Research Article

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Inverse scattering for three-dimensional quasi-linear biharmonic operator

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Abstract: We consider an inverse scattering problem of recovering the unknown coefficients of a quasi-linearly perturbed biharmonic operator in the three-dimensional case. These unknown complex-valued coefficients are assumed to satisfy some regularity conditions on their nonlinearity, but they can be discontinuous or singular in their space variable. We prove Saito's formula and uniqueness theorem of recovering some essential information about the unknown coefficients from the knowledge of the high frequency scattering amplitude.

Keywords: Inverse problem, scattering theory, biharmonic, quasi-linear

MSC 2010: 35R30, 47A40

1 Introduction

We consider a quasi-linear three-dimensional differential operator of order four defined by

$$H_4u(x) := \Delta^2 u(x) + \overrightarrow{W}(x, |u|) \cdot \nabla u(x) + V(x, |u|)u(x),$$

where Δ is the three-dimensional Laplacian and $\cdot$ denotes the dot-product in $\mathbb{R}^3$ for complex-valued vectors in $\mathbb{C}^3$. The bi-Laplacian is perturbed by first- and zero-order nonlinear perturbations, a vector-valued function $\overrightarrow{W}$ and a scalar function V that may be complex-valued and singular. Basic assumptions for the coefficients of H_4 are the following.

Assumption 1.1. We assume that the following conditions hold:

(i) $V(x, 1), \overrightarrow{W}(x, 1) \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$, where $\frac{3}{2} < p \leq \infty$.

(ii) There exists $R > 0$ such that for all $|x| \geq R$,

$$|V(x, 1)|, |\overrightarrow{W}(x, 1)| \leq \frac{C}{|x|^\mu},$$

with $\mu > 3$ and some constant $C > 0$.

(iii) For any $\rho > 0$, the functions $\overrightarrow{W}$ and V satisfy the following conditions:

$$|V(x, s_1) - V(x, s_2)| \leq C_\rho \beta_V(x)|s_1 - s_2|,$$
$$|\overrightarrow{W}(x, s_1) - \overrightarrow{W}(x, s_2)| \leq C'_\rho \beta_W(x)|s_1 - s_2|,$$

where $s_1, s_2 \leq \rho$ and the functions β_V and β_W satisfy conditions (i) and (ii), with some constants $C_\rho, C'_\rho > 0$.

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The motivation and the interest to study multi-dimensional quasi-linear operators of order four arise, for example, in the study of elasticity and in the theory of vibration of beams. As a concrete example, the non-linear beam equation (see [3])

$$\partial_t^2 U(x,t) + \Delta_x^2 U(x,t) + m(x)|U(x,t)|^p U(x,t) = 0$$

under time-harmonic assumptions $U(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t}$ leads to the equation

$$\Delta^2 u(x) + m(x)|u(x)|^p u(x) = \omega^2 u(x).$$

In particular, when we fix ω to be large enough and then apply a limiting process, some high-frequency scattering problems (for the potential equation) can be considered.

Some examples of scattering problems for biharmonic operators (including nonlinear equations) can be found in [8] and in the references therein. One can refer also to [13], where the fundamental result concerning the global uniqueness for an inverse boundary value problem was proved. For operators with vector potentials, we mention [12].

The present work follows in the footsteps of [2, 5, 10, 11, 16]. In [2, 5, 11], the inverse scattering problems for multi-dimensional nonlinear Schrödinger operators were considered. In [16], a similar study was carried out for a multi-dimensional biharmonic operator with linear perturbations of first- and zero-order (see also [15]). In [10], the fixed energy problem (the inverse scattering problem with fixed wave number) for nonlinear Schrödinger operators is studied. In [17], these problems were considered for biharmonic operators with first- and zero-order nonlinear perturbations on the line, while a general nonlinear Schrödinger operator on the line was investigated in [9]. The purpose of this work is to initiate similar studies in the multi-dimensional case.

The present work is concerned with the following scattering problem for the operator H_4:

$$\begin{cases} H_4 u(x,k,\theta) = k^4 u(x,k,\theta), & \\ u(x,k,\theta) = u_0(x,k,\theta) + u_{\mathrm{sc}}(x,k,\theta), & \text{where } u_0(x,k,\theta) = \mathrm{e}^{\mathrm{i}k(\theta,x)}, \\ \lim\limits_{r\to\infty} r\Big[\dfrac{\partial}{\partial r} f - \mathrm{i}kf\Big] = 0 & \text{for both } f = u_{\mathrm{sc}} \text{ and } f = \Delta u_{\mathrm{sc}}, \end{cases} \tag{1.1}$$

meaning that we are interested only in solutions that can be expressed as a sum of a plane-wave u_0 and an out-going wave u_{sc}. Here $k > 0$ corresponds to the wavenumber and it is inversely proportional to the wavelength, $\theta \in \mathbb{S}^2$ is the direction of the incident, and here (and later) $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^3$. The function u_{sc} is out-going in the sense that it satisfies the Sommerfeld radiation conditions for biharmonic operators as they were posed in [16].

We are looking for the scattering solution u_{sc} to the equation in (1.1) in the Sobolev space $W^1_\infty(\mathbb{R}^3)$. Under the Sommerfeld radiation conditions (see (1.1)), the scattering solutions to (1.1) are the unique solutions of the Lippmann–Schwinger integral equation (see [16] for details)

$$u_{\mathrm{sc}}(x) = -\int\limits_{\mathbb{R}^3} G_k^+(|x-y|)\big[\overrightarrow{W}(y,|u|)\cdot\nabla u + V(y,|u|)u\big]\,\mathrm{d}y, \tag{1.2}$$

where the function G_k^+ is the outgoing fundamental solution of the operator $\Delta^2 - k^4$ in $\mathbb{R}^3$, i.e., the kernel of the integral operator $(\Delta^2 - k^4 - \mathrm{i}0)^{-1}$. This function G_k^+ in $\mathbb{R}^3$ has the following form:

$$G_k^+(|x|) = \frac{\mathrm{e}^{\mathrm{i}k|x|} - \mathrm{e}^{-k|x|}}{8\pi k^2|x|}, \quad k > 0.$$

Once we have shown that the unique solution exists, by repeating the calculations that were done in [5, 16], we obtain that for fixed $k > 0$ the function u_{sc} has the following asymptotic behavior as $|x| \to \infty$:

$$u_{\mathrm{sc}}(x) = -\frac{\mathrm{e}^{\mathrm{i}k|x|}}{8\pi k^2|x|}A(k,\theta',\theta) + o\Big(\frac{1}{|x|}\Big),$$

where $\theta' = x/|x|$ is the angle of observation and the function A is called the scattering amplitude given via the formula

$$A(k,\theta',\theta) = \int\limits_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta',y)}\big[\overrightarrow{W}(y,|u|)\cdot\nabla u + V(y,|u|)u\big]\,\mathrm{d}y.$$

From the point of view of inverse problems, one regards this scattering amplitude as one possible scattering data. For these purposes, one requires the scattering amplitude to be known for all possible angles θ and θ' and all arbitrarily high frequencies $k > 0$.

To formulate the main result (Saito's formula), we need more conditions for the nonlinearities $\vec{W}$ and V, namely the following assumption.

Assumption 1.2. The functions $\vec{W}$ and V satisfy Assumption 1.1 with $3 < p \leq \infty$ in condition (i). We also assume that the function $\vec{W}$ has the following representation:

$$\vec{W}(x, 1 + s) = \vec{W}(x, 1) + \vec{W}^*(x)s + \vec{W}^{**}(x, s^*)O(s^2),$$

for some $|s^*| < |s|$, where the functions $\vec{W}^*$ and $\vec{W}^{**}$ satisfy Assumption 1.1 with some $6 < p \leq \infty$.

As the main result of this work, we prove Saito's formula. Similarly to other scattering problems, it allows us to obtain a uniqueness result for the inverse problem with full scattering data and a representation formula for the unknown combination β which appears in Saito's formula. What is more, it was shown in [14] and further demonstrated in [6] that Saito's formula can be inverted numerically by fixing a large value for $k > 0$ and solving the convolution-type equation for β. This recent development further underlines the importance of the formula.

Theorem 1.3 (Saito's formula). *Let the functions $\vec{W}$ and V satisfy Assumption 1.2 and, in addition, let the function $\nabla \cdot \vec{W}(x, 1)$ satisfy conditions* (i) *and* (ii) *in Assumption 1.1 with $3 < p \leq \infty$. Then*

$$\lim_{k\to\infty} k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} A(k, \theta', \theta)\, \mathrm{d}\theta'\, \mathrm{d}\theta = 8\pi^2 \int_{\mathbb{R}^3} \frac{\beta(y)}{|x - y|^2}\, \mathrm{d}y \tag{1.3}$$

holds in the sense of distributions, where $\beta(y) = -\frac{1}{2}\nabla \cdot \vec{W}(y, 1) + V(y, 1)$.

The most significant consequences of Saito's formula are contained in the following corollaries.

Corollary 1.4. *Let*

$$\beta_1(y) = -\frac{1}{2}\nabla \cdot \vec{W}_1(y, 1) + V_1(y, 1) \quad \text{and} \quad \beta_2(y) = -\frac{1}{2}\nabla \cdot \vec{W}_2(y, 1) + V_2(y, 1)$$

be as in Theorem 1.3 and let $A_1(k, \theta', \theta)$ and $A_2(k, \theta', \theta)$ be the corresponding scattering amplitudes arising from these two scattering problems. If these scattering amplitudes coincide for all angles θ, θ' and for some sequence $k_j \to \infty$ as $j \to \infty$, then $\beta_1 = \beta_2$ in the sense of tempered distributions.

Corollary 1.5. *If all conditions of Theorem 1.3 are satisfied, then*

$$\beta(y) = \frac{1}{16\pi^4} \lim_{k\to\infty} k^3 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} A(k, \theta', \theta)|\theta - \theta'| \mathrm{e}^{-\mathrm{i}k(\theta-\theta',y)}\, \mathrm{d}\theta\, \mathrm{d}\theta',$$

in the sense of tempered distributions

Proofs for these corollaries can be found, for example, in [5, 7].

The following notations are used throughout the text. The symbol $L^p_\delta(\mathbb{R}^3)$, $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$, denotes the p-based Lebesgue space over $\mathbb{R}^3$ with norm

$$\|f\|_{L^p_\delta} = \Big(\int_{\mathbb{R}^3} (1 + |x|)^{\delta p} |f(x)|^p \, \mathrm{d}x \Big)^{1/p}.$$

The weighted Sobolev spaces $W^m_{p,\delta}(\mathbb{R}^3)$ are defined as the spaces of functions whose weak derivatives up to order $m \geq 0$ belong to $L^p_\delta(\mathbb{R}^3)$ and the norm is defined by

$$\|f\|_{W^m_{p,\delta}} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p_\delta}.$$

For L^2-based spaces, we use the special notation

$$H^m_\delta(\mathbb{R}^3) = W^m_{2,\delta}(\mathbb{R}^3).$$

Throughout the text, the symbol C (compare with the constants C with some special index and special meaning) is used to denote generic positive constants whose value may change from line to line.

The paper is organized as follows. In Section 2, we study the direct scattering problem and establish its unique solvability under some suitable assumptions. Section 3 is devoted to proving the main result of the paper, i.e., Saito's formula.

2 Direct scattering problem

The goal of this section is to find sufficient conditions for nonlinearities $\vec{W}$ and V under which the direct scattering problem has a unique solution.

To this end, the following theorem holds.

Theorem 2.1. *Let functions V and $\vec{W}$ be as in Assumption 1.1. Let $\rho > 0$ satisfy the conditions*

$$\begin{cases} \sup\limits_{x\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|\beta_W(y)|}{|x-y|}\,dy < \frac{\pi}{C'_{\rho+1}}, \\ \sup\limits_{x\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|\vec{W}(y,1)|}{|x-y|}\,dy < \rho\Big(\pi - C'_{\rho+1} \sup\limits_{x\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|\beta_W(y)|}{|x-y|}\,dy\Big), \end{cases} \tag{2.1}$$

where the constant $C'_{\rho+1} > 0$ is as in Assumption 1.1. Then there exists $k_0 > 0$ such that equation (1.2) *has a unique solution in the ball*

$$B_\rho(0) = \{f \in W^1_\infty(\mathbb{R}^3) : \|f\|_{W^1_\infty} \le \rho\} \quad \text{for all } k \ge k_0.$$

Proof. We will use the Banach fixed-point theorem [18, p. 19] to prove this result. Since $B_\rho(0)$ is a closed subset of a Banach space $W^1_\infty(\mathbb{R}^3)$, this approach is well justified.

Let us start by defining an operator F by setting

$$F\varphi(x) = -\int\limits_{\mathbb{R}^3} G_k^+(|x-y|)[\vec{W}(y,|u_0+\varphi|)\cdot\nabla(u_0+\varphi) + V(y,|u_0+\varphi|)(u_0+\varphi)]\,dy,$$

where $\varphi \in W^1_\infty(\mathbb{R}^3)$. We will show that the operator F is a contraction from $B_\rho(0)$ to itself. For the fundamental solution G_k^+, the following estimates hold for all $k > 0$ and $x, y \in \mathbb{R}^3$:

$$|G_k^+(|x-y|)| \le \frac{1}{4\pi k^2|x-y|} \quad \text{and} \quad |\nabla_x G_k^+(|x-y|)| \le \frac{1}{\pi k|x-y|}. \tag{2.2}$$

For a function g satisfying conditions (i) and (ii) in Assumption 1.1, we introduce the following notation:

$$S(g) = \sup_{x\in\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|g(y)|}{|x-y|}\,dy.$$

Let now $\varphi \in B_\rho(0)$. By Assumption 1.1, we have

$$\begin{aligned} |F\varphi(x)| &\le \frac{1}{4\pi k^2} \int\limits_{\mathbb{R}^3} \frac{|\vec{W}(y,|u_0+\varphi|)|}{|x-y|}|\nabla(u_0+\varphi)|\,dy + \frac{1}{4\pi k^2} \int\limits_{\mathbb{R}^3} \frac{|V(y,|u_0+\varphi|)|}{|x-y|}|u_0+\varphi|\,dy \\ &\le \frac{1}{4\pi k^2} \int\limits_{\mathbb{R}^3} \frac{|\vec{W}(y,1)| + C'_{\rho+1}\beta_W(y)|\varphi|}{|x-y|}(k+\rho)\,dy + \frac{1}{4\pi k^2} \int\limits_{\mathbb{R}^3} \frac{|V(y,1)| + C'_{\rho+1}\beta_V(y)|\varphi|}{|x-y|}(1+\rho)\,dy \\ &\le \frac{C'}{k} + \frac{C''}{k^2}, \end{aligned} \tag{2.3}$$

where the constants C' and C'' are given by

$$\begin{aligned} C' &= \frac{1}{4\pi}(S(\vec{W}(\cdot,1)) + \rho C'_{\rho+1} S(\beta_W)), \\ C'' &= \frac{\rho}{4\pi}(S(\vec{W}(\cdot,1)) + \rho C'_{\rho+1} S(\beta_W)) + \frac{1+\rho}{4\pi}(S(V(\cdot,1)) + \rho C_{\rho+1} S(\beta_V)). \end{aligned}$$

In a similar fashion, for the gradient of $F\varphi$, we have

$$|\nabla F\varphi(x)| \le 4C' + \frac{4C''}{k}.$$

By assumption (2.1), we have $4C' < \rho$, and therefore there exists $\delta > 0$ such that $4C' + \delta < \rho$. Now,

$$\|F\varphi\|_{W^1_\infty} \le 4C' + \frac{5C'' + C'}{k} \le \rho$$

when

$$k \ge k_1 := \frac{5C'' + C'}{\delta},$$

and thus F maps from $B_\rho(0)$ to itself for all $k \ge k_1$.

Next we will show that F is a contraction. Let $\varphi, \psi \in B_\rho(0)$. We split the difference $F\varphi(x) - F\psi(x)$ into two parts

$$\begin{aligned} F\varphi(x) - F\psi(x) &= -\int_{\mathbb{R}^3} G_k^+(|x-y|)[\vec{W}(y, |u_0+\varphi|)\cdot\nabla(u_0+\varphi) - \vec{W}(y,|u_0+\psi|)\cdot\nabla(u_0+\psi)]\,dy \\ &\quad - \int_{\mathbb{R}^3} G_k^+(|x-y|)[V(y,|u_0+\varphi|)(u_0+\varphi) - V(y,|u_0+\psi|)(u_0+\psi)]\,dy \\ &= I_1 + I_2. \end{aligned}$$

Let us now consider the absolute values of both of these terms. Using estimate (2.2), we have

$$\begin{aligned} |I_1| &\le \int_{\mathbb{R}^3} |G_k^+(|x-y|)||\vec{W}(y,|u_0+\varphi|) - \vec{W}(y,|u_0+\psi|)||\nabla u_0|\,dy \\ &\quad + \int_{\mathbb{R}^3} |G_k^+(|x-y|)||\vec{W}(y,|u_0+\varphi|) - \vec{W}(y,|u_0+\psi|)||\nabla\varphi|\,dy \\ &\quad + \int_{\mathbb{R}^3} |G_k^+(|x-y|)||\vec{W}(y,|u_0+\psi|)||\nabla\varphi - \nabla\psi|\,dy \\ &\le \frac{C'_{\rho+1}}{4\pi k} S(\beta_W)\|\varphi-\psi\|_{W^1_\infty} + \frac{\rho C'_{\rho+1}}{4\pi k^2} S(\beta_W)\|\varphi-\psi\|_{W^1_\infty} \\ &\quad + \frac{1}{4\pi k^2} S(\vec{W}(\cdot,1))\|\varphi-\psi\|_{W^1_\infty} + \frac{\rho C'_{\rho+1}}{4\pi k^2} S(\beta_W)\|\varphi-\psi\|_{W^1_\infty}. \end{aligned}$$

Similar calculations show that

$$|I_2| \le \frac{1}{4\pi k^2}\|\varphi-\psi\|_{W^1_\infty}((\rho+1)C_{\rho+1}S(\beta_V) + S(V(\cdot,1)) + \rho C_{\rho+1}S(\beta_V)),$$

and thus

$$\|F\varphi - F\psi\|_\infty \le \Big(\frac{C_1}{k} + \frac{C_2}{k^2}\Big)\|\varphi-\psi\|_{W^1_\infty},$$

where

$$\begin{cases} C_1 := \dfrac{C'_{\rho+1}}{4\pi} S(\beta_W), \\ C_2 := \dfrac{1}{4\pi}[2\rho C'_{\rho+1}S(\beta_W) + S(\vec{W}(\cdot,1)) + (2\rho+1)C_{\rho+1}S(\beta_V) + S(V(\cdot,1))]. \end{cases} \tag{2.4}$$

Similarly, for the gradient of the difference, we have

$$|\nabla F\varphi(x) - \nabla F\psi(x)| \le \Big(4C_1 + \frac{4C_2}{k}\Big)\|\varphi-\psi\|_{W^1_\infty}.$$

By combining these estimates, we have that

$$\|F\varphi - F\psi\|_{W^1_\infty} \le \Big(4C_1 + \frac{5C_2 + C_1}{k}\Big)\|\varphi-\psi\|_{W^1_\infty}.$$

Since $4C_1 < 1$ by assumption (2.1), there exists $k_2 > 0$ such that $4C_1 + \frac{5C_2+C_1}{k} < 1$ for all $k \ge k_2$. Now by choosing $k_0 = \max\{k_1, k_2\}$, we have that F is a contraction from $B_\rho(0)$ for all $k \ge k_0$. This proves the theorem. □

Remark 2.2. The Banach fixed-point theorem gives us an iterative way of finding the solution. If we set $u_{\mathrm{sc}}^{(0)}(x) = 0$ and define $u_{\mathrm{sc}}^{(j)}(x) = Fu_{\mathrm{sc}}^{(j-1)}(x)$, then

$$u_{\mathrm{sc}}(x) := \lim_{j\to\infty} u_{\mathrm{sc}}^{(j)}(x)$$

is the unique solution to the Lippmann–Schwinger equation. Moreover, for all $j = 2, 3, \dots,$ we have the following a priori estimate for the error term:

$$\|u_{\mathrm{sc}}^{(j)} - u_{\mathrm{sc}}\|_{W^1_\infty} \le \Big(4C_1 + \frac{5C_2 + C_1}{k}\Big)^j \Big(1 - 4C_1 - \frac{5C_2 + C_1}{k}\Big)^{-1} \|u_{\mathrm{sc}}^{(1)}\|_{W^1_\infty},$$

where the constants C_1 and C_2 are given in (2.4) and

$$\|u_{\mathrm{sc}}^{(1)}\|_{W^1_\infty} \le \frac{1}{\pi} S(\overrightarrow{W}(\cdot\,, 1)) + \frac{1}{4\pi k}(S(\overrightarrow{W}(\cdot\,, 1)) + 5S(V(\cdot\,, 1))).$$

Lemma 2.3. *Let functions $\overrightarrow{W}$ and V be as in Theorem 2.1. Then for $k \ge k_0$, where k_0 is the same as in Theorem 2.1, the following norm estimates hold:*

$$\|u_{\mathrm{sc}}\|_\infty \le \frac{C}{k} \quad \textit{and} \quad \|\nabla u_{\mathrm{sc}}\|_\infty \le C,$$

where the constant $C > 0$ does not depend on k.

Proof. Clearly, we have that

$$\|\nabla u_{\mathrm{sc}}\|_\infty \le \|u_{\mathrm{sc}}\|_{W^1_\infty} \le \rho.$$

For the first estimate we can use (2.3), and we have that

$$\|u_{\mathrm{sc}}\|_\infty \le \frac{C_1}{k} + \frac{C_2}{k^2},$$

where the constants C_1 and C_2 are given in (2.4). Therefore, the lemma follows when we set $C = C_1 + C_2$. □

In what follows, we will also need norm estimates in weighted Sobolev spaces in the form of the following two results.

Lemma 2.4. *The integral operator with kernel G_k^+ maps from $L^2_\delta(\mathbb{R}^3)$ to $H^j_{-\delta}(\mathbb{R}^3)$ with norm-estimate*

$$\|\widehat{G_k^+}\varphi\|_{H^j_{-\delta}} \le \frac{C}{k^{3-j}}\|\varphi\|_{L^2_\delta}, \quad j = 0, 1, 2,$$

with some constant $C > 0$, where $\delta > \frac{1}{2}$.

Proof. This lemma follows from Agmon's estimate [1, Appendix A, Remark 2], which states that for $\delta > \frac{1}{2}$ and $k > 1$,

$$\sum_{|\alpha|\le 4} k^{3-|\alpha|}\|D^\alpha f\|_{L^2_{-\delta}(\mathbb{R}^n)} \le C_0\|(\Delta^2 - k^4)f\|_{L^2_\delta(\mathbb{R}^n)}$$

for all $f \in H^4(\mathbb{R}^n)$, where the constant $C_0 > 0$ only depends on n and δ. □

Corollary 2.5. *Under the assumptions of Theorem 2.1, the following norm estimates hold for the function u_{sc}:*

$$\|u_{\mathrm{sc}}\|_{H^j_{-\delta}} \le \frac{C}{k^{2-j}}, \quad j = 0, 1, 2,$$

with $\delta > \frac{1}{2}$ and some constant $C > 0$.

Proof. By Theorem 2.1, we have that u_{sc} is a solution to equation (1.2). Therefore, it suffices to show that

$$\|\overrightarrow{W}(\cdot\,, |u|) \cdot \nabla u + V(\cdot\,, |u|)u\|_{L^2_\delta} \le Ck$$

for some constant $C > 0$. For simplicity, we only consider the term $\overrightarrow{W}(y, |u|) \cdot \nabla u(y)$. Assumption 1.1 gives us that $\overrightarrow{W}(\cdot\,, |u|) \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$, where $\frac{3}{2} < p \le \infty$, and there exists $R > 0$ such that when $|y| \ge R$,

$$|\overrightarrow{W}(y, |u|)| \le \frac{C}{|y|^\mu}, \quad \text{where } \mu > 3,$$

uniformly in $|u| \le \rho$. Therefore, the Cauchy–Schwarz inequality gives that

$$
\begin{aligned}
\|\vec{W}(\cdot, |u|) \cdot \nabla u\|^2_{L^2_\delta} &\le \int_{\mathbb{R}^3} (1+|y|)^{2\delta} |\vec{W}(y, |u|)|^2 |\nabla(u_0 + u_{sc})|^2 \, dy \\
&\le \int_{|y|\le R} (1+|y|)^{2\delta} |\vec{W}(y, |u|)|^2 (k+\rho)^2 \, dy + C \int_{|y|>R} (1+|y|)^{2\delta} \frac{(k+\rho)^2}{|y|^{2\mu}} \, dy \\
&\le Ck^2,
\end{aligned}
$$

when $\delta < \mu - \frac{3}{2}$. Similarly, $\|V(\cdot, |u|)u\|_{L^2_\delta} \le C$, and therefore Lemma 2.4 gives us that

$$
\|u_{sc}\|_{H^j_{-\delta}} \le \frac{C}{k^{2-j}}, \quad j = 0, 1, 2.
$$

For a general $\delta > \frac{1}{2}$, the result follows straightforwardly from the inequality

$$
\|f\|_{H^j_{-\delta_1}} \le \|f\|_{H^j_{-\delta_2}} \quad \text{for all } \delta_1 \ge \delta_2.
$$

This finishes the proof. □

For the convenience of the reader, the asymptotic behavior of the solution is recorded in the following theorem.

Theorem 2.6. *Let u_{sc} be the solution of* (1.2) *obtained in Theorem 2.1. Then, for fixed $k > k_0$, it has the following asymptotic representation as $|x| \to \infty$:*

$$
u_{sc}(x) = -\frac{e^{ik|x|}}{8\pi|x|k^2} A(k, \theta', \theta) + o\Big(\frac{1}{|x|}\Big),
$$

where $\theta' \in \mathbb{S}^2$ is the angle of observation, i.e., $\theta' = \frac{x}{|x|}$ and the function A is called a scattering amplitude and it is given by

$$
A(k, \theta', \theta) = \int_{\mathbb{R}^3} e^{-ik(\theta', y)} [\vec{W}(y, |u|) \cdot \nabla u + V(y, |u|)u] \, dy.
$$

Proof. See [16, Theorem 5.2.] □

3 Saito's formula

The main goal of this section is to prove the main result of the paper, i.e., Theorem 1.3 (Saito's formula).

The proof of Saito's formula makes frequent use of the following technical lemma.

Lemma 3.1. *Let functions $\vec{W}$ and V satisfy Assumption 1.1 and let $\nabla \cdot \vec{W}(\cdot, 1)$ satisfy conditions* (i) *and* (ii) *in Assumption 1.1, with $3 < p \le \infty$. Then for $\delta > \frac{1}{2}$ and $k \ge k_0$, where $k_0 > 0$ is as in Theorem 2.1,*

$$
\int_{\mathbb{S}^2} e^{-ik(\theta, x)} u_{sc}(y) \, d\theta \in H^j_{-\delta}(\mathbb{R}^3)
$$

as a function of $y \in \mathbb{R}^3$, uniformly in $x \in \mathbb{R}^3$. Moreover, there exists a constant $C > 0$ such that the following estimates hold:

$$
\Big\| \int_{\mathbb{S}^2} e^{-ik(\theta, x)} u_{sc}(\cdot) \, d\theta \Big\|_{H^j_{-\delta}} \le \frac{C}{k^{3-j}}, \quad j = 0, 1, 2.
$$

Proof. Recall that the function u_{sc} can be presented as

$$
u_{sc} = \widehat{G_k^+}(\vec{W} \cdot \nabla u + Vu).
$$

By the Fubini theorem, we may change the order of integration and we have

$$
\int_{\mathbb{S}^2} e^{-ik(\theta, x)} u_{sc}(y) \, d\theta = \widehat{G_k^+}\Big(\int_{\mathbb{S}^2} e^{-ik(\theta, x)} [\vec{W}(z, |u|) \cdot \nabla u(z) + V(z, |u|)u(z)] \, d\theta \Big).
$$

Let us denote the argument of the operator $\widehat{G_k^+}$ above by h. We want to use Lemma 2.4, and therefore we need to estimate the L^2_δ-norm of the function $h(z)$. We start by splitting the function h into four parts:

$$\begin{aligned} h(z) &:= \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}[\overrightarrow{W}(z,|u|)\cdot\nabla u(z) + V(z,|u|)u(z)]\,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}\overrightarrow{W}(z,1)\cdot(\mathrm{i}k\theta)\mathrm{e}^{\mathrm{i}k(\theta,z)}\,\mathrm{d}\theta + \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}(\overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1))\cdot(\mathrm{i}k\theta)\mathrm{e}^{\mathrm{i}k(\theta,z)}\,\mathrm{d}\theta \\ &\quad + \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}\overrightarrow{W}(z,|u|)\cdot\nabla u_{\mathrm{sc}}(z)\,\mathrm{d}\theta + \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}V(z,|u|)u(z)\,\mathrm{d}\theta \\ &= h_1(z) + h_2(z) + h_3(z) + h_4(z). \end{aligned}$$

Let first $\frac{1}{2} < \delta < \mu - \frac{1}{2}$. Now we can estimate the L^2_δ-norm of each function h_i separately. First, we note that

$$h_1(z) = \overrightarrow{W}(z,1)\cdot\nabla_z \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\,\mathrm{d}\theta.$$

Here the integral over θ can be calculated precisely (see [4, Appendix D.3]) and we obtain

$$\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\,\mathrm{d}\theta = \frac{4\pi\sin(k|x-z|)}{k|x-z|}. \tag{3.1}$$

Therefore, we have that

$$\begin{aligned} |\nabla_z \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\,\mathrm{d}\theta| &= 4\pi\Big|\frac{x-z}{|x-z|^2}\cos(k|x-z|) + \frac{x-z}{k|x-z|^3}\sin(k|x-z|)\Big| \\ &\le 4\pi\Big[\frac{1}{|x-z|} + \Big|\frac{\sin(k|x-z|)}{k|x-z|^2}\Big|\Big] \\ &\le 8\pi\frac{1}{|x-z|}. \end{aligned} \tag{3.2}$$

Now by using the Cauchy–Schwarz inequality and the previous estimate, the L^2_δ-norm of h_1 can be estimated as

$$\begin{aligned} \|h_1\|^2_{L^2_\delta} &= \int_{\mathbb{R}^3} (1+|z|)^{2\delta}|h_1(z)|^2\,\mathrm{d}z \\ &= \int_{\mathbb{R}^3} (1+|z|)^{2\delta}|\overrightarrow{W}(z,1)\cdot\nabla_z \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-z)}\,\mathrm{d}\theta|^2\,\mathrm{d}z \\ &\le C\int_{\mathbb{R}^3} (1+|z|)^{2\delta}\frac{|\overrightarrow{W}(z,1)|^2}{|x-z|^2}\,\mathrm{d}z. \end{aligned} \tag{3.3}$$

This integral is finite uniformly in $x \in \mathbb{R}^3$ and $k \ge k_0$, since

$$\overrightarrow{W}(\cdot\,,1) \in W^1_{p,\mathrm{loc}}(\mathbb{R}^3) \hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R}^3),$$

and it satisfies the decay property (ii) in Assumption 1.1.

To estimate the L^2_δ-norm of h_2, we use Assumption 1.1 and Lemma 2.3 and we have

$$\begin{aligned} \|h_2\|^2_{L^2_\delta} &= \int_{\mathbb{R}^3} (1+|z|)^{2\delta}|\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-z)}(\overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1))\cdot(\mathrm{i}k\theta)\,\mathrm{d}\theta|^2\,\mathrm{d}z \\ &\le C\int_{\mathbb{R}^3} (1+|z|)^{2\delta}\Big(k\int_{\mathbb{S}^2} |\overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1)|\,\mathrm{d}\theta\Big)^2\,\mathrm{d}z \\ &\le C\int_{\mathbb{R}^3} (1+|z|)^{2\delta}\Big(k\int_{\mathbb{S}^2} \|u_{\mathrm{sc}}\|_\infty\,\mathrm{d}\theta|\beta_W(z)|\Big)^2\,\mathrm{d}z \\ &\le C\|\beta_W\|^2_{L^2_\delta}, \end{aligned}$$

where the constant C is independent of x and k.

For the functions h_3 and h_4, we substitute

$$\overrightarrow{W}(z,|u|) = \overrightarrow{W}(z,1) + \overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1) \quad \text{and} \quad V(z,|u|) = V(z,1) + V(z,|u|) - V(z,1),$$

respectively, and we have

$$\begin{aligned}\|h_3\|^2_{L^2_\delta} &\leq \int_{\mathbb{R}^3} (1+|z|)^{2\delta} \Big| \int_{\mathbb{S}^2} e^{-ik(\theta, x-z)} \overrightarrow{W}(z,1) \cdot \nabla u_{\mathrm{sc}}(z) \, d\theta \Big|^2 \, dz \\ &\quad + \int_{\mathbb{R}^3} (1+|z|)^{2\delta} \Big| \int_{\mathbb{S}^2} e^{-ik(\theta, x-z)} (\overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1)) \cdot \nabla u_{\mathrm{sc}}(z) \, d\theta \Big|^2 \, dz \\ &\leq C \int_{\mathbb{R}^3} (1+|z|)^{2\delta} |\overrightarrow{W}(z,1)|^2 \int_{\mathbb{S}^2} \|\nabla u_{\mathrm{sc}}\|^2_\infty \, d\theta \, dz \\ &\quad + C \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} (1+|z|)^{2\delta} |\overrightarrow{W}(z,|u|) - \overrightarrow{W}(z,1)|^2 \|\nabla u_{\mathrm{sc}}\|^2_\infty \, dz \, d\theta < C.\end{aligned}$$

Similarly,

$$\begin{aligned}\|h_4\|_{L^2_\delta} &\leq \int_{\mathbb{R}^3} (1+|z|)^{2\delta} \Big| \int_{\mathbb{S}^2} e^{-ik(\theta, x-z)} V(z,1) \cdot u(z) \, d\theta \Big|^2 \, dz \\ &\quad + \int_{\mathbb{R}^3} (1+|z|)^{2\delta} \Big| \int_{\mathbb{S}^2} e^{-ik(\theta, x-z)} (V(z,|u|) - V(z,1)) \cdot u(z) \, d\theta \Big|^2 \, dz \\ &\leq C \int_{\mathbb{R}^3} (1+|z|)^{2\delta} |V(z,1)|^2 \int_{\mathbb{S}^2} \|u\|^2_\infty \, d\theta \, dz \\ &\quad + C \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} (1+|z|)^{2\delta} |V(z,|u|) - V(z,1)|^2 \|u\|^2_\infty \, dz \, d\theta < C. \end{aligned} \tag{3.4}$$

All of these estimates hold uniformly in $x \in \mathbb{R}^3$ and $k \geq k_0$. To combine, estimates (3.3) and (3.4) give us that $h \in L^2_\delta(\mathbb{R}^3)$ with a norm estimate

$$\|h\|_{L^2_\delta} \leq C,$$

where the constant $C > 0$ does not depend on $x \in \mathbb{R}^3$ and $k \geq k_0$. Therefore, Lemma 2.4 gives us the claim for all $\frac{1}{2} < \delta < \mu - \frac{1}{2}$. For a general $\delta > \frac{1}{2}$, the result follows straightforwardly from the inequality

$$\|f\|_{H^j_{-\delta_1}} \leq \|f\|_{H^j_{-\delta_2}} \quad \text{for all } \delta_1 \geq \delta_2. \qquad \square$$

Proof of Theorem 1.3. We split the left-hand side of (1.3) into four parts as follows:

$$\begin{aligned} k^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} e^{-ik(\theta-\theta', x)} A(k,\theta',\theta) \, d\theta' \, d\theta &= k^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} e^{-ik(\theta-\theta', x)} \int_{\mathbb{R}^3} e^{-ik(\theta', y)} \overrightarrow{W}(y,|u|) \cdot \nabla u_{\mathrm{sc}}(y) \, dy \, d\theta' \, d\theta \\ &\quad + k^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} e^{-ik(\theta-\theta', x)} \int_{\mathbb{R}^3} e^{-ik(\theta'-\theta, y)} \overrightarrow{W}(y,|u|) \cdot (ik\theta) \, dy \, d\theta' \, d\theta \\ &\quad + k^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} e^{-ik(\theta-\theta', x)} \int_{\mathbb{R}^3} e^{-ik(\theta', y)} V(y,|u|) u_{\mathrm{sc}}(y) \, dy \, d\theta' \, d\theta \\ &\quad + k^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} e^{-ik(\theta-\theta', x)} \int_{\mathbb{R}^3} e^{-ik(\theta'-\theta, y)} V(y,|u|) \, dy \, d\theta' \, d\theta \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us first consider the term I_2. Assumption 1.2 allows us to write the function $\overrightarrow{W}(x,|u|)$ as

$$\overrightarrow{W}(x,|u|) = \overrightarrow{W}(x,1) + \frac{1}{2} \overrightarrow{W}^*(x) (\overline{u_0} u_{\mathrm{sc}} + u_0 \overline{u_{\mathrm{sc}}}) + \widetilde{W}(x,s^*) O(|u_{\mathrm{sc}}|^2). \tag{3.5}$$

Indeed, using the Taylor expansion for small values of z, i.e., $(1+z)^r = 1 + rz + O(z^2)$, for $|u|$ we have (see Lemma 2.3)

$$\begin{aligned}|u_0 + u_{sc}| &= (1 + u_0\overline{u_{sc}} + \overline{u_0}u_{sc} + |u_{sc}|^2)^{1/2}\\ &= 1 + \frac{1}{2}(u_0\overline{u_{sc}} + \overline{u_0}u_{sc} + |u_{sc}|^2) + O((u_0\overline{u_{sc}} + \overline{u_0}u_{sc} + |u_{sc}|^2)^2)\\ &= 1 + \frac{1}{2}(u_0\overline{u_{sc}} + \overline{u_0}u_{sc}) + O(|u_{sc}|^2).\end{aligned}$$

Therefore, (3.5) follows when we set

$$\widetilde{W}(x, s^*) = \overrightarrow{W}^*(x) + \overrightarrow{W}^{**}(x, s^*).$$

By substituting (3.5) into I_2, we have

$$\begin{aligned}I_2 &= k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}(\mathrm{i}k\theta)\cdot \overrightarrow{W}(y,1)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &\quad + k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}(\mathrm{i}k\theta)\cdot \overrightarrow{W}^*(y)(u_0\overline{u_{sc}} + \overline{u_0}u_{sc})\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &\quad + k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta',y)}(\mathrm{i}k\theta)\cdot \widetilde{W}(y)O(|u_{sc}|^2)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &= I_2^{(1)} + I_2^{(2)} + I_2^{(3)}.\end{aligned}$$

Now,

$$\begin{aligned}I_2^{(1)} &= k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\overrightarrow{W}(y,1)\cdot \mathrm{i}k(\theta-\theta')\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &\qquad + k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\overrightarrow{W}(y,1)\cdot(\mathrm{i}k\theta')\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &= k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \nabla_y \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\cdot \overrightarrow{W}(y,1)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta - I_2^{(1)}.\end{aligned} \tag{3.6}$$

Here we have used integration by parts and Assumption 1.1 for the function $\overrightarrow{W}$. The fact that $-I_2^{(1)}$ appears in (3.6) follows from the substitutions $y = -\theta'$ and $y' = -\theta$.

Rearranging equation (3.6) leads us to

$$\begin{aligned}2I_2^{(1)} &= -k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)}\nabla\cdot \overrightarrow{W}(y,1)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta\\ &= -k^2 \int_{\mathbb{R}^3} \nabla\cdot\overrightarrow{W}(y,1)\Big(\frac{4\pi\sin(k|x-y|)}{k|x-y|}\Big)^2\,\mathrm{d}y\\ &= -8\pi^2 \int_{\mathbb{R}^3} \frac{\nabla\cdot\overrightarrow{W}(y,1)}{|x-y|^2}\,\mathrm{d}y + 8\pi^2 \int_{\mathbb{R}^3} \frac{\nabla\cdot\overrightarrow{W}(y,1)}{|x-y|^2}\cos(2k|x-y|)\,\mathrm{d}y.\end{aligned}$$

By our assumptions,

$$\frac{\nabla\cdot\overrightarrow{W}(\cdot,1)}{|x-\cdot|^2} \in L^1(\mathbb{R}^3)$$

uniformly in $x \in \mathbb{R}^3$, and therefore the second term above tends to zero as $k \to \infty$ due to the Riemann–Lebesgue lemma. So, we have that

$$I_2^{(1)} = 8\pi^2 \int_{\mathbb{R}^3} \frac{-\frac{1}{2}\nabla\cdot\overrightarrow{W}(y,1)}{|x-y|^2}\,\mathrm{d}y + o(1)$$

for large values of $k > 0$.

Next we consider the term $I_2^{(2)}$. Using the Leibniz rule for differentiation together with (3.1), we have

$$
\begin{aligned}
I_2^{(2)} &= -k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{\mathrm{i}k(\theta',x)} \nabla_x \mathrm{e}^{-\mathrm{i}k(\theta,x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} \overrightarrow{W}^*(y)(\overline{u_0}u_{\mathrm{sc}} + u_0\overline{u_{\mathrm{sc}}})\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta \\
&= -k^2\nabla_x \int_{\mathbb{R}^3} \frac{4\pi\sin(k|x-y|)}{k|x-y|}\overrightarrow{W}^*(y) \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)}(\overline{u_0}u_{\mathrm{sc}} + u_0\overline{u_{\mathrm{sc}}})\,\mathrm{d}\theta\,\mathrm{d}y \\
&\quad + 4\pi k^2 \int_{\mathbb{R}^3} \nabla_x\Big(\frac{\sin(k|x-y|)}{k|x-y|}\Big)\overrightarrow{W}^*(y) \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)}(\overline{u_0}u_{\mathrm{sc}} + u_0\overline{u_{\mathrm{sc}}})\,\mathrm{d}\theta\,\mathrm{d}y \\
&= -\nabla_x L_1(x,k) + L_2(x,k).
\end{aligned}
$$

Let us consider now the term $L_1(x,k)$. Using first the Hölder inequality and then Lemma 3.1, we have the following estimate:

$$
\begin{aligned}
|L_1(x,k)| &\le 4\pi k^2 \int_{\mathbb{R}^3} |\overrightarrow{W}^*(y)|\Big|\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)}(\overline{u_0}u_{\mathrm{sc}} + u_0\overline{u_{\mathrm{sc}}})\,\mathrm{d}\theta\Big|\,\mathrm{d}y \\
&\le Ck^2\Big(\int_{\mathbb{R}^3}(1+|y|)^{2\delta}|\overrightarrow{W}^*(y)|^2\,\mathrm{d}y\Big)^{1/2}\Big[\Big\|\int_{\mathbb{S}^2}\mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(\,\cdot\,)\,\mathrm{d}\theta\Big\|_{L^2_{-\delta}} + \Big\|\int_{\mathbb{S}^2}\mathrm{e}^{-\mathrm{i}k(\theta,2\cdot-x)}u_{\mathrm{sc}}(\,\cdot\,)\,\mathrm{d}\theta\Big\|_{L^2_{-\delta}}\Big] \\
&\le \frac{C}{k}
\end{aligned}
$$

uniformly in $x \in \mathbb{R}^3$. Therefore, we have for any $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$
|\langle -\nabla L_1(\,\cdot\,,k),\varphi\rangle| = |\langle L_1(\,\cdot\,,k),\nabla\varphi\rangle| = \Big|\int_{\mathbb{R}^3} L_1(x,k)\overline{\nabla\varphi(x)}\,\mathrm{d}x\Big| \le \sup_{x\in\operatorname{supp}\varphi}|L_1(x,k)|\|\nabla\varphi\|_{L^1} \le \frac{C}{k}.
$$

Before we consider the term $L_2(x,k)$, we note that, if the function f satisfies Assumption 1.1 with $6 < p \le \infty$, then for $\delta < \mu - \frac{1}{2}$ the integral

$$
\int_{\mathbb{R}^3}(1+|y|)^{2\delta}\frac{|f(y)|^2}{|x-y|^2}\,\mathrm{d}y \tag{3.7}
$$

is bounded uniformly in $x \in \mathbb{R}^3$.

Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(1+|y|)^{2\delta}\frac{|f(y)|^2}{|x-y|^2}\,\mathrm{d}y \\
&\quad = \int_{|x-y|\le 1}(1+|y|)^{2\delta}\frac{|f(y)|^2}{|x-y|^2}\,\mathrm{d}y + \int_{|x-y|>1}(1+|y|)^{2\delta}\frac{|f(y)|^2}{|x-y|^2}\,\mathrm{d}y \\
&\quad \le C\|f\|_{L^p}^2\Big(\int_{|x-y|\le 1}|x-y|^{\frac{-2p}{p-2}}\,\mathrm{d}y\Big)^{\frac{p-2}{p}} + C\int_{|y|>R,|x-y|>1}|y|^{2\delta-2\mu}|x-y|^{-2}\,\mathrm{d}y + C\int_{|y|\le R}|f(y)|^2\,\mathrm{d}y.
\end{aligned}
$$

These integrals are all uniformly bounded when $6 < p \le \infty$ and $\delta < \mu - \frac{1}{2}$. Now we may estimate $L_2(x,k)$, using (3.2), Lemma 3.1 and (3.7), to obtain

$$
\begin{aligned}
|L_2(x,k)| &\le 8\pi k^2 \int_{\mathbb{R}^3}\frac{|\overrightarrow{W}^*(y)|}{|x-y|}\Big|\int_{\mathbb{S}^2}\mathrm{e}^{-\mathrm{i}k(\theta,x-y)}(\overline{u_0}u_{\mathrm{sc}} + u_0\overline{u_{\mathrm{sc}}})\,\mathrm{d}\theta\Big|\,\mathrm{d}y \\
&\le Ck^2\Big(\int_{\mathbb{R}^3}(1+|y|)^{2\delta}\frac{|\overrightarrow{W}^*(y)|^2}{|x-y|^2}\,\mathrm{d}y\Big)^{1/2}\Big[\Big\|\int_{\mathbb{S}^2}\mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(\,\cdot\,)\,\mathrm{d}\theta\Big\|_{L^2_{-\delta}} + \Big\|\int_{\mathbb{S}^2}\mathrm{e}^{-\mathrm{i}k(\theta,2\cdot-x)}u_{\mathrm{sc}}(\,\cdot\,)\,\mathrm{d}\theta\Big\|_{L^2_{-\delta}}\Big] \\
&\le \frac{C}{k}\Big(\int_{\mathbb{R}^3}(1+|y|)^{2\delta}\frac{|\overrightarrow{W}^*(y)|^2}{|x-y|^2}\,\mathrm{d}y\Big)^{1/2} \\
&\le \frac{C}{k}
\end{aligned}
$$

uniformly in $x \in \mathbb{R}^3$.

Combining the estimates for $-\nabla_x L_1(x,k)$ and $L_2(x,k)$, we have that

$$-\nabla_x L_1(x,k) + L_2(x,k) = o(1)$$

for $k \gg 1$, pointwise in $x \in \mathbb{R}^3$.

For $I_2^{(3)}$, we may use straightforward estimation, (3.7) and the fact that $\|u_{\mathrm{sc}}\|_\infty \le \frac{C}{k}$ to obtain

$$\begin{aligned}
|I_2^{(3)}| &\le Ck^2 \int_{\mathbb{R}^3} \frac{1}{|x-y|} \int_{\mathbb{S}^2} |\widetilde{W}(y,s^*)||u_{\mathrm{sc}}(y)|^2 \,\mathrm{d}\theta\,\mathrm{d}y \\
&\le Ck^2 \int_{\mathbb{S}^2} \|u_{\mathrm{sc}}\|_\infty \Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta} \frac{|\widetilde{W}(y,s^*)|^2}{|x-y|^2} \,\mathrm{d}y \Big)^{1/2} \Big(\int_{\mathbb{R}^3} (1+|y|)^{-2\delta} |u_{\mathrm{sc}}(y)|^2 \,\mathrm{d}y \Big)^{1/2} \mathrm{d}\theta \\
&\le \frac{C}{k} \int_{\mathbb{S}^2} \Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta} \frac{|\widetilde{W}(y,s^*)|^2}{|x-y|^2} \,\mathrm{d}y \Big)^{1/2} \mathrm{d}\theta \\
&\le \frac{C}{k}.
\end{aligned}$$

Let us next consider the term I_4. First, we split the integral into two parts:

$$\begin{aligned}
k^2 & \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta'-\theta,y)} V(y,|u|)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta \\
&= k^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)}\,\mathrm{d}\theta \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta',y-x)}\,\mathrm{d}\theta'\, V(y,1)\,\mathrm{d}y \\
&\quad + k^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta',y-x)}\,\mathrm{d}\theta' \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)} (V(y,|u|) - V(y,1))\,\mathrm{d}\theta\,\mathrm{d}y \\
&= J_1 + J_2.
\end{aligned}$$

In J_1, integrals with respect to both θ and θ' can be calculated using (3.1), and we obtain

$$\begin{aligned}
J_1 &= 16\pi^2 \int_{\mathbb{R}^3} \frac{\sin^2(k|x-y|)}{|x-y|^2} V(y,1)\,\mathrm{d}y \\
&= 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2}\,\mathrm{d}y + 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} \cos(2k|x-y|)\,\mathrm{d}y \\
&= 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2}\,\mathrm{d}y + o(1)
\end{aligned}$$

for large values of $k > 0$ due to the Riemann–Lebesgue lemma.

In order to estimate the term J_2, we use Corollary 2.5 and Lemma 2.3 to get

$$\begin{aligned}
|J_2| &= k^2 \Big| \int_{\mathbb{R}^3} \frac{4\pi \sin(k|x-y|)}{k|x-y|} \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x-y)} (V(y,|u|) - V(y,1))\,\mathrm{d}\theta\,\mathrm{d}y \Big| \\
&\le Ck^2 \int_{\mathbb{S}^2} \int_{|x-y|<\frac{1}{k}} |\beta_V(y)||u_{\mathrm{sc}}(y)|\,\mathrm{d}y\,\mathrm{d}\theta + Ck \int_{\mathbb{S}^2} \int_{|x-y|>\frac{1}{k}} \frac{|\beta_V(y)|}{|x-y|} |u_{\mathrm{sc}}(y)|\,\mathrm{d}y\,\mathrm{d}\theta \\
&\le Ck \int_{|x-y|<\frac{1}{k}} |\beta_V(y)|\,\mathrm{d}y + Ck^{1+\gamma} \int_{\mathbb{S}^2} \|u_{\mathrm{sc}}\|_{L^2_{-\delta}}\,\mathrm{d}\theta \Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta} \frac{|\beta_V(y)|^2}{|x-y|^{2(1-\gamma)}}\,\mathrm{d}y \Big)^{1/2} \\
&\le \frac{C}{k^{\frac{3}{p'}-1}} \|\beta_V\|_{L^p} + \frac{C}{k^{1-\gamma}} \Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta} \frac{|\beta_V(y)|^2}{|x-y|^{2(1-\gamma)}}\,\mathrm{d}y \Big)^{1/2},
\end{aligned}$$

where $p > 3$, $p' < \frac{3}{2}$ and $0 < \gamma < 1$ is chosen so close to 1 that the latter integral is finite under the conditions for β_V.

To combine what we have done thus far, we have shown that

$$I_2 + I_4 = 8\pi^2 \int_{\mathbb{R}^3} \frac{-\frac{1}{2}\nabla \cdot \overrightarrow{W}(y,1) + V(y,1)}{|x-y|^2}\,\mathrm{d}y + o(1),$$

in the sense of distributions, for large values of $k > 0$. Therefore, all that is left to show is that both I_1 and I_3 are at most $o(1)$ for large $k > 0$.

Let us first prove that $I_1 = O(\frac{1}{k})$. We once again start by substituting

$$\overrightarrow{W}(y,|u|) = \overrightarrow{W}(y,1) + \overrightarrow{W}(y,|u|) - \overrightarrow{W}(y,1)$$

and splitting the integral into two parts:

$$\begin{aligned} I_1 &= k^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta',y-x)}\,\mathrm{d}\theta' \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}\overrightarrow{W}(y,1)\cdot\nabla u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\,\mathrm{d}y \\ &\quad + k^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta',y-x)}\,\mathrm{d}\theta' \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}(\overrightarrow{W}(y,|u|) - \overrightarrow{W}(y,1))\cdot\nabla u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\,\mathrm{d}y \\ &= I' + I''. \end{aligned}$$

By the Cauchy–Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned} |I'| &= 4\pi k\Big|\int_{\mathbb{R}^3} \frac{4\pi\sin(k|x-y|)}{|x-y|}\overrightarrow{W}(y,1)\cdot\nabla_y \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\,\mathrm{d}y\Big| \\ &\le 4\pi k \int_{\mathbb{R}^3} \frac{|\overrightarrow{W}(y,1)|}{|x-y|}\Big|\nabla_y \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\Big|\,\mathrm{d}y \\ &\le 4\pi k\Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta}\frac{|\overrightarrow{W}(y,1)|^2}{|x-y|^2}\,\mathrm{d}y\Big)^{1/2}\Big(\int_{\mathbb{R}^3} (1+|y|)^{-2\delta}\Big|\nabla_y \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\Big|^2\,\mathrm{d}y\Big)^{1/2} \\ &\le Ck\Big\|\frac{\overrightarrow{W}(\cdot,1)}{|x-\cdot|}\Big\|_{L^2_\delta}\Big\|\int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)}u_{\mathrm{sc}}(\cdot)\,\mathrm{d}\theta\Big\|_{H^1_{-\delta}} \\ &\le \frac{C}{k}. \end{aligned}$$

Next, we consider the absolute value of the term I''. Under Assumption 1.2, we have that

$$|\overrightarrow{W}(y,|u|) - \overrightarrow{W}(y,1)| \le C|u_{\mathrm{sc}}(y)|\widetilde{\beta}_W(y,s^*),$$

where $\widetilde{\beta}(\cdot,s^*) \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$, with $6 < p \le \infty$. Now, using $\|\nabla u_{\mathrm{sc}}\|_\infty \le C$, we obtain

$$\begin{aligned} |I''| &= 4\pi k\Big|\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} \frac{\sin(k|x-y|)}{|x-y|}\mathrm{e}^{-\mathrm{i}k(\theta,x)}(\overrightarrow{W}(y,|u|) - \overrightarrow{W}(y,1))\cdot\nabla u_{\mathrm{sc}}(y)\,\mathrm{d}y\,\mathrm{d}\theta\Big| \\ &\le Ck\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|}|u_{\mathrm{sc}}(y)||\widetilde{\beta}_W(y,s^*)||\nabla u_{\mathrm{sc}}(y)|\,\mathrm{d}y\,\mathrm{d}\theta \\ &\le Ck\int_{\mathbb{S}^2} \|u_{\mathrm{sc}}\|_{L^2_{-\delta}}\Big(\int_{\mathbb{R}^3} (1+|y|)^{2\delta}\frac{|\widetilde{\beta}_W(y,s^*)|^2}{|x-y|^2}\,\mathrm{d}y\Big)^{1/2}\,\mathrm{d}\theta \\ &\le \frac{C}{k} \end{aligned}$$

uniformly in $x \in \mathbb{R}^3$ due to (3.7).

Finally, for I_3 we once again substitute

$$V(y,|u|) = V(y,1) + V(y,|u|) - V(y,1),$$

and we have

$$
\begin{aligned}
I_3 &= k^2 \int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int\limits_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta',y)} V(y,1) u_{\mathrm{sc}}(y)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta \\
&\quad + k^2 \int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} \int\limits_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}k(\theta',y)} (V(y,|u|) - V(y,1)) u_{\mathrm{sc}}(y)\,\mathrm{d}y\,\mathrm{d}\theta'\,\mathrm{d}\theta \\
&= K_1 + K_2.
\end{aligned}
$$

Now, by the Hölder inequality and Lemma 3.1, we have

$$
\begin{aligned}
|K_1| &\le k^2 \Big| \int\limits_{\mathbb{R}^3} \frac{\sin(k|x-y|)}{k|x-y|} V(y,1) \int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)} u_{\mathrm{sc}}(y)\,\mathrm{d}\theta\,\mathrm{d}y \Big| \\
&\le k^2 \int\limits_{\mathbb{R}^3} |V(y,1)| \Big| \int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)} u_{\mathrm{sc}}(y) \Big|\,\mathrm{d}y \\
&\le k^2 \Big(\int\limits_{\mathbb{R}^3} (1+|y|)^{2\delta} |V(y,1)|^2\,\mathrm{d}y \Big)^{1/2} \Big\| \int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta,x)} u_{\mathrm{sc}}(y)\,\mathrm{d}\theta \Big\|_{L^2_{-\delta}} \\
&\le \frac{C}{k}
\end{aligned}
$$

and

$$
|K_2| \le Ck \int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} |\beta_V(y)| |u_{\mathrm{sc}}(y)|^2\,\mathrm{d}y\,\mathrm{d}\theta \le Ck \int\limits_{\mathbb{S}^2} \|u_{\mathrm{sc}}\|_\infty^2\,\mathrm{d}\theta \le \frac{C}{k}
$$

uniformly in $x \in \mathbb{R}^3$. This finishes the proof of Theorem 1.3. □

Remark 3.2. If we assume that $\overrightarrow{W} = 0$, i.e., we have only zero-order nonlinear perturbations V in the operator H_4, then, under the same conditions for V as in Theorem 1.3, the proof of this theorem shows that the limit in Saito's formula holds uniformly in $x \in \mathbb{R}^3$ and it has the form

$$
\lim_{k\to\infty} k^2 \int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}k(\theta-\theta',x)} A(k,\theta',\theta)\,\mathrm{d}\theta'\,\mathrm{d}\theta = 8\pi^2 \int\limits_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2}\,\mathrm{d}y.
$$

4 Conclusions

The direct and inverse scattering problems for the first- and zero-order quasi-linear perturbations of the three-dimensional biharmonic operators in the frequency domain with singular coefficients (in the space coordinate) were considered. It is assumed that the nonlinearities depend on the modulus of the wave and that they may be complex-valued and singular. The linear case and many well-known (in physics) types of nonlinearities are included in the considerations. Under some additional regularity conditions for the nonlinearities, the classical Saito's formula and very important consequences for the inverse scattering problems are justified for this nonlinear operator of order four.

One could consider many types of data for solving the inverse scattering problem. For our purposes, we consider the scattering amplitude $A(k,\theta',\theta)$ for all angles θ,θ' and for arbitrary large frequencies k. Under this limited (with respect to frequency) data, we proved a formula (Saito's formula) which allows us to reconstruct analytically and numerically the combination of the nonlinearities.

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