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ON THE TOPOLOGY OF THE EVEN-DIMENSIONAL COMPLEX QUADRICS

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ABSTRACT. Neat proofs are given of the explicit structures of the homotopy groups and cohomology rings of the even-dimensional complex quadrics.

1. **Introduction.** It is well known that the complex quadric Q_{2n}, defined by the equation $z_1^2+\cdots+z_{2n+2}^2=0$ in homogeneous coordinates in complex projective space $\mathbf{C}\mathrm{P}^{2n+1}$, is diffeomorphic with the Grassmannian $\mathbf{R}_{2,2n}$ of oriented 2-planes through the origin in Euclidean space $\mathbf{R}^{2n+2}$. The homotopy sequence of the fibration $V_{2n+2,2}\to\mathbf{R}_{2,2n}$, where $V_{2n+2,2}$ is the Stiefel manifold of orthonormal 2-frames in $\mathbf{R}^{2n+2}$, easily yields the homotopy groups of $\mathbf{R}_{2,2n}$ in terms of the homotopy groups of spheres, but without giving any insight into the homotopy structure of $\mathbf{R}_{2,2n}$. On the other hand, the explicit generators of the (integral) cohomology ring of $\mathbf{R}_{2,2n}$ have been determined [3] using complicated calculations with Schubert varieties. In this paper, using the observation that the Hopf bundle $S^{2n+1}\to\mathbf{C}\mathrm{P}^n$ is induced from the bundle $V_{2n+2,2}\to\mathbf{R}_{2,2n}$, and employing standard sphere bundle and characteristic class techniques, we show that the homotopy groups and cohomology groups of $\mathbf{R}_{2,2n}$ are isomorphic to those of $S^{2n}\times\mathbf{C}\mathrm{P}^n$, where both S^{2n} and $\mathbf{C}\mathrm{P}^n$ are naturally embedded in $\mathbf{R}_{2,2n}$, and give the cup product formulas for the cohomology.

2. **Complex quadric and oriented Grassmannian.** Let $e_1,\cdots,e_{2n+2}$ be an oriented orthonormal basis of $\mathbf{R}^{2n+2}$, and consider $\mathbf{C}^{2n+2}$ as the complexification of $\mathbf{R}^{2n+2}$. A natural diffeomorphism between $\mathbf{R}_{2,2n}$ and Q_{2n} is given explicitly as follows: For any two orthonormal vectors v_1, v_2 in $\mathbf{R}^{2n+2}$, the oriented 2-plane spanned by v_1, v_2 corresponds to $\pi(v_1+iv_2)$ in Q_{2n}, where π is the natural projection $\mathbf{C}^{2n+2}\backslash\{0\}\to\mathbf{C}\mathrm{P}^{2n+1}$. For details see, for example, [2, p. 280]. We orient $\mathbf{R}_{2,2n}$ by the complex structure on Q_{2n}, and will identify the two manifolds.

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Introduce in $\mathbf{R}^{2n+2}$ an almost complex structure J by defining $Je_{2j-1} = e_{2j}$ for $j = 1, \cdots, n+1$. Then there is a natural embedding $i_1 : \mathbf{CP}^n \longrightarrow \mathbf{R}_{2,2n}$ which sends a J-complex line in $\mathbf{R}^{2n+2}$ to its underlying oriented 2-plane. Let K denote its image in $\mathbf{R}_{2,2n}$. Extend J to a complex linear automorphism of $\mathbf{C}^{2n+2}$. Take an oriented 2-plane ω spanned by orthonormal vectors v_1, v_2 in $\mathbf{R}^{2n+2}$. Clearly ω is in K if and only if $v_2 = Jv_1$, that is, if and only if $J(v_1 + iv_2) = -i(v_1 + iv_2)$. Following the convention in differential geometry, we can therefore say that K is the n-dimensional projective subspace of $\mathbf{CP}^{2n+1}$ defined by the $(n+1)$-dimensional linear subspace of vectors of type $(0,1)$ in $\mathbf{C}^{2n+2}$.

There is also a natural embedding $i_2 : S^{2n} \longrightarrow \mathbf{R}_{2,2n}$ obtained by taking the oriented 2-planes in $\mathbf{R}^{2n+2}$ that are spanned and oriented by e_1 and an orthogonal vector. At any $\omega \in i_2(S^{2n})$ defined by e_1 and v, where v is in the linear space $\mathbf{R}_*^{2n+1}$ spanned by $e_2, \cdots, e_{2n+2}$, the tangent space to $i_2(S^{2n})$ can be identified with the $2n$-dimensional subspace of $\mathbf{R}_*^{2n+1}$ that is orthogonal to v. Therefore we can orient $i_2(S^{2n})$ in such a way that its tangent bundle is induced from the natural oriented $2n$-plane bundle over $\mathbf{R}_{2,2n}$ whose fibre over $\omega \in \mathbf{R}_{2,2n}$ is the oriented orthogonal complement of ω in $\mathbf{R}^{2n+2}$. With this orientation, it is straightforward to show that the intersection number of K and $i_2(S^{2n})$ in $\mathbf{R}_{2,2n}$ is equal to $+1$.

We observe that there is a commutative diagram

$$\begin{tikzcd}
V_{2n+2,2} \arrow[d, "p"] & S^{2n+1} \arrow[l, "s"'] \arrow[d, "h"] \\
\mathbf{R}_{2,2n} & \mathbf{CP}^n \arrow[l, "i_1"']
\end{tikzcd}$$

where h is the Hopf map which sends a unit vector $v \in \mathbf{R}^{2n+2}$ into the J-complex line defined by it, p is the natural projection, and s is the inclusion map $s(v) = (v, Jv)$. The Stiefel manifold $V_{2n+2,2}$ can also be considered as the unit tangent bundle of S^{2n+1}, and s is a section.

3. The homotopy groups and cohomology ring of $\mathbf{R}_{2,2n}$.

Theorem 1. $\pi_k(\mathbf{R}_{2,2n}) = i_{1\#}\pi_k(\mathbf{CP}^n) \oplus i_{2\#}\pi_k(S^{2n})$ *for* $k \geq 1$, *where* $\#$ *denotes induced homomorphism.*

Proof. With the notations of §2, we can lift the inclusion $i_2 : S^{2n} \longrightarrow \mathbf{R}_{2,2n}$ over the projection p to an inclusion $i_3 : S^{2n} \longrightarrow V_{2n+2,2}$, by defining $i_3(v) = (e_1, v)$ for a unit vector v orthogonal to e_1. We note that i_3 is the inclusion of S^{2n} as a fibre of the bundle $V_{2n+2,2} \to S^{2n+1}$, which has a section s. Hence

$$\pi_k(V_{2n+2,2}) = i_{3\#}\pi_k(S^{2n}) \oplus s_\#\pi_k(S^{2n+1}).$$

Incorporating this into the homotopy sequences of the bundles $S^1 \to V_{2n+2,2} \to \mathbf{R}_{2,2n}$ and $S^1 \to S^{2n+1} \to \mathbf{CP}^n$, we get the following commutative diagram in which each three-term sequence is exact:

$$\begin{tikzcd}
 & & \pi_k(S^1) \arrow[r, equal] \arrow[d] & \pi_k(S^1) \arrow[d] & \\
0 \arrow[r] & \pi_k(S^{2n}) \arrow[r, "i_{3\#}"] \arrow[d, equal] & \pi_k(V_{2n+2,2}) \arrow[d, "p_\#"] & \pi_k(S^{2n+1}) \arrow[l, "s_\#"'] \arrow[d, "h_\#"] & 0 \arrow[l] \\
 & \pi_k(S^{2n}) \arrow[r, "i_{2\#}"] & \pi_k(\mathbf{R}_{2,2n}) \arrow[d] & \pi_k(\mathbf{CP}^n) \arrow[l, "i_{1\#}"'] \arrow[d] & \\
 & & \pi_{k-1}(S^1) \arrow[r, equal] & \pi_{k-1}(S^1) &
\end{tikzcd}$$

The theorem then follows easily.

There are a natural oriented $2n$-plane bundle ξ and a natural oriented 2-plane bundle $\tilde{\xi}$ over $\mathbf{R}_{2,2n}$, where the fibre of $\tilde{\xi}$ over a point $\omega \in \mathbf{R}_{2,2n}$ is the 2-plane ω, and the corresponding fibre of ξ is the orthogonal complement of ω in $\mathbf{R}^{2n+2}$. Let $\Omega, \tilde{\Omega}$ be their Euler classes. Let κ be the Poincaré dual of the $2n$-dimensional homology class of $\mathbf{R}_{2,2n}$ represented by K. Thus $\tilde{\Omega} \in H^2(\mathbf{R}_{2,2n})$ and $\Omega, \kappa \in H^{2n}(\mathbf{R}_{2,2n})$.

Theorem 2. *The cohomology groups of $\mathbf{R}_{2,2n}$ are isomorphic to those of $\mathbf{CP}^n \times S^{2n}$, namely, for $0 \le k \le 4n$,*

$$H^k(\mathbf{R}_{2,2n}) = \begin{cases} \mathbf{Z} & \textit{if } k \textit{ is even and } k \neq 2n, \\ \mathbf{Z} \oplus \mathbf{Z} & \textit{if } k = 2n, \\ 0 & \textit{if } k \textit{ is odd.} \end{cases}$$

As a ring, $H^*(\mathbf{R}_{2,2n})$ is generated by $\tilde{\Omega}$ and κ, with the relation $\tilde{\Omega}^{n+1} = 2\kappa \cup \tilde{\Omega}$. Moreover, $\Omega + \tilde{\Omega}^n = 2\kappa$, $\kappa \cup \tilde{\Omega}^n = (-1)^n$, $\kappa \cup \Omega = 1$, $\kappa \cup \kappa = (1 + (-1)^n)/2$, $\tilde{\Omega}^{2n} = 2(-1)^n$, and $\Omega \cup \Omega = 2$.

Proof. The Gysin cohomology sequence of the $2n$-sphere bundle $V_{2n+2,2} \to S^{2n+1}$ splits since the bundle has a section s. It follows easily that

$$H^k(V_{2n+2,2}) = \begin{cases} \mathbf{Z} & \text{if } k = 0,\, 2n,\, 2n+1 \text{ or } 4n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the Gysin cohomology sequence for the 1-sphere bundle $V_{2n+2,2} \to \mathbf{R}_{2,2n}$ shows that the cohomology groups of $\mathbf{R}_{2,2n}$ are as stated in the theorem, and that cup product with $\widetilde{\Omega}$ gives isomorphisms $H^{2j}(\mathbf{R}_{2,2n}) \cong H^{2j+2}(\mathbf{R}_{2,2n})$ for $j = 0, 1, \cdots, n-2, n+1, \cdots, 2n-1$.

Since $H^{2n}(\mathbf{R}_{2,2n}) = \mathbf{Z} \oplus \mathbf{Z}$ and $\Omega, \widetilde{\Omega}^n, \kappa$ are three elements in it, there is a linear relation

$$(1) \qquad p\Omega + q\widetilde{\Omega}^n = r\kappa, \quad \text{where } p, q, r \in \mathbf{Z}.$$

We will calculate the values of i_1^* and i_2^* on the cohomology classes $\Omega, \widetilde{\Omega}$ and κ and so solve for $p : q : r$. Let $\alpha \in H^2(\mathbf{CP}^n)$ be the generator of $H^*(\mathbf{CP}^n)$ which is the Poincaré dual of the homology class represented by a projective hyperplane in $\mathbf{CP}^n$. Then the Euler class of the Hopf bundle $(S^{2n+1}, h, \mathbf{CP}^n)$ is $-\alpha$ [1], so $i_1^*(\widetilde{\Omega}) = -\alpha$, or $i_1^*(\widetilde{\Omega}^n) = (-1)^n\alpha^n$. On the other hand, $i_1^*(\Omega)$ is the Euler class of the bundle $i_1^*\xi$ over $\mathbf{CP}^n$ which is the normal bundle to the Hopf bundle. By the duality theorem on Chern classes, $c(i_1^*\xi) = (1-\alpha)^{-1}$, and in particular, $i_1^*(\Omega) = c_n(i_1^*\xi) = \alpha^n$.

Lastly, to calculate $i_1^*\kappa$, we make use of a well-known fact, that for any embedding $j : X \to Y$ of a compact orientable manifold into another compact orientable manifold, if DX denotes the cohomology class in Y which is the Poincaré dual of the homology class represented by $j(X)$, and Ω_ν denotes the Euler class of the oriented normal bundle of X in Y, then $\Omega_\nu = j^*(DX)$ (see [1, p. 72]). We apply this fact to the embedding $i_1 : \mathbf{CP}^n \to Q_{2n}$, and conclude that $i_1^*(\kappa)$ is equal to the Euler class of the normal bundle ν of $\mathbf{CP}^n$ in Q_{2n}. We know

$$c(\nu) = c(i_1^*(T(Q_{2n})))c(\mathbf{CP}^n)^{-1} = i_1^*c(T(Q_{2n}))(1+\alpha)^{-(n+1)}.$$

Using the above quoted fact on normal Euler classes again, we see that the Euler class of the normal bundle ν' of Q_{2n} in $\mathbf{CP}^{2n+1}$ is $j^*(DQ_{2n}) = j^*(2\overline{\alpha})$, where $j : Q_{2n} \to \mathbf{CP}^{2n+1}$ is the inclusion map and $\overline{\alpha}$ is defined in $\mathbf{CP}^{2n+1}$ as α is defined in $\mathbf{CP}^n$. We note that $j \circ i_1$ is the inclusion of $\mathbf{CP}^n$ in $\mathbf{CP}^{2n+1}$, and since by §2 this is a projective subspace in standard position, $(j \circ i_1)^*\overline{\alpha} = \alpha$. We therefore know that the total Chern class of $i_1^*(\nu')$ is $1 + i_1^*j^*(2\overline{\alpha}) = 1 + 2\alpha$. Hence

$$i_1^*c(T(Q_{2n})) = (j \circ i_1)^*c(T(\mathbf{CP}^{2n+1}))(1+2\alpha)^{-1} = (1+\alpha)^{2n+2}(1+2\alpha)^{-1}.$$

Finally,

$$c(\nu) = (1+\alpha)^{2n+2}(1+2\alpha)^{-1}(1+\alpha)^{-(n+1)} = (1+\alpha)^{n+1}(1-2\alpha+4\alpha^2-8\alpha^3+\cdots).$$

In particular, the Euler class of ν is

$$\begin{aligned} c_n(\nu) &= \left[(-1)^n 2^n + \binom{n+1}{1}(-1)^{n-1}2^{n-1} + \cdots \right. \\ &\qquad \left. + \binom{n+1}{n-r}(-1)^r 2^r + \cdots + \binom{n+1}{n}\right]\alpha^n \\ &= -\frac{1}{2}\left[\sum_{r=0}^{n}(-1)^{r+1}2^{r+1}\binom{n+1}{n-r}\right]\alpha^n \\ &= -\frac{1}{2}\,[(1-2)^{n+1}-1]\alpha^n = \sigma(n)\alpha^n, \end{aligned}$$

where

$$\sigma(n) = \begin{cases} 1, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

Thus we have shown that $i_1^*\kappa = \sigma(n)\alpha^n$. This, together with the earlier formulas for $i_1^*\Omega$ and $i_1^*\tilde{\Omega}^n$ and the relation (1), shows that

$$p + (-1)^n q = \sigma(n) r. \tag{2}$$

We now calculate i_2^*. From §2 we know that $i_2^*\xi$ is just the tangent bundle of S^{2n}, so its Euler class $i_2^*(\Omega) = 2\beta$, where β is the standard generator of $H^{2n}(S^{2n})$. On the other hand, the bundle $i_2^*\tilde{\xi}$ has a nowhere zero section defined by e_1, so its Euler class $i_2^*(\tilde{\Omega})$ vanishes. To evaluate $i_2^*\kappa$, we take the fundamental class $\zeta \in H_{2n}(S^{2n})$ and calculate the Kronecker pairing

$$\langle i_2^*\kappa, \zeta\rangle = \langle \kappa, i_{2*}\zeta\rangle = \text{intersection number of } K \text{ and } i_2(S^{2n}) = 1.$$

This means that $i_2^*\kappa = \beta$. Therefore from (1) we get

$$2p = r. \tag{3}$$

The equations (2), (3) then yield $p : q : r = 1 : 1 : 2$, so (1) reduces to

$$\Omega + \tilde{\Omega}^n = 2\kappa. \tag{4}$$

We now calculate the cup products. Let $\eta \in H_{4n}(\mathbf{R}_{2,2n})$ be the fundamental class of $R_{2,2n}$ defined by the complex structure on Q_{2n}, and $\langle\ ,\ \rangle$ the Kronecker pairing. Then

$$\langle \kappa \cup \Omega, \eta\rangle = \langle \Omega, \kappa \cap \eta\rangle = \langle \Omega, i_{1*}(\mathbf{CP}^n)\rangle = \langle i_1^*\Omega, \mathbf{CP}^n\rangle = 1,$$

as shown earlier. We can therefore write

$$\kappa \cup \Omega = 1. \tag{5}$$

Similarly, we prove $\kappa \cup \kappa = \sigma(n)$ and

$$\kappa \cup \tilde{\Omega}^n = (-1)^n. \tag{6}$$

From these formulas and the fact that $\Omega \cup \tilde{\Omega} = 0$, we can take the cup products of (4) with $\tilde{\Omega}, \tilde{\Omega}^n$ and Ω, respectively, and obtain $\tilde{\Omega}^{n+1} = 2\kappa \cup \tilde{\Omega}$, $\tilde{\Omega}^{2n} = 2(-1)^n$ and $\Omega \cup \Omega = 2$.

From (6) it follows that $\kappa \cup \tilde{\Omega}^n$ is a generator of $H^{4n}(R_{2,2n})$, so a fortiori $\kappa \cup \tilde{\Omega}$ is a generator of $H^{2n+2}(R_{2,2n})$. Also, using the above formulas for the cup products, we can calculate the determinant

$$\begin{vmatrix} \tilde{\Omega}^n \cup \tilde{\Omega}^n & \tilde{\Omega}^n \cup \kappa \\ \kappa \cup \tilde{\Omega}^n & \kappa \cup \kappa \end{vmatrix} = (-1)^n,$$

so that $\tilde{\Omega}^n$, κ together generate $H^{2n}(R_{2,2n})$. We can now write down a set of generators for the cohomology groups of $R_{2,2n}$, $\{1, \tilde{\Omega}, \tilde{\Omega}^2, \cdots, \tilde{\Omega}^n, \kappa, \kappa \cup \tilde{\Omega}, \cdots, \kappa \cup \tilde{\Omega}^n\}$.

4. Some remarks. (1) In [4, Theorem 3.3], the cohomology ring structure of $R_{2,2n}$ is applied to find a lower bound for the number of parallel trangents of an embedding of a $2n$-dimensional compact orientable manifold M in R^{2n+2}. Now a proof without using calculations with Schubert varieties can be given by noticing that, if $d: R_{2,2n} \to R_{2,2n}$ is the map which reverses orientations of 2-planes, we have $d^*(\Omega) = -\Omega$ and $d^*(\tilde{\Omega}) = -\tilde{\Omega}$.

(2) Calculation of the normal geodesics to K in $R_{2,2n}$ shows that there are two embedded $\mathbf{CP}^n$ in $R_{2,2n}$, denoted by K and K^-, such that $R_{2,2n}\setminus K^-$ is an open disc bundle over K. This is a more general structure than a sphere bundle over K with section, and one can ask the question whether the product structure of homotopy groups holds for any space with this kind of decomposition.

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