(\mathcal{D})$ and so in $H^{0,\gamma}(\mathcal{D})$. According to the weak compactness principle we may assume that it converges weakly to zero in $H^{0,\gamma}(\mathcal{D})$.

On the other hand, if $\tilde{a}_k = 2\gamma\, \rho^{-1} \overline{X_k \rho}$ for all $1 \leq k \leq m$, $t = 0$ and $\Delta B_0 = 0$, then

$$(\Delta L u_\nu, v)_\gamma = (F_\nu, e \rho^{\varepsilon - s} \iota_s v)_{H^{0,\gamma}(\mathcal{D})} = (e^* F_\nu, \rho^{\varepsilon - s} \iota_s v)_{H^{s,\gamma+\varepsilon-s}(\mathcal{D})}.$$

Using embedding (9.9) and Corollary 3.3 we conclude that

$$\begin{array}{rcl} |(\Delta L u_\nu, v)_\gamma| & \leq & c \, \|e^* F_\nu\|_{H^{s,\gamma+\varepsilon-s}(\mathcal{D})} \|\iota_s v\|_{H^{s,\gamma}(\mathcal{D})} \\ & \leq & c \, \|e^* F_\nu\|_{H^{s,\gamma+\varepsilon-s}(\mathcal{D})} \|v\|_{H^{+,\gamma}(\mathcal{D})} \end{array}$$

with c a constant independent on ν and v. Hence

$$\|\Delta L u_\nu\|_{-,\gamma} \leq c \, \|e^* F_\nu\|_{H^{s,\gamma+\varepsilon-s}(\mathcal{D})} \to 0,$$

as $\nu \to \infty$, i.e. ΔL is compact. $\square$

Lemma 12.6. *Let for some $1/2 < s < 1$ there be a continuous embedding (9.9). Suppose*

$$\tilde{a}_k = 2\gamma\, \rho^{-1} \overline{X_k \rho}$$

for $1 \leq k \leq m$, $t = 0$ and there is a number $\varepsilon > 0$ such that either ΔB_0 is given by multiplication with a function Δb_0 satisfying $\rho^{2s-1-\varepsilon} \Delta b_0 / b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$ or $b_1^{-1} \Delta B_0$ maps the space $H^{s-1/2,\gamma}(\partial\mathcal{D})$ compactly to $H^{1/2-s,\gamma}(\partial\mathcal{D})$. Then the operator

$$\Delta L = L - L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$$

is compact.

Proof. Let the operator ΔB_0 be given by multiplication with a function Δb_0 satisfying $\rho^{2s-1-\varepsilon} \Delta b_0 / b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$.

Pick a bounded sequence $\{u_\nu\}$ in $H^{+,\gamma}(\mathcal{D})$. By Lemma 4.16, the trace operator $t_s : H^{s,\gamma}(\mathcal{D}) \to H^{s-1/2,\gamma}(\partial\mathcal{D})$ is bounded. Then the sequence

$$U_\nu = \begin{cases} \rho^{2s-1-\varepsilon} b_1^{-1} \Delta b_0 \, t_s u_\nu & \text{on} \quad \partial\mathcal{D} \setminus S, \\ 0 & \text{on} \quad S \end{cases}$$

is bounded in $\mathcal{H}^{0,\gamma+s-1/2}(\partial\mathcal{D})$. Again we may assume that it converges weakly to zero in $\mathcal{H}^{0,\gamma+s-1/2}(\partial\mathcal{D}\setminus S)$. Write

$$\begin{aligned}(\Delta L u_\nu, v)_\gamma &= (b_1^{-1}\Delta b_0\, t_s u_\nu, t_s v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}\\ &= (\rho^{2s-1-\varepsilon} b_1^{-1}\Delta b_0\, t_s u_\nu, \rho^{\varepsilon} t_s v)_{H^{0,\gamma+s-1/2}(\partial\mathcal{D}\setminus S)}.\end{aligned}$$

According to Corollary 3.3 the operator $\mathrm{Op}\,(\rho^\varepsilon)$ maps $H^{s-1/2,\gamma}(\partial\mathcal{D})$ continuously to $H^{s-1/2,\gamma+\varepsilon}(\partial\mathcal{D}) = \mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})$. But Corollary 4.9 implies that the embedding

$$e : \mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D}) \hookrightarrow \mathcal{H}^{0,\gamma+s-1/2}(\partial\mathcal{D})$$

is compact. We thus get

$$\begin{aligned}(\Delta L u_\nu, v)_\gamma &= (U_\nu, e\rho^\varepsilon t_s v)_{\mathcal{H}^{s-1/2,\gamma+s-1/2}(\partial\mathcal{D})}\\ &= (e^* U_\nu, \rho^\varepsilon t_s v)_{\mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})},\end{aligned}$$

where $e^* : \mathcal{H}^{0,\gamma+s-1/2}(\partial\mathcal{D}) \to \mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})$ is the Hilbert space adjoint of e. As the operator

$$\mathrm{Op}\,(\rho^\varepsilon) t_s : H^{s,\gamma}(\mathcal{D}) \to \mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})$$

is bounded (see Lemma 4.16 and Corollary 3.3), it follows that

$$\begin{aligned}|(\Delta L u_\nu, v)_\gamma| &\le c\,\|e^* U_\nu\|_{\mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})} \|v\|_{H^{s,\gamma}(\mathcal{D})}\\ &\le c\,\|e^* U_\nu\|_{\mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})} \|v\|_{H^{+,\gamma}(\mathcal{D})},\end{aligned}$$

where c stands for a constant independent on u and v and different in diverse applications. Summarising we get

$$\|\Delta L u_\nu\|_{-,\gamma} \le \|e^* U_\nu\|_{\mathcal{H}^{s-1/2,\gamma+s-1/2+\varepsilon}(\partial\mathcal{D})} \to 0,$$

as $\nu\to\infty$, i.e. ΔL is compact in this case.

Finally, assume that the operator $b_1^{-1}\Delta B_0$ maps $H^{s-1/2,\gamma}(\partial\mathcal{D})$ compactly to $H^{1/2-s,\gamma}(\partial\mathcal{D})$. As the trace operator $t_s : H^{s,\gamma}(\mathcal{D}) \to H^{s-1/2,\gamma}(\partial\mathcal{D})$ is bounded, the sequence

$$U_\nu = b_1^{-1}\Delta B_0\, t_s u_\nu$$

is precompact in $H^{0,1/2-s}(\partial\mathcal{D})$. Hence, as the sequence $\{u_\nu\}$ converges weakly to zero in $H^{s-1/2,\gamma}(\partial\mathcal{D})$, we deduce that $\{U_\nu\}$ converges to zero in $H^{1/2-s,\gamma}(\partial\mathcal{D})$. By duality and Theorem 4.13,

$$\begin{aligned}|(\Delta L u_\nu, v)_\gamma| &= |(b_1^{-1}\Delta B_0\, t_s u_\nu, t_s v)_\gamma|\\ &\le \|b_1^{-1}\Delta B_0\, t_s u_\nu\|_{H^{1/2-s,\gamma}(\partial\mathcal{D})} \|t_s v\|_{H^{s-1/2,\gamma}(\partial\mathcal{D})}\\ &\le c\,\|U_\nu\|_{H^{1/2-s,\gamma}(\partial\mathcal{D})} \|v\|_{H^{s,\gamma}(\partial\mathcal{D})}\\ &\le c\,\|U_\nu\|_{H^{1/2-s,\gamma}(\partial\mathcal{D})} \|v\|_{+,\gamma}\end{aligned}$$

with c a constant independent of ν and v. Therefore,

$$\|\Delta L u_\nu\|_{-,\gamma} \le c\,\|U_\nu\|_{H^{s-1/2,\gamma}(\partial\mathcal{D})} \to 0,$$

as $\nu\to\infty$, i.e. ΔL is compact, as desired. □

We can not use Lemma 12.6 for $0 < s \le 1/2$. However, we may extract the compactness property of the operator ΔL, related to the boundary term ΔB_0, from estimate (8.10).

Lemma 12.7. *Let*
$$\tilde{a}_k = 2\gamma\, \rho^{-1}\overline{X_k \rho}$$
for $1 \le k \le m$, $t = 0$ *and (8.10) hold for some* $r \in [-1/2, 1/2]$. *If* $\rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^{\gamma}$ *maps* $H^r(\partial\mathcal{D}, S)$ *compactly to* $H^{-r}(\partial\mathcal{D})$, *then* $\Delta L = L - L_0 : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ *is compact.*

In the case of regular singularities the condition on $b_1^{-1}\Delta B_0$ may be equivalently reformulated by saying that $b_1^{-1}\Delta B_0$ maps $H^{r,\gamma}(\partial\mathcal{D}, S)$ compactly to $H^{-r,\gamma}(\partial\mathcal{D})$. By the Rellich theorem, the operator $\rho^{-\gamma} b_1^{-1}\Delta B_0 \rho^{\gamma}$ maps $H^r(\partial\mathcal{D}, S)$ compactly to $H^{-r}(\partial\mathcal{D})$ provided it maps the space $H^r(\partial\mathcal{D}, S)$ continuously to $H^{-r+\varepsilon}(\partial\mathcal{D})$ for some $\varepsilon > 0$.

Proof. Fix a bounded sequence $\{u_\nu\}$ in $H^{+,\gamma}(\mathcal{D})$. It follows from (8.31) that the sequence $\{\rho^{-\gamma} u_\nu\}$ is bounded in $H^r(\partial\mathcal{D})$. Then the sequence
$$U_\nu = \rho^{-\gamma} b_1^{-1} \Delta B_0\, u_\nu = \rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^{\gamma}\, \rho^{-\gamma} u_\nu$$
is precompact in $H^{-r}(\partial\mathcal{D})$. We may assume without restriction of generality that $\{u_\nu\}$ converges weakly to zero in $H^r(\partial\mathcal{D})$ and then $\{U_\nu\}$ converges to zero in $H^{-r}(\partial\mathcal{D})$. By duality, we obtain
$$\begin{array}{rcl} |(\Delta L u_\nu, v)_\gamma| & = & |(b_1^{-1}\Delta B_0\, u_\nu, v)_{H^{0,\gamma}(\partial\mathcal{D})}| \\ & = & |(\rho^{-\gamma} b_1^{-1} \Delta B_0 \rho^{\gamma}\, \rho^{-\gamma} u_\nu, \rho^{-\gamma} v)_{L^2(\partial\mathcal{D})}| \\ & \le & c\, \|U_\nu\|_{H^{-r}(\partial\mathcal{D})} \|v\|_{H^{r,\gamma}(\partial\mathcal{D})} \end{array}$$
whence
$$\|\Delta L u_\nu\|_{-,\gamma} \le c\, \|U_\nu\|_{H^{-r}(\partial\mathcal{D})} \to 0,$$
as $\nu \to \infty$, i.e. ΔL is compact, as desired. □

As before, we split
$$\tilde{a}_k = 2\gamma\rho^{-1}\overline{X_k\rho} + \Delta_c \tilde{a}_k + \Delta_s \tilde{a}_k$$
and
$$\begin{array}{rcl} \Delta a_0 & = & \Delta_c a_0 + \Delta_s a_0, \\ \Delta B_0 & = & \Delta_c B_0 + \Delta_s B_0, \end{array}$$
where the terms $\Delta_c \tilde{a}_k$, $\Delta_c a_0$ and $\Delta_c B_0$ induce compact operators and the terms $\Delta_s \tilde{a}_k$, $\Delta_s a_0$ and $\Delta_s B_0$ "small" operators,
$$|\sum_{j=k}^{m} (\Delta_s \tilde{a}_k X_k u + \Delta_s a_0 u, v)_{H^{0,\gamma}(\mathcal{D})} + (\Delta_s B_0 u, v)_{H^{0,\gamma}(\partial\mathcal{D}\setminus S)}| \le \tilde{M}\, \|u\|_{+,\gamma} \|v\|_{+,\gamma} \tag{12.7}$$
for all $u, v \in H^{+,\gamma}(\mathcal{D})$, with $\tilde{M} > 0$ a constant independent of u and v.

Corollary 12.8. *Under the hypotheses of Corollary 8.12, let estimate (8.10) hold with some* $-1/2 < r < 1/2$ *and let* $t = 0$. *Let* $\rho^{-\gamma}\Delta_s B_0 \rho^{\gamma}$ *map* $H^r(\partial\mathcal{D}, S)$ *continuously to* $H^{-r}(\partial\mathcal{D})$ *and the constant* $\tilde{M}$ *in (12.7) be less than one. If there is a number* $\varepsilon > 0$ *such that* $\rho^{1-\varepsilon}\Delta_c \tilde{a}_k \in L^\infty(\mathcal{D})$, *for* $1 \le k \le m$, $\rho^{2-\varepsilon}\Delta_c a_0 \in L^\infty(\mathcal{D})$ *and either the operator* $\Delta_c B_0$ *is given by the multiplication with a function* $\Delta_c b_0$ *satisfying* $\rho^{2r-\varepsilon}\Delta_c b_0 / b_1 \in L^\infty(\partial\mathcal{D} \setminus S)$ *or* $\rho^{-\gamma} b_1^{-1} \Delta_c B_0 \rho^{\gamma}$ *maps* $H^r(\partial\mathcal{D}, S)$ *compactly to* $H^{-r}(\partial\mathcal{D})$, *then problem (8.47) is Fredholm of index zero. Moreover, if* $\Delta_c \tilde{a}_k = 0$, *for* $1 \le k \le m$, $\Delta_c a_0 = 0$, $\Delta_c B_0 = 0$, *then problem (8.47) is uniquely solvable and the inverse operator* $L^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D}) \subset H^{s,\gamma}(\mathcal{D}, S)$ *is bounded (where* s *is given by (8.12).*

Proof. Corollary 8.12 yields that $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^s(\mathcal{D})$ with $s > 0$ given by (8.12). Besides, the index s satisfies $1/2 < s = r + 1/2 < 1$ if $0 < r < 1/2$.

It follows from Lemmas 8.13, 12.1 and 12.4 that (8.44) is fulfilled under the hypotheses of the corollary.

Write $\Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ for the operator determined by the terms $\Delta_s \tilde{a}_k$, $\Delta_s a_0$, and $\Delta_s B_0$, as is described in Section 8. From estimate (12.7) we conclude that

$$\|\Delta L\|_{\mathcal{L}(H^{+,\gamma}(\mathcal{D}),H^{-,\gamma}(\mathcal{D}))} < 1 = \|L_0^{-1}\|_{\mathcal{L}(H^{-,\gamma}(\mathcal{D}),H^{+,\gamma}(\mathcal{D}))},$$

and so a familiar argument shows that the operator $L_0 + \Delta L : H^{+,\gamma}(\mathcal{D}) \to H^{-,\gamma}(\mathcal{D})$ is invertible.

According to Lemmas 12.4, 12.5 and 12.6 (for $0 < r < 1/2$) and 12.7 (for $-1/2 < r < 1/2$), the operator $C = L - L_0 - \Delta L$ is compact. Therefore, problem (8.47) is equivalent to the Fredholm-type operator equation

$$(I + (L_0 + \Delta L)^{-1} C) u = (L_0 + \Delta L)^{-1} f$$

in $H^{+,\gamma}(\mathcal{D})$ with compact operator $(L_0 + \Delta L)^{-1} C : H^{+,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D})$. This establishes the corollary. □

Corollary 12.9. *Under the hypotheses of Corollary 12.8, if moreover the constant $\tilde{M}$ from (12.7) satisfies $\tilde{M} < \sin \pi(2r+1)/2n$, then the system of root functions of the corresponding closed operator T in $H^{-,\gamma}(\mathcal{D})$ is complete in the spaces $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{s,\gamma}(\mathcal{D}, S)$ (with s given by (8.12)), and, for any $\delta > 0$, all eigenvalues of T (except for a finite number) lie in the corner $|\arg \lambda| < \delta + \arcsin \tilde{M}$ in $\mathbb{C}$. Besides, if $\Delta_c \tilde{a}_k = 0$, for $1 \le k \le m$, $\Delta_c a_0 = 0$, $\Delta_c B_0 = 0$, then all eigenvalues of T belong to the corner $|\arg \lambda| \le \arcsin \tilde{M}$ in $\mathbb{C}$.*

Proof. This is a consequence of Theorem 10.6 and Corollary 12.8. For $\tilde{M} = 0$ it suffices to apply Theorem 9.10 combined with Corollary 12.8. The last statement follows from Theorem 10.5 and Corollary 12.8. □

13. An example of non-coercive problems

We now wish to consider a typical non-coercive problem in weighted Sobolev spaces in a bounded Lipschitz domain $\mathcal{D} \subset \mathbb{R}^{2n}$. The space $\mathbb{R}^{2n}$ bears an additional complex structure.

Let the complex structure in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ be given by $z^j = x^j + \sqrt{-1} x^{n+j}$, for $j = 1, \ldots, n$. Denote by $\bar{\partial}$ the Cauchy-Riemann operator corresponding to this structure in $\mathbb{C}^n$, i.e. the column of n complex derivatives

$$\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial x^{n+j}}\Big)$$

for $1 \le j \le n$.

The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ with respect to the standard Hermitian structure of the space $L^2(\mathbb{C}^n)$ is the line of n operators

$$-\frac{1}{2}\Big(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial x^{n+j}}\Big) =: -\frac{\partial}{\partial z^j}.$$

An easy computation shows that $\bar{\partial}^* \bar{\partial}$ just amounts to the $-1/4$ multiple of the (non-positive) Laplace operator

$$\Delta = \sum_{j=1}^{2n} \partial^2_{x^j}$$

in $\mathbb{R}^{2n}$.

We take A to be

$$A = -\Delta + \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}^j} + a_0,$$

where $\rho a_1, \ldots, \rho a_n$ and $\rho^2 a_0$ are assumed to be bounded functions in $\mathcal{D}$. The complex matrix

$$(a_{i,j}(x))_{\substack{i=1,\ldots,2n \\ j=1,\ldots,2n}}$$

has the form

$$\begin{pmatrix} E_n & \sqrt{-1} E_n \\ -\sqrt{-1} E_n & E_n \end{pmatrix}$$

where E_n is the unity $(n \times n)$-matrix. Obviously, the matrix is Hermitian, and the corresponding conormal derivative is

$$\partial_c = \frac{\partial}{\partial \nu} + \sqrt{-1} \sum_{j=1}^{n} \Big(\nu_j \frac{\partial}{\partial x^{n+j}} - \nu_{n+j} \frac{\partial}{\partial x^j} \Big),$$

which is known as (the 2 multiple of) complex normal derivative $\bar{\partial}_\nu$ at the boundary of $\mathcal{D}$.

Consider the following boundary value problem. Given a function f in $\mathcal{D}$, find a function u in $\mathcal{D}$ satisfying

$$\left\{ \begin{array}{rcll} -\Delta u + \displaystyle\sum_{j=1}^{n+d} a_j \frac{\partial u}{\partial \bar{z}^j} + a_0 u & = & f & \text{in} \quad \mathcal{D}, \\ \partial_c u + B_0 u & = & 0 & \text{at} \quad \partial \mathcal{D}. \end{array} \right. \tag{13.1}$$

In this case S is empty, $b_1 = 1$ and $t = 0$. Set $a_{0,0}(z) := a \rho^{-2}$ in $\mathcal{D}$ with a real constant a which is assumed to be positive, if $Y \neq \emptyset$, and non-negative, if $Y = \emptyset$. Then the corresponding Hermitian form $(\cdot, \cdot)_{+,\gamma}$ is given by

$$(u,v)_{+,\gamma} = 4\, (\bar{\partial} u, \bar{\partial} v)_{H^{0,\gamma}(\mathcal{D})} + a\, (u,v)_{H^{0,\gamma+1}(\mathcal{D})} + (\,\Psi(\rho^{-\gamma} u), \Psi(\rho^{-\gamma} v))_{L^2(\partial \mathcal{D})}$$

and the space $H^{+,\gamma}(\mathcal{D})$ is defined to be the completion of $C^\infty_{\mathrm{comp}}(\overline{\mathcal{D}} \setminus Y)$ with respect to the norm

$$\|u\|_{+,\gamma} := \sqrt{(u,u)_{+,\gamma}}.$$

Denote by $\mathcal{H}(\mathcal{D})$ the subspace of $L^2(\mathcal{D})$ consisting of those functions u which are holomorphic, i.e. satisfy $\bar{\partial} u = 0$ in $\mathcal{D}$.

Lemma 13.1. *The inclusion $\iota : H^{+,\gamma}(\mathcal{D}) \to H^{0,\gamma}(\mathcal{D})$ is continuous. If (8.10) holds for some $-1/2 < r \le 1/2$, then it is compact. More precisely,*

1) If (8.10) holds for some $-1/2 < r \le 1/2$, then there are continuous embeddings $H^{1,\gamma}(\mathcal{D}) \hookrightarrow H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{s,\gamma}(\mathcal{D})$, where $s > 0$ is given by (8.12).

2) In particular, if Ψ is given by multiplication with a non-zero constant, then there are continuous embeddings $H^{1,\gamma}(\mathcal{D}) \hookrightarrow H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1/2-\varepsilon,\gamma}(\mathcal{D})$ for any $\varepsilon > 0$.

3) If $\rho \equiv 1$, $\Psi = 0$ and $\mathcal{D}$ is a strictly pseudoconvex domain with C^∞ boundary, then there are continuous embeddings

$$H^1(\mathcal{D}) \hookrightarrow H^{+,\gamma}(\mathcal{D}) \hookrightarrow \mathcal{H}(\mathcal{D}) \oplus (\mathcal{H}(\mathcal{D})^\perp \cap H^{1/2}(\mathcal{D})).$$

Proof. The continuity of the embedding ι has been proved in Lemma 8.1. The part 1) and 2) of the lemma follow from Corollary 8.12.

The proof of 3) can be given within the framework of complex analysis. Indeed, as $a > 0$, the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $L^2(\mathcal{D})$, which just amounts to $H^{0,\gamma}(\mathcal{D})$ in the case under study. If $\mathcal{D}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$, then holomorphic functions in a neighborhood of $\overline{\mathcal{D}}$ are dense in $\mathcal{H}(\mathcal{D})$. Since $\Psi = 0$, we conclude that $\mathcal{H}(\mathcal{D}) \subset H^{+,\gamma}(\mathcal{D})$. Besides, if $\mathcal{D}$ is a strictly pseudoconvex domain with C^∞ boundary, then there is a constant $c > 0$ with the property that

$$\|\bar{\partial} u\|_{L^2(\mathcal{D})} \geq c\, \|u\|_{H^{1/2}(\mathcal{D})}$$

for all $u \in L^2(\mathcal{D})$ orthogonal to the subspace $\mathcal{H}(\mathcal{D})$ of $L^2(\mathcal{D})$ (see [Koh79]). Using the orthogonal decomposition

$$L^2(\mathcal{D}) = \mathcal{H}(\mathcal{D}) \oplus \mathcal{H}(\mathcal{D})^\perp$$

we conclude that the completion $H^{+,\gamma}(\mathcal{D})$ of $H^1(\mathcal{D})$ with respect to the norm $\|\cdot\|_{+,\gamma}$ lies in $\mathcal{H}(\mathcal{D}) \oplus (\mathcal{H}(\mathcal{D})^\perp \cap H^{1/2}(\mathcal{D}))$, as desired. □

Example 13.2. Let $\rho \equiv 1$ and Ψ be given by multiplication with a non-zero constant. By Lemma 13.1, the space $H^{+,\gamma}(\mathcal{D})$ is continuously embedded into $H^{1/2-\varepsilon}(\mathcal{D})$ for any $\varepsilon > 0$ or even into $H^{1/2}(\mathcal{D})$, if $\partial\mathcal{D} \in C^2$. However, whatever $\varepsilon > 0$ is, there is no continuous embedding $H^{+,\gamma}(\mathcal{D}) \hookrightarrow H^{1/2+\varepsilon}(\mathcal{D})$. Indeed, if $\mathcal{D}$ is the unit disc in $\mathbb{C}$ then a direct computation using Lemma 1.4 of [Shl96] shows that the series

$$u(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)^{(1+\varepsilon)/2}}$$

converges in the space $H^{+,\gamma}(\mathcal{D})$ but diverges in $H^{1/2+\varepsilon}(\mathcal{D})$. This means that the coercive estimate (8.5) does not hold for problem (13.1). In particular, the form $(\cdot,\cdot)_{+,\gamma}$ is not coercive in this case, and so the Shapiro-Lopatinskii condition fails to hold for problem (13.1). Besides, as the monomials z^k are L^2-orthogonal on the circles $|z| = r$, we see that in this case the term induced by multiplication with constant $\Delta B_0 \in \mathbb{C}$ fails to be a compact operator from $H^{+,\gamma}(\mathcal{D})$ to $H^{-,\gamma}(\mathcal{D})$ (cf. [PS13]).

Example 13.3. Let $\mathcal{D}$ be the unit ball around the origin in $\mathbb{C}^n$, the boundary of $\mathcal{D}$ being the unit sphere $\mathbb{S}^{2n-1}$. The Laplace-Beltrami operator $\Delta_{\mathbb{S}^{2n-1}}$ on the sphere is non-negative and gives rise to the family

$$\Psi_r = (1 + \Delta_{\mathbb{S}^{2n-1}})^{r/2}$$

of invertible pseudodifferential operators of order r on $\mathbb{S}^{2n-1}$, parametrised by real $r \in \mathbb{R}$. By the invertibility is meant that (8.10) is fulfilled for $\Psi = \sqrt{B_{0,0}}\,\Psi_r$, where $B_{0,0}$ is a non-negative constant. Then, the selfadjoint version of problem (13.1) reads

$$\left\{ \begin{array}{rcll} -\Delta u + a_0 u & = & f & \text{in } \ \mathcal{D}, \\ \bar{\partial}_\nu u + B_{0,0}\rho^\gamma\, \Psi_r^* \Psi_r(\rho^{-\gamma} u) & = & 0 & \text{at } \ \partial\mathcal{D}. \end{array} \right. \tag{13.2}$$

For any $r \in [-1/2, 1/2]$, the operator $\Delta B_0 := \rho^\gamma \Psi_\delta \rho^{-\gamma}$ maps $H^r(\mathbb{S}^{2n-1})$ continuously to $H^{r-\delta}(\mathbb{S}^{2n-1})$. Hence, it maps $H^r(\mathbb{S}^{2n-1})$ compactly to $H^{-r}(\mathbb{S}^{2n-1})$, if $\delta < 2r$. In particular, if Ψ is given by multiplication with a non-zero constant, then $r = 0$ and $\rho^{-\gamma} \Delta B_0 \rho^\gamma = \Psi_\delta$ is a compact selfmapping of $L^2(\mathbb{S}^{2n-1})$, if $\delta < 0$. We may use the so-called spherical harmonics to make the action of Ψ_r more illustrative. Namely, spherical harmonics h_k are eigenfunctions of $\Delta_{\mathbb{S}^{2n-1}}$ of degree k, i.e.

$$\Delta_{\mathbb{S}^{2n-1}} h_k = k\,(2n+k-2)\,h_k. \tag{13.3}$$

The harmonic extension of h_k into the unit ball gives a harmonic homogeneous polynomial of degree k. The number of linearly independent spherical harmonics of degree k is finite and equals

$$J(k) = \frac{(2n+2k-2)(2n+k-3)!}{(2n-2)!\,k!}.$$

Thus, we may build an orthonormal basis $\{h_k^{(j)}\}$ of them in $L^2(\mathbb{S}^{2n-1})$. It is easy to check that, for $r \geq 0$, the operator Ψ_r is given by

$$(1 + \Delta_{\mathbb{S}^{2n-1}})^{r/2} u = \sum_{k=0}^{\infty} (1 + k(2n+k-2))^{r/2} \sum_{j=1}^{J(k)} (u, h_k^{(j)})_{L^2(\mathbb{S}^{2n-1})} h_k^{(j)}$$

for $u \in L^2(\mathbb{S}^{2n-1})$. By duality, this formula extends to all $r \in \mathbb{R}$ while the functions $u \in H^s(\mathbb{S}^{2n-1})$ with arbitrary $s \in \mathbb{R}$ are specified within the framework of series expansions

$$u = \sum_{k=0}^{\infty} \sum_{j=1}^{J(k)} c_k^{(j)} h_k^{(j)},$$

where $c_k^{(j)} \in \mathbb{C}$ satisfy

$$\sum_{k=0}^{\infty} (1 + k(2n+k-2))^s \sum_{j=1}^{J(k)} |c_k^{(j)}|^2 < \infty,$$

cf. [Pla86, Ch. 1, § 5] or [Shl96]. In particular, under this identification, we get readily

$$\Psi^* \Psi = B_{0,0} (1 + \Delta_{\mathbb{S}^{2n-1}})^r. \tag{13.4}$$

Recall that estimate (8.44) in the particular case under considerations becomes explicitly

$$\Big| \Big(\sum_{j=1}^{n} a_j \frac{\partial u}{\partial \bar{z}^j} + \Delta a_0 u, v \Big)_{H^{0,\gamma}(\mathcal{D})} + (\Delta B_0 u, v)_{H^{0,\gamma}(\partial \mathcal{D})} \Big| \leq c\, \|u\|_{+,\gamma} \|v\|_{+,\gamma}$$

for all $u, v \in H^{1,\gamma}(\mathcal{D})$, with c a constant independent of u and v. As $b_1 \equiv 1$, it follows from the definition of $\|\cdot\|_{+,\gamma}$ that (8.45) and (12.1) are valid if the functions $\rho^2 \Delta a_0$ and ρa_j, for $1 \leq j \leq n$, are of class $L^\infty(\mathcal{D})$ and either the operator ΔB_0 is given by multiplication with a function Δb_0 satisfying $\rho^{2r} \Delta b_0 \in L^\infty(\partial \mathcal{D})$, for $0 < r \leq 1/2$, or $\rho^{-\gamma} \Delta B_0 \rho^\gamma$ maps $H^r(\partial \mathcal{D})$ continuously to $H^{-r}(\partial \mathcal{D})$, for $-1/2 \leq r \leq 1/2$ (see Lemma 8.13).

Now, if $a > 0$, we deduce that estimate (8.44) is true under the conditions above with constant

$$c = a^{-1} \Big(\sum_{j=1}^{n} \| \rho a_j \|^2_{L^\infty(\mathcal{D})} \Big)^{1/2} + a^{-2} \| \rho^2 \Delta a_0 \|_{L^\infty(\mathcal{D})} + c' \qquad (13.5)$$

where $c' = \| \rho^{2r} \Delta b_0 \|_{L^\infty(\partial\mathcal{D})}$ or $c' = \| \rho^{-\gamma} \Delta B_0 \rho^{\gamma} \|$ depending upon the case under discussion.

The operator L_0 corresponds to the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u + 8\gamma\, \rho^{-1} \displaystyle\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial u}{\partial \bar{z}^j} + a \rho^{-2} u & = & f & \text{in } \ \mathcal{D}, \\ \partial_c u + B_{0,0}\, u & = & 0 & \text{at } \ \partial\mathcal{D}. \end{array} \right. \qquad (13.6)$$

Corollary 13.4. *Let (8.10) hold with some $-1/2 < r \leq 1/2$. Then the inverse L_0^{-1} of the operator L_0 induces compact positive selfadjoint operators*

$$\begin{array}{rcllcl} Q_1 & = & \iota' \iota\, L_0^{-1} & : & H^{-,\gamma}(\mathcal{D}) & \to & H^{-,\gamma}(\mathcal{D}), \\ Q_2 & = & \iota\, L_0^{-1}\, \iota' & : & H^{0,\gamma}(\mathcal{D}) & \to & H^{0,\gamma}(\mathcal{D}), \\ Q_3 & = & L_0^{-1}\, \iota' \iota & : & H^{+,\gamma}(\mathcal{D}) & \to & H^{+,\gamma}(\mathcal{D}) \end{array}$$

which have the same systems of eigenvalues and eigenvectors. Besides, all eigenvalues are positive and there are orthonormal bases in $H^{+,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ and $H^{-,\gamma}(\mathcal{D})$ consisting of the eigenvectors.

Proof. For the proof it suffices to combine Lemmas 9.1 and 13.1 with Theorem 4.5. □

We are in a position to evaluate the eigenfunctions and eigenvalues of selfadjoint problem (13.2).

Example 13.5. Let $\mathcal{D}$ be the unit ball around the origin in $\mathbb{C}^n$, $\rho \equiv 1$, $a_{0,0} \geq 0$ and $a_{0,0}^2 + B_{0,0}^2 \neq 0$. We pass to spherical coordinates $x = r\, S(\varphi)$ in $\mathbb{R}^{2n}$, where $r = |x|$ and φ are coordinates on the unit sphere $\mathbb{S}^{2n-1}$. The Laplace operator Δ takes the form

$$\Delta = \frac{1}{r^2} ((r\partial_r)^2 + (2n-2)(r\partial_r) - \Delta_{\mathbb{S}^{2n-1}}), \qquad (13.7)$$

where $\Delta_{\mathbb{S}^{2n-1}}$ is the Laplace-Beltrami operator on the unit sphere. Furthermore, since $\partial\mathcal{D} = \mathbb{S}^{2n-1}$, we get

$$\begin{array}{rcl} \dfrac{\partial}{\partial \nu} & = & r\partial_r, \\ \bar{\partial}_\nu & = & \displaystyle\sum_{j=1}^{n} \bar{z}^j \frac{\partial}{\partial \bar{z}^j} \ = \ \frac{1}{2} \left((r\partial_r) + B_{\mathbb{S}^{2n-1}} \right) \end{array}$$

where $B_{\mathbb{S}^{2n-1}}$ depends on the coordinates on the unit sphere only. If for instance $n = 1$, then

$$\bar{\partial}_\nu = \frac{1}{2} \left((r\partial_r) + \sqrt{-1}\, \partial_\varphi \right)$$

in polar coordinates in the plane.

To solve the homogeneous equation $(-\Delta + a)u = 0$ we apply the Fourier method of separation of variables with $a \in \mathbb{R}$. Writing $u(r, \varphi) = g(r) h(\varphi)$ we get two separate equations for g and h, namely

$$\begin{array}{rcl} \left(-(r\partial_r)^2 + (2-2n)(r\partial_r) + ar^2\right) g & = & c\,g \\ \varDelta_{\mathbb{S}^{2n-1}} h & = & c\,h, \end{array}$$

where c is an arbitrary constant. The second equation possesses non-zero solutions if and only if $c = k(2n+k-2)$ and $h = h_k$, a spherical harmonic of degree k.

It is known that we may choose the harmonics h_k in accordance with complex structure. Namely, there is an orthonormal basis $\{h^{(j)}_{p,q}\}$ in $L^2(\mathbb{S}^{2n-1})$ consisting of polynomials of the form

$$h^{(j)}_{p,q}(z,\overline{z}) = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} c^{(j)}_{\alpha,\beta} z^\alpha \overline{z}^\beta$$

with complex coefficients $c^{(j)}_{\alpha,\beta}$ (see for instance [AK91]). Let $J(p,q)$ stand for the number of polynomials of bidegree (p,q) in the basis; of course $J(p,q) \le J(p+q)$. Clearly,

$$\begin{array}{rcl} \bar{\partial}_\nu h^{(j)}_{p,q} & = & q h^{(j)}_{p,q}, \\ B_{\mathbb{S}^{2n-1}} h^{(j)}_{p,q} & = & (q-p)\, h^{(j)}_{p,q} \end{array} \tag{13.8}$$

and

$$\varPsi_r^* \varPsi_r\, h^{(j)}_{p,q} = (1 + \varDelta_{\mathbb{S}^{2n-1}})^r\, h^{(j)}_{p,q}) = (1 + (p+q)(2n+p+q-2))^r h^{(j)}_{p,q},$$

which is due to (13.4).

Consider the Sturm-Liouville problem for the ordinary differential equation with respect to the variable r in the interval $(0,1)$,

$$\left\{ \begin{array}{rcll} \dfrac{1}{r^2}\left(-(r\partial_r)^2 + (2-2n)(r\partial_r) + ar^2 + (p+q)(2n+p+q-2)\right) g & = & \lambda\, g & \text{in } (0,1) \\ g \text{ is bounded} & & & \text{at } 0, \\ \left((r\partial_r) + (q-p) + 2B_{0,0}\,(1+(p+q)(2n+p+q-2))^r\right) g & = & 0 & \text{at } 1 \end{array} \right. \tag{13.9}$$

see [TS72, Suppl. II, P. 1, § 2]. Actually, if a_0 and λ are real numbers then (13.9) is a particular case of the Bessel equation. Its (real-valued) solution $g(r)$ is a Bessel function defined on $(0,+\infty)$, and the space of all solutions is two-dimensional. For example, if $\lambda = a_0$ then $g(r) = \alpha r^{p+q} + \beta r^{2-p-q-n}$ with arbitrary constants α and β is a general solution to (13.9). In the general case the space of solutions to (13.9) contains a one-dimensional subspace of functions bounded at the point $r=0$, cf. [TS72].

For any p, q and j, fix a non-trivial solution $g^{(j,k)}_{p,q}(r)$ of problem (13.9) corresponding to an eigenvalue $\lambda^{(j,k)}_{p,q}$. Then the function

$$u^{(j,k)}_{p,q} = g^{(j,k)}_{p,q}(r) H^{(j)}_{p,q}(\varphi)$$

satisfies

$$\left\{ \begin{array}{rcll} \left(-\varDelta + (a - \lambda^{(j,k)}_{p,q})\right) u^{(j,k)}_{p,q} & = & 0 & \text{in } \mathbb{C}^n, \\ \left(\bar{\partial}_\nu + B_{0,0}(1+\varDelta_{\mathbb{S}^{2n-1}})^r\right) u^{(j,k)}_{p,q} & = & 0 & \text{at } \partial\mathcal{D}. \end{array} \right. \tag{13.10}$$

Indeed, by (13.3), (13.7), (13.9) and the discussion above we conclude that this equality holds in $\mathbb{C}^n \setminus \{0\}$. We now use the fact that $u^{(j,k)}_{p,q}$ is bounded at the origin to see that the differential equation of (13.10) holds in all of $\mathbb{C}^n$. On the other hand, boundary equation of (13.10) follows from (13.8) immediately, as already mentioned.

Let us show that the system

$$\{u^{(j,k)}_{p,q}\}_{\substack{p,q\in\mathbb{Z}_{\geq 0}\\ j=1,\dots,J(p,q)\\ k=1,2,\dots}}$$

coincides with the system of all eigenvectors of Sturm-Liouville problem (13.2) in the unit ball $\mathbb{B}^{2n}$ around the origin in $\mathbb{C}^n$. In particular, it is an orthogonal basis in $H^{+,\gamma}(\mathbb{B}^{2n})$, $H^{0,\gamma}(\mathbb{B}^{2n})$ and $H^{-,\gamma}(\mathbb{B}^{2n})$. Really, as $a^2_{0,0}+B^2_{0,0}\neq 0$, Theorem 8.4 implies that $H^{+,\gamma}(\mathbb{B}^{2n})$ is continuously embedded into $L^2(\mathbb{B}^{2n})$. From (13.4) it follows that the system $\{u^{(j,k)}_{p,q}\}$ consists of eigenvectors of Sturm-Liouville problem (13.2) in the ball. Moreover, by Lemma 7.1 of [ST03], the system $\{u^{(j,k)}_{p,q}\}$ is orthogonal with respect to each of the Hermitian forms $(\cdot,\cdot)_{L^2(\mathbb{S}^{2n-1})}$, $(\cdot,\cdot)_{L^2(\mathbb{B}^{2n})}$ and $(\bar\partial\cdot,\bar\partial\cdot)_{L^2(\mathbb{B}^{2n})}$. In particular, it is orthogonal in $H^{+,\gamma}(\mathbb{B}^{2n})$. On the other hand, the orthogonality of the system in $H^{-,\gamma}(\mathbb{B}^{2n})$ is fulfilled because (9.1) and Lemma 9.1 imply

$$\begin{aligned}(u^{(j,k)}_{p,q},u^{(j',k')}_{p',q'})_{-,\gamma} &= (\lambda^{(j,k)}_{p,q})^{-1}(\iota'\iota L_0^{-1}u^{(j,k)}_{p,q},u^{(j',k')}_{p',q'})_{-,\gamma}\\ &= \lambda^{(j',k')}_{p',q'}\,(u^{(j,k)}_{p,q},u^{(j',k')}_{p',q'})_{L^2(\mathbb{B}^{2n})}.\end{aligned}$$

By construction, the system $\{h^{(j)}_{p,q}\}$, where $p,q\in\mathbb{Z}_{\geq 0}$ and $1\leq j\leq J(p,q)$, is an orthonormal basis in $L^2(\mathbb{S}^{2n-1})$. For any fixed p, q and j, we have $\lambda^{(j,k)}_{p,q}\geq a_{0,0}$ and the countable system $\{g^{(j,k)}_{p,q}\}_{k\in\mathbb{N}}$ of eigenfunctions is an orthogonal basis in the weighted space $L^2((0,1),r)$ of complex-valued functions with scalar product $(\sqrt{r}\cdot,\sqrt{r}\cdot)_{L^2(0,1)}$ (see [TS72, Suppl. II, P. 1, § 2]). Hence, a familiar argument now shows that the system $\{u^{(j,k)}_{p,q}\}$ is an orthogonal basis in $L^2(\mathbb{B}^{2n})$. As $\{u^{(j,k)}_{p,q}\}$ is an orthogonal basis in $L^2(\mathbb{B}^{2n})$, there are no other eigenvalues of problem (13.2) but the already mentioned $\lambda^{(j,k)}_{p,q}$. Hence, there are no eigenvectors corresponding to an eigenvalue λ which fail to be finite linear combinations of the eigenfunctions already constructed.

By the above, the space $L^2(\mathbb{B}^{2n})$ is dense in $H^{-,\gamma}(\mathbb{B}^{2n})$. It follows that the system $\{u^{(j,k)}_{p,q}\}$ is complete in $H^{-,\gamma}(\mathbb{B}^{2n})$, too. Finally, let a function $u\in H^{+,\gamma}(\mathbb{B}^{2n})$ be orthogonal to each vector $u^{(j,k)}_{p,q}$ with respect to $(\cdot,\cdot)_{+,\gamma}$. Then, using Lemma 9.1 and (9.2) we conclude that

$$\begin{aligned}(u,u^{(j,k)}_{p,q})_{L^2(\mathbb{B}^{2n})} &= (u,L_0^{-1}\iota'\iota u^{(j,k)}_{p,q})_{+,\gamma}\\ &= \lambda^{(j,k)}_{p,q}\,(u,u^{(j,k)}_{p,q})_{+,\gamma}\\ &= 0,\end{aligned}$$

i.e. u is orthogonal to each vector $u^{(j,k)}_{p,q}$ in $L^2(\mathbb{B}^{2n})$. Therefore, $u=0$ in $L^2(\mathbb{B}^{2n})$ and so in the space $H^{+,\gamma}(\mathbb{B}^{2n})$, too. This means precisely that the system $\{u^{(j,k)}_{p,q}\}$ is complete in $H^{+,\gamma}(\mathbb{B}^{2n})$.

Corollary 9.8 and Lemma 13.1 actually show that the operators Q_1, Q_2 and Q_3 are of Schatten class $\mathfrak{S}_{n+\varepsilon}$ for any $\varepsilon>0$, and so their orders are finite. Moreover, in this case any eigenvalue has finite multiplicity. From this point of view the case where $\Psi\equiv 0$ is of certain interest.

Theorem 13.6. *Let $\rho\equiv 1$ and $\Psi\equiv 0$. If $\mathcal{D}$ is strictly pseudoconvex domain with C^∞ boundary then the inverse L_0^{-1} of the operator L_0 induces positive selfadjoint* *operators*

$$\begin{array}{rcccrcl} Q_1 & = & \iota'\iota\, L_0^{-1} & : & H^{-,\gamma}(\mathcal{D}) & \to & H^{-,\gamma}(\mathcal{D}), \\ Q_2 & = & \iota\, L_0^{-1}\,\iota' & : & L^2(\mathcal{D}) & \to & L^2(\mathcal{D}), \\ Q_3 & = & L_0^{-1}\,\iota'\iota & : & H^{+,\gamma}(\mathcal{D}) & \to & H^{+,\gamma}(\mathcal{D}) \end{array}$$

which have the same systems of eigenvalues and eigenvectors. Moreover, all eigenvalues are larger than or equal to 1 *and there are orthonormal bases in the spaces* $H^{+,\gamma}(\mathcal{D})$, $L^2(\mathcal{D})$ *and* $H^{-,\gamma}(\mathcal{D})$ *consisting of the eigenvectors. Besides, the eigenvalue* $\lambda = 1$ *has infinite multiplicity and the multiplicities of all other eigenvalues are finite.*

Proof. The fact that the operators Q_1, Q_2 and Q_3 are selfadjoint and non-negative follows from Lemma 9.1.

Consider the operator equation $(Q_2 - \lambda I)u = 0$ with $\lambda \in \mathbb{C}$. The corresponding weak identity has the form

$$4\,(\bar{\partial} u, \bar{\partial} v)_{L^2(\mathcal{D})} + (u,v)_{L^2(\mathcal{D})} = \lambda (u,v)_{L^2(\mathcal{D})}$$

for all $v \in H^{+,\gamma}(\mathcal{D})$. In particular, we get

$$4\,\|\bar{\partial} u\|_{L^2(\mathcal{D})} = (\lambda - 1)\,\|u\|^2_{L^2(\mathcal{D})}$$

for its solution. Hence, non-trivial solutions are only possible if $\lambda \geq 1$.

A function $u \in H^{+,\gamma}(\mathcal{D})$ satisfies this equation for $\lambda = 1$ if and only if u is holomorphic in $\mathcal{D}$. As such functions constitute $\mathcal{H}(\mathcal{D})$, we see that the eigenvalue $\lambda = 1$ has infinite multiplicity.

Write $\mathcal{H}(\mathcal{D})^\perp = L^2(\mathcal{D}) \ominus \mathcal{H}(\mathcal{D})$ for the subspace of $L^2(\mathcal{D})$ consisting of all functions which orthogonal to $\mathcal{H}(\mathcal{D})$. Clearly, if $u \in \mathcal{H}(\mathcal{D})^\perp$ then

$$(Q_2 u, v)_{L^2(\mathcal{D})} = (u, Q_2 v)_{L^2(\mathcal{D})} = (u,v)_{L^2(\mathcal{D})} = 0$$

for all $v \in \mathcal{H}(\mathcal{D})$. Therefore, Q_2 maps $\mathcal{H}(\mathcal{D})^\perp$ to $\mathcal{H}(\mathcal{D})^\perp$.

According to Lemma 13.1, the restriction of the operator Q_2 to $\mathcal{H}(\mathcal{D})^\perp$ is compact and selfadjoint. By the Hilbert-Schmidt theorem, there is an orthonormal basis $\{b_\nu\}$ in $\mathcal{H}(\mathcal{D})^\perp$ which consists of eigenvectors of this operator. On choosing an orthonormal basis $\{h_\nu\}$ in $\mathcal{H}(\mathcal{D})$ we obtain the orthonormal basis $\{h_\nu\} \cup \{b_\nu\}$ in $L^2(\mathcal{D})$ consisting of the eigenvectors of Q_2.

Finally, the fact that the operators Q_1, Q_2, Q_3 have the same systems of eigenvalues and eigenvectors follows from Lemma 9.1. □

The Fredholm property and theorem on the completeness of root functions of problem (13.1) read as follows.

Corollary 13.7. *Let* $a_{0,0} \equiv \rho^{-2}$, $\Psi = \sqrt{B_{0,0}}\,\Psi_r$ *with* $B_{0,0} > 0$ *and* $t = 0$. *If there is a number* $\varepsilon > 0$ *such that* $\rho^{1-\varepsilon}\left(a_j - 8\gamma\,\rho^{-1}\partial_{z_j}\rho\right) \in L^\infty(\mathcal{D})$, *for* $1 \leq j \leq n$, $\rho^{2-\varepsilon}\Delta a_0 \in L^\infty(\mathcal{D})$ *and either the operator* ΔB_0 *is given by multiplication with a function* Δb_0 *satisfying* $\rho^{2r-\varepsilon}\Delta b_0 \in L^\infty(\partial\mathcal{D})$, *for* $0 < r \leq 1/2$, *or* $\rho^{-\gamma}\Delta B_0 \rho^\gamma$ *maps* $H^r(\partial\mathcal{D}, S)$ *compactly to* $H^{-r}(\partial\mathcal{D})$, *for* $-1/2 < r \leq 1/2$, *then problem (13.1) is Fredholm. If moreover the constant* c *given by (13.5) is less than* 1, *then problem (13.1) is uniquely solvable and the inverse* $L^{-1} : H^{-,\gamma}(\mathcal{D}) \to H^{+,\gamma}(\mathcal{D}) \subset H^{s,\gamma}(\mathcal{D}, S)$ *is bounded, where* s *is given by (8.12).*

Corollary 13.8. *Let* $a_{0,0} \equiv \rho^{-2}$, $\Psi = \sqrt{B_{0,0}}\,\Psi_r$ *with* $B_{0,0} > 0$ *and* $t = 0$. *If the constant* c *of (13.5) is less than* $\sin(r+1/2)\pi/2n$, *then the system of root functions of the closed operator* T *in* $H^{-,\gamma}(\mathcal{D})$ *corresponding to problem (13.1) is complete in* *the spaces* $H^{-,\gamma}(\mathcal{D})$, $H^{0,\gamma}(\mathcal{D})$ *and* $H^{1,\gamma}(\mathcal{D},S)$, *and, for any* $\delta > 0$, *all eigenvalues of* T *(except for a finite number) lie in the corner* $|\arg\lambda| < \arcsin c + \delta$ *in* $\mathbb{C}$. *Besides, if* $a_j = 8\gamma\,\rho^{-1}\partial_{z_j}\rho$, *for* $1 \le j \le n$, $\Delta a_0 = 0$ *and* $\Delta B_0 = 0$, *then all eigenvalues of* T *belong to the corner* $|\arg\lambda| \le \arcsin c$.

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