ON THE PROBLEM OF THE INVARIANCE OF HOMOTOPICAL STABILITY OF POINTS UNDER CARTESIAN MULTIPLICATION

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1. **Introduction.** The principal purpose of this paper is to give a negative solution for the problem of Borsuk and Jaworowski [**3**, p. 164],[1] that is to say, in §4 we prove that Begle's example [**2**] shows that the homotopical stability of points is not invariant under Cartesian multiplication (Theorem 4).

In §2 we investigate set-theoretic properties of the joins of spaces. The relation between our problem and the join is also described in §2 (Theorem 1). In §3 homology groups and fundamental groups of the joins are considered.

2. **Set-theoretic properties of the joins.** Throughout the paper all spaces are finite complexes (=finite polyhedra) and nonvacuous, unless the contrary is explicitly stated. The equality between two spaces implies they are homeomorphic.

DEFINITION 1. Let A and B be spaces, their *join* $A*B$ is the space obtained from $A\times B\times I$ by identifying each set of the form $a\times B\times 1$ with $a\in A$ and each set of the form $A\times b\times 0$ with $b\in B$, where I denotes the interval $0\leqq t\leqq 1$. Let either A or B, say B, be vacuous, their join is A itself. Let either A or B, say B, be a point b, their join will be called a *cone* $\hat{A}$ of the base A and the vertex b. Let either A or B, say B, be the $(n-1)$-sphere S^{n-1} $(n\geqq 1)$, their join will be called an *n-fold suspension* of A.

Under the identification ω_{A*B}, A and B may be regarded as subsets $\omega_{A*B}(A\times B\times 1)$ and $\omega_{A*B}(A\times B\times 0)$ of $A*B$ respectively.

LEMMA 1. (i) $A*B=B*A$,

(ii) *if* $A=\bigcup_{i=1}^{n} A_i$, $A*B=\bigcup_{i=1}^{n}(A_i*B)$,

(iii) *if* $A=\bigcap_{i=1}^{n} A_i$, $A*B=\bigcap_{i=1}^{n}(A_i*B)$, *where n is a positive integer.*

PROOF. (i) Let h: $A\times B\times I\to B\times A\times I$ be a homeomorphism defined by taking $h(a, b, t)=(b, a, 1-t)$ and let $\psi\equiv\omega_{B*A}h\omega_{A*B}^{-1}$. It is easily checked that ψ is a 1:1-transformation. From Lemma 3 of [**8**], ψ is a homeomorphism. (ii) Since $\omega_{A*B}|A_i\times B\times I=\omega_{A_i*B}$, it follows that

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$A*B\supset\bigcup_{i=1}^{n}(A_i*B)$. For each point $x\in A*B$ we may find $a\in A_i$ such that $x=\omega_{A_i*B}(a, b, t)$. Hence we have $A*B\subset\bigcup_{i=1}^{n}(A_i*B)$. (iii) It is similar to (ii).

LEMMA 2. *$A*B$ is arcwise-connected.*

PROOF. If both A and B are arcwise-connected, from Definition 1 $A*B$ is arcwise-connected. In the general case, let x and y be arbitrary points of $A*B$. From Lemma 1 there are arcwise-connected components A', A'' of A and B', B'' of B such that $x\in A'*B'$, $y\in A''*B''$ and $A'*B'\cap A''*B'\supset B'$, $A''*B'\cap A''*B''\supset A''$. It follows that x and y are in an arcwise-connected component of $A*B$. This completes the proof.

LEMMA 3. $\hat{A}\times\hat{B}=(A\times\hat{B}\cup\hat{A}\times B)^\frown$.

PROOF. Let $\hat{A}=A*a_0$, $\hat{B}=B*b_0$ and $(A\times\hat{B}\cup\hat{A}\times B)^\frown=(A\times\hat{B}\cup\hat{A}\times B)*c$. From Lemma 4 of [8], $\omega_{\hat{A}\times\hat{B}}\equiv(\omega_{\hat{A}}, \omega_{\hat{B}})$, $\omega_{A\times\hat{B}}\equiv$(identity, $\omega_{\hat{B}}$) and $\omega_{\hat{A}\times B}\equiv(\omega_{\hat{A}}$, identity) are identifications. Let P be a space obtained from $A\times B\times b_0\times I\cup A\times a_0\times I\times B$ by identifying each point $(a, b, b_0, 1)$ with $(a, a_0, 1, b)$ and ω_1 be the identification. Then from Lemma 3 of [8] we have an identification $\omega_2: P\to A\times\hat{B}\cup\hat{A}\times B$ such that $\omega_2\omega_1|A\times B\times b_0\times I=\omega_{A\times\hat{B}}$, $\omega_2\omega_1|A\times a_0\times I\times B=\omega_{\hat{A}\times B}$. Let ω_3 be the identification $\omega_3:(A\times\hat{B}\cup\hat{A}\times B)\times c\times I\to(A\times\hat{B}\cup\hat{A}\times B)^\frown$. Let $f: A\times a_0\times I\times B\times b_0\times I\to(A\times\hat{B}\cup\hat{A}\times B)^\frown$ be a transformation defined by taking

$$f(a, a_0, t, b, b_0, t') = \begin{cases}\omega_3(\omega_2\omega_1(a, b, b_0, t'/t), (c, t)) & \text{for } t \geqq t', \\ \omega_3(\omega_2\omega_1(a, a_0, t/t', b), (c, t')) & \text{for } t' \geqq t,\end{cases}$$

where $0/0$ means 1.

Let $\phi\equiv f\omega_{\hat{A}\times\hat{B}}^{-1}: \hat{A}\times\hat{B}\to(A\times\hat{B}\cup\hat{A}\times B)^\frown$. It is easily checked that f is a map and ϕ is a 1:1-transformation. Therefore the lemma follows from Lemma 3 of [8].

LEMMA 4. $A\times\hat{B}\cup\hat{A}\times B=A*B$.

PROOF. Let $g: A\times B\times b_0\times I\cup A\times a_0\times I\times B\to A\times B\times I$ be a map defined by taking

$$(g\,|\,(A\times B\times b_0\times I))(a, b, b_0, t) = (a, b, 1-t/2),$$
$$(g\,|\,(A\times a_0\times I\times B))(a, a_0, t, b) = (a, b, t/2),$$

and let $\kappa\equiv\omega_{A*B}g\omega_1^{-1}\omega_2^{-1}: A\times\hat{B}\cup\hat{A}\times B\to A*B$, it is easily checked that κ is a 1:1-transformation. From Lemma 3 of [8] κ is a homeomorphism.

THEOREM 1. *Let K, L be complexes and a, a vertex of K, A a neighborhood complex of a and b, a vertex of L, B a neighborhood complex of b. Then $A * B$ is a neighborhood complex of (a, b) in $K \times L$.*

PROOF. From Lemma 3 and the definition of the neighborhood complex [6, §3], $A \times \hat{B} \cup \hat{A} \times B$ is a neighborhood complex of (a, b) in $K \times L$. Therefore the theorem follows from Lemma 4.

3. Homology groups and fundamental groups of the joins.

DEFINITION 2. Let A, B be complexes augmented by the null-simplex. The *join* $A * B$ of A and B is the augmented complex whose vertices form the union of the vertices of A and B respectively. A set of vertices are those of a simplex if the join of its subsets in A and B span a simplex there [5, p. 137].

From Lemma 1 it readily follows that Definition 2 and Definition 1 are equivalent.

We already know the boundary relation for the join;

$$F(\xi^p * \eta^q) = (F\xi^p) * \eta^q + (-1)^{p+1}\xi^p * (F\eta^q),$$

where ξ^p is a p-dimensional chain of A, η^q is a q-dimensional chain of B, and F denotes the boundary operation [5, p. 139].

Let $H_r(A)$ and $T_r(A)$, $r \geqq 1$, be the integral r-dimensional homology group of A and the integral r-dimensional torsion group of A respectively and $H_0(A)$ be the reduced (berandungsfähigen) 0-dimensional homology group of A [1, p. 209].

Then by Lefschetz's method [5, p. 137] and Künneth's theorem [1, p. 308] we can deduce;

THEOREM 2. *$H_n(A * B) \approx H_{n-1}(A \times B) \approx \sum_{p+q=n-1}(H_p(A) \otimes H_q(B)) \oplus \sum_{r+s=n-2} (T_r(A) \otimes T_s(B))$ for $n \geqq 0$, where the symbols $\sum$ and $\oplus$ signify the direct sum of groups and $\otimes$ signifies the tensor product of groups.*

COROLLARY. *The homology groups of the n-fold suspension of A are*

$$H_p(A * S^{n-1}) \approx H_{p-n}(A) \otimes H_{n-1}(S^{n-1}) \approx H_{p-n}(A) \qquad \text{for } p \geqq n,$$

$$H_p(A * S^{n-1}) \approx 0 \qquad \text{for } 0 \leqq p < n.$$

THEOREM 3. *Let either A or B be arcwise-connected, then the fundamental group of the join of A and B is trivial, that is to say, $\pi_1(A * B) \approx$ unity.*

PROOF. In the first place we assume that both A and B are arcwise-connected. From Lemma 2 and the assumption complexes $A \times \hat{B}$, $\hat{A} \times B$, and $A \times B$ are arcwise-connected. From Theorem 1 of [7, p.

177] and Lemma 4, $\pi_1(A*B)=\pi_1(\kappa(A\times\hat{B}\cup\hat{A}\times B))$ is a factor group of the free product $\pi_1(\kappa(A\times\hat{B}))\circ\pi_1(\kappa(\hat{A}\times B))$. Since $\pi_1(A\times\hat{B})\approx\pi_1(A\times b)$ and $\pi_1(\hat{A}\times B)\approx\pi_1(a\times B)$, $\pi_1(A*B)$ is a factor group of $\pi_1(\kappa(A\times b))\circ\pi_1(\kappa(a\times B))$, where a, b are fixed points of A, B respectively. It is easily seen that each closed path of $\kappa(A\times b)$ which passes $\kappa(a, b)$ and $\kappa(a, b)$ are homotopic in the cone $\omega_{A*B}(A\times b\times I)$ relative to $\kappa(a, b)$. Since each generator of $\pi_1(\kappa(A\times b))$ is null-homotopic in $A*B$ and the same is also true for $\pi_1(\kappa(a\times B))$, then $\pi_1(A*B)\approx$ unity.

In the next place we assume that either A or B, say A, is arcwise-connected. Let $B_1, \cdots, B_n$ be arcwise-connected components of B, from Lemma 1, $A*B=A*B_1\cup\cdots\cup A*B_n$ and $A*B_i\cap A*B_j=A$, where $i\neq j$ and $1\leqq i$, $j\leqq n$. Since $A*B_1, \cdots, A*B_n$ are arcwise-connected, $\pi_1(A*B)$ is a factor group of the free product $\pi_1(A*B_1)\circ\cdots\circ\pi_1(A*B_n)$. Thus the theorem follows from the first consideration.

Corollary. *$A*B$ is noncontractible if, and only if, there exists $p\geqq 0$ such that $H_p(A\times B)$ not ≈ 0.*

Proof. *Necessity*: If $A*B$ is noncontractible, either $\pi_1(A*B)$ not $\approx$ unity or $H_n(A*B)$ not ≈ 0, $n\geqq 2$. If $\pi_1(A*B)$ not $\approx$ unity, from Theorem 3 neither A nor B is arcwise-connected. Then we have $H_0(A\times B)$ not ≈ 0. If $H_n(A*B)$ not ≈ 0, $n\geqq 2$, from Theorem 2, $H_{n-1}(A\times B)$ not ≈ 0. *Sufficiency*: From Theorem 2 it is trivial.

Remark. From the above facts we can easily deduce:

Let either A or B be arcwise-connected and $H_i(A)\approx 0$, $H_j(B)\approx 0$ for $i<p$, $j<q$. Then $\pi_{p+q+1}(A*B)\approx H_p(A)\otimes H_q(B)$, where $\pi_r(A)$ denotes r-dimensional homotopy group of A, $r\geqq 2$.

4. Application to the homotopical stability of points.

Example. Let A be a complex which is obtained from a Poincaré's sphere with nonvanishing fundamental group by omitting an open 3-simplex and let $K=(a_0\cup a_1)*A$ and let $L=(b_0\cup b_1)*S^{n-1}$. Then from Theorem 1, $A*S^{n-1}$ is a neighborhood complex of (a_0, b_0) in $K\times L$. From Theorems 3.1 and 3.2 of [6] and corollaries of Theorems 2 and 3, we have that a_0 and b_0 are homotopically stabile but (a_0, b_0) is homotopically labile. When $S^{n-1}=S^0$, then the example is the one of Begle [2].

Thus we have;

Theorem 4. *The homotopical stability of points is not invariant under Cartesian multiplication.*

From Theorem 3.2 of [6] and corollary of Theorem 3 we have:

THEOREM 5. *A point (a, b) of $K^\frown L$ is homotopically stabile if, and only if, there exist neighborhood complexes A of a in K and B of b in L such that $H_p(A \times B)$ not ≈ 0 for some $p \geqq 0$.*

REMARK. We may propose an analogous problem for the stability of points owing to Hopf and Pannwitz [**4**]. From Theorem 6 of [**4**], the boundary relation for the join in §3 and Theorem 1, we can deduce;

For homogeneous complexes the stability of points owing to Hopf and Pannwitz is invariant under Cartesian multiplication.

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