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Some fractional integral formulas for the Mittag-Leffler type function with four parameters

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Abstract: In this paper we present some results from the theory of fractional integration operators (of Marichev-Saigo-Maeda type) involving the Mittag-Leffler type function with four parameters ${}_{\zeta,\gamma}E_{\mu,\nu}[z]$ which has been recently introduced by Garg *et al*. Some interesting special cases are given to fractional integration operators involving some Special functions.

Keywords: Marichev-Saigo-Maeda type fractional integral operators, Mittag-Leffler type function with four parameters, Generalized Wright function

MSC: 26A33, 33E12, 33C60, 33E20

1 Introduction and Preliminaries

The fractional calculus and its various applications have become a very popular subject between mathematicians and engineers. New era in the development of this branch of science began 40-50 years ago due to numerous application of fractional-type models and is continued up to now (see [54] and [55]). One can mention a large list of areas of application, in particular, continuum mechanics [8, 39] (including viscoelasticity [27], thermodynamics [17] and anomalous diffusion [38]), astrophysics [30], nuclear physics [53], nanophysics and cosmic physics [57, 58], statistical mechanics [60], fractional order systems and control [7], finance and economics [5], solutions of differential equations [4].

Among the monographs developing the theory of fractional calculus and presenting some applications we have to point out monographs by Diethelm [11], Gorenflo and Mainardi [15], Kiryakova [21], Kilbas, Srivastava and Trujillo [20], Miller and Ross [32], Oldham and Spanier [35], Podlubny [36], and of *course the Bible of fractional calculus*, monograph by Samko, Kilbas and Marichev [43]. Interested reader can find in these books an extended list of publications on the theory and applications of fractional calculus (see also [56]).

Recently, Mittag-Leffler functions show its close relation to Fractional Calculus and especially to fractional problems which come from applications. This new era of research attract many scientists from different point of view (see [2, 6, 9, 12, 16, 18, 19, 22, 23, 37, 40, 44, 46, 52]). In 1899 G. Mittag-Leffler began the publication of a series of articles under the common title *Sur la representation analytique d'une branche uniforme d'une fonction monogµene* (*On the analytic representation of a single-valued branch of a monogene function*) published mainly at *Acta Mathematica*. Nowadays this function and its numerous generalizations are involved in the different fractional

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models (see monographs listed above). Motivated by the above works Kiryakova [25, 26] for the first time pointed out the special role of the Mittag-Leffler function and included it into the class of Special Functions for Fractional Calculus. Moreover, based on the role of the Mittag-Leffler function in application, Mainardi called it The Queen of Fractional Calculus (see [27]).

Here, our investigation are based on the so-called Marichev-Saigo-Maeda type generalized fractional operator, *i.e.* integral transform of the Mellin convolution type with the Appell (or Horn) function F_3 developed by Marichev [28] and studied in some recent papers, including the papers by Agarwal *et al* [2], Choi and Agarwal [10], Saigo and Maeda [42], Saigo and Saxena [45]. The aim of our paper is to present formulas of the Marichev-Saigo-Maeda generalized fractional integration of the generalized Mittag-Leffler type function with four parameters ${}_{\zeta,\gamma}E_{\mu,\nu}[z]$ which has been recently introduced by Garg *et al.* [13], and study its various properties, which mainly motivated our present investigation. Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^+$, $\mathbb{N}$ be the sets of complex numbers, real and positive real numbers, nonpositive integers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 Definitions and earlier works

For the present investigation, we consider the following definitions and earlier works.

Definition 2.1. *The Mittag-Leffler type function with four parameters is defined and studied by Garg* et al. *[13] in the following manner :*

$$ {}_{\zeta,\gamma}E_{\mu,\nu}[z] = \sum_{n=0}^{\infty} \frac{(\zeta)_{\gamma n}}{\Gamma(\mu n+\nu)}.z^n \qquad (\mu,\nu,\zeta,\gamma,\, z \in \mathbb{C},\ \Re(\mu) > \Re(\nu) > 0), \tag{1} $$

where $(\zeta)_\gamma$ is the Pochhammer symbol defined, for $\gamma \in \mathbb{C}$, as follows (see, *e.g.*, [49, p. 2 and p. 5])):

$$ (\zeta)_\gamma = \begin{cases} 1, & \gamma = 0, \\ \zeta(\zeta+1)\cdots(\zeta+\gamma-1), & \gamma \in \mathbb{N} \end{cases} = \frac{\Gamma(\zeta+\gamma)}{\Gamma(\zeta)} \quad \left(\zeta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) $$

and Γ being the familiar Gamma function (see, *e.g.*, [48, Section 1.1] and [49, Section 1.1]).

Detail account of several results which included integral representations, recurrence relations, differential formula, fractional derivative and integral, Mellin Barnes integral representation and fractional calculus integral operator involving (1) can be found in the article [13]. Some important special cases of this function are enumerated below:

(i) When $\gamma = 1$ with $(\mu,\nu,\zeta,z \in \mathbb{C},\ \Re(\mu) > \Re(\nu) > 0)$, the function (1) reduces to the one that has been considered by Garg *et al.* [13]:

$$ {}_{\zeta,1}E_{\mu,\nu}[z] = \sum_{n=0}^{\infty} \frac{(\zeta)_n}{\Gamma(\mu n+\nu)}.z^n. \tag{2} $$

(i1) If we set $\gamma = 0$ with $\min\{\Re(\mu),\Re(\nu)\} > 0$, then (1) reduces to the generalized Mittag-Leffler function considered by Wiman [59] (see also [14, p. 39, Eqn. (3.1)]). The case when $\gamma = 0$ and $\nu = 1$ can be found in [33] (see also [14, p. 39]).

In recent years, the Mittag-Leffler function and its various generalizations have become a very popular subject of mathematics and its applications. Among the large number of works regarding the Mittag-Leffler function, for a remarkably clear, insightful, and systematic exposition of the investigations carried out by various authors in the field of mathematical analysis and its applications, the interested reader should refer also to a survey-cum-expository Book by Gorenflo *et al.*, which contains a fairly comprehensive bibliography of as many as 170 *further* references on the subject.

Definition 2.2. *The H-function is defined in terms of a Mellin-Barnes integral in the following manner ([29]):*

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathfrak{L}}\Theta(s)\,z^{-s}\mathrm{d}s, \tag{3}$$

where

$$\Theta(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\prod_{i=1}^{n}\Gamma\left(1-a_i-\alpha_i s\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+\alpha_i s\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)}, \tag{4}$$

where $m,\ n,\ p,\ q$ are integers such that $0\leq m\leq q$, $0\leq n\leq p$, and for parameters $a_i,\ b_i\in\mathbb{C}$ and for parameters $\alpha_i,\ \beta_j\in\mathbb{R}^+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$ with the contour $\mathfrak{L}$ suitably chosen, and an empty product, if it occurs, is taken to be unity. The theory of the H-function are well explained in the book of Srivastava, Gupta and Goyal ([50], Ch.1) (see also [30]).

Definition 2.3. *The generalized Wright's function is defined as follows (see, e.g., [20, p.56, Eqns. (1.11.14) and (1.11.15)]):*

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\ A_1),\ \ldots,\ (\alpha_p,\ A_p);\\(\beta_1,\ B_1),\ \ldots,\ (\beta_q,\ B_q);\end{matrix}z\right]=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_j\,k\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_j\,k\right)}\frac{z^k}{k!}, \tag{5}$$

where the coefficients $A_1,\ \ldots,\ A_p\in\mathbb{R}^+$ and $B_1,\ \ldots,\ B_q\in\mathbb{R}^+$ with

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geqq 0. \tag{6}$$

Here, in this paper, our main results are obtained by applying the ${}_{\xi,\gamma}E_{\mu,\nu}[z]$ to the fractional integration operators (of Marichev-Saigo-Maeda type) given in (7) and (8), respectively . So we continue to recall the following definitions.

$$\left(\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt,\quad(\Re(\eta)>0), \tag{7}$$

and

$$\left(\mathcal{I}_{-}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\eta)}\int_x^\infty(t-x)^{\eta-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt,\quad(\Re(\eta)>0). \tag{8}$$

These operators (integral transforms) were introduced by Marichev [28] as Mellin type convolution operators with a special function $F_3(.)$ in the kernel. These operators were rediscovered and studied by Saigo in [41] as generalization of so-called Saigo fractional integral operators, see [24]. The properties of these operators were studied by Saigo and Maeda [42], in particular, relations of operators with the Mellin transforms, hypergeometric operators (or Saigo fractional integral operators), their decompositions and acting properties in the McBride spaces $F_{p;\mu}$ (see [31]).

In (7), (8) the symbol $F_3(.)$ denotes so-called 3rd Appell function (known also as Horn function) (see [34, p. 413]):

$$F_3(\alpha,\alpha';\beta,\beta';\eta;x;y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\eta)_{m+n}m!n!}x^m y^n\quad\max\{|x|,|y|\}<1). \tag{9}$$

The properties of this function are discussed in [34, p. 412-415]. In particular, its relation to the Gauss hypergeometric function is presented:

$$F_3(\alpha,\eta-\alpha,\beta,\eta-\beta;\eta;x;y)={}_2F_1(\alpha,\beta;\eta;x+y-xy). \tag{10}$$

Moreover, it is easily observed that

$$F_3(\alpha,0,\beta,\beta';\eta;x;y)=F_3(\alpha,\alpha',\beta;\eta;x;y)={}_2F_1(\alpha,\beta;\eta;x), \tag{11}$$

and

$$F_3(0,\alpha',\beta,\beta';\eta;x;y)=F_3(\alpha,\alpha',\beta';\eta;x;y)={}_2F_1(\alpha',\beta';\eta;x). \tag{12}$$

It is known that the 3rd Appell function cannot be expressed as a product of two ${}_2F_1$ functions, and satisfy pairs of linear partial differential equations of the second order.

3 Left-sided fractional integration of generalized Mittag-Leffler functions with four parameters

Our results in this Section are based on the preliminary assertions giving composition formula of fractional integral (7) with a power function.

Lemma 3.1 ([42, p. 394]). *Let* $\alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}$ *and*

$$\Re(\eta) > 0,\ \Re(\rho) > \max\left\{0, \Re(\alpha+\alpha'+\beta-\eta), \Re(\alpha'-\beta')\right\},$$

then the following relation holds

$$\left(\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\eta-\alpha-\alpha')\Gamma(\rho+\eta-\alpha'-\beta)} x^{\rho+\eta-\alpha-\alpha'-1}. \tag{13}$$

The value of the left-sided Marichev-Saigo-Maeda fractional integral (7) for the generalized Mittag-Leffler function (1) is given by the following theorem.

Theorem 3.2. *Let the parameters* $\alpha, \alpha', \beta, \beta', \eta, \zeta, \gamma, \rho, \mu, \nu, x \in \mathbb{C}$ *and* $\Re(\mu) > \Re(\nu) > 0$ *be such that*

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) > max\left[0, \Re(\alpha+\alpha'+\beta-\eta), \Re(\alpha'-\beta')\right],$$

then for all $x > 0$ *the following relation is valid*

$$\begin{aligned}\left(\mathcal{I}_{0+}^{(\alpha,\alpha',\beta,\beta',\eta)}\left[t^{\rho-1}\, {}_{\zeta,\gamma}E_{\mu,\nu}(ct^{\sigma})\right]\right)(x) &= \frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\times \\ \times\, {}_5\Psi_4\Bigg[&(\zeta,\gamma),(\rho,\sigma),(\rho+\eta-\alpha-\alpha'-\beta,\sigma),(\rho+\beta'-\alpha',\sigma),(1,1) \\ &\Big(\nu,\mu\Big),(\rho+\beta',\sigma\Big),(\rho+\eta-\alpha-\alpha',\sigma),(\rho+\eta-\beta-\alpha',\sigma)\Big| cx^{\sigma}\Bigg].\end{aligned} \tag{14}$$

Proof. For convenience, let the left-hand side of the formula (14) be denoted by $\mathfrak{I}$. We apply (1) and use definition of the integral operator (7) and the representation of (1) in terms of generalized Wright function (5). We use then series form definition of the generalized Wright function (5). Finally, we change the order of integration and summation and find

$$\begin{aligned}\mathfrak{I} &= \left(\mathcal{I}_{0+}^{(\alpha,\alpha',\beta,\beta',\gamma)}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\zeta)_{\gamma n}}{\Gamma(\mu n+\nu)}c^n t^{\sigma n}\right]\right)(x), \\ &= \sum_{n=0}^{\infty}\frac{(\zeta)_{\gamma n}}{\Gamma(\mu n+\nu)}c^n\left(\mathcal{I}_{0+}^{(\alpha,\alpha',\beta,\beta',\gamma)}\left\{t^{\rho+\sigma n-1}\right\}\right)(x).\end{aligned}$$

Due to the convergence conditions of Theorem 3.2, for any $n \in \mathbb{N}_0$, we have $\Re(\rho+\sigma n) \geq \Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$

Therefore we can apply Lemma 3.1 and use (13) with ρ replaced by $(\rho+\sigma n)$:

$$\begin{aligned}\mathfrak{I} &= \sum_{n=0}^{\infty}\frac{\Gamma(\zeta+\gamma n)}{\Gamma(\zeta)\Gamma(\mu n+\nu)}\frac{\Gamma(\rho+\sigma n)\,\Gamma(\rho+\sigma n+\eta-\alpha-\alpha'-\beta)}{\Gamma(\rho+\sigma n+\beta')\,\Gamma(\rho+\sigma n+\eta-\alpha-\alpha')}\times \\ &\qquad\times\frac{\Gamma(\rho+\sigma n+\beta'-\alpha')\,\Gamma(1+n)}{\Gamma(\rho+\sigma n+\eta-\alpha'-\beta)}\frac{c^n x^{\rho+\sigma n+\eta-\alpha-\alpha'-1}}{n!} \\ &= \frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\sum_{n=0}^{\infty}\frac{\Gamma(\zeta+\gamma n)}{\Gamma(\mu n+\nu)}\frac{\Gamma(\rho+\sigma n)\,\Gamma(\rho+\sigma n+\eta-\alpha-\alpha'-\beta)}{\Gamma(\rho+\sigma n+\beta')\,\Gamma(\rho+\sigma n+\eta-\alpha-\alpha')}\times \\ &\qquad\times\frac{\Gamma(\rho+\sigma n+\beta'-\alpha')\,\Gamma(1+n)}{\Gamma(\rho+\sigma n+\eta-\alpha'-\beta)}\frac{c^n x^{\sigma n}}{n!}.\end{aligned} \tag{15}$$

This, in accordance with (5), completes the proof. □

For $\gamma = 1$ in (14), Theorem 3.2, yields to the following result:

Corollary 3.3. *Let the parameters* $\alpha, \alpha', \beta, \beta', \eta, \zeta, \rho, \mu, \nu, x \in \mathbb{C}$ *and* $\Re(\mu) > \Re(\nu) > 0$ *be such that*

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) > max\left[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\right],$$

then for all $x > 0$ *the following result holds:*

$$\begin{aligned}\left(\mathcal{I}_{0+}^{(\alpha,\alpha',\beta,\beta',\eta)}\left[t^{\rho-1}\,{}_{\zeta,1}E_{\mu,\nu}(ct^{\sigma})\right]\right)(x) &= \frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\times \\ \times\,{}_5\Psi_4\Bigg[&(\zeta,1),(\rho,\sigma),(\rho+\eta-\alpha-\alpha'-\beta,\sigma),(\rho+\beta'-\alpha',\sigma),(1,1) \\ &\Big(\nu,\mu\Big),(\rho+\beta',\sigma\Big),(\rho+\eta-\alpha-\alpha',\sigma),(\rho+\eta-\beta-\alpha',\sigma)\Big|cx^{\sigma}\Bigg],\end{aligned} \tag{16}$$

where ${}_{\zeta,1}E_{\mu,\nu}[z]$ is another new generalized Mittag–Leffler type function defined as (see [13, p.4, Eqn. (2.2)]):

$${}_{\zeta,1}E_{\mu,\nu}[z] = \sum_{n=0}^{\infty}\frac{(\zeta)_n}{\Gamma(\mu n+\nu)}.z^n (\mu,\nu,\zeta,\, z\in\mathbb{C},\ \Re(\mu)>\Re(\nu)>0),$$

Remark 3.4. *It is easily seen that setting* $\gamma \to 0$ *in equation (14) with some suitable parametric replacements in the resulting identities yields the corresponding known integral formulas in Agarwal* et al. *[1].*

4 Right-sided fractional integration of generalized Mittag-Leffler functions with four parameters

In this Section, our results are based on the preliminary assertions giving composition formula of fractional integral (8) with a power function.

Lemma 4.1 ([42, p. 394])**.** *Let* $\alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}$ *and*

$$\Re(\eta) > 0,\ \Re(\rho) < 1 + \min\left\{\Re(-\beta), \Re(\alpha+\alpha'-\eta), \Re(\alpha+\beta'-\eta)\right\},$$

then the following relation holds

$$\left(\mathcal{I}_{-}^{\alpha,\alpha',\beta,\beta',\eta}x^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\eta+\alpha+\alpha')\Gamma(1-\rho+\alpha+\beta'-\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\eta)\Gamma(1-\rho+\alpha-\beta)}x^{\rho+\eta-\alpha-\alpha'-1}. \tag{17}$$

The value of the right-sided Marichev-Saigo-Maeda fractional integral (8) for the generalized Mittag-Leffler function (1) is given by the following theorem.

Theorem 4.2. *Let the parameters* $\alpha, \alpha', \beta, \beta', \eta, \zeta\gamma, \sigma, \rho, \mu, \nu, x \in \mathbb{C}$ *and* $\Re(\mu) > \Re(\nu) > 0$ *be such that*

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha'-\eta), \Re(\alpha+\beta'-\eta)\right],$$

then for all $x > 0$ *the following relation is valid*

$$\begin{aligned}\left(\mathcal{I}_{-}^{(\alpha,\alpha',\beta,\beta',\eta)}\left[t^{\rho-1}\,{}_{\zeta,\gamma}E_{\mu,\nu}\left(\frac{c}{t^{\sigma}}\right)\right]\right)(x) &= \frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\times \\ \times\,{}_5\Psi_4\Bigg[&(\zeta,\gamma),(1-\rho-\beta,\sigma),(1-\rho-\eta+\alpha-\beta',\sigma),(1-\rho-\eta+\alpha+\alpha',\sigma),(1,1) \\ &\Big(\nu,\mu\Big),(1-\rho,\sigma),(1-\rho-\eta+\alpha+\alpha'+\beta',\sigma),(1-\rho-\beta+\alpha,\sigma)\Big|cx^{\sigma}\Bigg].\end{aligned} \tag{18}$$

Proof. For convenience, let the left-hand side of the formula (18) be denoted by $\mathcal{J}$. We apply (1) and use definition of the integral operator (8) and the representation of (1) in terms of generalized Wright function (5). We use then series form definition of the generalized Wright function (5). Finally, we change the order of integration and summation and find

$$\begin{aligned}\mathcal{J} &= \left(\mathcal{I}_{-}^{(\alpha,\alpha',\beta,\beta',\gamma)}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\zeta)_{\gamma n}}{\Gamma(\mu n+\nu)}c^n t^{-\sigma n}\right]\right)(x)\\ &= \sum_{n=0}^{\infty}\frac{(\zeta)_{\gamma n}}{\Gamma(\mu n+\nu)}c^n\left(\mathcal{I}_{-}^{(\alpha,\alpha',\beta,\beta',\gamma)}\left\{t^{\rho-\sigma n-1}\right\}\right)(x).\end{aligned}$$

Due to the convergence conditions of Theorem 4.2, for any $n \in \mathbb{N}_0$, we have $\Re(\rho-\sigma n-1) \leq \Re(\rho-1) < 1-\min[\Re(-\beta),\Re(\alpha+\alpha'-\eta),\Re(\alpha+\beta'-\eta)]$

Therefore we can apply Lemma 4.1 and use (17) with ρ replaced by $(\rho-\sigma n)$:

$$\begin{aligned}\mathcal{J} &= \sum_{n=0}^{\infty}\frac{\Gamma(\zeta+\gamma n)}{\Gamma(\zeta)\Gamma(\mu n+\nu)}\frac{\Gamma(1-\rho-\eta+\sigma n+\alpha+\alpha')\,\Gamma(1-\rho-\sigma n-\eta+\alpha+\beta')}{\Gamma(1-\rho+\sigma n)\,\Gamma(1-\rho+\sigma n-\eta+\alpha+\alpha'+\beta')}\times\\ &\quad\times\frac{\Gamma(1-\rho+\sigma n-\beta)\,\Gamma(1+n)}{\Gamma(1-\rho+\sigma n+\alpha-\beta)}\frac{c^n x^{\rho-\sigma n+\eta-\alpha-\alpha'-1}}{n!}\\ &=\frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\sum_{n=0}^{\infty}\frac{\Gamma(\zeta+\gamma n)}{\Gamma(\mu n+\nu)}\frac{\Gamma(1-\rho-\eta+\alpha+\alpha'+\sigma n)\,\Gamma(1-\rho+\sigma n-\eta+\alpha-\beta')}{\Gamma(1-\rho+\sigma n)\,\Gamma(1-\rho+\sigma n-\eta+\alpha+\alpha'+\beta')}\times\\ &\quad\times\frac{\Gamma(1-\rho+\sigma n-\beta)\,\Gamma(1+n)}{\Gamma(1-\rho+\sigma n+\alpha-\beta)}\frac{c^n x^{-\sigma n}}{n!}.\end{aligned}\tag{19}$$

This, in accordance with (5), completes the proof. □

For $\gamma = 1$ in (18), Theorem 4.2, yields to the following result:

Corollary 4.3. *Let the parameters $\alpha,\alpha',\beta,\beta',\eta,\zeta,\sigma,\rho,\mu,\nu,x \in \mathbb{C}$ and $\Re(\mu) > \Re(\nu) > 0$ be such that*

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1+\min\left[\Re(-\beta),\Re(\alpha+\alpha'-\eta),\Re(\alpha+\beta'-\eta)\right],$$

then for all $x > 0$ the following relation is valid

$$\begin{aligned}&\left(\mathcal{I}_{-}^{(\alpha,\alpha',\beta,\beta',\eta)}\left[t^{\rho-1}\,{}_{\zeta,1}E_{\mu,\nu}\left(\frac{c}{t^{\sigma}}\right)\right]\right)(x) = \frac{x^{\rho+\eta-\alpha-\alpha'-1}}{\Gamma(\zeta)}\times\\ &\times{}_5\Psi_4\left[(\zeta,1),(1-\rho-\beta,\sigma),(1-\rho-\eta+\alpha-\beta',\sigma),(1-\rho-\eta+\alpha+\alpha',\sigma),(1,1)\right.\\ &\left.\left(\nu,\mu\right),(1-\rho,\sigma),(1-\rho-\eta+\alpha+\alpha'+\beta',\sigma),(1-\rho-\beta+\alpha,\sigma)\,\middle|\,cx^{\sigma}\right].\end{aligned}\tag{20}$$

Remark 4.4. *It is easily seen that setting $\gamma \to 0$ in equation (18) with some suitable parametric replacements in the resulting identities yields the corresponding known integral formulas in Agarwal* et al. *[1].*

5 Further special cases and concluding remarks

In view of the obvious reduction formula (11), the fractional integration operators (of Marichev-Saigo-Maeda type) given in (7) and (8) reduces to the aforementioned Saigo operators $I_{0+}^{(\alpha,\beta,\eta)}$ and $I_{-}^{(\alpha,\beta,\eta)}$ defined by (see, for details, [41]; see also [24] and [51] and the references cited therein)

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt,\quad (\Re(\alpha)>0),\tag{21}$$

and

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)dt,\quad(\Re(\alpha)>0). \tag{22}$$

respectively. In the light of above definitions, we have the following relationships (see [47, p.338, Eqns. (2.9) and (2.10)]:

$$\left(I_{0+}^{(\alpha,0,\beta,\beta',\eta)}\right)(x)=\left(I_{0+}^{(\eta,\alpha-\eta,-\beta)}\right)(x),\quad(\eta\in\mathbb{C}), \tag{23}$$

$$\left(\mathcal{I}_{-}^{(\alpha,0,\beta,\beta',\eta)}\right)(x)=\left(\mathcal{I}_{-}^{(\eta,\alpha-\eta,-\beta)}\right)(x),\quad(\eta\in\mathbb{C}). \tag{24}$$

By setting $\alpha'=0$ in Theorems 3.2 and 4.2 and in Corollaries 3.3 and 4.3, if we use the relationships (23) and (24), we can deduce the following interesting corollaries involving the generalized Mittag-Leffler type function with four parameters ${}_{\zeta,\gamma}E_{\mu,\nu}[z]$ defined by (1) and the Saigo fractional integral operators defined by (21) and (22), respectively.

Corollary 5.1. *Let the parameters $\alpha,\beta,\eta,\zeta,\gamma,\rho,\mu,\nu,x\in\mathbb{C}$ and $\Re(\mu)>\Re(\nu)>0$ be such that*

$$\Re(\sigma)>0,\Re(\eta)>0,\Re(\rho)>max\left[0,\Re(\eta-\alpha-\beta)\right].$$

Then each of the following fractional integral formulas holds true for all $x>0$

$$\begin{aligned}\left(\mathcal{I}_{0+}^{(\eta,\alpha-\eta,-\beta)}\left[t^{\rho-1}\,{}_{\zeta,\gamma}E_{\mu,\nu}(ct^{\sigma})\right]\right)(x)&=\frac{x^{\rho+\eta-\alpha-1}}{\Gamma(\zeta)}\times\\ \times\,{}_4\Psi_3\Bigg[&(\zeta,\gamma),(\rho,\sigma),(\rho+\eta-\alpha-\beta,\sigma),(1,1)\\ &\Big(\nu,\mu\Big),(\rho+\eta-\alpha,\sigma),(\rho+\eta-\beta,\sigma)\Big|cx^{\sigma}\Bigg].\end{aligned} \tag{25}$$

Corollary 5.2. *Let the parameters $\alpha,\beta,\eta,\zeta,\gamma,\rho,\mu,\nu,x\in\mathbb{C}$ and $\Re(\mu)>\Re(\nu)>0$ be such that*

$$\Re(\sigma)>0,\Re(\eta)>0,\Re(\rho)>max\left[0,\Re(\eta-\alpha-\beta)\right].$$

Then each of the following fractional integral formulas holds true for all $x>0$

$$\begin{aligned}\left(\mathcal{I}_{0+}^{(\eta,\alpha-\eta,-\beta)}\left[t^{\rho-1}\,{}_{\zeta,1}E_{\mu,\nu}(ct^{\sigma})\right]\right)(x)&=\frac{x^{\rho+\eta-\alpha-1}}{\Gamma(\zeta)}\times\\ \times\,{}_4\Psi_3\Bigg[&(\zeta,1),(\rho,\sigma),(\rho+\eta-\alpha-\beta,\sigma),(1,1)\\ &\Big(\nu,\mu\Big),(\rho+\eta-\alpha,\sigma),(\rho+\eta-\beta,\sigma)\Big|cx^{\sigma}\Bigg].\end{aligned} \tag{26}$$

Corollary 5.3. *Let the parameters $\alpha,\beta,\eta,\zeta\gamma,\sigma,\rho,\mu,\nu,x\in\mathbb{C}$ and $\Re(\mu)>\Re(\nu)>0$ be such that*

$$\Re(\sigma)>0,\Re(\eta)>0,\Re(\rho)<1+\min\left[\Re(-\beta),\Re(\alpha-\eta)\right].$$

Then each of the following fractional integral formulas holds true for all $x>0$

$$\begin{aligned}\left(\mathcal{I}_{-}^{(\eta,\alpha-\eta,-\beta)}\left[t^{\rho-1}\,{}_{\zeta,\gamma}E_{\mu,\nu}\left(\frac{c}{t^{\sigma}}\right)\right]\right)(x)&=\frac{x^{\rho+\eta-\alpha-1}}{\Gamma(\zeta)}\times\\ \times\,{}_4\Psi_3\Bigg[&(\zeta,\gamma),(1-\rho-\beta,\sigma),(1-\rho-\eta+\alpha,\sigma),(1,1)\\ &\Big(\nu,\mu\Big),(1-\rho,\sigma),(1-\rho-\beta+\alpha,\sigma)\Big|cx^{\sigma}\Bigg].\end{aligned} \tag{27}$$

Corollary 5.4. *Let the parameters* $\alpha, \beta, \eta, \zeta, \sigma, \rho, \mu, \nu, x \in \mathbb{C}$ *and* $\Re(\mu) > \Re(\nu) > 0$ *be such that*

$$\Re(\sigma) > 0, \Re(\eta) > 0, \Re(\rho) < 1 + \min\left[\Re(-\beta), \Re(\alpha - \eta)\right].$$

Then each of the following fractional integral formulas holds true for all $x > 0$

$$\begin{aligned}\left(\mathcal{I}_{-}^{(\eta,\alpha-\eta,-\beta)}\left[t^{\rho-1}\,{}_{\zeta,\gamma}E_{\mu,\nu}\left(\frac{c}{t^{\sigma}}\right)\right]\right)(x) &= \frac{x^{\rho+\eta-\alpha-1}}{\Gamma(\zeta)}\times\\ &\times {}_4\Psi_3\left[(\zeta,\gamma),(1-\rho-\beta,\sigma),(1-\rho-\eta+\alpha,\sigma),(1,1)\right.\\ &\left.\left(\nu,\mu\right),(1-\rho,\sigma),(1-\rho-\beta+\alpha,\sigma)\middle| cx^{\sigma}\right].\end{aligned} \tag{28}$$

It is noted that if we set $\beta = -\alpha$ and $\beta = 0$ (21) and (22) yields the Erdélyi-Kober fractional integral operators $\mathcal{E}_{0+}^{\alpha,\eta}$ and $\mathcal{K}_{-}^{\alpha,\eta}$, the Riemann-Liouville fractional integral operator $\mathcal{R}_{0+}^{\alpha}$, and the Weyl fractional integral operator$\mathcal{W}_{-}^{\alpha}$. Therefore the results presented here are easily shown to be converted to those corresponding to the above well known fractional operators.

We conclude our present investigation by remarking further that several further consequences of Theorems 3.2 and 3.2 and Corollaries 3.3–5.4 can easily be derived by using some known and new relationships between Mittag-Leffler type function with four parameters ${}_{\zeta,\gamma}E_{\mu,\nu}[z]$, which is an elegant unification of various special functions (see [13]), and Fox H-function as given in Definition 2.2, after some suitable parametric replacements, which are more simpler fractional integration operators (of Marichev-Saigo-Maeda type), can be deduced from Theorems 3.2 and 3.2, and Corollaries 3.3–5.4 by appropriately applying the following relationships:

$${}_{\zeta,\gamma}E_{\mu,\nu}[z] = \frac{1}{\Gamma(\gamma)}H_{2,2}^{1,2}\left[z\middle|\begin{matrix}(1-\zeta,\eta),(0,1)\\(0,1),(1-\nu,\mu)\end{matrix}\right]. \tag{29}$$

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