ON CONVEX TOPOLOGICAL LINEAR SPACES

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Communicated September 9, 1943

In an earlier note[1] the author introduced and discussed the notion of linear system. It is the purpose of the present note to apply this notion to the study of convex topological linear spaces.[2] Let X be an abstract linear space. Let X^* be the space of all linear functionals defined on X. With each convex topology t in X we associate the subspace L of X^* consisting of those linear functionals continuous with respect to t. In general, there will be many t's associated with a single L and we obtain in this way a natural many-to-one correspondence between convex topological linear spaces and linear systems. It is our purpose here to correlate the properties of a convex topological linear space with those of its linear system and with the strength of its topology relative to that of the other convex topological linear spaces associated with the same linear system.

Our principal tool is a reformulation of von Neumann's[3] observation that the topology of a convex topological linear space may be described by means of pseudo-norms. This reformulation, whose proof is a consequence of Fichtenholz's[4] theorem on the relationship between norms and linear functionals, is as follows. Let L be an arbitrary total[5] subspace of X^*. Let $\mathfrak{J}$ be any family of pseudo-norm sets which has the following three properties. (a) For each two members of $\mathfrak{J}$ there is a third member of $\mathfrak{J}$ which contains them both. (b) If a pseudo-norm set is contained in a member of $\mathfrak{J}$ then it is itself a member of $\mathfrak{J}$. (c) The set theoretic union of all members of $\mathfrak{J}$ is L. Then there is a unique convex topology in X whose continuous linear functionals are precisely the members of $\mathfrak{J}$. Conversely, every convex topology in X associated with L may be obtained in this way. In other words, there is a natural one-to-one correspondence between the convex topologies associated with X_L and the "ideals of pseudo-norm sets which span L."

As will be proved in the author's forthcoming paper on linear systems, every finite dimensional subspace of an X^* is a pseudo-norm set and the linear union of any two pseudo-norm sets is again a pseudo-norm set. Thus for any L the family of all pseudo-norm sets in L and the family of all finite dimensional pseudo-norm sets in L are both ideals which span L. As an immediate consequence we have:

THEOREM 1. *Let L be an arbitrary total subspace of X^*. Then the family of convex topologies in X associated with L not only is not empty but also contains a weakest member and a strongest member.*

This theorem suggests the following definitions. A *relatively weak* (*relatively strong*) convex topological linear space is a convex topological linear space which has a weaker (stronger) topology than any other such space with the same linear system. Normed linear spaces in their weak topologies are relatively weak convex topological linear spaces and in their norm topologies are relatively strong convex topological linear spaces. Because of the latter fact one can regard the notion of relatively strong convex topological linear space as a natural generalization of that of normed linear space. Theorem 3 below is of interest in this connection. In general, of course, a convex topological linear space will be neither relatively weak nor relatively strong but on the other hand, may be both.

The standard notion of boundedness[6] in topological linear spaces coincides in the convex case with the boundedness for linear systems introduced in "IDS." That is, if X is a convex topological linear space and L is its family of continuous linear functionals then a subset A of X is bounded if and only if $\text{l.u.b.}_{(x \in A)} \left| l(x) \right| < \infty$ for each l in L. This has as an immediate consequence the fact that two convex topologies in X generate the same bounded sets if and only if their families of continuous linear functionals have identical bounded closures. Thus it is clear that not only are there, in general, many convex topologies with the same continuous linear functionals but also many convex topologies with different continuous linear functionals and the same bounded sets. These considerations lead at once to a proof of the following theorem and hence show that the converse of a certain theorem of Wehausen[7] is not true.

THEOREM 2. *Let X be a convex topological linear space. Then every linear transformation from X to another convex topological linear space which takes bounded sets into bounded sets is continuous if and only if X is relatively strong and has a boundedly closed linear system.*

It follows from Theorem 2 and a slight extension of a theorem of Wehausen[8] that every metrizable convex topological linear space is relatively strong and has a boundedly closed linear system. The question as to which boundedly closed linear systems are such that their associated relatively strong convex topological linear spaces are metrizable is answered at once by the Birkhoff-Kakutani[9] group metrizability criterion and we have:

THEOREM 3. *Let X be a convex topological linear space and let X_L be its linear system. Then X is metrizable if and only if it is relatively strong and L is the union of an ascending sequence of pseudo-norm sets. (Such an L is automatically boundedly closed.)*

Thus given a linear space X there is a natural one-to-one correspondence between the metrizable convex topologies in X and the ascending sequences of pseudo-norm sets with total unions. It follows from a theorem in the theory of linear systems that such a union is a norm set if and only if all members past a certain one are identical. Thus the metrizable but nonnormable convex topologies correspond to the strictly ascending sequences. Using simple theorems about pseudo-norm sets it is possible to construct many examples of such sequences.

As a consequence of Theorem 3 one may prove:

THEOREM 4. *A relatively weak convex topological linear space is normable if and only if it is finite dimensional and is metrizable if and only if it is isomorphic to a subspace of the space (s) of Banach.*[10]

A topological linear space X, being a topological group, has a natural uniform structure.[11] Hence one may speak of its totally bounded subsets, its Cauchy directed systems, and of whether or not it is complete. It turns out that completeness in this sense is relatively rare and various authors have introduced several weaker notions which we shall now formulate. X is C_4 complete if it is complete as a uniform structure with respect to the convergence of directed systems. X is C_3 complete if every closed and bounded subset is C_4 complete. X is C_2 complete if every closed and totally bounded subset is C_4 complete. X is C_1 complete if it is complete as a uniform structure with respect to the convergence of sequences. X is T_2 complete if every closed and totally bounded subset is bicompact. X is T_1 complete if every closed and totally bounded subset is compact. It is more or less obvious that for $i = 2, 3$ or 4, C_i completeness implies C_{i-1} completeness and that T_2 completeness implies T_1 completeness. It follows from the generalization to uniform structures of a well-known theorem on metric spaces[12] that C_2 completeness and T_2 completeness are equivalent. Finally von Neumann[3] has shown that T_1 completeness implies C_1 completeness. Thus the five possibly distinct notions of completeness among those described above may be arranged in order so that each is weaker than or equivalent to its predecessor.

The principal theorems relating the completeness of convex topological linear spaces to their relative strength and to the properties of their linear systems are as follows.

THEOREM 5. *If $i = 1, 2$ or 3 and X is a C_i complete convex topological linear space then X is C_i complete in every relatively stronger convex topology.*

THEOREM 6. *If X is a C_1 complete convex topological linear space then the linear system of X is a complete linear system.*

THEOREM 7. *If X_L is a complete linear system whose conjugate system is boundedly closed then every convex topological linear space associated with X_L is C_3 complete.*

It is not known whether or not the converse of Theorem 7 is true. However, the following partial converse may be proved.

THEOREM 8. *If X_L is a regular linear system whose associated relatively weak convex topological linear space is C_3 complete and if X_L is relatively bounded then the conjugate of X_L is boundedly closed.*

It is interesting to note that the truth of the strict converse of Theorem 7 would imply that every linear system of the form X^*_X is boundedly closed and hence answer the measure theory question of Ulam mentioned in "IDS."

It is an obvious consequence of the definition that the linear system of a normed linear space is relatively bounded. Furthermore, it is readily verified that a normed linear space is reflexive if and only if its linear system is complete and has a boundedly closed conjugate. Finally, for relatively weak convex topological linear spaces, bounded subsets and totally bounded subsets are identical[13] so that C_3 completeness and T_2 completeness are equivalent. Thus Theorems 7 and 8 have the following known corollary.

THEOREM 9. *For a normed linear space X the following are equivalent:* (*a*) *X is reflexive.* (*b*) *X is* C_3 *complete in its weak topology.* (*c*) *X is* T_2 *complete in its weak topology.*

Wehausen[14] has pointed out that a T_1 complete topological linear space need not be of the second category. The first statement of the following combined with Theorem 7 shows that this may be extended to C_3 completeness and furnishes a wide class of examples including Wehausen's.

THEOREM 10. *If a convex topological linear space is of the second category then it is relatively strong and its linear system is uniform.*

One may also prove:

THEOREM 11. *If X is a convex topological linear space of the second category whose linear system is almost relatively bounded then X is normable.*

[1] Mackey, G. W., "On Infinite Dimensional Linear Spaces," these PROCEEDINGS, **29**, 216 (1943). In the sequel we shall refer to this paper as "IDS" and shall use the definitions and notations introduced in it without comment or explanation.

[2] By a topological linear space we mean a real linear space which is at the same time a T_1 space in the sense of Alexandroff and Hopf (*Topologie I*, J. Springer, Berlin, 1935) and in which the topology is related to the algebra in such a manner that the operations of addition and multiplication by reals are continuous in both variables together. By a convex topological linear space we mean a topological linear space in which every point has a complete system of neighborhoods each of which is a convex set. These notions have been introduced in slightly different ways by various authors. See Wehausen, J. V., "Transformations in Linear Topological Spaces," *Duke Math. Jour.*, **4**, 157 (1938), for a discussion. Also see Whitney, H., "Tensor Products of Abelian Groups," *Ibid.*, **5**, 518 (1939), footnote 22, for a discussion of a popular misconception.

[3] von Neumann, J., "On Complete Topological Spaces," *Trans. Amer. Math. Soc.*, **37**, 1 (1935).

[4] Fichtenholz, G., "Sur les fonctionelles linéaires, continues au sens généralisé," Rec. Math. (Mat. Sbornik), N. S., **4**, 192 (1938).

[5] Wehausen has shown that the family of continuous linear functionals on a convex topological linear space X is always total; that is, for each non-zero member of X there is a continuous linear functional which does not take it into zero, loc. cit., Theorem 8.

[6] See Hyers, D., "A Note on Linear Topological Spaces." *Bull. Amer. Math. Soc.*, **44**, 76 (1938), for statements of the two standard definitions of boundedness and a proof of their equivalence.

[7] Loc. cit., Theorem 2.
[8] Loc. cit., Theorem 3'.
[9] Birkhoff, G., "A Note on Topological Groups," Comp. Math., 3, 427 (1936); Kakutani, S., "Ueber die Metrization der topologische Gruppen," Proc. Im. Ac. Jap., 12, 82 (1936).
[10] Banach, S., *Théorie des opérations linéaires* Warsaw, 1932, p. 10. By an isomorphism between two topological linear spaces we mean an algebraic isomorphism which is at the same time a homeomorphism.
[11] For a discussion of the notion of uniform structure see Weil, A., *Sur les espaces á structure uniforme et sur la topologie genérale*, Hermann, Paris 1937.
[12] The theorem that a totally bounded metric space is complete if and only if it is compact. The generalization is essentially contained in G. Birkhoff's proof that C_4 completeness implies T_2 completeness. Birkhoff, G., "Moore-Smith Convergence in General Topology," *Ann. Math.*, 38, 39 (1937).
[13] Various authors have proved this fact in special cases and the general case offers no new difficulties.
[14] Loc. cit., Theorem 15.

NATURAL LOGARITHMS OF SMALL PRIME NUMBERS

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Communicated September 30, 1943

PART I. EXTENSION OF J. C. ADAMS' TABLE ABOVE Log_e 7

In the summer of the year 1938 the author began the computation of the Napierian logarithms of 11, 13, 17, 19, 23, 29 and 31 with the object of extending J. C. Adams' basic 273-place table[1] which gives the logarithms of the four prime numbers falling between 1 and 11. In the following spring these calculations were interrupted intentionally in order to carry out a prerequisite investigation[2] of the perfection of the published records of Adams' constants. It was not until the early summer of the present year that the opportunity arose for effectuating the conviction that, aside from whatever intrinsic importance may attach to the project, the earlier sporadic work had progressed so far and had involved so much labor that it deserved to be revived, definitively checked and made accessible to other arithmeticians.

The calculations were usually performed with the aid of the formula $\ln(\alpha/\beta) = 2\sum_{1}^{\infty}\{(\alpha - \beta)^{2m-1}(\alpha + \beta)^{-2m+1}(2m - 1)^{-1}\}$. The composite numbers α and β were chosen to satisfy the simplifying condition $\alpha - \beta = 1$ and to contain as factors only the small prime numbers under consideration