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REMARKS ON SMALL SETS ON THE REAL LINE

Małgorzata Filipczak — Elżbieta Wagner-Bojakowska

ABSTRACT. We consider two kinds of small subsets of the real line: the sets of strong measure zero and the microscopic sets. There are investigated the properties of these sets. The example of a microscopic set, which is not a set of strong measure zero, is given.

The notion of strong measure zero set was introduced by E. Borel [Bo]. Properties of these sets were investigated by W. Sierpiński, A. S. Besicovitch, F. Galvin, J. Mycielski, R. M. Solovay, and others.

Definition 1. A set $E \subset \mathbb{R}$ is a strong measure zero set if for each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and $m(I_n) < \epsilon_n$ for $n \in \mathbb{N}$.

Sometimes, in the definition of strong measure zero set, instead of $E \subset \bigcup_{n=1}^{\infty} I_n$, one demands that $E \subset \limsup_n I_n$, where

$$\limsup_n I_n = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} I_n.$$

The following theorem shows that both conditions are equivalent.

Theorem 1. *The following conditions are equivalent:*

(i) *E is a strong measure zero set;*

(ii) *for each sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of positive real numbers there exists a sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals such that $E \subset \limsup_n J_n$ and $\sum_{k=n}^{\infty} m(J_k) < \eta_n$ for $n \in \mathbb{N}$;*

(iii) *for each sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of positive real numbers there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals such that $E \subset \limsup_n I_n$ and $m(I_n) < \delta_n$ for $n \in \mathbb{N}$.*

Proof. $(i) \implies (ii)$. Let E be a strong measure zero set and let $\{\eta_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers. Put

$$\theta_m = \min\{\eta_1, \dots, \eta_m\} \qquad \text{for} \quad m \in \mathbb{N}.$$

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Obviously, the sequence $\{\theta_m\}_{m\in\mathbb{N}}$ is nonincreasing and $\theta_m \le \eta_m$ for $m \in \mathbb{N}$.

Let $[a_{kn}]_{k\in\mathbb{N},n\in\mathbb{N}}$ be an arbitrary infinite matrix of positive real numbers such that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{kn} \le 1 \tag{1}$$

(for example, $a_{kn} = \frac{1}{2^{k+n}}$ for $k, n \in \mathbb{N}$). Let us consider a function $\psi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined as follows:

$$\psi(k,n) = 2^{k-1}(2n-1) \qquad \text{for} \quad (k,n) \in \mathbb{N}\times\mathbb{N}.$$

It is not difficult to prove that ψ is a bijection. Put

$$\epsilon_n^{(k)} = a_{kn}\theta_{\psi(k,n)} \qquad \text{for} \quad k,n \in \mathbb{N}. \tag{2}$$

Let k be a fixed positive integer. From (i) it follows that for the sequence $\left\{\epsilon_n^{(k)}\right\}_{n\in\mathbb{N}}$ there exists a sequence $\left\{I_n^{(k)}\right\}_{n\in\mathbb{N}}$ of intervals such that

$$E \subset \bigcup_{n=1}^{\infty} I_n^{(k)} \quad \text{and} \quad m\left(I_n^{(k)}\right) < \epsilon_n^{(k)} \tag{3}$$

for $n \in \mathbb{N}$. Let $m \in \mathbb{N}$. Since ψ is a one-to-one correspondence between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$, there exists exactly one pair $(k,n) \in \mathbb{N}\times\mathbb{N}$ such that $\psi(k,n) = m$. Then put

$$J_m = I_n^{(k)}.$$

Obviously, $E \subset \limsup_m J_m$, since each point of E belongs to infinitely many of intervals $I_n^{(k)}$, $n, k \in \mathbb{N}$.

Let $p \in \mathbb{N}$. Put $A_p = \big\{(k,n) \in \mathbb{N}\times\mathbb{N} : \psi(k,n) \ge p\big\}$. Using (3), (2), and (1) we obtain

$$\begin{aligned}\sum_{m=p}^{\infty} m(J_m) &= \sum_{(k,n)\in A_p} m\left(I_n^{(k)}\right) < \sum_{(k,n)\in A_p} \epsilon_n^{(k)}\\ &= \sum_{(k,n)\in A_p} a_{kn}\theta_{\psi(k,n)}\\ &= \sum_{m=p}^{\infty} a_{\psi^{-1}(m)}\theta_m \le \theta_p \sum_{m=p}^{\infty} a_{\psi^{-1}(m)} \le \theta_p \le \eta_p.\end{aligned}$$

The other implications are obvious. □

The notion of microscopic set was introduced by J. Appell in [A1]. The properties of these sets were investigated by J. Appell, E. D'Aniello, and M. Väth in [AAV] and [A2].

Definition 2. A set $E \subset \mathbb{R}$ is microscopic if for each $\epsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad m(I_n) < \epsilon^n \qquad\qquad \text{for} \quad n \in \mathbb{N}.$$

Theorem 2. *The following conditions are equivalent:*

(i) *E is a microscopic set;*

(ii) *for each positive number η there exists a sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals such that*

$$E \subset \limsup_n J_n \qquad \textit{and} \qquad \sum_{k=n}^{\infty} m(J_k) < \eta^n \qquad \textit{for} \quad n \in \mathbb{N};$$

(iii) *for each positive number δ there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of intervals such that*

$$E \subset \limsup_n I_n \qquad \textit{and} \qquad m(I_n) < \delta^n \qquad\qquad \textit{for} \quad n \in \mathbb{N}.$$

Proof. $(i) \implies (ii)$. Suppose that E is a microscopic set and $\eta \in (0,1)$. Let $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function considered in the proof of Theorem 1. Put

$$\theta = \frac{\eta}{1+\eta} \tag{4}$$

and

$$\epsilon_k = \theta^{2^k} \qquad \text{for} \quad k \in \mathbb{N}.$$

Let k be a fixed positive integer. From (i) it follows that there exists a sequence $\{I_n^{(k)}\}_{n\in\mathbb{N}}$ of intervals such that

$$E \subset \bigcup_{n=1}^{\infty} I_n^{(k)} \quad \text{and} \quad m\left(I_n^{(k)}\right) < (\epsilon_k)^n . \tag{5}$$

Let $m \in \mathbb{N}$. There exists a unique pair $(k,n) \in \mathbb{N} \times \mathbb{N}$ such that $\psi(k,n) = m$. Put

$$J_m = I_n^{(k)}.$$

Then $E \subset \limsup_m J_m$. Let $p \in \mathbb{N}$ and $A_p = \{(k,n) \in \mathbb{N} \times \mathbb{N} : \psi(k,n) \geq p\}$.

Using (5) and (4) we obtain

$$\begin{aligned}\sum_{m=p}^{\infty} m\left(J_m\right) &= \sum_{(k,n)\in A_p} m\left(I_n^{(k)}\right) \quad < \sum_{(k,n)\in A_p} \left(\epsilon_k\right)^n \\ &= \sum_{(k,n)\in A_p} \theta^{2^k n} \\ &= \sum_{(k,n)\in A_p} \theta^{2^{k-1}\cdot 2n} \quad < \sum_{(k,n)\in A_p} \theta^{2^{k-1}(2n-1)} \\ &= \sum_{(k,n)\in A_p} \theta^{\psi(k,n)} \\ &= \sum_{m=p}^{\infty} \theta^m \\ &= \frac{\theta^p}{1-\theta} \leq \left(\frac{\theta}{1-\theta}\right)^p = \eta^p,\end{aligned}$$

since $0 < 1-\theta < 1$, so $(1-\theta)^p \leq 1-\theta$.

The other implications are obvious. $\square$

Denote by $\mathcal{P}, \mathcal{S}, \mathcal{M}, \mathcal{N}$ the family of countable sets, strong measure zero sets, microscopic sets, and Lebesgue measure zero sets, respectively. It is easy to see (compare [BJ] and [AAV]) that both families $\mathcal{S}$ and $\mathcal{M}$ are the σ-ideals situated between countable sets and sets of Lebesgue measure zero. Obviously, each strong measure zero set is microscopic, so

$$\mathcal{P} \subset \mathcal{S} \subset \mathcal{M} \subset \mathcal{N}.$$

We have $\mathcal{M} \neq \mathcal{N}$, because the classical Cantor set has Lebesgue measure zero but is not microscopic (see [AAV]). If we assume CH, then $\mathcal{P} \neq \mathcal{S}$, since every Luzin set is a strong measure zero set which is uncountable (see [BJ]). Recall that a Luzin set is an uncountable subset of a real line having countable intersection with every set of the first category. The construction of such a set using the continuum hypothesis was given first by L u z i n (1914) and M a h l o (1913), independently. It is easy to see that each Luzin set is a strong measure zero set. Indeed, suppose that A is a Luzin set. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers and let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers. Then the set

$$\bigcup_{n=1}^{\infty}\left(r_n - \frac{\epsilon_{2n}}{3},\, r_n + \frac{\epsilon_{2n}}{3}\right)$$

is open and dense, so its complement is a set of the first category. Consequently, the set

$$B = A \setminus \bigcup_{n=1}^{\infty} \left(r_n - \frac{\epsilon_{2n}}{3},\, r_n + \frac{\epsilon_{2n}}{3} \right)$$

is countable. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of all elements of B. Then

$$A \subset \bigcup_{n=1}^{\infty} \left(r_n - \frac{\epsilon_{2n}}{3},\, r_n + \frac{\epsilon_{2n}}{3} \right) \cup \bigcup_{n=1}^{\infty} \left(x_n - \frac{\epsilon_{2n-1}}{3},\, x_n + \frac{\epsilon_{2n-1}}{3} \right),$$

so A is a strong measure zero set.

Now we will construct an example of some set $A \in \mathcal{M} \setminus \mathcal{S}$.

Example 1. Let $\{r_k\}_{k\in\mathbb{N}}$ be a following sequence of all rational numbers from the interval $(0,1)$:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$$

Let I_k be a closed interval centered at a point r_k with a length equal to

$$\frac{1}{(k+1)^{2^{k-1}}} \qquad \text{for} \quad k \in \mathbb{N}.$$

Put

$$A_n = I_{\frac{n^2-n+2}{2}} \cup \cdots \cup I_{\frac{n^2+n}{2}} \qquad \text{for} \quad n \in \mathbb{N}$$

and

$$A \ = \limsup_n A_n.$$

Then

$$A \ = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} A_n = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} I_n.$$

First we will prove that A is a microscopic set. Let ϵ be an arbitrary positive number. There exists $k_0 \in \mathbb{N}$ such that

$$\frac{1}{(k_0+1)^{2^{k_0-1}}} < \epsilon.$$

Obviously,

$$A \subset \bigcup_{n=k_0}^{\infty} I_n = \bigcup_{n=1}^{\infty} I_{k_0+n-1}.$$

We will show that

$$m\left(I_{k_0+n-1}\right) < \epsilon^n \qquad \text{for} \quad n \in \mathbb{N}.$$

We have

$$\begin{aligned} 2^{k_0+n-2} &= 2^{k_0-1} \cdot 2^{n-1} \\ &\geq 2^{k_0-1} \cdot n \qquad\qquad \text{for} \quad n \in \mathbb{N}, \end{aligned}$$

so

$$m\left(I_{k_0+n-1}\right) = \frac{1}{(k_0+n)^{2^{k_0+n-2}}} \leq \left(\frac{1}{(k_0+1)^{2^{k_0-1}}}\right)^n < \epsilon^n \qquad \text{for} \quad n \in \mathbb{N}.$$

Consequently, A is a microscopic set.

Now we will prove that A is not a strong measure zero set. We will construct a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that for each sequence $\{J_n\}_{n\in\mathbb{N}}$ of intervals with $m\left(J_n\right) < \epsilon_n$ for $n \in \mathbb{N}$, we have

$$A \setminus \bigcup_{n=1}^{\infty} J_n \neq \emptyset.$$

Put

$$\delta_n = \min\{m\left(I_i\right) : I_i \subset A_n\}$$

and

$$A_n^* = \left\{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\right\} \qquad \text{for} \quad n \in \mathbb{N}.$$

Obviously,

$$n^2 + n \geq 2n \qquad \text{for} \quad n \in \mathbb{N},$$

so

$$\delta_n = m\left(I_{\frac{n^2+n}{2}}\right) = \frac{1}{\left(\frac{n^2+n+2}{2}\right)^{2^{\frac{n^2+n-2}{2}}}} \leq \frac{1}{n+1}$$

and the set A_n^* is the ϵ-net of the interval $[0,1]$ for $\epsilon = 1/\left(n+1\right)$.

We will define by induction two sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ and $\{i_n\}_{n\in\mathbb{N}}$ in a following way. Put $\epsilon_1 = \frac{1}{2}\delta_3$. There exists a positive integer $i_1 > 1$ such that

$$\frac{1}{i_1+1} < \frac{1}{7}\epsilon_1.$$

Put $\epsilon_2 = \delta_{i_1}$. Now suppose that the positive real numbers $\epsilon_1, \ldots, \epsilon_n$ and positive integers $i_1, \ldots, i_{n-1}$ are chosen. There exists $i_n > i_{n-1}$ such that

$$\frac{1}{i_n+1} < \frac{1}{7}\epsilon_n.$$

Let us put $\epsilon_{n+1} = \delta_{i_n}$.

Now, let $\{J_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of intervals such that $m(J_n) < \epsilon_n$ for $n \in \mathbb{N}$. We will find the descending subsequence $\{I_{k_n}\}_{n\in\mathbb{N}}$ of the sequence $\{I_k\}_{k\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$

$$\left(\bigcup_{j=1}^{n} J_j\right) \cap I_{k_n} = \emptyset.$$

We have

$$m(J_1) < \epsilon_1 = \frac{1}{2}\delta_3 \leq \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

Simultaneously, $A_3 = I_4 \cup I_5 \cup I_6$ and the distance between I_4 and I_6 is greater than $\frac{1}{4}$ because $m(I_6) < m(I_4) < \frac{1}{8}$. So, there exists the component interval of the set A_3 which is disjoint with J_1. Denote it by I_{k_1} (if there is more than one such component interval, we take the longest one).

Let us consider the interval I_{k_1} and the set $A^*_{i_1}$. Since $m(I_{k_1}) \geq \delta_3$, in the interval I_{k_1} we can find at least six points of the set $A^*_{i_1}$. We have

$$m(J_2) < \epsilon_2 = \delta_{i_1} \leq \frac{1}{i_1+1},$$

so, $\operatorname{card}\left(J_2 \cap A^*_{i_1}\right) \leq 1$ and J_2 has non-empty intersection with at most three component intervals of A_{i_1}. Consequently, there exists a component interval I_{k_2} of the set A_{i_1} such that

$$I_{k_2} \subset I_{k_1} \quad \text{and} \quad I_{k_2} \cap J_2 = \emptyset.$$

Suppose that the intervals $I_{k_1}, I_{k_2}, \ldots, I_{k_n}$ such that

$$I_{k_1} \supset I_{k_2} \supset \cdots \supset I_{k_n},$$

I_{k_p} is a component interval of the set $A_{i_{p-1}}$ for $p = 1, \ldots, n$ and

$$\left(\bigcup_{j=1}^{n} J_j\right) \cap I_{k_n} = \emptyset$$

are chosen. Let us consider the interval $I_{k_n} \subset A_{i_{n-1}}$ and the set $A^*_{i_n}$. We have

$$m(J_{n+1}) < \epsilon_{n+1} = \delta_{i_n} \leq \frac{1}{i_n+1}$$

so $\operatorname{card}\left(J_{n+1} \cap A^*_{i_n}\right) \leq 1$. Hence, there exists a component interval $I_{k_{n+1}}$ of the set A_{i_n} such that

$$I_{k_{n+1}} \subset I_{k_n} \quad \text{and} \quad I_{k_{n+1}} \cap J_{n+1} = \emptyset.$$

Consequently,

$$\left(\bigcup_{j=1}^{n+1} J_j\right) \cap I_{k_{n+1}} = \emptyset.$$

Let $x_0 \in \bigcap_{n=1}^{\infty} I_{k_n}$. Then $x_0 \in \limsup_n I_n = A$. On the other hand, $x_0 \in I_{k_n}$, so $x_0 \notin \bigcup_{j=1}^{n} J_j$ for each $n \in \mathbb{N}$. Consequently, $x_0 \notin \bigcup_{j=1}^{\infty} J_j$, i.e., $x_0 \in A \setminus \bigcup_{n=1}^{\infty} J_n$.

In the previous example we have shown straightly from the definition that A is not a strong measure zero set. Remark that it is sufficient to observe that A is a perfect set because no perfect set has strong measure zero (compare [BJ]).

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Małgorzata Filipczak
University of Łódź
Faculty of Mathematics and Computer Science
Banacha 22
PL–90-238 Łódź
POLAND
E-mail: malfil@math.uni.lodz.pl

Elżbieta Wagner-Bojakowska
The College of Computer Science
Chair of Mathematics
Rzgowska 17A
PL–93-008 Łódź
POLAND

University of Łódź
Faculty of Mathematics and Computer Science
Banacha 22
PL–90-238 Łódź
POLAND
E-mail: wagner@uni.lodz.pl