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A note on Jeśmanowicz' conjecture for non-primitive Pythagorean triples

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Abstract: Let (a, b, c) be a primitive Pythagorean triple parameterized as $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$, where $u > v > 0$ are co-prime and not of the same parity. In 1956, L. Jeśmanowicz conjectured that for any positive integer n, the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. In this connection we call a positive integer solution $(x, y, z) \neq (2, 2, 2)$ with $n > 1$ exceptional. In 1999 M.-H. Le gave necessary conditions for the existence of exceptional solutions which were refined recently by H. Yang and R.-Q. Fu. In this paper we give a unified simple proof of the theorem of Le-Yang-Fu. Next we give necessary conditions for the existence of exceptional solutions in the case $v = 2$, u is an odd prime. As an application we show the truth of the Jeśmanowicz conjecture for all prime values $u < 100$.

Keywords: Diophantine equations; Non-primitive Pythagorean triples; Jeśmanowicz conjecture.

MSC: 11D61; 11D41.

1. Introduction

Let (a, b, c) be a primitive Pythagorean triple. Clearly for such a triple with $2 \mid b$ one has the following parameterization

$$a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2$$

with

$$u > v > 0,\ \gcd(u, v) = 1,\ u + v \equiv 1 \pmod 2. \tag{1}$$

In 1956 L. Jeśmanowicz ([1]) made the following conjecture:

Conjecture 1. *For any positive integer n, the Diophantine equation*

$$(an)^x + (bn)^y = (cn)^z \tag{2}$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

The primitive case of the conjecture ($n = 1$) was investigated thoroughly. Although the conjecture is still open, many special cases are shown to be true. We refer to a recent survey [2] for a detailed account.

Much less known about the non-primitive case ($n > 1$). A positive integer solution (x, y, z, n) of (2) is called exceptional if $(x, y, z) \neq (2, 2, 2)$ and $n > 1$. For a positive integer t, let $\mathcal{P}(t)$ denote the set of distinct prime factors of t and $P(t)-$ their product. The first known result in this direction was obtained in 1998 by M.-J. Deng and G.L. Cohen ([3]), namely if $u = v + 1$, a is a prime power, and either $P(b) \mid n$, or $P(n) \nmid b$, then (2) has only positive integer solution $(x, y, z) = (2, 2, 2)$. In 1999, M.-H. Le gave necessary conditions for (2) to have exceptional solutions.

Theorem 1. [4] *If (x, y, z, n) is an exceptional solution of* (2)*, then one of the following three conditions is satisfied:*

(i) $\max\{x, y\} > \min\{x, y\} > z$, $\mathcal{P}(n) \subsetneq \mathcal{P}(c)$;

(ii) $x > z > y$, $\mathcal{P}(n) \subset \mathcal{P}(b)$;
(iii) $y > z > x$, $\mathcal{P}(n) \subset \mathcal{P}(a)$.

However, as noted in [5] by H. Yang and R.-Q. Fu, the case $x = y > z$ is not completely handled by the arguments used in [4]. Furthermore they completed the unhandled case ([5], Theorem 1) based on a powerful result of Zsigmondy ([6], *cf.* [7,8]). In fact one can give a unified simple proof of Theorem of Le-Yang-Fu (Theorem 1) by using a weaker version of the Zsigmondy theorem as stated in Lemma 3 of [3].

Since many works [3,4] intensively investigated the first interesting family of primitive triples:

$$v = 1,\ u = 2^k,\ k = 1, 2, \ldots. \tag{3}$$

Most recently, X.-W. Zhang and W.-P. Zhang [9], and T. Miyazaki [10] independently proved Conjecture 1 for the (infinite) family (3).

It is natural to treat the next interesting case: $v = 2$, u is an *odd prime* which was known recently for few values u: $u = 3$ ([3]), $u = 5$ – by Z. Cheng, C.-F. Sun and X.-N. Du, $u = 7$ – by C.-F. Sun, Z. Cheng, and by G. Tang, $u = 11$ – by W.-Y. Lu, L. Gao and H.-F. Hao (*cf.* [2] for references). Let's formulate our main results. We rewrite (2) as

$$[(u^2-4)n]^x + (4un)^y = [(u^2+4)n]^z. \tag{4}$$

An arithmetical argument (given in Lemma 7 below) shows that $u^2 - 4$ admits a proper decomposition $u^2 - 4 = u_1u_2$, $\gcd(u_1, u_2) = 1$, so that there are three possibilities to consider: $u_1 \equiv \pm 1, 5 \pmod 8$.

Theorem 2. *If (x, y, z, n) is an exceptional solution of (4) and $u_1 \equiv \pm 1 \pmod 8$, then y is even.*

In view of Theorem 2 the possibility $u_1 \equiv -1 \pmod 8$ is eliminated, because in this case x, y, z are even, which is in general impossible by an auxiliary argument (Lemma 8 below).

Let $\nu_q(t)$, for a prime q, denote the exponent of q in the prime factorization of t, and let $\left(\frac{-}{m}\right)$ denote the Jacobi quadratic residue symbol.

Theorem 3. *If (x, y, z, n) is an exceptional solution of (4), then one of the following cases is satisfied*

(1) $\nu_2(u_1 - 1) = 3$: $(\nu_2(x), \nu_2(y), \nu_2(z)) = (0, \geq 2, 1)$; *$u_1$ admits a proper decomposition $u_1 = t_1t_2$, $\gcd(t_1, t_2) = 1$ and $t_1, t_2 \equiv 5 \pmod 8$ satisfying certain special Diophantine equations;*
(2) $u_1 \equiv 5 \pmod 8$, $u_2 = w^{2^s}$, *where $s = \nu_2(z - x) - \nu_2(x)$ and either of the following*
  (2.1) $w \equiv \pm 3 \pmod 8$: $(\nu_2(x), \nu_2(y), \nu_2(z)) = (0, \geq 1, 0)$; $u \equiv 1 \pmod 4$; $\left(\frac{u_1}{p}\right) = \left(\frac{w}{p}\right)$, $\forall\, p \mid (u^2 + 4)$ *and* $\left(\frac{w}{p}\right) = \left(\frac{u^2+4}{p}\right)$, $\forall\, p \mid u_1$;
  (2.2) $w \equiv \pm 1 \pmod 8$: $(\nu_2(x), \nu_2(y), \nu_2(z)) = (\beta, 0, \beta)$, $\beta \geq 1$; $u \equiv \pm 3 \pmod 8$; $\left(\frac{w}{p}\right) = 1$, $\forall\, p \mid (u^2 + 4)$ *and* $\left(\frac{w}{p}\right) = \left(\frac{u}{p}\right)$, $\forall\, p \mid u_1$. *Moreover, if $u \equiv 3 \pmod 8$, then w can not be a square.*

Corollary 1. *Conjecture 1 is true for $v = 2$, u – an odd prime < 100.*

Let's explain the ideas in proving our main results. As for Theorem 2 and Theorem 3 we exploit a total analysis of Jacobi quadratic and quartic residues. In the case $u_1 \equiv 1 \pmod 8$ we have a further proper decomposition $u_1 = t_1t_2$, which leads to certain special Diophantine equations. Theorem 3 helps us substantially in reducing the verification process, as the possibility $u_1 \equiv 5 \pmod 8$ occurs quite sparsely. We demonstrate this for $u < 100$ in proving Corollary 1.

The paper is organized as follows. In Section 2 we give a unified simple proof of Theorem 1. Section 3 provides some reduction of the problem and preliminary results. Theorem 2 will be proved in Section 4. The case $u_1 \equiv 5 \pmod 8$ and Theorem 3 will be treated in Section 5. The verification for $u < 100$ in Corollary 1 will be given in the last Section 6.

2. A Simple Proof of Theorem 1

We shall use the following weaker version of Zsigmondy's theorem.

Lemma 1. (*cf.* [3], Lemma 3) *For $X > Y > 0$ co-prime integers,*

(1) if q is a prime, then
$$\gcd\left(X-Y, \frac{X^q - Y^q}{X-Y}\right) = 1, \text{ or } q;$$
(2) if q is an odd prime, then
$$\gcd\left(X+Y, \frac{X^q + Y^q}{X+Y}\right) = 1, \text{ or } q.$$

Proof. Part (2) is Lemma 3 of [3]. As for part (1) one argues similarly: if ℓ^r is a common prime power divisor of $X-Y$ and $(X^q - Y^q)/(X-Y)$. Clearly
$$\frac{X^q - Y^q}{X-Y} \equiv 0 \pmod{\ell^r}. \tag{5}$$
On the other hand from the fact that $X \equiv Y \pmod{\ell^r}$ it follows
$$\frac{X^q - Y^q}{X-Y} = X^{q-1} + X^{q-2}Y + \cdots + XY^{q-2} + Y^{q-1} \equiv qY^{q-1} \pmod{\ell^r}. \tag{6}$$
Since $\ell \nmid Y$, (5)-(6) imply that $\ell = q$, and $r = 1$. □

Remark 1. Part (1) of Lemma 1 is a special case of Theorem IV in [7].

Lemma 2. *For a prime divisor q of $(X-Y)$ and positive integer β*
$$\nu_q(X^{q^\beta} - Y^{q^\beta}) = \beta + \nu_q(X-Y). \tag{7}$$

Proof. Applying part (1) of Lemma 1 β times one has
$$\gcd\left(X^{q^{\beta-1}} - Y^{q^{\beta-1}}, \frac{X^{q^\beta} - Y^{q^\beta}}{X^{q^{\beta-1}} - Y^{q^{\beta-1}}}\right) = q;$$
$$\cdots$$
$$\gcd\left(X-Y, \frac{X^q - Y^q}{X-Y}\right) = q.$$
Hence the formula (7). □

In view of Lemma 2 of [3] there are no exceptional solutions with $z \geq \max\{x, y\}$, so as in [4] we have to eliminate the following three cases:

(I) $x > y = z$;
(II) $y > x = z$;
(III) $x = y > z$.

(I) $\underline{x > y = z}$: Dividing both sides of (2) by n^y one gets
$$a^x n^{x-y} = c^y - b^y. \tag{8}$$
By considering mod $c+b$, and taking into account $(c+b)(c-b) = a^2$, one sees that y must be even, say $y = 2y_1$. Now put $X = c^2$, $Y = b^2$, so $X \equiv Y (\bmod\, a^2)$, $\gcd(Y, a) = 1$. Taking mod a and in view of (8)
$$0 \equiv \frac{X^{y_1} - Y^{y_1}}{X-Y} = X^{y_1-1} + X^{y_1-2}Y + \cdots + XY^{y_1-2} + Y^{y_1-1} \equiv y_1 Y^{y_1-1} \pmod{a},$$
one concludes that $a \mid y_1$.

For any $q \in \mathcal{P}(a)$ let $\beta = \nu_q(y_1)$, so that $y_1 = q^\beta y_2$ with $q \nmid y_2$. Putting $U = X^{q^\beta}$, $V = Y^{q^\beta}$ for short, we have
$$X^{y_1} - Y^{y_1} = (U-V)(U^{y_2-1} + U^{y_2-2}V + \cdots + UV^{y_2-2} + V^{y_2-1}), \tag{9}$$

and

$$U^{y_2-1}+U^{y_2-2}V+\cdots+UV^{y_2-2}+V^{y_2-1}\equiv y_2V^{y_2-1}\not\equiv 0 \pmod q. \tag{10}$$

Lemma 2 and (9), (10) imply that

$$\nu_q(X^{y_1}-Y^{y_1})=\nu_q(U-V)=\beta+2\nu_q(a). \tag{11}$$

In view of (8) the equality (11) means that $a^{x-2}\mid y_1$ in contradiction with $y_1=y/2<a^{x-2}$ as $x>y,\ a>1$.

(II) $\underline{y>x=z}$: Similarly dividing both sides of (2) by n^z one gets

$$b^y n^{y-x}=c^x-a^x. \tag{12}$$

Arguing as above with mod $c+a$, one sees that x must be even, say $x=2x_1$. Put $X=c^2,\ Y=a^2$. Considering mod b and from (12) it follows that $b\mid x_1$. So $\nu_q(X^{x_1}-Y^{x_1})=\nu_q(x_1)+2\nu_q(b)$ for any $q\in\mathcal{P}(b)$, therefore $b^{y-2}\mid x_1$ in contradiction with $x_1=x/2<b^{y-2}$ as $y>x,\ b>1$.

(III) $\underline{x=y>z}$: Dividing both sides of (2) by n^z one gets

$$(a^x+b^x)n^{x-z}=c^z. \tag{13}$$

First we claim that x must be *even*. Indeed, if x is odd, then from (13) it follows that there is an odd prime $q\in\mathcal{P}(a+b)\cap\mathcal{P}(c)$, so $q\in\mathcal{P}(ab)$, as $c^2=a^2+b^2$. A contradiction with $\gcd(a,b)=1$.

Writing now $x=2x_1$ one sees that x_1 must be *odd*. Since otherwise for an odd prime $q\in\mathcal{P}(a^x+b^x)\cap\mathcal{P}(c)$ taking mod q and by (13)

$$0\equiv a^x+b^x=a^{2x_1}+(c^2-a^2)^{x_1}\equiv 2a^{2x_1} \pmod q,$$

one gets a contradiction with $\gcd(a,c)=1$.

Now from (13) we see that

$$\frac{(a^2)^{x_1}+(b^2)^{x_1}}{a^2+b^2}=\frac{c^{z-2}}{n^{x-z}}>1. \tag{14}$$

as $x>z\geq 2$. So there is an odd prime $q\in\mathcal{P}(c)$ dividing $((a^2)^{x_1}+(b^2)^{x_1})/(a^2+b^2)$. Considering mod q and taking into account $a^2\equiv -b^2 \bmod q,\ q\nmid a$ one has

$$0\equiv\frac{(a^2)^{x_1}+(b^2)^{x_1}}{a^2+b^2}=(a^2)^{x_1-1}-(a^2)^{x_1-2}b^2+\cdots-a^2(b^2)^{x_1-2}+(b^2)^{x_1-1}\equiv x_1a^{2x_1-2} \pmod q.$$

Hence $q\mid x_1$, and so $((a^2)^q+(b^2)^q)\mid((a^2)^{x_1}+(b^2)^{x_1})$. Applying part (1) of Lemma 1 we get

$$\gcd\left(a^2+b^2,\frac{(a^2)^q+(b^2)^q}{a^2+b^2}\right)=q. \tag{15}$$

On the other hand from (14) one knows that $((a^2)^q+(b^2)^q)/(a^2+b^2)$ is a product of primes in $\mathcal{P}(c)$. It is easy to see that $((a^2)^q+(b^2)^q)/(a^2+b^2)>q$. So either $\nu_q\big(((a^2)^q+(b^2)^q)/(a^2+b^2)\big)\geq 2$ and $\nu_q(a^2+b^2)\geq 2$, or both of them must have another common prime factor in $\mathcal{P}(c)$, a contradiction with (15).

3. Preliminary reduction

We need some reduction of the problem. The following result is due to N. Terai [11].

Lemma 3. *Conjecture 1 is true for $n=1,\ v=2$.*

Because of Lemma 3 we will assume henceforth $n>1$.

M.-J. Deng ([12], from the proof of Lemma 2), and H. Yang, R.-Q. Fu ([5]) showed that we can remove the condition (i) in Theorem 1.

Lemma 4. *If (x,y,z,n) is an exceptional solution, then either $x>z>y$, or $y>z>x$.*

Note that the proof of Lemma 4 relies essentially on the condition $n > 1$. It could be interesting to find a proof of this result for the case $n = 1$.

Furthermore, in the case when u is an odd prime and $v = 2$, H. Yang, R.-Q. Fu [13] succeeded to eliminate the possibility (ii) in Theorem 1.

Lemma 5. *Suppose that u is an odd prime and $v = 2$. Then equation (2) has no exceptional solutions (x, y, z, n) with $x > z > y$.*

Lemma 6. *For a positive integer w*

(1) if $\nu_2(w) \geq 2$, then $\nu_2[(1+w)^x - 1] = \nu_2(w) + \nu_2(x)$;
(2) if $\nu_2(w) = 1$ and x is odd, then $\nu_2[(1+w)^x - 1] = 1$;
(3) if $\nu_2(w) = 1$ and x is even, then $\nu_2[(1+w)^x - 1] = \nu_2(2+w) + \nu_2(x)$.

In particular $\nu_2[(1+w)^x - 1] = 2 + \nu_2(x)$, if $w \equiv 4 \pmod 8$; or if $w \equiv 2 \pmod 8$ and x is even.

Proof. (1) The conclusions of Lemma 6 are true trivially for $x = 1$. Assuming now $x \geq 2$ we have

$$(1+w)^x - 1 = w(C_x^1 + C_x^2 w + \cdots + C_x^{x-1} w^{x-2} + C_x^x w^{x-1}). \tag{16}$$

Clearly $\nu_2(j) \leq j - 1$ for $j = 2, \cdots, x$, and so

$$\nu_2(C_x^j w^{j-1}) = \nu_2\left(\frac{x}{j} C_{x-1}^{j-1} w^{j-1}\right) \geq \nu_2(x) + j - 1 > \nu_2(x),$$

as $\nu_2(w) \geq 2$. Hence the conclusion follows from taking $\nu_2(\cdot)$ on both sides of (16).

(2) Obvious from (16), since $C_x^1 + C_x^2 w + \cdots + C_x^{x-1} w^{x-2} + C_x^x w^{x-1}$ is *odd* in this case.

(3) Writing $x = 2x_1$ we have

$$(1+w)^x - 1 = [(1+w)^{x_1} - 1][(1+w)^{x_1} + 1]. \tag{17}$$

If x_1 is *odd*, *i.e.*, $\nu_2(x) = 1$, then $\nu_2[(1+w)^{x_1} - 1] = 1$ by the part (2) above, and $\nu_2[(1+w)^{x_1} + 1] = \nu_2(2+w)$, as

$$(1+w)^{x_1} + 1 = (2+w)[(1+w)^{x_1-1} - (1+w)^{x_1-2} + \cdots - (1+w) + 1]$$

and $(1+w)^{x_1-1} - (1+w)^{x_1-2} + \cdots - (1+w) + 1$ is *odd*.

If x_1 is *even*, then $\nu_2[(1+w)^{x_1} + 1] = 1$, since

$$(1+w)^{x_1} + 1 = 2 + C_{x_1}^1 w + C_{x_1}^2 w^2 + \cdots + C_{x_1}^{x_1-1} w^{x_1-1} + w^{x_1}.$$

Therefore $\nu_2[(1+w)^x - 1] = \nu_2[(1+w)^{x_1} - 1] + 1$ by (17). Now the descending argument yields the conclusion.

□

The following claims play a central role in the next sections.

Lemma 7. *If (x, y, z, n) is an exceptional solution of (4), then $u^2 - 4$ admits a proper decomposition $u^2 - 4 = u_1 u_2$, $\gcd(u_1, u_2) = 1$ and with one of the following conditions satisfied:*

(1) $u_1 \equiv 1 \pmod 8$ and $\nu_2(z) = \nu_2(u_1 - 1) + \nu_2(x) - 2$;
(2) $u_1 \equiv 7 \pmod 8$, $\nu_2(z) = \nu_2(u_1 + 1) + \nu_2(x) - 2$, and $\nu_2(x) \geq 1$;
(3) $u_1 \equiv 5 \pmod 8$, u_2 is a square and and $\nu_2(z) = \nu_2(x)$.

Proof. In view of Lemmas 4, 5 we may assume the existence of an exceptional solution with $y > z > x$ (the case (iii) of Theorem 1). Dividing both sides of (4) by n^x one gets

$$(u^2 - 4)^x = [(u^2 + 4)^z - (4u)^y n^{y-z}] n^{z-x}. \tag{18}$$

It is easy to see that $\gcd(u^2+4,n)=1$. So (18) is equivalent to the following system

$$\begin{cases} (u^2+4)^z-(4u)^y n^{y-z}=u_1^x \\ n^{z-x}=u_2^x \end{cases} \tag{19}$$

with $u^2-4=u_1u_2,\ \gcd(u_1,u_2)=1$. The system (19) can be rewritten as

$$(u^2+4)^z-2^{2y}u^y n^{y-z}=u_1^x, \tag{20}$$

or equivalently

$$[(u^2+4)^z-1]-(u_1^x-1)=2^{2y}u^y n^{y-z}, \tag{21}$$

with $k(z-x)=mx$, and $n^m=u_2^k$.

Clearly $u_2>1$. Assume now $u_1=1$. As $u^2\equiv 1 \pmod 8$, by comparing $\nu_2(\cdot)$ both sides of (20) and by (1) of Lemma 6 we have $\nu_2[(u^2+4)^z-1]=2+\nu_2(z)<2y$. So (21) is inconsistent. So $u_1>1$ and

$$\nu_2(z)=\nu_2(u_1^x-1)-2. \tag{22}$$

If $u_1\equiv 1 \bmod 8$, then by (1) of Lemma 6 we get $\nu_2(z)=\nu_2(u_1-1)+\nu_2(x)-2$.

If $u_1\equiv 7 \bmod 8$ and x is *odd*, then by (2) of Lemma 6: $\nu_2(u_1^x-1)=1$, impossible by (22). Thus (21) is inconsistent.

If $u_1\equiv 7 \bmod 8$ and x is *even*, then by (3) of Lemma 6: $\nu_2(u_1^x-1)=\nu_2(u_1+1)+\nu_2(x)$. Hence by (22) one gets $\nu_2(z)=\nu_2(u_1+1)+\nu_2(x)-2$.

For $u_1\equiv 3 \bmod 8$, we have $\nu_2(u_1^x-1)=1$, if x is *odd* (by (2) of Lemma 6), and $\nu_2(u_1^x-1)=2+\nu_2(x)$, if x is *even* (by (3) of Lemma 6). Hence for (20) to be consistent one has necessarily $\nu_2(z)=\nu_2(x)$, which implies $\nu_2(z-x)\geq \nu_2(x)+1$. So from the second equation of (19): $n^{z-x}=u_2^x$ it follows that u_2 must be a square, hence $u_2\equiv 1 \bmod 8$. Thus $u_1u_2\equiv 3 \bmod 8$, a contradiction with $u_1u_2=u^2-4\equiv 5 \bmod 8$.

Similarly, for $u_1\equiv 5 \bmod 8$, by using (1) of Lemma 6 we have $\nu_2(u_1^x-1)=2+\nu_2(x)$, and by the same reason $\nu_2(z)=\nu_2(x)$. Hence the system (19) is inconsistent, if u_2 is not a square. □

Lemma 8. *In the notations above if x,y,z are even, then (20) is inconsistent.*

Proof. In this case we can rewrite (20) in the form of Pythagorian equation

$$\left(u_1^{x/2}\right)^2+\left[2^y u^{y/2} n^{(y-z)/2}\right]^2=\left[(u^2+4)^{z/2}\right]^2.$$

Hence (*cf.* (1)) there are integers X,Y, say with $2\mid Y$ such that

$$(u^2+4)^{z/2}=X^2+Y^2, \tag{23}$$

$$2^y u^{y/2} n^{(y-z)/2}=2XY. \tag{24}$$

In view of Lemma 2.2 of [9], Equation (23) has solutions

$$u^2+4=A^2+B^2,\ \ 2\mid B, \tag{25}$$

$$\nu_2(Y)=\nu(z/2)+\nu_2(B). \tag{26}$$

Since $u^2+4\equiv 5 \bmod 8$ it follows from (25) that $\nu_2(B)=1$. From (24) we have $\nu_2(Y)=y-1$ which together with (26) implies

$$y=\nu_2(z)+1,$$

a contradiction with $y>z$. □

Corollary 2. *In the notations above if y,z are even and (20) is consistent, then x is odd and $u_1\equiv 1 \pmod 8$. Moreover u_1 admits a proper decomposition $u_1=t_1t_2$ such that $\gcd(t_1,t_2)=1$ and*

$$t_2^x+t_1^x=2(u^2+4)^{z/2}, \tag{27}$$

$$t_2^x - t_1^x = 2^{y+1}u^{y/2}n^{(y-z)/2}, \tag{28}$$

$$\nu_2(t_1^x - 1) = \nu_2(t_2^x - 1) = \nu_2(u_1^x - 1) - 1. \tag{29}$$

Proof. By Lemma 8 x is odd. In fact one can rewrite (20) as

$$A \cdot B = u_1^x \text{ with } \gcd(A, B) = 1,$$

where

$$A = (u^2+4)^{z/2} - 2^y u^{y/2} n^{(y-z)/2}, \quad B = (u^2+4)^{z/2} + 2^y u^{y/2} n^{(y-z)/2}.$$

Hence

$$A = t_1^x,\ B = t_2^x \text{ with } u_1 = t_1 t_2 \text{ and } \gcd(t_1, t_2) = 1. \tag{30}$$

If $t_1 = 1$, then by (1) of Lemma 6: $\nu_2[(u^2+4)^{z/2} - 1] = 2 + \nu_2(z/2) < y = \nu_2(2^y u^{y/2} n^{(y-z)/2})$. So $A = 1$ is impossible.

Now from (30) we have two possibilities:

(1) $z/2$ is odd: $t_1 \equiv t_2 \equiv 5 \pmod 8$;
(2) $z/2$ is even: $t_1 \equiv t_2 \equiv 1 \pmod 8$;

both of them imply $u_1 \equiv 1 \pmod 8$.

Also (27)-(29) follow immediately from (30). □

Corollary 3. *In the situation of Corollary 2 we have $t_1, t_2 \equiv 5 \pmod 8$ and $\nu_2(u_1 - 1) = 3$.*

Proof. We will show that $z/2$ must be odd, from which the conclusion immediately follows by the proof above, noting that $\nu_2(u_1 - 1) = \nu_2(u_1^x - 1) = \nu_2(A - 1) + 1 = 3$.

Assume on the contrary that $\nu_2(z) \geq 2$. In view of (30) one has $x \geq 3$, as $t_1 < t_2 < u^2 - 4$. We claim that $x > 3$. Indeed, if $x = 3$, then $n = u_2^3$ by (19), noting that $z = 4$ by $B = t_2^x$ of (30), so $y = 6$ as $A = t_1^x > 0$. Now from the equation $t_1^x = A$ in (30) we see that $(t_1, 4uu_2, u^2+4)$ is a primitive solution of

$$X^3 + Y^3 = Z^2. \tag{31}$$

Euler ([14], pp. 578–579) indicated a primitive parameterization for the Diophantine Equation (31) with $3 \nmid Z$, $2 \mid Y$ as follows

$$X = (s-t)(3s-t)(3s^2+t^2), \quad Y = 4st(3s^2 - 3st + t^2),$$

with s, t co-prime, $3 \nmid t$ and $s \not\equiv t \pmod 2$. Hence $8 \mid Y$ which shows that $t_1^x = A$ in (30) is impossible.

Furthermore, if $x \geq 4$, then by Theorem 1.1 of [15], (27) is again impossible. □

4. Proof of Theorem 2

The aim of this section is to show that the case $u_1 \equiv 7 \pmod 8$ in Lemma 7 is not realized. We refer the reader to [16] for basic properties of Jacobi quadratic and quartic residue symbols $\left(\frac{-}{m}\right)$, $\left(\frac{-}{m}\right)_4$ we shall use in the following lemmas.

Lemma 9. *For a prime $p \mid (u^2+4)$ one has $p \equiv 1 \pmod 4$ and $\left(\frac{u}{p}\right) = 1$.*

Proof. Since $u^2 \equiv -4 \pmod p$, so $\left(\frac{-1}{p}\right) = 1$, *i.e.*, $p \equiv 1 \pmod 4$. Furthermore we include the following simple argument due to the referee instead of ours in the original version:

$$\left(\frac{u}{p}\right) = \left(\frac{4u}{p}\right) = \left(\frac{4u + u^2 + 4}{p}\right) = \left(\frac{(u+2)^2}{p}\right) = 1.$$

□

Lemma 10. *If (20) is consistent and $u_1 \equiv \pm 1 \pmod 8$, then $\left(\frac{n}{p}\right) = \left(\frac{u_2}{p}\right)$ for any prime p.*

Proof. Indeed, in this case by Lemma 7 $\nu_2(z) > \nu_2(x)$. Hence $\nu_2(z-x) = \nu_2(x)$, so we have in (21) $n^m = u_2^k$ with k, m odd, and therefore the conclusion of Lemma 10. $\square$

We are ready now to prove Theorem 2. Let $p \mid (u^2+4)$. By taking $\left(\frac{-}{p}\right)$ on (20) and using Lemmas 9, 10 one sees that

$$\left(\frac{u_1}{p}\right)^x = \left(\frac{n}{p}\right)^{y-z} = \left(\frac{u_2}{p}\right)^{y-z} = \left(\frac{u_2}{p}\right)^y, \tag{32}$$

(as z is even). Now taking the product of (32) over all (not necessarily distinct) prime divisors $p \mid (u^2+4)$ we have

$$\left(\frac{u_1}{u^2+4}\right)^x = \prod_{p|(u^2+4)} \left(\frac{u_1}{p}\right)^x = \prod_{p|(u^2+4)} \left(\frac{u_2}{p}\right)^y = \left(\frac{u_2}{u^2+4}\right)^y. \tag{33}$$

By the quadratic reciprocity law

$$\left(\frac{u_1}{u^2+4}\right) = \left(\frac{u^2+4}{u_1}\right) = \left(\frac{2}{u_1}\right) = 1, \tag{34}$$

$$\left(\frac{u_2}{u^2+4}\right) = \left(\frac{u^2+4}{u_2}\right) = \left(\frac{2}{u_2}\right) = -1, \tag{35}$$

as $u_1 \equiv \pm 1 \pmod 8$, $u_2 \equiv \pm 5 \pmod 8$. Altogether (33)-(35) imply that $\left(\frac{u_2}{p}\right)^y = (-1)^y = 1$, *i.e.*, y must be even.

Corollary 4. *The possibility $u_1 \equiv 7 \pmod 8$ in Lemma 7 is not realized.*

Proof. Indeed, in this case $\nu_2(z) > \nu_2(x) \geq 1$, so (20) is inconsistent by Lemma 8. $\square$

Corollary 5. *In the case $u_1 \equiv 1 \pmod 8$ of Lemma 7 we have*

$$(\nu_2(x), \nu_2(y), \nu_2(z)) = (0, \geq 2, 1).$$

Proof. By Lemma 7 and Theorem 2: y, z are even, hence x is odd by Lemma 8. From the proof of Corollary 3 it follows that $\nu_2(z) = 1$. For a prime $p \mid (u^2+4)$ by taking $\left(\frac{-}{p}\right)$ on $A = t_1^x$ of (30) and using Lemma 9 one gets

$$\left(\frac{t_1}{p}\right) = \left(\frac{n}{p}\right)^{(y-z)/2}. \tag{36}$$

By the same reason of (35) we have $\left(\frac{t_1}{u^2+4}\right) = -1$, as $t_1 \equiv 5 \pmod 8$ by Corollary 3. Hence there exists a prime $p_0 \mid (u^2+4)$ such that

$$\left(\frac{t_1}{p_0}\right) = -1. \tag{37}$$

From (36), (37) one concludes that $(y-z)/2$ must be odd (and $\left(\frac{n}{p_0}\right) = -1$), so the conclusion of Corollary 5 follows. $\square$

Remark 2. One can have another proof of Lemma 8 as shown in several steps below. Assuming y, z even, and arguing as in the proof of Corollary 2 one gets Equation (30) together with (27)-(29).

1) If $u_1 \equiv 5 \pmod 8$ we have four possibilities for (t_1, t_2):

   (i) $t_1 \equiv 1 \pmod 8$, $t_2 \equiv 5 \pmod 8$;
   (ii) $t_1 \equiv 5 \pmod 8$, $t_2 \equiv 1 \pmod 8$;
   (iii) $t_1 \equiv 3 \pmod 8$, $t_2 \equiv 7 \pmod 8$;
   (iv) $t_1 \equiv 7 \pmod 8$, $t_2 \equiv 3 \pmod 8$;

   all of them violate (29).

(2) Assume now $u_1 \equiv \pm 1 \pmod 8$ and x *even*, hence $\nu_2(z) \geq 2$ by Lemma 7. We will shows that $\nu_2(y) = 1$. Indeed, considering $p \mid (u^2+4)$ and taking $\left(\frac{-}{p}\right)_4$ on (20) one has by using Lemmas 9, 10

$$\left(\frac{u_1}{p}\right)^{x/2} = \left(\frac{-1}{p}\right)_4 \left(\frac{n}{p}\right)^{(y-z)/2} = \begin{cases} \left(\frac{u_2}{p}\right)^{y/2}, & p \equiv 1 \pmod 8 \\ -\left(\frac{u_2}{p}\right)^{y/2}, & p \equiv 5 \pmod 8 \end{cases} \tag{38}$$

as $z/2$ is even. Let r denote the number of prime divisors $p \mid (u^2+4)$, $p \equiv 5 \pmod 8$. Clearly r is *odd*, as $u^2+4 \equiv 5 \pmod 8$. In a similar way as in (33)-(35), taking the product of (38) over all (not necessarily distinct) prime divisors $p \mid (u^2+4)$ we get

$$1 = \left(\frac{u_1}{u^2+4}\right)^{x/2} = (-1)^r \left(\frac{u_2}{u^2+4}\right)^{y/2} = -(-1)^{y/2}.$$

Hence $y/2$ must be odd, so $(y-z)/2$ is odd. For any prime $p \mid (u^2+4)$ taking $\left(\frac{-}{p}\right)$ on equation $A = t_1^x$ from (30) now gives us

$$\left(\frac{n}{p}\right) = 1 \left(= \left(\frac{u_2}{p}\right) \text{ by Lemma 10} \right) \tag{39}$$

On the other hand from (35) it follows that there exists a prime $p_0 \mid (u^2+4)$ such that $\left(\frac{u_2}{p_0}\right) = -1$, a contradiction with (39). Thus (30) (and hence (20)) is inconsistent.

5. The case $u_1 \equiv 5 \pmod 8$

In this case by (3) of Lemma 7 we have $\nu_2(z) = \nu_2(x)$, hence from (19) it follows that $u_2 = w^{2^s}$, where $s = \nu_2(z-x) - \nu_2(x)$. The following lemma can be proved similarly as Lemma 10.

Lemma 11. *If* (20) *is consistent and* $u_1 \equiv 5 \pmod 8$, *then* $\left(\frac{n}{p}\right) = \left(\frac{w}{p}\right)$ *for any prime* p.

Proof. Indeed, in this case $n^m = w^k$ with k, m odd by the above argument, and therefore the conclusion of Lemma 11. □

Lemma 12. *If* x, z *are even and* (20) *is consistent, then* y *is odd and* $u_1 \equiv 5 \pmod 8$. *Moreover* n *admits a decomposition* $n = n_1 n_2$ *such that* $\gcd(n_1, n_2) = 1$ *and*

$$\begin{cases} u_1^{x/2} = u^y n_2^{y-z} - 2^{2y-2} n_1^{y-z}; \\ (u^2+4)^{z/2} = u^y n_2^{y-z} + 2^{2y-2} n_1^{y-z}. \end{cases} \tag{40}$$

Proof. By Lemma 8 y is odd. In view of Lemma 7 and Theorem 2 we are in the situation (3) of Lemma 7. Now one rewrites (20) as

$$C_1 \cdot D_1 = 2^{2y} u^y n^{y-z} \text{ with } \gcd(C_1, D_1) = 2,\ 2 \| D_1,$$

where

$$C_1 = (u^2+4)^{z/2} - u_1^{x/2},\ D_1 = (u^2+4)^{z/2} + u_1^{x/2}.$$

As $2 \| D_1$ we obtain either

$$C_1 = 2^{2y-1} n_1^{y-z}, \quad D_1 = 2u^y n_2^{y-z}, \tag{41}$$

or

$$C_1 = 2^{2y-1} u^y n_1^{y-z}, \quad D_1 = 2n_2^{y-z}, \tag{42}$$

where $n = n_1 n_2$, $\gcd(n_1, n_2) = 1$ and

$$w = w_1 w_2,\ n_1^m = w_1^k,\ n_2^m = w_2^k, \tag{43}$$

with k, m odd from Lemma 11. Note that this is not used in the proof here, we label it for convenience in proving Proposition 1 below.

Clearly (41) is equivalent to (40). It remains to show that (42) can't happen by rewriting it as

$$\begin{cases} u_1^{x/2} &= n_2^{y-z} - 2^{2y-2}u^y n_1^{y-z}, \\ (u^2+4)^{z/2} &= n_2^{y-z} + 2^{2y-2}u^y n_1^{y-z}, \end{cases} \tag{44}$$

which is impossible, since $(u^2+4)^{z/2} < 2^{2y-2}u^y$. $\square$

Lemma 13. *If* $\left(\frac{u_1}{u}\right) = 1$ *and* u_2 *is a square, then* $u \equiv 1 \pmod 4$.

Proof. We have obviously

$$1 = \left(\frac{u_1}{u}\right) = \left(\frac{u_1 u_2}{u}\right) = \left(\frac{u^2-4}{u}\right) = \left(\frac{-1}{u}\right),$$

so the conclusion of the lemma. $\square$

Lemma 14. *In the notations of Lemma 11 we have*

(1) if $w \equiv \pm 3 \pmod 8$, *then* x, z *are odd,* y *is even;*
(2) if $w \equiv \pm 1 \pmod 8$, *then* x, z *are even,* y *is odd.*

Proof. For a prime $p \mid (u^2+4)$ by taking $\left(\frac{-}{p}\right)$ on (20) and using Lemmas 9, 11 one sees that

$$\left(\frac{u_1}{p}\right)^x = \left(\frac{n}{p}\right)^{y-z} = \left(\frac{w}{p}\right)^{y-z}. \tag{45}$$

By taking the product of both sides of (45) over all (not necessarily distinct) prime divisors $p \mid (u^2+4)$ and using the reciprocity law we have

$$\prod_{p \mid (u^2+4)} \left(\frac{u_1}{p}\right)^x = \left(\frac{u_1}{u^2+4}\right)^x = \left(\frac{u^2+4}{u_1}\right)^x = \left(\frac{2}{u_1}\right)^x = (-1)^x, \tag{46}$$

$$\prod_{p \mid (u^2+4)} \left(\frac{w}{p}\right)^{y-z} = \left(\frac{w}{u^2+4}\right)^{y-z} = \left(\frac{u^2+4}{w}\right)^{y-z} = \left(\frac{2}{w}\right)^{y-z} = \begin{cases} (-1)^{y-z}, & w \equiv \pm 3 \pmod 8, \\ 1, & w \equiv \pm 1 \pmod 8. \end{cases} \tag{47}$$

Hence if $w \equiv \pm 3 \pmod 8$, then by equalizing (46), (47): $(-1)^x = (-1)^{y-z}$. Thus y must be even, as $\nu_2(z) = \nu_2(x)$. In view of Lemma 8 x, z are odd.

In the case $w \equiv \pm 1 \pmod 8$, again equalizing (46), (47) we see that $(-1)^x = 1$, therefore x is even, and so is z. By Lemma 8 y must be odd. $\square$

Proposition 1. *In the situation of Lemma 14 we have*

(1) if $w \equiv \pm 3 \pmod 8$, *then* $u \equiv 1 \pmod 4$;
(2) if $w \equiv \pm 1 \pmod 8$, *then* $u \equiv \pm 3 \pmod 8$. *Moreover, if* $u \equiv 3 \pmod 8$, *then* w *can not be a square.*

Proof. (1) If $w \equiv \pm 3 \pmod 8$, then x, z are odd in view of Lemma 14. So by taking $\left(\frac{-}{u}\right)$ on (20) one gets $\left(\frac{u_1}{u}\right) = 1$, hence $u \equiv 1 \pmod 4$ by Lemma 13.

(2) In the case $w \equiv \pm 1 \pmod 8$: x, z are even, y is odd by Lemma 14. There are two subcases to consider.

I. $x/2, z/2$ are odd. For a prime $p \mid (u^2+4)$ by taking $\left(\frac{-}{p}\right)$ on $D_1 = 2u^y n_2^{y-z}$ from (41), (43) and using Lemmas 9, 11 one sees that

$$\left(\frac{u_1}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{n_2}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{w_2}{p}\right) = \begin{cases} \left(\frac{w_2}{p}\right), & p \equiv 1 \pmod 8, \\ -\left(\frac{w_2}{p}\right), & p \equiv 5 \pmod 8. \end{cases} \tag{48}$$

Recall that the number of (not necessarily distinct) prime divisors $p \mid (u^2+4)$, $p \equiv 5 \pmod 8$ is odd, so $\prod_{p \mid (u^2+4)} \left(\frac{2}{p}\right) = -1$. Now taking the product of both sides of (48) over all (not necessarily distinct) prime divisors $p \mid (u^2+4)$ and using the reciprocity law one has

$$\prod_{p \mid (u^2+4)} \left(\frac{u_1}{p}\right) = \left(\frac{u_1}{u^2+4}\right) = \left(\frac{u^2+4}{u_1}\right) = \left(\frac{2}{u_1}\right) = -1, \tag{49}$$

and

$$\prod_{p \mid (u^2+4)} \left(\frac{2}{p}\right)\left(\frac{w_2}{p}\right) = -\prod_{p \mid (u^2+4)} \left(\frac{w_2}{p}\right) = -\left(\frac{w_2}{u^2+4}\right) = -\left(\frac{u^2+4}{w_2}\right) = -\left(\frac{2}{w_2}\right). \tag{50}$$

Equalizing (49), (50) we get $w_2 \equiv \pm 1 \pmod 8$, so in view of (43): $n_2 \equiv \pm 1 \pmod 8$. From this and (40) it follows that $u \equiv \pm 3 \pmod 8$. Moreover, if $u \equiv 3 \pmod 8$, then $w_2 \equiv -1 \pmod 8$, hence by (43) w can not be a square.

II. $x/2, z/2$ are even. If one takes $\left(\frac{-}{u}\right)$ on the second equation of (40), then $\left(\frac{n_1}{u}\right) = 1$. Now taking $\left(\frac{-}{u}\right)$ on the first equation of (40) we get $1 = \left(\frac{-1}{u}\right)\left(\frac{n_1}{u}\right)$. Thus $u \equiv 1 \pmod 4$.

The proof of Proposition 1 is completed. □

As for Theorem 3 notice that the case $u_1 \equiv \pm 1 \pmod 8$ follows from Corollaries 2, 3, 4 and 5. The rest of Theorem 3, *i.e.*, the case $u_1 \equiv 5 \pmod 8$, follows from Lemma 14 and Proposition 1.

The equalities for Jacobi symbols are immediate from (20) and Lemma 11.

6. Proof of Corollary 1

In this section we shall apply results of previous parts for establishing the truth of Jeśmanowicz' conjecture for $u < 100$ and $v = 2$. In view of Theorem 3 one has to consider only two cases: $u_1 \equiv 1 \pmod 8$ and $u_1 \equiv 5 \pmod 8$.

Observation 1. *If $u_1 \equiv 1 \pmod 8$ and (20) is consistent, then $u > 183$.*

Proof. Indeed, it was noted that $x \geq 3$ by (30). On the other hand from the proof of Corollary 3 we have $\nu_2(z) = 1$, so $z \geq 6$, hence $y \geq 8$. From (28) it follows that $2^{y+1} \mid t_2 - t_1$, as x is odd. Since t_1, u_2 are co-prime and $\equiv 5 \pmod 8$, so $t_1 u_2 \geq 5 \cdot 13$. Therefore $u > \sqrt{t_1 t_2 u_2} \geq \sqrt{(2^9+5) \cdot 65} > 183$. □

Observation 2. *If $u_1 \equiv 1 \pmod 8$ and (20) is consistent, then in fact $u > 729$.*

Proof. By Corollary 5 one knows $4 \mid y$. We claim that $y \geq 12$. Assuming on the contrary $y = 8$, then by the above $z = 6$. In view of (27) and [17] we must have $x > 3$, so $x = 5$, which gives us a non-trivial solution of $X^5 + Y^5 = 2Z^3$. This is impossible by [18] (Theorem 1.5).

Therefore $y \geq 12$, and by the argument above $u > \sqrt{(2^{13}+5) \cdot 65} > 729$. □

It remains to consider the case $u_1 \equiv 5 \pmod 8$. In the range of odd primes < 100 there are ten possibilities with $u^2 - 4 = u_1 u_2$ and u_2 is a square, namely $u = 7, 11, 23, 43, 47, 61, 73, 79, 83, 97$. In view of Proposition 1 we shall exclude the possibilities $u = 7, 23, 47, 79$.

Observation 3. *For $(u, u_1, u_2) = (11, 13, 3^2), (43, 5 \cdot 41, 3^2), (83, 5 \cdot 17, 3^4)$ we have $w \equiv \pm 3 \pmod 8$, hence $u \equiv 1 \pmod 4$ by Proposition 1, a contradiction. Note that in the original version to eliminate the possibility $(83, 5 \cdot 17, 3^4)$ and $w = 9$ we used implicitly the fact that if $u \equiv 3 \pmod 8$, then w can not be a square, which we include a proof in the revised version (cf. Proposition 1 above). The referee provides another argument by choosing $p = 5 \mid u_1$ which leads also to a contradiction as follows*

$$1 = \left(\frac{9}{5}\right) = \left(\frac{w}{p}\right) \neq \left(\frac{u}{p}\right) = \left(\frac{83}{5}\right) = -1.$$

Observation 4. *For $(u, u_1, u_2) = (61, 7 \cdot 59, 3^2)$ one has $w = 3$, so*

$$-1 = \left(\frac{w}{7}\right) \neq \left(\frac{u^2+4}{7}\right) = 1,$$

a contradiction with (2.1) *of Theorem* 3.

Observation 5. *For* $(u, u_1, u_2) = (73, 3 \cdot 71, 5^2)$ *we have* $w = 5$, *hence* x, z *are odd and* y *is even by* (2.1) *of Theorem* 3. *Taking modulo* 73 *on* (20) *one gets*

$$4^z \equiv (-6)^x \pmod{73}. \tag{51}$$

Working in $\mathbb{F}_{73}^*$ *we have*

$$\operatorname{ord}(4) = 9, \quad \operatorname{ord}(-6) = 36. \tag{52}$$

Therefore from (51), (52) *it follows that* $36 \mid 9x$, *so* $4 \mid x$, *a contradiction.*

Observation 6. *For* $(u, u_1, u_2) = (97, 5 \cdot 11 \cdot 19, 3^2)$ *one has* $w = 3$, *so*

$$1 = \left(\frac{w}{11}\right) \neq \left(\frac{u^2+4}{11}\right) = -1,$$

again a contradiction with (2.1) *of Theorem* 3.

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