J. Math. Soc. Japan
Vol. 29, No. 1, 1977

Arithmetic triangle groups

By Kisao TAKEUCHI

(Received Jan. 13, 1976)

§ 1. Introduction.

We can find some examples of arithmetic triangle Fuchsian groups of type $(2, e_2, e_3)$ in the book written by Fricke-Klein [1]. Mr. T. Kaise has proved some results on arithmetic triangle groups of type (e, e, e) ([2]). In the paper [5] we have given a characterization of arithmetic Fuchsian groups. As an application of this result, we shall determine in the present paper all arithmetic triangle groups explicitly. In § 3 we shall give a necessary and sufficient condition for a triangle group to be arithmetic (Theorem 1, § 3). Making use of this condition we shall prove that there exist only finitely many arithmetic triangle groups up to $SL_2(\boldsymbol{R})$-conjugation (Theorem 2 in § 4). In § 5 by making use of a computer we shall give a complete list of all arithmetic types (e_1, e_2, e_3) (Theorem 3, § 5).

The author is grateful to professor G. Shimura for many valuable suggestions.

§ 2. Triangle Fuchsian groups.

Let $SL_2(\boldsymbol{R})$ be the special linear group of degree 2 over the real number field $\boldsymbol{R}$. Then $SL_2(\boldsymbol{R})$ operates on the upper half plane $H=\{z\in\boldsymbol{C}\,|\,\mathrm{Im}\,(z)>0\}$ by fractional linear transformations. This gives a homomorphism π of $SL_2(\boldsymbol{R})$ onto the group $\mathrm{Aut}\,(H)$ of all analytic automorphisms on H. For any $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{R})$ put $\bar{g}=\pi(g)$. Then we have $\bar{g}(z)=(az+b)/(cz+d)$. The kernel of π is $\{\pm 1_2\}$.

Let Γ be a Fuchsian group of the first kind (i. e. a discrete subgroup of $SL_2(\boldsymbol{R})$ such that quotient space $H/\pi(\Gamma)$ is of finite volume). Then $\pi(\Gamma)$ is generated by $2g$ hyperbolic elements $\{\bar{\alpha}_i\}$, $\{\bar{\beta}_i\}$ $(1\leqq i\leqq g)$, s elliptic elements $\{\bar{\gamma}_j\}$ $(1\leqq j\leqq s)$, and t parabolic elements $\{\bar{\gamma}_j\}$ $(s+1\leqq j\leqq s+t)$, which satisfy the fundamental relations

$$\begin{cases} \bar{\alpha}_1\bar{\beta}_1\bar{\alpha}_1^{-1}\bar{\beta}_1^{-1}\cdots\bar{\alpha}_g\bar{\beta}_g\bar{\alpha}_g^{-1}\bar{\beta}_g^{-1}\bar{\gamma}_1\cdots\bar{\gamma}_{s+t}=\bar{1}_2\,, \\ \bar{\gamma}_j^{e_j}=\bar{1}_2 \qquad (1\leqq j\leqq s)\,, \end{cases} \tag{1}$$

where e_j $(1\leqq j\leqq s)$ is a positive integer $\geqq 2$. Put $e_i=\infty$ for $s+1\leqq j\leqq s+t$. Then $(g;e_1,\cdots,e_{s+t})$ is called *the signature of* Γ. This satisfies the inequality

$$2g-2+\sum_{j=1}^{s+t}(1-1/e_j)>0\,, \tag{2}$$

where $1/e_j=0$ for $e_j=\infty$.

In the case where $g=0$ and $s+t=3$, Γ is called *a triangle group of type* (e_1,e_2,e_3). If $t=0$ (resp. $t\geqq 1$), then we say that Γ is *of compact* (resp. *non-compact*) *type*. By (1) there exist elliptic or parabolic elements $\{\gamma_j\}$ $(1\leqq j\leqq 3)$ of Γ which generate $\pi(\Gamma)$ and satisfy the fundamental relations

$$\begin{cases} \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3=\bar{1}_2\,, & \\ \bar{\gamma}_j^{e_j}=\bar{1}_2 & (1\leqq j\leqq s)\,. \end{cases} \tag{3}$$

By (2) we have the inequality

$$1/e_1+1/e_2+1/e_3<1\,. \tag{4}$$

By changing generators we may assume that

$$2\leqq e_1\leqq e_2\leqq e_3\leqq\infty\,. \tag{5}$$

Now we shall determine all triangle groups of given type (e_1,e_2,e_3) up to $SL_2(\boldsymbol{R})$-conjugation.

PROPOSITION 1. *Notations being as above, let* (e_1,e_2,e_3) *be a triple satisfying* (4) *and* (5). *Then the following assertions hold:*

(i) *If* $s\geqq 1$ *and at least one of* e_j $(1\leqq j\leqq s)$ *is even, then there exists a triangle group* Γ_0 *of type* (e_1,e_2,e_3) *such that any triangle group of type* (e_1,e_2,e_3) *is* $SL_2(\boldsymbol{R})$*-conjugate to* Γ_0. Γ_0 *contains* -1_2.

(ii) *If* $2\leqq s\leqq 3$ *and all* e_j $(1\leqq j\leqq s)$ *are odd, then there exist two triangle groups* Γ_0 *and* Γ_1 *of type* (e_1,e_2,e_3) *such that any triangle group of this type is* $SL_2(\boldsymbol{R})$*-conjugate to one of these groups.* Γ_0 *contains* -1_2 *and* Γ_1 *does not. In particular,* Γ_0 *and* Γ_1 *are not* $SL_2(\boldsymbol{R})$*-conjugate to each other.* Γ_1 *is a subgroup of* Γ_0 *of index* 2.

(iii) *If either* $s=1$ *and* e_1 *is odd or* $s=0$, *then there exist three triangle groups* Γ_0, Γ_1 *and* Γ_2 *of type* (e_1,e_2,e_3) *such that any triangle group of this type is* $SL_2(\boldsymbol{R})$*-conjugate to one of these groups.* Γ_0 *contains* -1_2 *and* Γ_i $(2\leqq i\leqq 3)$ *does not.* Γ_i *is a subgroup of* Γ_0 *of index* 2.

PROOF. We need the following well-known lemma (cf. [3]).

LEMMA 1. *Let* Γ *and* Γ' *be two triangle groups of the same type. Then* $\pi(\Gamma)$ *and* $\pi(\Gamma')$ *are* Aut(H)*-conjugate to each other.*

It is shown in [3] that for any triple (e_1,e_2,e_3) satisfying (4) and (5) there exists a triangle group Γ_0 of type (e_1,e_2,e_3) generated by $\{\gamma_{0j}\}$ $(1\leqq j\leqq 3)$ and $\{-1_2\}$ such that

$$\mathrm{tr}\,(\gamma_{0j})=2\cos(\pi/e_j)\qquad(1\leqq j\leqq 3)\,, \tag{6}$$

where $\cos(\pi/e_j)=1$ for $e_j=\infty$.

The fundamental relations are given by

$$\begin{cases} \gamma_{01}\gamma_{02}\gamma_{03}(-1_2)=1_2\,, \\ \gamma_{0j}^{e_j}(-1_2)=1_2 \quad (1\leqq j\leqq s)\,, \qquad (-1_2)^2=1_2\,, \\ \gamma_{0j}(-1_2)\gamma_{0j}^{-1}(-1_2)=1_2 \qquad (1\leqq j\leqq 3)\,. \end{cases} \tag{7}$$

(i) Suppose that e_j is even. Let Γ be any triangle group of type (e_1, e_2, e_3). We shall show that Γ contains -1_2. Γ contains an element γ such that $\bar{\gamma}$ is of order e_j. Hence we have $\gamma^{e_j}=\pm 1_2$. Assume that $\gamma^{e_j}=1_2$. Since e_j is even, we have $(\gamma^{e_j/2})^2=1_2$. Hence $\gamma^{e_j/2}=\pm 1_2$. This means that $\bar{\gamma}$ is of the smaller order than e_j, which is a contradiction. Therefore, we see that $\gamma^{e_j}=-1_2$. This shows that Γ contains -1_2. It implies that $\Gamma=\pi^{-1}(\pi(\Gamma))$. By Lemma 1, Γ is $SL_2(\boldsymbol{R})$-conjugate to Γ_0.

(ii) Suppose that all e_j $(1\leqq j\leqq s)$ are odd. Put $\gamma_{1j}=-\gamma_{0j}$ $(1\leqq j\leqq 3)$. Let Γ_1 be the subgroup of Γ_0 generated by $\{\gamma_{1j}\,|\,1\leqq j\leqq 3\}$. By (7) these satisfy the relations:

$$\begin{cases} \gamma_{11}\gamma_{12}\gamma_{13}=1_2\,, \\ \gamma_{1j}^{e_j}=1_2 \end{cases} \qquad (1\leqq j\leqq s)\,. \tag{8}$$

Since $\pi(\Gamma_1)=\pi(\Gamma_0)$, Γ_1 is of type (e_1, e_2, e_3). We shall show that Γ_1 is of index 2 in Γ_0 and that Γ_1 does not contain -1_2. Since $\pi(\Gamma_1)$ $(=\pi(\Gamma_0))$ is presented by (3), in view of (8) there exists a homomorphism ρ of $\pi(\Gamma_1)$ onto Γ_1 such that $\rho(\bar{\gamma}_{1j})=\gamma_{1j}$ $(1\leqq j\leqq 3)$. It is easy to see that $(\pi|_{\Gamma_1})\circ\rho=$the identity and that $\rho\circ(\pi|_{\Gamma_1})=$the identity. It follows that $\pi|_{\Gamma_1}$ is an isomorphism of Γ_1 onto $\pi(\Gamma_1)$. This shows that Γ_1 does not contain -1_2 and that $[\Gamma_0:\Gamma_1]=2$. In particular, Γ_1 is not $SL_2(\boldsymbol{R})$-conjugate to Γ_0.

Suppose that $2\leqq s\leqq 3$. Let Γ be any triangle group of type (e_1, e_2, e_3). We shall show that Γ is $SL_2(\boldsymbol{R})$-conjugate to Γ_0 or Γ_1. Suppose that Γ contains -1_2. Then by Lemma 1 Γ is $SL_2(\boldsymbol{R})$-conjugate to Γ_0. Suppose that Γ does not contain -1_2. By Lemma 1 we may assume that $\pi(\Gamma)=\pi(\Gamma_0)$. Since Γ is isomorphic to $\pi(\Gamma_0)$, there exists a set of generators $\{\gamma_j\,|\,1\leqq j\leqq 3\}$ of Γ such that $\bar{\gamma}_j=\bar{\gamma}_{1j}$ $(1\leqq j\leqq 3)$. Hence we have $\gamma_j=\varepsilon_j\gamma_{1j}$, where $\varepsilon_j=\pm 1$ $(1\leqq j\leqq 3)$. Since Γ and Γ_1 do not contain -1_2, we have $\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=\varepsilon_j^{e_j}=1$ $(1\leqq j\leqq s)$. Since all e_j are odd, we have $\varepsilon_j=1$ $(1\leqq j\leqq s)$. By the assumption that $2\leqq s\leqq 3$ we see that $\varepsilon_j=1$ $(1\leqq j\leqq 3)$. This shows $\Gamma=\Gamma_1$.

(iii) First consider the case (e_1, ∞, ∞) such that e_1 is odd. Let Γ_0 and Γ_1 be the same as in (ii). Put $\gamma_{21}=-\gamma_{01}$, $\gamma_{22}=\gamma_{02}$ and $\gamma_{23}=\gamma_{03}$. Let Γ_2 be the subgroup of Γ_0 generated by $\{\gamma_{2j}\,|\,1\leqq j\leqq 3\}$. Then by the same argument as in the case of Γ_1, we see that Γ_2 is isomorphic to $\pi(\Gamma_2)$ and that Γ_2 does not contain -1_2. In particular, Γ_2 is not conjugate to Γ_0. We shall show that

Γ_2 is not conjugate to Γ_1. Assume that Γ_2 is conjugate of Γ_1. Since γ_{23} is a primitive parabolic element of Γ_2 such that $\mathrm{tr}(\gamma_{23})=2$, Γ_1 also contains a primitive parabolic element γ such that $\mathrm{tr}(\gamma)=2$. Consequently, there exists an element $\delta\in\Gamma_1$ such that $\bar{\gamma}=\bar{\delta}^{-1}\cdot\bar{\gamma}_{12}^{\nu}\cdot\bar{\delta}$ or $\bar{\gamma}=\bar{\delta}^{-1}\cdot\bar{\gamma}_{13}^{\nu}\cdot\bar{\delta}$, where $\nu=\pm1$. Since Γ_1 does not contain -1_2, we have $\gamma=\delta^{-1}\cdot\gamma_{12}^{\nu}\cdot\delta$ or $\gamma=\delta^{-1}\cdot\gamma_{13}^{\nu}\cdot\delta$. Since $\mathrm{tr}(\gamma_{12})=\mathrm{tr}(\gamma_{13})=-2$, this is a contradiction.

Let Γ be any triangle group of type (e_1, ∞, ∞). We shall show that Γ is $SL_2(\boldsymbol{R})$-conjugate to one of Γ_i $(0\leqq i\leqq 2)$. By Lemma 1 we may assume that $\pi(\Gamma)=\pi(\Gamma_0)$. If Γ contains -1_2, then $\Gamma=\Gamma_0$. Suppose that Γ does not contain -1_2. Since Γ is isomorphic to $\pi(\Gamma)$, Γ is generated by $\{\gamma_j \mid 1\leqq j\leqq 3\}$ such that $\bar{\gamma}_j=\bar{\gamma}_{0j}$ $(1\leqq j\leqq 3)$. Hence we have $\gamma_j=\varepsilon_j\gamma_{0j}$, where $\varepsilon_j=\pm1$ $(1\leqq j\leqq 3)$. By the fundamental relations of Γ and Γ_0 we see that $\varepsilon_1\varepsilon_2\varepsilon_3=\varepsilon_1^{e_1}=-1$. Since e_1 is odd, we have $\varepsilon_1=-1$. Therefore, we have $\varepsilon_2=\varepsilon_3=1$ or -1. Hence we see that $\Gamma=\Gamma_1$ or Γ_2.

Now consider the type (∞, ∞, ∞). We shall give Γ_0 explicitly. Let $\Gamma(1)$ be the modular group $SL_2(\boldsymbol{Z})$. It is easy to see that Γ_0 can be given as the group generated by

$$\gamma_{01}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \qquad \gamma_{02}=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \qquad \gamma_{03}=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix}.$$

Put

$$\Gamma(2)=\left\{\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(1)\,\middle|\, a-1\equiv b\equiv c\equiv d-1 \pmod 2\right\}.$$

Then $\Gamma(2)$ contains Γ_0. Since $\Gamma(1)$ is of type $(2, 3, \infty)$, comparing the indices of Γ_0 and $\Gamma(2)$ in $\Gamma(1)$ we see that $\Gamma_0=\Gamma(2)$. Let Γ_1 and Γ_2 be the subgroups of Γ_0 generated by $\{-\gamma_{01}, -\gamma_{02}, -\gamma_{03}\}$ and $\{-\gamma_{01}, \gamma_{02}, \gamma_{03}\}$ respectively. Then Γ_i does not contain -1_2. Since γ_{02} is a primitive parabolic element of Γ_2 such that $\mathrm{tr}(\gamma_{02})=2$, Γ_2 is not conjugate to Γ_1.

Let Γ be any triangle group of type (∞, ∞, ∞). We shall show that Γ is $SL_2(\boldsymbol{R})$-conjugate to one of Γ_i $(0\leqq i\leqq 2)$. By Lemma 1 we may assume that $\pi(\Gamma)=\pi(\Gamma_0)$. If Γ contains -1_2, then we see that $\Gamma=\Gamma_0$. Suppose that Γ does not contain -1_2. Then Γ is generated by $\{\gamma_j \mid 1\leqq j\leqq 3\}$ such that $\bar{\gamma}_j=\bar{\gamma}_{0j}$ $(1\leqq j\leqq 3)$. Hence we have $\gamma_j=\varepsilon_j\cdot\gamma_{0j}$ $(1\geqq j\geqq 3)$, where $\varepsilon_j=\pm1$. Since Γ does not contain -1_2, we have $\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=-1$. Hence $(\varepsilon_1, \varepsilon_2, \varepsilon_3)=(-1, -1, -1)$ or $(-1, 1, 1)$ or $(1, -1, 1)$ or $(1, 1, -1)$. By the following relations:

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix},$$

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},$$

$$\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

we see that the group Γ for $(\varepsilon_1, \varepsilon_2, \varepsilon_3)=(1, -1, 1)$ or $(1, 1, -1)$ is $SL_2(\boldsymbol{R})$-conjugate to Γ_2. This completes the Proof of Proposition 1.

PROPOSITION 2. *Let Γ be a triangle group of type (e_1, e_2, e_3). Then* $\mathrm{tr}(\Gamma)$ *is contained in the ring $\boldsymbol{Z}[2, 2\cos(\pi/e_1), 2\cos(\pi/e_2), 2\cos(\pi/e_3)]$ generated by $\{2, 2\cos(\pi/e_1), 2\cos(\pi/e_2), 2\cos(\pi/e_3)\}$ over $\boldsymbol{Z}$, where $\pi/e_j=0$ for $e_j=\infty$. In particular, the field $k_1=\boldsymbol{Q}(\mathrm{tr}(\gamma)\,|\,\gamma\in\Gamma)$ coincides with $\boldsymbol{Q}(\cos(\pi/e_1), \cos(\pi/e_2), \cos(\pi/e_3))$.*

PROOF. Clearly we may assume that Γ contains -1_2. Hence it is sufficient to verify the assertion for Γ_0 defined in the Proof of Proposition 1. Now we need the following

LEMMA 2. *Let Γ be a finitely generated subgroup of $SL_2(\boldsymbol{R})$. Let $\{\delta_1, \cdots, \delta_r\}$ be a set of generators of Γ. For any subset $\{i_1, \cdots, i_s\}$ of $\{1, \cdots, r\}$ put $t_{i_1\cdots i_s}=\mathrm{tr}(\delta_{i_1}\cdots\delta_{i_s})$. Then* $\mathrm{tr}(\Gamma)$ *is contained in the ring $\boldsymbol{Z}[t_{i_1\cdots i_s}\,|\,\{i_1, \cdots, i_s\}\subset\{1, \cdots, r\}]$.*

PROOF OF LEMMA 2. This lemma is given in the book [4] p. 148, without proof. We shall sketch the proof. For any $\gamma\in\Gamma$ we have $\gamma^2-t\cdot\gamma+1_2=0$, where $t=\mathrm{tr}(\gamma)$. Hence we have for any integer n,

$$\gamma^n=f_n(t)\cdot\gamma+g_n(t)\cdot 1_2\,, \tag{9}$$

where $f_n(T)$ and $g_n(T)$ are monic polynomials in $\boldsymbol{Z}[T]$. Moreover, for any $\alpha, \beta, \gamma\in\Gamma$ we have

$$\begin{cases} \mathrm{tr}(\alpha)\cdot\mathrm{tr}(\beta)=\mathrm{tr}(\alpha\cdot\beta)+\mathrm{tr}(\alpha\cdot\beta^{-1})\,, \\ \mathrm{tr}(\alpha\beta\alpha\gamma)=\mathrm{tr}(\alpha\beta)\cdot\mathrm{tr}(\alpha\gamma)-\mathrm{tr}(\beta\cdot\gamma^{-1})\,. \end{cases} \tag{10}$$

For any $\gamma\in\Gamma$ we can express

$$\gamma=\delta_{i_1}^{n_1}\cdots\delta_{i_s}^{n_s}\,.$$

Let $m(\gamma)$ be the minimum of $\sum_{j=1}^{s}|n_j|$ for all such expressions. Making use of (9) and (10) by induction on $m(\gamma)$ we can verify the assertion of Lemma 2.

Since Γ_0 is generated by $\{\gamma_{01}, \gamma_{02}, -1_2\}$, by Lemma 1 we can prove Proposition 2. Q. E. D.

Let Γ be a Fuchsian group of the first kind. Let $k_1=\boldsymbol{Q}(\mathrm{tr}(\gamma)\,|\,\gamma\in\Gamma)$ be the field generated by the set $\mathrm{tr}(\Gamma)$ over $\boldsymbol{Q}$. Let $A(\Gamma)$ be the vector space generated by Γ over k_1 in $M_2(\boldsymbol{R})$. It is shown in [5] that $A(\Gamma)$ is a quaternion algebra over k_1.

PROPOSITION 3. *Let Γ be a triangle group of type (e_1, e_2, e_3) $(2\leqq e_1\leqq e_2\leqq e_3\leqq\infty)$. Let $\{\gamma_j\}$ $(1\leqq j\leqq 3)$ be the elements of Γ such that $\{\bar{\gamma}_j\}$ $(1\leqq j\leqq 3)$ satisfy (3). Then $\{1_2, \gamma_1, \gamma_2, \gamma_3\}$ is a basis of $A(\Gamma)$ over k_1. For any $\xi=x_0 1_2+x_1\cdot\gamma_1+x_2\cdot\gamma_2+x_3\cdot\gamma_3\in A(\Gamma)$ the reduced norm $n_{A(\Gamma)}(\xi)$ can be expressed as follows:*

$$n_{A(\Gamma)}(\xi)=(x_0, x_1, x_2, x_3)\cdot D\cdot {}^t(x_0, x_1, x_2, x_3)\,,$$

where

$$D=\begin{pmatrix} 1 & \mathrm{tr}(\gamma_1)/2 & \mathrm{tr}(\gamma_2)/2 & \mathrm{tr}(\gamma_3)/2 \\ \mathrm{tr}(\gamma_1)/2 & 1 & \mathrm{tr}(\gamma_2\cdot\gamma_1^{-1})/2 & \mathrm{tr}(\gamma_3\cdot\gamma_1^{-1})/2 \\ \mathrm{tr}(\gamma_2)/2 & \mathrm{tr}(\gamma_2\cdot\gamma_1^{-1})/2 & 1 & \mathrm{tr}(\gamma_3\cdot\gamma_2^{-1})/2 \\ \mathrm{tr}(\gamma_3)/2 & \mathrm{tr}(\gamma_3\cdot\gamma_1^{-1})/2 & \mathrm{tr}(\gamma_3\cdot\gamma_2^{-1})/2 & 1 \end{pmatrix}. \tag{11}$$

PROOF. Assume that γ_1 commutes with γ_2. Then we see that Γ is abelian, which is a contradiction. Hence γ_1 does not commute with γ_2. First consider the case where γ_1 is elliptic. We can find an element $g\in SL_2(\boldsymbol{C})$ such that $g^{-1}\cdot\gamma_1\cdot g=\begin{pmatrix} w & 0 \\ 0 & 1/w\end{pmatrix}$, where w is a complex number such that $w^2\neq 1, \neq 0$. Put $g^{-1}\cdot\gamma_2\cdot g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Since γ_1 does not commute with γ_2, we see that $bc\neq 0$. Making use of (4), we have $g^{-1}\cdot\gamma_3\cdot g=\pm\begin{pmatrix} -d/w & bw \\ c/w & -aw\end{pmatrix}$. By the equation

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ w & 0 & 0 & 1/w \\ a & b & c & d \\ -d/w & bw & c/w & -aw \end{vmatrix} = bc(w-1/w)^2\neq 0\,,$$

we see that $\{1_2, g^{-1}\cdot\gamma_1\cdot g, g^{-1}\cdot\gamma_2\cdot g, g^{-1}\cdot\gamma_3\cdot g\}$ are linearly independent over $\boldsymbol{C}$. Hence $\{1_2, \gamma_1, \gamma_2, \gamma_3\}$ is linearly independent over k_1. Suppose that γ_1 is parabolic. By the same argument as in the elliptic case we can verify the assertion.

Now we shall give the reduced norm $n_{A(\Gamma)}(\xi)$ of $A(\Gamma)$ with respect to the basis $\{1_2, \gamma_1, \gamma_2, \gamma_3\}$. For any $\xi\in A(\Gamma)$ denote by $\tilde{\xi}$ the image of ξ under the main involution of $A(\Gamma)$. Then we have $\xi\cdot\tilde{\xi}=n_{A(\Gamma)}(\xi)\cdot 1_2=\det(\xi)\cdot 1_2$. It follows that for any $\xi=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in A(\Gamma)$ we have $\tilde{\xi}=\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$. In particular, we have $\tilde{\gamma}=\gamma^{-1}$ for any $\gamma\in\Gamma$. Therefore, for the expression $\xi=x_0 1_2+x_1\cdot\gamma_1+x_2\cdot\gamma_2+x_3\cdot\gamma_3$ we have $\tilde{\xi}=x_0 1_2+x_1\gamma_1^{-1}+x_2\gamma_2^{-1}+x_3\cdot\gamma_3^{-1}$. Now it is easy to obtain the explicit form of the reduced norm. Q. E. D.

PROPOSITION 4. *Let Γ be a triangle group of type (e_1, e_2, e_3) $(2\leqq e_1\leqq e_2\leqq e_3 \leqq\infty)$. Let k_0 be the field*

$$\boldsymbol{Q}((\cos(\pi/e_1))^2, (\cos(\pi/e_2))^2, (\cos(\pi/e_3))^2, \cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3))\,.$$

Let $\{\gamma_j \mid 1\leqq j\leqq 3\}$ be a set of generators of Γ such that $\{\bar{\gamma}_j\}$ $(1\leqq j\leqq 3)$ satisfy (3). *Let A_0 be the vector space generated by $\{1_2, \gamma_2^2, \gamma_3^2, \gamma_2^2\cdot\gamma_3^2\}$ in $M_2(\boldsymbol{R})$ over k_0. Then A_0 is a quaternion algebra over k_0 such that $A_0\otimes_{k_0} k_1=A(\Gamma)$.*

PROOF. In view of Proposition 1 we may assume that Γ contains -1_2 and that $\{\gamma_j \mid 1 \leqq j \leqq 3\}$ satisfy (6) and (7). By the equations $\gamma_j^2 - t_j \cdot \gamma_j + 1_2 = 0$, where $t_j = \mathrm{tr}(\gamma_j) = 2\cos(\pi/e_j)$ $(1 \leqq j \leqq 3)$, we have

$$\gamma_2^2 \cdot \gamma_3^2 = (1 - t_1 t_2 t_3) 1_2 - t_2 \cdot \gamma_2 - t_3 \cdot \gamma_3 + t_2 t_3 \cdot \gamma_1 .$$

Hence we see that

$$[1_2, \gamma_2^2, \gamma_3^2, \gamma_2^2 \cdot \gamma_3^2] = [1_2, \gamma_2, \gamma_3, \gamma_1] Q , \tag{12}$$

where

$$Q = \begin{pmatrix} 1 & -1 & -1 & 1 - t_1 t_2 t_3 \\ 0 & t_2 & 0 & -t_2 \\ 0 & 0 & t_3 & -t_3 \\ 0 & 0 & 0 & t_2 t_3 \end{pmatrix} . \tag{13}$$

By (4) we see that $3 \leqq e_2 \leqq e_3 \leqq \infty$. Hence $t_2 t_3 \neq 0$. Therefore, the matrix Q is non-singular. By Proposition 3, $\{1_2, \gamma_2^2, \gamma_3^2, \gamma_2^2 \cdot \gamma_3^2\}$ is a basis of $A(\Gamma)$ over k_1. Now we shall show that A_0 is a k_0-algebra. Constructing the multiplication table of A_0 with respect to $\{1_2, \gamma_2^2, \gamma_3^2, \gamma_2^2 \cdot \gamma_3^2\}$, it is sufficient to show that γ_2^4, γ_3^4 and $\gamma_3^2 \cdot \gamma_2^2$ are contained in A_0. Since $\mathrm{tr}(\gamma_j^2)$ $(= t_j^2 - 2)$ is contained in k_0, γ_j^4 $(= \mathrm{tr}(\gamma_j^2) \cdot \gamma_j^2 - 1_2)$ is contained in A_0. By the following calculations:

$$\begin{aligned} \gamma_3^2 \cdot \gamma_2^2 &= -(\gamma_3^2 \cdot \gamma_2^2)^{-1} + \mathrm{tr}(\gamma_3^2 \cdot \gamma_2^2) 1_2 = -(-\gamma_2^2 + (t_2^2 - 2) 1_2) \\ &\quad \cdot (-\gamma_3^2 + (t_3^2 - 2) 1_2) + \mathrm{tr}((t_2 \cdot \gamma_2 - 1_2)(t_3 \cdot \gamma_3 - 1_2)) 1_2 \\ &= (t_2^2 + t_3^2 - t_2^2 t_3^2 - t_1 t_2 t_3 - 2) 1_2 + (t_3^2 - 2) \cdot \gamma_2^2 + (t_2^2 - 2) \cdot \gamma_3^2 - \gamma_2^2 \cdot \gamma_3^2 , \end{aligned}$$

we see that $\gamma_3^2 \cdot \gamma_2^2$ belongs to A_0. Clearly $A_0 \otimes_{k_0} k_1 = A(\Gamma)$. Q. E. D.

Let Γ be a Fuchsian group of the first kind. Let $\Gamma^{(2)}$ be the subgroup of Γ generated by $\{\gamma^2 \mid \gamma \in \Gamma\}$. Then $\Gamma^{(2)}$ is a normal subgroup of Γ such that the quotient group $\Gamma/\Gamma^{(2)}$ is a finite elementary abelian group of type $(2, 2, \cdots, 2)$. Let k_2 be the field $\boldsymbol{Q}((\mathrm{tr}(\gamma))^2 \mid \gamma \in \Gamma)$. Then it can be proved that k_2 coincides with the field $\boldsymbol{Q}(\mathrm{tr}(\gamma) \mid \gamma \in \Gamma^{(2)})$ (cf. [**5**]).

PROPOSITION 5. *Notations being the same as above, k_2 coincides with k_0 and $A(\Gamma^{(2)})$ coincides with A_0.*

PROOF. In view of Proposition 1 we may assume that Γ contains -1_2. Therefore, we can take $\{\gamma_j \mid 1 \leqq j \leqq 3\}$ satisfying (6) and (7). Clearly t_j^2 $(= (\mathrm{tr}(\gamma_j))^2)$ is contained in k_2. By the equations

$$\begin{aligned} \mathrm{tr}(\gamma_1^2 \cdot \gamma_2^2 \cdot \gamma_3^2) &= \mathrm{tr}((t_1 \cdot \gamma_1 - 1_2)(t_2 \cdot \gamma_2 - 1_2)(t_3 \cdot \gamma_3 - 1_2)) \\ &= t_1^2 + t_2^2 + t_3^2 + t_1 t_2 t_3 - 2 , \end{aligned}$$

we see that $t_1 t_2 t_3$ is contained in k_2. This shows that k_0 is contained in k_2 and that A_0 is contained in $A(\Gamma^{(2)})$. In order to verify the converse inclusion we distinguish three cases.

(i) Suppose that at least two of $\{e_j \mid 1 \leqq j \leqq 3\}$ are odd. If e_1, e_2 are odd, then by (7) we see that $\gamma_1 \equiv \gamma_2 \equiv -1_2 \pmod{\Gamma^{(2)}}$. Hence $\gamma_3 \equiv -1_2 \pmod{\Gamma^{(2)}}$. This implies that Γ is generated by $\Gamma^{(2)}$ and -1_2. It follows that $k_2 = k_1$ and that $A(\Gamma^{(2)}) = A(\Gamma)$. Since e_1 and e_2 are odd, we see that $\boldsymbol{Q}((\cos(\pi/e_j))^2) = \boldsymbol{Q}(\cos(\pi/e_j))$ $(1 \leqq j \leqq 2)$ and that $\cos(\pi/e_1)\cos(\pi/e_2) \neq 0$. Therefore, $k_2 = k_1 = k_0$ and $A(\Gamma^{(2)}) = A_0$. In the other cases we can similarly verify our assertions.

(ii) Suppose that one of $\{e_j \mid 1 \leqq j \leqq 3\}$ is odd and the others are all even or ∞. If e_1 is odd, then $\gamma_1 \equiv -1_2 \pmod{\Gamma^{(2)}}$ and $\gamma_2 \equiv \gamma_3 \equiv -1_2 \pmod{\Gamma^{(2)}}$. Now we shall define a homomorphism ν_{23} of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$. We can express any $\gamma \in \Gamma$ as follows:

$$\gamma = \pm \gamma_{i_1}^{n_1} \cdots \gamma_{i_r}^{n_r}.$$

Put $\nu_{23}(\gamma) = \sum_{i_j = 2, 3} n_j \pmod 2$. Since e_2 and e_3 are even or ∞, in view of (7) this is well-defined. It is easy to see that ν_{23} is a homomorphism of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$. Let Γ_{23} be the kernel of ν_{23}. Then it is a subgroup of Γ of index 2. Γ_{23} contains $\Gamma^{(2)}$ and -1_2. Since Γ is generated by $\Gamma^{(2)}$ and $\{\gamma_2, -1_2\}$, we see that $\Gamma = \Gamma_{23} + \gamma_2^{-1} \cdot \Gamma_{23}$ and that $\Gamma_{23} = \Gamma^{(2)} \cup (-1_2) \cdot \Gamma^{(2)}$. We need the following well-known lemma (e.g. cf. [**6**], p. 96).

LEMMA 3. *Let G be a group generated by $\{a_i \mid i \in I\}$. Let H be a subgroup of G. Let $\{b_j \mid j \in J\}$ be a complete set of representatives of the right cosets G/H. Let c_{ij} be an element of H uniquely determined by $a_i b_j = b_k c_{ij}$ for any pair (a_i, b_j). Then $\{c_{ij} \mid i \in I, j \in J\}$ generates H.*

Since Γ is generated by $\{\gamma_1, \gamma_2, -1_2\}$ and $\Gamma = \Gamma_{23} + \gamma_2^{-1} \cdot \Gamma_{23}$, applying Lemma 3 to Γ, we see that Γ_{23} is generated by $\{\gamma_1, \gamma_2 \cdot \gamma_1 \cdot \gamma_2^{-1}, \gamma_2^2, -1_2\}$. We shall show that Γ_{23} is contained in A_0. Clearly γ_2^2 and -1_2 are contained in A_0. By the equations $\gamma_j^{-1} = (1/t_j)(\gamma_j^{-2} + 1_2)$ $(2 \leqq j \leqq 3)$ we have

$$\gamma_1 = -\gamma_3^{-1} \cdot \gamma_2^{-1} = (-t_1/(t_1 t_2 t_3))(\gamma_3^{-2} + 1_2)(\gamma_2^{-2} + 1_2).$$

Since e_1 is odd, $\boldsymbol{Q}(t_1^2) = \boldsymbol{Q}(t_1)$. This shows that γ_1 is contained in A_0. Since $\gamma_1(\gamma_2 \cdot \gamma_1 \cdot \gamma_2^{-1}) \cdot \gamma_2^2 \cdot \gamma_3^2 = 1_2$, $\gamma_2 \cdot \gamma_1 \cdot \gamma_2^{-1}$ is also contained in A_0. Therefore, $\Gamma^{(2)}$ is contained in A_0. Consequently, $k_2 \subset k_0$ and $A(\Gamma^{(2)}) \subset A_0$. Therefore, we see that $k_2 = k_0$ and that $A(\Gamma^{(2)}) = A_0$. In the case where e_2 (resp. e_3) is odd and the others are all even or ∞, we can define a homomorphism ν_{31} (resp. ν_{12}) of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$. In the same way we can verify our assertions.

(iii) Suppose that all e_j $(1 \leqq j \leqq 3)$ are even or ∞. In this case we can define homomorphisms $\nu_{23}, \nu_{31}, \nu_{12}$ of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$ in the same way as in (ii). Put $\Gamma_{ij} = \mathrm{Ker}(\nu_{ij})$. Then Γ_{ij} is a subgroup of Γ of index 2 and contains the group $\Gamma_0^{(2)}$ generated by $\Gamma^{(2)}$ and -1_2. Since Γ is generated by $\Gamma_0^{(2)}$ and $\{\gamma_1, \gamma_2\}$, we see that $[\Gamma : \Gamma_0^{(2)}] = 2$ or 4. Since Γ_{23}, Γ_{31} and Γ_{12} are different subgroups of Γ, we see that $[\Gamma_{31} : \Gamma_0^{(2)}] = 2$ and that $\Gamma_{31} = \Gamma_0^{(2)} + \gamma_2^{-1} \cdot \Gamma_0^{(2)}$. Applying Lemma

3 to Γ and Γ_{31} we see that Γ_{31} is generated by $\{\gamma_2, \gamma_3\cdot\gamma_2\cdot\gamma_3^{-1}, \gamma_3^2, -1_2\}$. Applving again Lemma 3 to Γ_{31} and $\Gamma_0^{(2)}$ we see that $\Gamma_0^{(2)}$ is generated by

$$\{\gamma_2^2, \gamma_2\cdot\gamma_3\cdot\gamma_2\cdot\gamma_3^{-1}, \gamma_3\cdot\gamma_2\cdot\gamma_3^{-1}\cdot\gamma_2^{-1}, \gamma_3^2, \gamma_2\cdot\gamma_3^2\cdot\gamma_2^{-1}, -1_2\}\ .$$

By the equations

$$\gamma_2\cdot\gamma_3\cdot\gamma_2\cdot\gamma_3^{-1}=(1/(t_2^2t_3^2))(\gamma_2^2+1_2)(\gamma_3^2+1_2)(\gamma_2^2+1_2)(\gamma_3^{-2}+1_2)\ ,$$

$$\gamma_3\cdot\gamma_2\cdot\gamma_3^{-1}\cdot\gamma_2^{-1}=(1/(t_2^2t_3^2))(\gamma_3^2+1_2)(\gamma_2^2+1_2)(\gamma_3^{-2}+1_2)(\gamma_2^{-2}+1_2)\ ,$$

$$\gamma_2\cdot\gamma_3^2\cdot\gamma_2^{-1}=(1/t_2^2)(\gamma_2^2+1_2)\cdot\gamma_3^2\cdot(\gamma_2^{-2}+1_2)\ ,$$

we see that $\Gamma_0^{(2)}$ is contained in A_0. Hence $k_2\subset k_0$ and $A(\Gamma^{(2)})\subset A_0$. This shows that $k_2=k_0$ and that $A(\Gamma^{(2)})=A_0$. Q. E. D.

§ 3. Arithmetic triangle groups.

Let k be a totally real algebraic number field of finite degree. Let A be a quaternion algebra over k such that there exists an $\boldsymbol{R}$-isomorphism ρ

$$\rho:\quad A\otimes_{\boldsymbol{Q}}\boldsymbol{R}\longmapsto M_2(\boldsymbol{R})\oplus\boldsymbol{H}\oplus\cdots\oplus\boldsymbol{H}\ , \tag{14}$$

where $\boldsymbol{H}$ is the Hamilton quaternion algebra over $\boldsymbol{R}$. Then there exists a k-isomorphism ρ_1 of A into $M_2(\boldsymbol{R})$. Let O be an order of A. Put $U=\{\varepsilon\in O\,|\,\varepsilon O=O,\ n_A(\varepsilon)=1\}$, where $n_A(\ )$ is the reduced norm of A. Then U is called the unit group of O of norm 1. Let $\Gamma(A, O)$ be the image of U under ρ_1. Then $\Gamma(A, 0)$ is a subgroup of $SL_2(\boldsymbol{R})$. It is well-known that $\Gamma(A, O)$ is a Fuchsian group of the first kind.

DEFINITION 1. Let Γ be a Fuchsian group of the first kind. Γ is called *arithmetic* if Γ is commensurable with $\Gamma(A, O)$. If Γ is a subgroup of $\Gamma(A, O)$ of finite index, we say that *Γ is derived from a quaternion algebra.*

REMARK. The isomorphism ρ is not unique. If we take another isomorphism ρ', then ρ_1 is changed into the composite of ρ_1 with an inner automorphism of $M_2(\boldsymbol{R})$. Therefore, if Γ is arithmetic, then the conjugate group $g\cdot\Gamma\cdot g^{-1}$ of Γ by $g\in SL_2(\boldsymbol{R})$ is also arithmetic.

DEFINITION 2. If a triangle group of type (e_1, e_2, e_3) is arithmetic, we say that *the triple (e_1, e_2, e_3) is arithmetic.*

By Proposition 1 and the above remark, if the triple (e_1, e_2, e_3) is arithmetic, then all triangle groups of this type are arithmetic. Now we shall prove

THEOREM 1. *Let Γ be a triangle group of type (e_1, e_2, e_3) $(2\leqq e_1\leqq e_2\leqq e_3\leqq\infty)$. Let k_0 be the field*

$$\boldsymbol{Q}((\cos(\pi/e_1))^2, (\cos(\pi/e_2))^2, (\cos(\pi/e_3))^2, \cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3))\ .$$

Then the following assertions hold:

(i) *Suppose that Γ is of compact type. Then Γ is arithmetic if and only if either $k_0=\boldsymbol{Q}$ or $k_0\supsetneqq\boldsymbol{Q}$ and for any non-identity isomorphism σ of k_0 into $\boldsymbol{R}$*

the following inequality holds:

$$\sigma((\cos(\pi/e_1))^2+(\cos(\pi/e_2))^2+(\cos(\pi/e_3))^2 \\ +2\cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3)-1)<0\,. \qquad (15)$$

(ii) *Suppose that Γ is of non-compact type. Then Γ is arithmetic if and only if k_0 coincides with $\boldsymbol{Q}$.*

PROOF. (i) We may assume that Γ contains -1_2. Hence Γ is generated by $\{\gamma_1, \gamma_2, \gamma_3, -1_2\}$ which satisfy (6) and (7). By Proposition 5 we have $k_2=k_0$ and $A(\Gamma^{(2)})=A_0$. By (11), (12) and (13), for any $\xi=y_0 1_2+y_1\cdot\gamma_2^2+y_2\cdot\gamma_3^2+y_3\cdot\gamma_2^2\cdot\gamma_3^2 \in A_0$, the reduced norm $n_{A_0}(\xi)$ can be written as

$$n_{A_0}(\xi)=(y_0, y_1, y_2, y_3)\cdot D_0\cdot {}^t(y_0, y_1, y_2, y_3)\,,$$

where $D_0={}^tQDQ$.

Let d_i (resp. d_{0i}) be the principal minor determinant of D (resp. D_0) of degree i ($1\leqq i\leqq 4$). Then we have

$$\begin{cases} d_1=1\,, \\ d_2=1-(\cos(\pi/e_2))^2\,, \\ d_3=-(\cos(\pi/e_1))^2-(\cos(\pi/e_2))^2-(\cos(\pi/e_3))^2 \\ \qquad -2\cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3)+1\,, \\ d_4=d_3^2\,. \end{cases}$$

Since Q is an upper triangular matrix, we see easily that

$$\begin{cases} d_{01}=1\,, \\ d_{02}=2^2(\cos(\pi/e_2))^2(1-(\cos(\pi/e_2))^2)\,, \\ d_{03}=-2^4(\cos(\pi/e_2)\cos(\pi/e_3))^2((\cos(\pi/e_1))^2+(\cos(\pi/e_2))^2+(\cos(\pi/e_3))^2 \\ \qquad +2\cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3)-1)\,, \\ d_{04}=d_{03}^2\,. \end{cases}$$

Suppose that Γ is commensurable with $\Gamma(A, O)$. Then it is shown in [**5**] that $k_2=k$ and that $A(\Gamma^{(2)})=A$. By Proposition 5 we see that $k=k_0$ and that $A=A_0$. Suppose that $k_0\supsetneqq\boldsymbol{Q}$. Since A satisfies (14), for any non-identity isomorphism σ of k_0 into $\boldsymbol{R}$ the conjugate form $\sigma(n_{A_0}(\xi))$ of $n_{A_0}(\xi)$ must be positive definite. Since d_{01}, d_{02} and d_{04} are totally positive, $\sigma(n_{A_0}(\xi))$ is positive definite if and only if $\sigma(d_{03})$ is positive. This is equivalent to (15).

Conversely suppose that either $k_0=\boldsymbol{Q}$ or $k\supsetneqq\boldsymbol{Q}$ and for any non-identity isomorphism σ of k_0 into $\boldsymbol{R}$ the inequality (15) holds. Then $A_0=A(\Gamma^{(2)})$ satisfies (14). Since $\Gamma^{(2)}\subset A_0\cap SL_2(\boldsymbol{R})$, for any non-identity isomorphism σ of k_0 into $\boldsymbol{R}$ we see that $\sigma(\mathrm{tr}(\Gamma^{(2)})\subset[-2, 2]$. On the other hand, by Proposition 2 $\mathrm{tr}(\Gamma^{(2)})$

is contained in the ring of integers in k_0. It follows from Theorem 2 in [5] that $\Gamma^{(2)}$ is a Fuchsian group derived from a quaternion algebra. Since $\Gamma^{(2)}$ is of finite index in Γ, Γ is arithmetic.

(ii) Suppose that Γ is of non-compact type commensurable with $\Gamma(A, O)$. Then we see that $k_2=k_0$ and that $A_0=A(\Gamma^{(2)})=A$. We shall show that $k_0=\boldsymbol{Q}$. Assume that k_0 is a proper extension of $\boldsymbol{Q}$. Then there exists a non-identity isomorphism σ of k_0 into $\boldsymbol{R}$. Suppose that $e_2=\infty$. Then we have $d_{02}=0$. So that the conjugate matrix $\sigma(D_0)$ of D_0 by σ is not positive definite, which is a contradiction. Suppose that $e_2<\infty$. Then we have $e_3=\infty$. Hence $\cos(\pi/e_3)=1$. In this case we have $d_{03}=-2^4(\cos(\pi/e_2))^2(\cos(\pi/e_1)+\cos(\pi/e_2))^2$. This shows that $\sigma(d_{03})$ is negative, which is a contradiction. This proves that $k_0=\boldsymbol{Q}$.

Conversely suppose that $k_0=\boldsymbol{Q}$. Then we see that $\mathrm{tr}(\Gamma)$ is contained in $\boldsymbol{Z}$. It follows from Theorem 2 in [5] that $\Gamma^{(2)}$ is a Fuchsian group derived from a quaternion algebra over $\boldsymbol{Q}$. This implies that Γ is arithmetic. Q. E. D.

§4. Finiteness of arithmetic triangle groups up to $SL_2(\boldsymbol{R})$-conjugation.

Let $\mathfrak{J}$ be the set of all triples (e_1, e_2, e_3) of positive integers e_j $(1\leqq j\leqq 3)$ such that $2\leqq e_1\leqq e_2\leqq e_3<\infty$ and $1/e_1+1/e_2+1/e_3<1$.

DEFINITION 3. Let p_n $(n=1, 2, \cdots)$ be the n-th odd prime number in order of magnitude. Let (e_1, e_2, e_3) be an element of $\mathfrak{J}$. Let e be the least common multiple of $\{e_j\}$. If $p_1p_2\cdots p_{n-1}$ divides e and p_n does not divide e, we say that (e_1, e_2, e_3) *is of the n-th type*. Let $\mathfrak{J}_n$ be the set of all $(e_1, e_2, e_3)\in\mathfrak{J}$ of the n-th type. Furthermore, put

$$\mathfrak{J}_{n,1}=\{(e_1, e_2, e_3)\in\mathfrak{J}_n \mid 2p_n<e_1\leqq e_2\leqq e_3\},$$
$$\mathfrak{J}_{n,2}=\{(e_1, e_2, e_3)\in\mathfrak{J}_n \mid e_1<2p_n<e_2\leqq e_3\},$$
$$\mathfrak{J}_{n,3}=\{(e_1, e_2, e_3)\in\mathfrak{J}_n \mid e_1\leqq e_2<2p_n<e_3\},$$
$$\mathfrak{J}_{n,4}=\{(e_1, e_2, e_3)\in\mathfrak{J}_n \mid e_1\leqq e_2\leqq e_3<2p_n\}.$$

Then we have

$$\mathfrak{J}=\bigcup_{n=1}^{\infty}\mathfrak{J}_n, \qquad \mathfrak{J}_n=\bigcup_{i=1}^{4}\mathfrak{J}_{n,i}.$$

Let $\mathfrak{A}$ be the set of all (e_1, e_2, e_3) of arithmetic type. Put

$$\mathfrak{A}_n=\mathfrak{A}\cap\mathfrak{J}_n, \qquad \mathfrak{A}_{n,i}=\mathfrak{A}\cap\mathfrak{J}_{n,i} \qquad (1\leqq i\leqq 4).$$

Then we have

$$\mathfrak{A}=\bigcup_{n=1}^{\infty}\mathfrak{A}_n, \qquad \mathfrak{A}_n=\bigcup_{i=1}^{4}\mathfrak{A}_{n,i}.$$

Now we shall prove

PROPOSITION 6. *Notations being as above, $\mathfrak{A}_n$ is a finite set for each positive integer n. More precisely, the following assertions hold:*

(i) *If* (e_1, e_2, e_3) *is contained in* $\mathfrak{A}_{n,1}$, *then*

$$e_1<3p_n\,,\qquad e_2<4p_n\,,\qquad e_3<p_n(2p_n+1)\,.$$

(ii) *If* (e_1, e_2, e_3) *is contained in* $\mathfrak{A}_{n,2}$, *then either*

$$e_1<p_n\,,\qquad e_2<2p_n^2\,,\qquad e_3<2p_n^4\,,$$

or

$$p_n<e_1<2p_n\,,\qquad e_2<2p_n(p_n+1)\,,\qquad e_3<4p_n^3(p_n+1)\,.$$

(iii) *If* (e_1, e_2, e_3) *is contained in* $\mathfrak{A}_{n,3}$, *then*

$$e_1\leqq e_2<2p_n\,,\qquad e_3<4p_n^3\,.$$

(iv) *If* (e_1, e_2, e_3) *is contained in* $\mathfrak{A}_{n,4}$, *then*

$$e_1\leqq e_2\leqq e_3<2p_n\,.$$

PROOF. The assertion (iv) is trivial by definition of $\mathfrak{A}_{n,4}$. Let (e_1, e_2, e_3) be an element of $\mathfrak{A}$. Let e be the least common multiple of $\{e_j \mid 1\leqq j\leqq 3\}$. Then the field $\boldsymbol{Q}(\cos(\pi/e))$ is a normal extension of $\boldsymbol{Q}$ whose Galois group is isomorphic to $(\boldsymbol{Z}/2e\boldsymbol{Z})^\times/\{\pm1\}$. For any integer a prime to $2e$ we can give the corresponding element σ_a of $\mathrm{Gal}(\boldsymbol{Q}(\cos(\pi/e))/\boldsymbol{Q})$ by $\sigma_a(\cos(\pi/e))=\cos(\pi a/e)$. There exists a unique integer a_j $(1\leqq j\leqq 3)$ such that $\sigma_a(\cos(\pi/e_j))=\cos(\pi a/e_j)$ $=\cos(\pi a_j/e_j)$, $1\leqq a_j\leqq e_j-1$. Now the condition (15) in Theorem 1 is equivalent to the following conditions:

For any integer a prime to $2e$ such that

$$\begin{aligned}&((\cos(\pi a_1/e_1))^2, (\cos(\pi a_2/e_2))^2, (\cos(\pi a_3/e_3))^2,\\ &\qquad \cos(\pi a_1/e_1)\cos(\pi a_2/e_2)\cos(\pi a_3/e_3))\\ &\neq((\cos(\pi/e_1))^2, (\cos(\pi/e_2))^2, (\cos(\pi/e_3))^2,\\ &\qquad \cos(\pi/e_1)\cos(\pi/e_2)\cos(\pi/e_3))\,,\end{aligned}\tag{16}$$

the inequality holds:

$$\begin{aligned}&(\cos(\pi a_1/e_1))^2+(\cos(\pi a_2/e_2))^2+(\cos(\pi a_3/e_3))^2\\ &\qquad +2\cos(\pi a_1/e_1)\cos(\pi a_2/e_2)\cos(\pi a_3/e_3)-1<0\,.\end{aligned}\tag{17}$$

By an easy calculation we have

$$\cos\pi(1-|a_1/e_1-a_2/e_2|)<\cos(\pi a_3/e_3)<\cos\pi(|a_1/e_1+a_2/e_2-1|)\,.$$

Since

$$|a_1/e_1-a_2/e_2|<1 \quad\text{and}\quad |a_1/e_1+a_2/e_2-1|<1\,,$$

we have

$$|a_1/e_1+a_2/e_2-1|<a_3/e_3<1-|a_1/e_1-a_2/e_2|\,.\tag{18}$$

This is equivalent to the inequalities:

$$\begin{cases} a_1/e_1+a_2/e_2+a_3/e_3>1\,, \\ -a_1/e_1+a_2/e_2+a_3/e_3<1\,, \\ a_1/e_1-a_2/e_2+a_3/e_3<1\,, \\ a_1/e_1+a_2/e_2-a_3/e_3<1\,. \end{cases} \tag{19}$$

We need the following

LEMMA 4. *Let* (e_1, e_2, e_3) *be an element of* $\mathfrak{A}_n$. *Let* a_j $(1\leqq j\leqq 3)$ *be the integer defined above for* $a=p_n$. *Then the following inequalities hold:*

$$a_j\leqq p_n \quad (1\leqq j\leqq 3)\,, \qquad |e_1e_2-a_1e_2-a_2e_1|\geqq 1.$$

PROOF. Suppose that $e_j<p_n$. Since $a_j\leqq e_j-1$, we see that $a_j<p_n$. Suppose that $p_n<e_j$. Then by definition of a_j we have $a_j=p_n$. This proves the first set of inequalities.

Assume that $e_1e_2-a_1e_2-a_2e_1=0$. Since $a_j\equiv\pm p_n \pmod{e_j}$, we have $p_ne_1\equiv 0 \pmod{e_2}$. Since p_n is prime to e_2, we have $e_1\equiv 0 \pmod{e_2}$. Similarly, we have $e_2\equiv 0 \pmod{e_1}$. Consequently, we see that $e_1=e_2$ and that $a_1=a_2$. Hence $e_1^2-2a_1e_1=0$. Hence $e_1=2a_1$. It implies that $2p_n\equiv 0 \pmod{e_1}$. Hence $2\equiv 0 \pmod{e_1}$. This means that $e_1=e_2=2$, which contradicts (4). Q.E.D.

In the cases of $\mathfrak{A}_{n,1}, \mathfrak{A}_{n,2}, \mathfrak{A}_{n,3}$ by the inequality $2p_n<e_3$ we see that σ_{p_n} is not the identity on k_0. Therefore, (18) and (19) are valid for $a=p_n$.

(i) Let (e_1, e_2, e_3) be an element of $\mathfrak{A}_{n,1}$. Then for $a=p_n$ we have $a_1=a_2=a_3=p_n$. By (19) we have $3p_n/e_1\geqq p_n/e_1+p_n/e_2+p_n/e_3>1$. Hence $e_1<3p_n$. Furthermore, we have $2p_n/e_2\geqq p_n/e_2+p_n/e_3>1-p_n/e_1>1/2$. Hence $e_2<4p_n$. By the inequalities $p_n/e_3>1-p_n/e_1-p_n/e_2\geqq 1-2p_n/(2p_n+1)=1/(2p_n+1)$, we see that $e_3<p_n(2p_n+1)$.

(ii) Let (e_1, e_2, e_3) be an element of $\mathfrak{A}_{n,2}$. Then for $a=p_n$ we have $a_2=a_3=p_n$. By definition we have $e_1<2p_n$. Suppose that $e_1<p_n$. Then $2p_n/e_2\geqq p_n/e_2+p_n/e_3>1-a_1/e_1\geqq 1/e_1$. Hence $e_2<2p_ne_1<2p_n^2$. By the inequalities $p_n/e_3>|e_1e_2-a_1e_2-a_2e_1|/(e_1e_2)\geqq 1/(e_1e_2)$, we have $e_3<p_ne_1e_2<2p_n^4$. Suppose that $p_n<e_1<2p_n$. Then we have $a_1=p_n$. By the inequalities $2p_n/e_2>1-p_n/e_1\geqq 1/(p_n+1)$, we have $e_2<2p_n(p_n+1)$ and $e_3<p_ne_1e_2<4p_n^3(p_n+1)$.

(iii) Let (e_1, e_2, e_3) be an element of $\mathfrak{A}_{n,3}$. Then by definition we have $e_1\leqq e_2<2p_n$. In the same way as in (ii) we have $e_3<p_ne_1e_2<4p_n^3$. Q.E.D.

PROPOSITION 7. *The notations being as above, the following assertions hold:*

(i) $\mathfrak{A}_{n,1}$ *is empty for all* $n\geqq 9$;

(ii) $\mathfrak{A}_{n,2}$ *is empty for all* $n\geqq 12$;

(iii) $\mathfrak{A}_{n,3}$ *is empty for all* $n\geqq 10$;

(iv) $\mathfrak{A}_{n,4}$ *is empty for all* $n\geqq 7$.

PROOF. Making use of the results of Proposition 6, by definition of $\mathfrak{A}_{n,i}$ we see that

(i) If $\mathfrak{A}_{n,1}$ is non-empty, then $p_1 \cdots p_{n-1} < 12p_n^3(2p_n+1)$;

(ii) If $\mathfrak{A}_{n,2}$ is non-empty, then either $p_1 \cdots p_{n-1} < 4p_n^7$ or $p_1 \cdots p_{n-1} < 16p_n^5(p_n+1)^2$;

(iii) If $\mathfrak{A}_{n,3}$ is non-empty, then $p_1 \cdots p_{n-1} < 16p_n^5$;

(iv) If $\mathfrak{A}_{n,4}$ is non-empty, then $p_1 \cdots p_{n-1} < 8p_n^3$.

Now Proposition 7 is a direct consequence of the following

LEMMA 5. *For any positive integer n we denote by p_n the n-th odd prime number in order of magnitude. Then the following assertions hold:*

(i) $p_1 \cdots p_{n-1} < 12p_n^3(2p_n+1)$ *if and only if* $n \leqq 8$;

(ii) (a) $p_1 \cdots p_{n-1} < 4p_n^7$ *if and only if* $n \leqq 11$;

(b) $p_1 \cdots p_{n-1} < 16p_n^5(p_n+1)^2$ *if and only if* $n \leqq 11$;

(iii) $p_1 \cdots p_{n-1} < 16p_n^5$ *if and only if* $n \leqq 9$;

(iv) $p_1 \cdots p_{n-1} < 8p_n^3$ *if and only if* $n \leqq 6$.

PROOF. (i) Suppose that $p_1 \cdots p_{n-1} < 12p_n^3(2p_n+1)$. Then we have $p_1 \cdots p_{n-1} < 32p_n^4$. By Cebyšev's theorem on the distribution of prime numbers we have $p_{n-1} < p_n < 2p_{n-1}$. Hence $p_n^4 < 2^{10}p_{n-4} \cdots p_{n-1}$. Therefore, we have $p_1 \cdots p_{n-5} < 2^{15}$. By an easy calculation we see that $n \leqq 10$. Now we can easily verify the assertion (i).

In the similar way we can also verify the assertions (ii), (iii) and (iv).

Q. E. D.

This completes the Proof of Proposition 7.

Now we can prove the following

THEOREM 2. *There exist only finitely many arithmetic triangle groups up to $SL_2(\boldsymbol{R})$-conjugation.*

PROOF. First consider the compact case. In this case our assertion is a direct consequence of Proposition 1, 6 and 7.

We turn to the non-compact case. Let Γ be a triangle group of non-compact type (e_1, e_2, e_3). Then by Theorem 1, Γ is arithmetic if and only if $k_0 = \boldsymbol{Q}((\cos(\pi/e_1))^2, (\cos(\pi/e_2))^2, (\cos(\pi/e_3))^2, \cos(\pi/e_1)\cdot\cos(\pi/e_2)\cdot(\cos\pi/e_3))$ coincides with $\boldsymbol{Q}$. It follows that $e_j = 2$ or 3 or 4 or 6 or ∞. By Proposition 1 we can verify our assertion. Q. E. D.

§5. Determination of all arithmetic types (e_1, e_2, e_3).

5.1. Non-compact types.

Let (e_1, e_2, e_3) be a triple of non-compact type. Then by an argument in the Proof of Theorem 2, we see that $e_j = 2$ or 3 or 4 or 6 or ∞ $(1 \leqq j \leqq 3)$. Considering all the conditions for (e_1, e_2, e_3) to be arithmetic, we see that (e_1, e_2, e_3) must be one of the following triples:

$$(2, 3, \infty),\ (2, 4, \infty),\ (2, 6, \infty),\ (2, \infty, \infty),\ (3, 3, \infty),\ (3, \infty, \infty),$$
$$(4, 4, \infty),\ (6, 6, \infty),\ (\infty, \infty, \infty).$$

5.2. Compact types

In order to make use of a computer we shall derive some inequalities. By Proposition 6 and 7 we see that $\mathfrak{A}=\bigcup_{n=1}^{11}\mathfrak{A}_n$ and we obtain the absolute bounds:

$$e_1\leqq 73\,,\qquad e_2\leqq 2811\,,\qquad e_3\leqq 10^7\,.$$

Let (e_1, e_2, e_3) be an element of $\mathfrak{A}_n$ such that σ_{p_n} is not the identity on k_0. Then for $a=p_n$ we have

$$2p_n/e_2\geqq a_2/e_2+a_3/e_3>1-a_1/e_1\,.$$

Hence we have

$$e_1\leqq e_2<c_2=2p_n e_1/(e_1-a_1)\,. \tag{20}$$

By the inequalities $p_n/e_3>|1-a_1/e_1-a_2/e_3|\geqq 1/(e_1e_2)$, we have

$$e_3<c_3=p_n e_1e_2/|e_1e_2-a_1e_2-a_2e_1|\leqq p_n e_1e_2<2p_n^2e_1/(e_1-a_1)\,. \tag{21}$$

Hence $e_2e_3<4p_n^3e_1^3/(e_1-a_1)^2$. Let $q(e_1, n)$ be the product of all p_m $(1\leqq m\leqq n-1)$ such that p_m divides e_1, where $q(e_1, n)=1$ if no such p_m exists. Since $p_1\cdots p_{n-1}$ divides $e_1e_2e_3$, we have

$$p_1\cdots p_{n-1}/q(e_1, n)<4p_n^3e_1^3/(e_1-a_1)^2\,.$$

Put

$$A(e_1, n)=q(e_1, n)e_1^3/(e_1-a_1)^2\,,\qquad B(n)=p_1\cdots p_{n-1}/(4p_n^3)\,.$$

Then we have

$$B(n)<A(e_1, n)\,. \tag{22}$$

Let $d_3(e_1e_2, n)$ be the product of all p_m $(1\leqq m\leqq n-1)$ such that p_m does not divide e_1e_2. Then e_3 must be a multiple of $d_3(e_1e_2, n)$. By making use of a computer for all triples (e_1, e_2, e_3) satisfying (20), (21) and (22) we check the condition (19) for any integer a prime to $2e$ satisfying (16). In this way we can obtain all (e_1, e_2, e_3) in $\mathfrak{A}_n$ such that σ_{p_n} is not the identity on k_0.

On the other hand, if σ_{p_n} is the identity on k_0, then (e_1, e_2, e_3) is contained in $\mathfrak{A}_{n,4}$. Hence we have $e_1\leqq e_2\leqq e_3<34$. By making use of a computer for all triples (e_1, e_2, e_3) such that $e_1\leqq e_2\leqq e_3\leqq 33$ we check the condition (19) for any integer a prime to $2e$ satisfying (16). Making use of the computer TOSBAC-3400, Saitama University, we have the following

THEOREM 3. *The complete list of all triples (e_1, e_2, e_3) of arithmetic type is as follows:*

(i) *Compact types.*

(2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10), (2, 3, 11), (2, 3, 12), (2, 3, 14), (2, 3, 16),
(2, 3, 18), (2, 3, 24), (2, 3, 30), (2, 4, 5), (2, 4, 6), (2, 4, 7), (2, 4, 8), (2, 4, 10),
(2, 4, 12), (2, 4, 18), (2, 5, 5), (2, 5, 6), (2, 5, 8), (2, 5, 10), (2, 5, 20), (2, 5, 30),
(2, 6, 6), (2, 6, 8), (2, 6, 12), (2, 7, 7), (2, 7, 14), (2, 8, 8), (2, 8, 16), (2, 9, 18),

(2, 10, 10), (2, 12, 12), (2, 12, 24), (2, 15, 30), (2, 18, 18),

(3, 3, 4), (3, 3, 5), (3, 3, 6), (3, 3, 7), (3, 3, 8), (3, 3, 9), (3, 3, 12), (3, 3, 15), (3, 4, 4), (3, 4, 6), (3, 4, 12), (3, 5, 5), (3, 6, 6), (3, 6, 18), (3, 8, 8), (3, 8, 24), (3, 10, 30), (3, 12, 12),

(4, 4, 4), (4, 4, 5), (4, 4, 6), (4, 4, 9), (4, 5, 5), (4, 6, 6), (4, 8, 8), (4, 16, 16),

(5, 5, 5), (5, 5, 10), (5, 5, 15), (5, 10, 10),

(6, 6, 6), (6, 12, 12), (6, 24, 24), (7, 7, 7), (8, 8, 8), (9, 9, 9), (9, 18, 18), (12, 12, 12), (15, 15, 15).

(ii) *Non-compact types.*

$(2, 3, \infty)$, $(2, 4, \infty)$, $(2, 6, \infty)$, $(2, \infty, \infty)$, $(3, 3, \infty)$, $(3, \infty, \infty)$, $(4, 4, \infty)$, $(6, 6, \infty)$, (∞, ∞, ∞).

REMARK. As to the triples of types $(2, 3, e_3)$, $(2, 4, e_3)$ and $(2, 6, e_3)$, our result coincides with the list of [1] pp. 610-611. It remains to classify all triples listed in Theorem 3 with respect to the commensurability. In the non-compact case this is trivial because these groups are all commensurable with some conjugate group of the modular group.

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Kisao TAKEUCHI
Department of Mathematics
Faculty of Science and Engineering
Saitama University
Urawa, Saitama,
Japan