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Torsion-free groups with indecomposable holonomy group. I

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Dedicated to Professor L.G. Kovács on his 65th birthday

Abstract. We study the torsion-free generalized crystallographic groups with indecomposable holonomy group which is isomorphic to either C_{p^s} or $C_p \times C_p$.

1 Introduction

A classical *crystallographic group* is a discrete cocompact subgroup of $I(\mathbb{R}^m)$, the isometry group of $\mathbb{R}^m$. Torsion-free crystallographic groups are called *Bieberbach groups*. The present state of the theory of crystallographic groups and a historical overview, as well as its connections to other branches of mathematics, are described in [17, 18].

In this paper we consider generalized torsion-free crystallographic groups with indecomposable holonomy groups isomorphic to either C_{p^s} or $C_p \times C_p$.

It was shown in [7, 8, 14] that the description of the n-dimensional crystallographic groups for arbitrary n is of wild type, in the sense that it is related to the classical unsolvable problem of describing the canonical forms of pairs of linear operators on finite-dimensional vector spaces.

Using Diederichsen's classification of integral representations of cyclic groups of prime order (see [6]), Charlap [5] gave a full classification of Bieberbach groups with cyclic holonomy group G of prime order. Hiss and Szczepánski [13] proved that there are no Bieberbach groups with non-trivial irreducible holonomy group. Kopcha and Rudko [14] studied torsion-free crystallographic groups with indecomposable cyclic holonomy group of order p^n, the classification of which for $n \geqslant 5$ also has wild type.

Cobb [5] constructed an infinite family of compact flat manifolds with first Betti number zero and holonomy group isomorphic to $C_2 \times C_2$. In [19, 20, 21] Rossetti

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and Tirao described the torsion-free crystallographic groups whose holonomy groups are direct sums of indecomposable subgroups of $\mathrm{GL}(n, \mathbb{Z})$ ($n \leqslant 5$) and isomorphic to $C_2 \times C_2$.

Further interesting results on this topic were obtained in the research of Gupta and Sidki [9, 10].

We need the following definitions and notation for the statement of our results.

Let K be a principal domain, let F be a field containing K and let G be a finite group. Let M be a KG-module of a faithful matrix K-representation Γ of G and let FM be a vector space over F in which M is a full lattice. Let $\hat{M} = FM^+/M^+$ be the quotient group of the additive group FM^+ of FM by the additive group M^+ of M. Then FM is an FG-module and $\hat{M}$ is a KG-module with operations defined by

$$g.(\alpha m) = \alpha g(m), \quad g.(x + M) = g(x) + M,$$

for $g \in G$, $\alpha \in F$, $m \in M$, $x \in FM$.

Let $T : G \to \hat{M}$ be a 1-cocycle of G with values in $\hat{M}$; thus each $T(g)$ is a coset of the form $x + M$. We define the group

$$\mathfrak{Crys}(G; M; T) = \{(g, x) \mid g \in G, x \in T(g)\}$$

with the operation

$$(g, x).(g', x') = (gg', g'x + x'),$$

for $g, g' \in G$, $x \in T(g)$, $x' \in T(g')$.

The purpose of this paper is to study the group $\mathfrak{Crys}(G; M; T)$, and in particular to determine when it is torsion-free. We note that if $K = \mathbb{Z}$ and $F = \mathbb{R}$ then $\mathfrak{Crys}(G; M; T)$ is isomorphic to an n-dimensional classical crystallographic group, where $n = \mathrm{rank}_{\mathbb{Z}} M$.

We use the terminology of the theory of group representations. The group $\mathfrak{Crys}(G; M; T)$ is called *irreducible* (resp. *indecomposable*) if M is an irreducible (resp. indecomposable) KG-module and the cocycle T is not cohomologous to zero.

A cocycle $T : G \to \hat{M}$ is called a *coboundary* if there exists an $x \in FM$ such that $T(g) = (g - 1)x + M$ for every $g \in G$. Cocycles $T_1 : G \to \hat{M}$ and $T_2 : G \to \hat{M}$ are called *cohomologous* if $T_1 - T_2$ is a coboundary.

Let $C^1(G, \hat{M})$, $B^1(G, \hat{M})$ and $H^1(G, \hat{M}) = C^1(G, \hat{M})/B^1(G, \hat{M})$ be respectively the group of cocycles, the group of coboundaries and the cohomology group of G with values in $\hat{M}$. The group $\mathfrak{Crys}(G; M; T)$ is an extension of M^+ by G; the extension splits if and only if $T \in B^1(G, \hat{M})$. Therefore $\mathfrak{Crys}(G; M; T)$ splits for all T if and only if $H^1(G, \hat{M})$ is trivial.

Throughout the paper, we write $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ respectively for the ring of rational integers, the localization of $\mathbb{Z}$ at the prime p and the ring of p-adic integers.

2 Main results

Using results from [2, 3, 11, 12, 15], we prove the following three theorems. Lemma 12 is also of independent interest.

Theorem 1. *Let K be one of the rings $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$ and let $G \cong C_{p^s}$ be a cyclic group of order p^s. If $s \geqslant 3$, then the set of K-dimensions of the indecomposable KC_{p^s}-modules M for which there exist torsion-free groups $\mathfrak{Crys}(C_{p^s}; M; T)$, is unbounded.*

Theorem 2. *Let K be $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ and let $G = \langle a \rangle \cong C_{p^2}$. Up to isomorphism, all torsion-free indecomposable groups $\mathfrak{Crys}(C_{p^2}; M; T)$ are described in terms of the following indecomposable KC_{p^2}-modules M and cocycles T of C_{p^2} with values in the groups $\hat{M} = FM^+/M^+$:*

(1) $M = X_i = \langle (a-1)\Phi(a^p), \Phi(a) + (a-1)^{i+1} \rangle$ *and* $T = T_i$, *where*

$$\Phi(x) = x^{p-1} + \cdots + x + 1, \quad T_i(a) = p^{-2}\Phi(a)\Phi(a^p) + X_i,$$

for $i = 0, 1, \ldots, p-2$;

(2) $M = U_j = \langle ((a-1)^{j+1} + \Phi(a), (a-1)^j), \Phi(a^p)(a-1, 1) \rangle$, *a* KC_{p^2}-*submodule of* $(KC_{p^2})^{(2)} = \{(x_1, x_2) \mid x_1, x_2 \in KC_{p^2}\}$, *and* $T = f_j$, *where*

$$f_j(a) = p^{-2}\Phi(a)\Phi(a^p)(1, 0) + U_j,$$

for $p > 2$ *and* $j = 1, \ldots, p-2$.

The number of these groups $\mathfrak{Crys}(C_{p^2}; M; T)$ is equal to $2p - 3$.

Corollary 1. *There exist at least $2p - 3$ Bieberbach groups (in the classical sense) with cyclic indecomposable holonomy group of order p^2.*

Theorem 3. *Let $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$ and let K be one of the rings $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_{(2)}$. Let F be a field containing K, let M be a KG-module corresponding to the indecomposable K-representation Γ of G, and let $f : G \to \hat{M} = FM^+/M^+$ be a cocycle. The following table lists the choices of Γ and f which define, up to isomorphism, all torsion-free indecomposable groups $\mathfrak{Crys}(G; M; f)$.*

N:	m	Γ	$f(a) = (x_1, \ldots, x_m) + M$, $f(b) = (y_1, \ldots, y_m) + M$	t_m
1	$4n+1$ $(n \geqslant 1)$	Δ_n	$x_{n+1} = \frac{1}{2},\ x_i = 0\ (i \neq n+1)$, $y_1 = \frac{1}{2},\ 2y_2 = \cdots = 2y_{n+1} = 0$, $y_2 + \cdots + y_{n+1} = \frac{1}{2}$, $y_{n+2} = \cdots = y_{4n+1} = 0$	2^{n-1}
2	$4n+4$ $(n \geqslant 0)$	W_n^*	$x_{2n+3} = \frac{1}{2},\ x_i = 0\ (i \neq 2n+3)$, $y_1 = 0,\ y_2 = \frac{1}{2},\ y_3 = \cdots = y_{3n+3} = 0$, $2y_{3n+4} = \cdots = 2y_{4n+3} = 0,\ y_{4n+4} = \frac{1}{2}$	2^n
3	5	Δ_1^*	$f(a) = (0, \frac{1}{2}, 0, 0, 0),\ f(b) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, 0)$	1
4	8	W_1	$f(a) = (0, 0, 0, 0, \frac{1}{2}, 0, 0, 0),\ f(b) = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{2})$	1

Here, m is the degree of the representation Γ, f is the cocycle, and t_m is the number of torsion-free groups.

3 Preliminary results and Theorem 1

Let $K = \mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ as above. We point out that in these cases the group $H^1(G, \hat{M})$ is finite. Denote by $C_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$ the cyclic group of order p^n. The following three lemmas and Corollary 2 are well known and they can be found for example in [1].

Lemma 1. *Let K be one of the rings $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$. For $i = 1, 2$, let G_i be a finite group and Γ_i, M_i, T_i be the representation, the module and the cocycle associated with G_i as in the Introduction. The groups $\mathfrak{Crys}(G_1; M_1; T_1)$ and $\mathfrak{Crys}(G_2; M_2; T_2)$ are isomorphic if and only if there exist a group isomorphism $\varepsilon : G_1 \to G_2$ and a K-module isomorphism $\tau : M_1 \to M_2$ which satisfy the following conditions*:

(1) $\varepsilon(g)\tau = \tau g$ *in* M_1, *for all* $g \in G_1$;

(2) *the cocycles T_2 and T_1^{ε} are cohomologous (here, $T_1^{\varepsilon}(g) = \tau' T_1(\varepsilon^{-1} g)$ for all $g \in G_2$, where $\tau' : \hat{M}_1 \to \hat{M}_2$ is the homomorphism induced by τ).*

Lemma 2. *Suppose that the character of the K-representation Γ of C_n does not contain the trivial character as a summand. Then $H^1(C_n, \hat{M})$ is trivial.*

Proof. Since 1 is not an eigenvalue of the operator a, which acts on FM, the operator $a - 1$ is a unit. This means that $T(a) = (a-1)x + M$ for some $x \in FM$, i.e. $B^1(C_n, \hat{M}) = C^1(C_n, \hat{M})$.

Lemma 3. *Let $G \cong C_{p^s}$ and M be a projective KG-module. Then $H^1(C_{p^s}, \hat{M})$ is trivial.*

Proof. Since some direct sum $M \oplus \cdots \oplus M$ of copies of M is a free KC_{p^s}-module, it is sufficient to prove the lemma for $M = KC_{p^s}$. Let $T(a) = x + M$ where

$$x = \lambda(1 + a + \cdots + a^{p^s - 1}) + u_1(a-1) \in FM,$$

and where $\lambda \in F$ and $u_1 \in FC_{p^s}$. From the condition $(1 + a + \cdots + a^{p^s-1})T(a) \subset M$ it follows that $\lambda p^s \in K$. Then $x - \lambda p^s = u_2(a-1)$, where $u_2 \in FC_{p^s}$. Therefore T is a coboundary.

Corollary 2. *Suppose that the K-representation Γ of C_p does not contain the trivial K-representation as a summand. Then $H^1(C_p, \hat{M})$ is trivial.*

Proof. The K-representation Γ of C_p is a direct sum $\Gamma = \Gamma_1 \oplus \Gamma_2$, where Γ_1 is a sum of copies of the irreducible K-representation of degree $p-1$ and Γ_2 is a K-representation corresponding to a projective KC_p-module. The proof follows by applying Lemma 1 to Γ_1 and Lemma 3 to Γ_2.

For the proof of Theorem 1 we consider certain K-representations of the group $\langle a\rangle = C_{p^s}$. Let ξ_t be a primitive p^tth root of unity and set $\xi_{t-1} = \xi_t^p$ for $t \geqslant 1$. Put

$$B_0 = \{1\}, \quad B_1 = \{1, \xi_1, \dots, \xi_1^{p-2}\}, \quad B_j = \bigcup_{i=0}^{p-1} \xi_j^i B_{j-1} \quad (j \geqslant 2).$$

Thus for each $t \leqslant s$ the set B_t is a K-basis of the ring $K_t = K[\xi_t]$, which is a KC_{p^s}-module with action defined by $a(\alpha) = \xi_t\alpha$ for $\alpha \in K_t$. The set B_t is also an F-basis of the space FK_t for each t.

Let δ_t be the matrix K-representation of C_{p^s} corresponding to the K-basis B_t of the module K_t. We note that δ_t is an irreducible K-representation of C_{p^s} and $\delta_t^p(a) = E_p \otimes \delta_{t-1}(a)$, where E_p is the $p \times p$ identity matrix. Let

$$\Delta_1 = \delta_0^{(n)} + \delta_1^{(n)}, \quad \Delta_2 = \delta_2^{(n)} + \delta_s^{(n)}$$

be sums of $2n$ irreducible K-representations of C_{p^s}, where

$$\delta_i^{(n)} = \underbrace{\delta_i + \cdots + \delta_i}_{n}.$$

Consider the K-representation Δ of C_{p^s} defined by

$$\Delta(a) = \begin{pmatrix} \Delta_1(a) & U(a) \\ 0 & \Delta_2(a) \end{pmatrix},$$

where

$$U(a) = \begin{pmatrix} E_n \otimes \langle 1\rangle_0 & J_n \otimes \langle 1\rangle_0 \\ E_n \otimes \langle 1\rangle_1 & E_n \otimes \langle 1\rangle_1 \end{pmatrix}.$$

Here, J_n is the Jordan block of order n with entries 1 on the main diagonal and $\langle\omega\rangle_t$ denotes the matrix with all columns zero except the last one, which consists of the coordinates of the element $\omega \in K_t$ written in the basis B_t $(t = 0, 1)$.

Lemma 4. (see [2, 3]). *The K-representation Δ of C_{p^s} is indecomposable.*

Lemma 5. *Let $x \in FK_t$ (where $t > 0$) and suppose that $(a-1)x \in K_t$. Then $px \in K_t$ and all coordinates of the vector px are multiples of the last coordinate.*

Proof. The K-basis B_t in K_t is an F-basis in FK_t. Consider the coordinates of the column vectors in FK_t and the matrix $\delta_t(a)$ of the operator a in this basis. The lemma is easily checked successively for $t = 1, 2, \dots$.

Let B be a K-basis of the K-module M_Δ affording the matrix K-representation Δ

of $\langle a \rangle = C_{p^s}$. Denote the first basis element by v. It is easy to see that B is an F-basis in FM_Δ. We define the function

$$T_\Delta : C_{p^s} \to \widehat{M_\Delta} = FM_\Delta^+ / M_\Delta^+,$$

by setting $T_\Delta(a^j) = jp^{-s}v + M_\Delta$ for $j = 0, 1, \ldots, p^s - 1$.

Lemma 6. *The function T_Δ is a 1-cocycle of C_{p^s} with values in $\widehat{M_\Delta}$, and it is not cohomologous to the zero cocycle at the element $b = a^{p^{s-1}}$ of order p.*

Proof. The first assertion follows from the definition of T_Δ. To prove the second assertion, consider the pth power $\Delta^p(a)$ of Δ. We note that

$$\Delta^p(a) = \begin{pmatrix} \Delta_1^p(a) & U'(a) \\ 0 & \Delta_2^p(a) \end{pmatrix} \quad \text{and} \quad \Delta_1^p(a) = E.$$

Clearly the first row in $U'(a)$ has the form

$$(\langle 1 \rangle_0, \ldots, \langle 1 \rangle_0, \langle 1 \rangle_0, \ldots, \langle 1 \rangle_0),$$

and the row of matrices corresponding to the first of the representations δ_1^p will take the form of the following matrix:

$$(\langle 1 \rangle_1, \langle \xi_1 \rangle_1, \ldots, \langle \xi_1^{p-1} \rangle_1, \langle 1 \rangle_1, \ldots, \langle \xi_1^{p-1} \rangle_1).$$

Subtracting the rows of this matrix from the first row in $U'(a)$, we obtain a row in which all the entries are multiples of p. This transformation of rows in $U'(a)$ corresponds to the replacement of some basis elements $u \in B$ $(u \neq v)$ by $u' = u \pm v$. We carry out this replacement; let Δ' be the K-representation of C_{p^s} in the new K-basis of M_Δ. It is easy to see that the change of basis does not change the values of the function T_Δ.

Let $H = \langle b \mid b = a^{p^{s-1}} \rangle$ and let Δ'_H be the restriction of the representation Δ' to H. Then

$$\Delta'_H(b) = \begin{pmatrix} \delta_0^{(m_1)}(a) & U''(b) \\ 0 & \delta_1^{(m_2)}(a) \end{pmatrix},$$

where, as shown above, all entries of the first row in $U''(b)$ are multiples of p. Let $M_\Delta = M_1 \oplus M_2$ be the decomposition of M_Δ as a direct sum corresponding to the representations $\delta_0^{(m_1)}$ and $\delta_1^{(m_2)}$.

Suppose that T_Δ is cohomologous to the trivial cocycle at H. Then there exists a vector $x \in FM_\Delta$ such that

$$T_\Delta(b) = (b - 1)x + M_\Delta.$$

Let $x = x_1 + x_2$ with $x_i \in FM_i$ for $i = 1, 2$. Since the projection of $T_\Delta(b)$ on FM_2 is equal to zero (modulo M_Δ), the projection of $(b-1)x = (b-1)x_2$ on FM_2 is also equal to zero. From Lemma 5 it follows that $px_2 \in M_2$. Let λ be the coefficient of the basis vector v in $(b-1)x$.

It is easy to see that λ is a sum of products of the entries of the first row in $U''(b)$ (these entries are multiples of p) on the column which consists of coordinates of the vector x_2. Since $px_2 \in M_2$ it follows that $\lambda \in K$. Since $T_\Delta(b) = p^{-1}v + M_\Delta$, we have $\lambda = p^{-1}$. But $p^{-1} \notin K$, and so $\lambda \notin K$. This contradiction proves that T_Δ is not cohomologous to zero at H. The lemma is proved.

Proof of Theorem 1. Let us consider the group $\mathfrak{Crys}(C_{p^s}; M_\Delta; T_\Delta)$. If this group has an element of prime order, then this order can only be p and, moreover, the cocycle T_Δ must be cohomologous to the zero cocycle at the element $b = a^{p^{s-1}}$ in C_{p^s}. By Lemma 6 this is impossible. Therefore $\mathfrak{Crys}(C_{p^s}; M_\Delta; T_\Delta)$ is torsion-free. Moreover this group is indecomposable (see Lemma 4).

4 Theorem 2

Now let $\langle a \rangle = C_{p^2}$. We want now to find all groups $\mathfrak{Crys}(C_{p^2}; M; T)$ which are torsion-free. Put

$$\Phi(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

There exists a unit θ in KC_{p^2} such that

$$(a-1)^p \Phi(a^p) = p(a-1)\theta\Phi(a^p).$$

For $0 \leqslant i \leqslant p-2$, let X_i be the KC_{p^2}-submodule of KC_{p^2} generated by the following elements:

$$u = \Phi(a)\Phi(a^p), \quad \omega = (a-1)\Phi(a^p), \quad v = \Phi(a) + (a-1)^{i+1}.$$

It is easy to see that

$$(a-1)u = 0, \quad \Phi(a)\omega = 0, \quad \Phi(a^p)v = u + (a-1)^i\omega.$$

From these equations it follows that the K-representation Γ_i of the group C_{p^2} in the K-basis

$$u; \quad \omega, a\omega, a^2\omega, \ldots, a^{p-2}\omega; \quad a^l v, a^{l+p}v, \ldots, a^{l+p(p-2)}v,$$

$(l = 0, 1, \ldots, p-1)$ corresponding to the module X_i, has the following form:

$$\Gamma_i(a) = \begin{pmatrix} 1 & 0 & \langle 1 \rangle_0 \\ & \delta_1(a) & \langle \alpha_i \rangle_1 \\ & & \delta_2(a) \end{pmatrix},$$

where $\alpha_i = (\xi_1 - 1)^i$ and $i = 0, 1, \ldots, p-2$.

Lemma 7. *Let $H = \langle b \rangle$ where $b = a^p$. The KH-module $X_i|_H$ is a direct sum of two KH-submodules, one of which coincides with Ku.*

Proof. Consider the K-submodule X_i' in X_i generated by the following $p^2 - 1$ elements of X_i:

$$V = \{v, bv, \ldots, b^{p-2}v\}, \quad (a-1)V, \ldots, (a-1)^{p-2}V, \quad v' = (a-1)^{p-1}v + \theta u,$$
$$bv', \ldots, b^{p-2}v', \quad \theta\omega_1, \ldots, \theta\omega_i, \quad u + \omega_{i+1}, \omega_{i+2}, \ldots, \omega_{p-1},$$

where $\omega_j = (a-1)^j \Phi(b) = (a-1)^{j-1}\omega$ for $j = 1, \ldots, p-1$.

Clearly X_i is the direct sum of Ku and X_i'. To prove the lemma it is sufficient to show that X_i' is a KH-module. We have

$$\begin{array}{lll} \Phi(b)v & = u + \omega_{i+1} & \in X_i', \\ \Phi(b)(a-1)v & = \omega_{i+2} & \in X_i', \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ \cdots\cdots\cdots & \cdots\cdots & \cdots \\ \Phi(b)(a-1)^{p-i-2}v & = \omega_{p-1} & \in X_i', \\ \Phi(b)(a-1)^{p-r-2+j}v & = p\theta\omega_j & \in X_r', \end{array}$$

for $0 \leqslant r \leqslant p-2, \ j = 1, \ldots, r$ and

$$\begin{aligned} \Phi(b)v' &= (a-1)^{p-1}\Phi(b)v + p\theta u = (a-1)^{p+i}\Phi(b) + p\theta u \\ &= p\theta(a-1)^{1+i}\Phi(b) + p\theta u = p\theta(\omega_{i+1} + u) \in X_i'. \end{aligned}$$

These equations show that X_i' is a KH-submodule of X_i.

For $i = 0, 1, \ldots, p-2$ we introduce the cocycle

$$T_i : C_{p^2} \to \widehat{X_i} = FX_i^+/X_i^+, \tag{1}$$

defined by $T_i(a) = p^{-2}u + X_i$.

Lemma 8. *For $i = 0, 1, \ldots, p-2$ the group $\mathfrak{Crys}(C_{p^2}; X_i; T_i)$ is torsion-free.*

Proof. Since $T_i(a^p) = p^{-1}u + X_i \neq X_i$ it follows from Lemma 7 that

$$((a^p - 1)FX_i + X_i) \cap (Fu + X_i) = X_i.$$

These conditions show that the cocycle T_i is not cohomologous to the zero cocycle at the element a^p. This means that $\mathfrak{Crys}(C_{p^2}; X_i; T_i)$ is torsion-free.

For $i = 0, 1, \dots, p-1$ let Y_i be the KC_{p^2}-submodule $\langle \Phi(a), (a-1)^i \rangle$ of KC_{p^2}. The K-representation Γ_i' corresponding to Y_i has the following form:

$$\Gamma_i'(a) = \begin{pmatrix} 1 & \langle 1 \rangle_0 & 0 \\ & \delta_1(a) & \langle \alpha_i \rangle_1 \\ & & \delta_2(a) \end{pmatrix},$$

where $\alpha_i = (\xi_1 - 1)^i$.

Lemma 9. *For each cocycle* $T : C_{p^2} \to \hat{Y}_i = FY_i^+ / Y_i^+$ *the group* $\mathfrak{Crys}(C_{p^2}; Y_i; T)$ *has an element of order* p.

Proof. It is easy to see that each cocycle of C_{p^2} with a value in $\hat{Y}_i$ will be cohomologous to a cocycle T such that $T(a) = \lambda p^{-2}u + Y_i$, where $\lambda \in K$, $u = \Phi(a)\Phi(b)$. Thus $T(a^p) = pT(a) = \lambda p^{-1}u + Y_i$, and so to prove the lemma it is sufficient to show that $p^{-1}u \in (a^p - 1)FY_i + Y_i$. It is easy to see that

$$(a^{p-1} + a^{p-2} + \dots + a + 1) - (a-1)^{p-1} = p\omega_1(a), \tag{2}$$

where $\omega_1(a) \in KC_{p^2}$.

Let $v_1 = (a-1)^i$. Then from (2) it follows that

$$(\Phi(a^p) - p)(a-1)^{p-i-1}v_1 = u - p\omega_1(a)\Phi(a^p) - p(a-1)^{p-i-1}v = u + py,$$

where $y \in Y_i$. Since $\Phi(a^p) - p = (a^p - 1)z$, where $z \in KC_{p^2}$, we have

$$p^{-1}u + Y_i = (a^p - 1)p^{-1}z + Y_i,$$

which completes the proof of the lemma.

Let $p \neq 2$. In the free KC_{p^2}-module

$$(KC_{p^2})^{(2)} = \{(x_1, x_2) \mid x_1, x_2 \in KC_{p^2}\}$$

we consider the KC_{p^2}-submodule

$$U_j = \langle ((a-1)^{j+1} + \Phi(a), (a-1)^j), \Phi(a^p)(a-1, 1) \rangle,$$

for $1 \leqslant j \leqslant p-2$. The K-representation of C_{p^2} corresponding to U_j has the form

$$\Gamma_j'' : a \mapsto \begin{pmatrix} 1 & 0 & 0 & \langle 1 \rangle_0 \\ & 1 & \langle 1 \rangle_0 & 0 \\ & & \delta_1(a) & \langle \alpha_j \rangle_1 \\ & & & \delta_2(a) \end{pmatrix}, \tag{3}$$

where $\alpha_j = (\xi_1 - 1)^j$ and $j = 1, 2, \ldots, p-2$. Define the cocycle

$$f_j : C_{p^2} \to \widehat{U_j} = FU_j^+ / U_j^+$$

by $f_j(a) = p^{-2}\Phi(a)\Phi(a^p)(1,0) + U_j$.

Lemma 10. *For $j = 1, \ldots, p-2$ the group $\mathfrak{Crys}(C_{p^2}; U_j; f_j)$ is torsion-free.*

Proof. Let $u_1 = \Phi(a)\Phi(a^p)(1,0)$ and $u_2 = \Phi(a)\Phi(a^p)(0,1)$. It is easy to see that the sequence of KC_{p^2}-modules

$$0 \to Ku_2 \to U_j \to X_j \to 0 \tag{4}$$

is exact. The cocycle f_j induces the cocycle $T_j : C_{p^2} \to \widehat{X_j}$ defined in (1), which is not equal to the zero cocycle on the group $H = \langle a^p \rangle$ by Lemma 8. Therefore f_j is also non-cohomologous to the zero cocycle in H. This means that $\mathfrak{Crys}(C_{p^2}; U_j; f_j)$ has no elements of order p.

We consider one more KC_{p^2}-module, namely the submodule U_0 of KC_{p^2} generated by $\Phi(a)$. The corresponding K-representation of C_{p^2} has the form

$$a \mapsto \begin{pmatrix} 1 & \langle 1 \rangle_0 \\ 0 & \delta_2(a) \end{pmatrix}.$$

Lemma 11. *For each cocycle $T : C_{p^2} \to U_0$ the group $\mathfrak{Crys}(C_{p^2}; U_0; T)$ has an element of order p.*

Proof. It is easy to see that any cocycle of C_{p^2} with values in $\widehat{U_0} = FU_0^+ / U_0^+$ is cohomologous to a cocycle T of the form

$$T(a) = \lambda p^{-2}\Phi(a)\Phi(a^p) + U_0,$$

with $\lambda \in K$. Replacing a by a^p in (2) we have

$$p^{-1}\Phi(a)\Phi(a^p) = p^{-1}(a^p - 1)^{p-1}\Phi(a) + \omega_1(a^p)\Phi(a).$$

Then $T(a^p) = (a-1)z + U_0$, where $z \in FU_0$, and this proves the lemma.

Proof of Theorem 2. From the description in [2] of the K-representations of C_{p^2} it follows that the indecomposable KC_{p^2}-modules corresponding to the faithful K-representations of C_{p^2} whose characters contain the trivial character are the following:

$$X_i \ (i = 0, 1, \dots, p-2), \quad Y_j \ (j = 0, 1, \dots, p-1),$$
$$U_0, \quad U_k \ (k = 1, \dots, p-2).$$

By Lemmas 9 and 11 we are interested only in the modules X_i and U_j. Let us consider the module X_i where $0 \leqslant i \leqslant p-2$. It is easy to see that Lemma 2 can be applied to the factor module X_i/Kv, where $v = \Phi(a)\Phi(a^p)$. Therefore any cocycle of C_{p^2} with the values in $\widehat{X_i}$ will be cohomologous to a cocycle T of the form

$$T(a) = \lambda p^{-2} v + X_i, \tag{5}$$

with $\lambda \in K$. We claim that if in this equation $\lambda \equiv 0 \pmod p$ then T is cohomologous to the trivial cocycle. From (2) we have

$$p^{-1}\Phi(a)\Phi(a^p) + p^{-1}\Phi(a^p)(a-1)^{i+1} = p^{-1}(a^p - 1)^{p-1}\theta_i + \omega_1(a^p)\theta_i, \tag{6}$$

where $\theta_i = \Phi(a) + (a-1)^{i+1} \in X_i$. We will use the equation

$$\Phi(a^p)(a-1)^p = p(a-1)\Phi(a^p)\omega_2,$$

where ω_2 is a unit in KC_{p^2}. From (6) it follows that

$$p^{-1}(a-1)^{i+1}\Phi(a^p) = p^{-2}\Phi(a^p)(a-1)^{p+i}\omega_2^{-1} \in (a-1)FX_i,$$

for $i = 0, 1, \dots, p-2$. Then from (6) one finds that

$$p^{-1}\Phi(a)\Phi(a^p) \in (a-1)FX_i + X_i$$

for $i = 0, 1, \dots, p-2$. Our claim follows.

From the above it follows that $H^1(C_{p^2}, \widehat{X_i})$ is cyclic of order p and that all elements of this group can be represented by the cocycles T defined in (5) with $\lambda = 0, 1, \dots, p-1$.

We will show that each non-zero cocycle T defines up to isomorphism the group $\mathfrak{Crys}(C_{p^2}; X_i; T_i)$.

Let ε be an automorphism of the group C_{p^2} and X_i^{ε} be the KC_{p^2}-module X_i twisted by this automorphism, i.e.

$$X_i^{\varepsilon} = X_i, \quad a.x = \varepsilon(a)x, \quad \text{for } x \in X_i.$$

It is not difficult to show the existence of an automorphism τ of the K-module X_i such that $\varepsilon(a)\tau = \tau a$ in X_i and $\tau(v) = v$, where $v = \Phi(a)\Phi(a^p)$.

Let $\varepsilon^{-1}(a) = a^s$, with $(s, p) = 1$. Since we have $aT_i(a) = T_i(a)$ and $\tau'(\bar{v}) = \bar{v}$, where $\bar{v} = v + X_i$, we obtain

$$T_i^{\varepsilon}(a) = \tau' T_i(\varepsilon^{-1}(a)) = sT_i(a) = sp^{-2}v + X_i.$$

From Lemma 1 it follows that $\mathfrak{Crys}(C_{p^2}; X_i; T_i)$ is isomorphic to $\mathfrak{Crys}(C_{p^2}; X_i; T)$, where $T(a) = sp^{-2}v + X_i$. We have shown that each group $\mathfrak{Crys}(C_{p^2}; X_i; T)$ with $T \not\equiv 0$ is isomorphic to $\mathfrak{Crys}(C_{p^2}; X_i; T_i)$ for some i.

Now consider groups of the form $\mathfrak{Crys}(C_{p^2}; U_j; T)$. First we remark that $H^1(C_{p^2}, \widehat{Y_j})$ is cyclic of order p, for $j = 1, \dots, p-1$. The proof of this is similar to the proof for the group $H^1(C_{p^2}, \widehat{X_i})$. Since $Y_0 = KC_{p^2}$, we have $H^1(C_{p^2}, \widehat{Y_0}) = 0$ (see Lemma 3).

Let $u_1, u_2, \dots, u_{p^2+1}$ be a K-basis in U_j such that

$$u_1 = \Phi(a)\Phi(a^p)(1,0) \quad \text{and} \quad u_2 = \Phi(a)\Phi(a^p)(0,1),$$

and for $0 \leqslant \alpha, \beta \leqslant p-1$ let the cocycle $T_{\alpha,\beta}$ satisfy

$$T_{\alpha,\beta}(a) = p^{-2}(\alpha u_1 + \beta u_2) + U_j.$$

We use the exact sequence (4) and the exact sequence

$$0 \to Ku_1 \to U_j \to Y_j \to 0. \tag{7}$$

This enables us to show that any cocycle $T : C_{p^2} \to U_j$ is cohomologous to some cocycle $T_{\alpha,\beta}$ with $0 \leqslant \alpha, \beta \leqslant p-1$.

By Lemma 9 and (7), the cocycle $T_{0,\beta}$ is cohomologous to the zero cocycle at the element a^p of C_{p^2} and therefore $\mathfrak{Crys}(C_{p^2}; U_j; T_{0,\beta})$ has an element of order p.

Now let $\alpha \neq 0$. Then α is a unit in K and the map τ defined by $\tau(x) = \alpha x$ is an automorphism of the KC_{p^2}-module U_j. It follows that the cocycle $T_{\alpha,\beta}$ $(\alpha \neq 0)$ can be replaced by $T_{1,\alpha^{-1}\beta}$. So it is enough to consider the cocycles $T_{1,\beta}$, where $\beta = 0, 1, \dots, p-1$. We will show that $\mathfrak{Crys}(C_{p^2}; U_j; T_{1,\beta})$ is isomorphic to $\mathfrak{Crys}(C_{p^2}; U_j; f_j)$ (note that $f_j = T_{1,0}$).

We replace the basis element u_1 by $u_1' = u_1 + \beta u_2$ in U_j. Then

$$T_{1,\beta}(a) = p^{-2}u_1' + U_j.$$

Let $Y_j' = U_j/Ku_1'$. Then the K-representation Γ_j''' corresponding to Y_j' is

$$\Gamma_j''' : a \to \begin{pmatrix} 1 & \langle 1 \rangle_0 & \langle -\beta \rangle_0 \\ & \delta_1(a) & \langle \alpha_j \rangle_1 \\ & & \delta_2(a) \end{pmatrix}.$$

This representation is equivalent to Γ_j'. Because of this equivalence we will replace the basis elements $u_2, \dots, u_{p^2+1}$ by $u_2', \dots, u_{p^2+1}'$. Then in the K-basis $u_1', u_2', \dots, u_{p^2+1}'$ the operator a has the same matrix (3) as in the basis $u_1, u_2, \dots, u_{p^2+1}$. Define an automorphism $\tau : U_j \to U_j$ of the K-module U_j by $\tau(u_i') = u_i$ for $i = 1, \dots, p^2+1$. We have $\tau a = a\tau$ and moreover

$$\tau' T_{1,\beta}(a) = \tau'(p^{-2}u_1' + U_j) = p^{-2}u_1 + U_j = f_j(a).$$

It follows from Lemma 1 that $\mathfrak{Crys}(C_{p^2}; U_j; T_{1,\beta})$ and $\mathfrak{Crys}(C_{p^2}; U_j; f_j)$ are isomorphic. So among the groups $\mathfrak{Crys}(C_{p^2}; M; T)$ the ones which can be indecomposable and torsion-free are isomorphic to those for which the module M and cocycle T were listed in this theorem. Now Lemmas 8 and 10 complete the proof.

5 Theorem 3

Let $G \cong C_p \times C_p$ with generators a, b and let K be one of the rings $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Z}_{(p)}$. In the case when $p = 2$ we will give a full description of the indecomposable torsion-free groups $\mathfrak{Crys}(C_2 \times C_2; M; T)$. We will use the classification of the indecomposable K-representations of $C_2 \times C_2$ given by Nazarova in [15, 16].

Lemma 12. *Let M be the $K[C_p \times C_p]$-submodule of the free $K[C_p \times C_p]$-module $(K[C_p \times C_p])^{(2)}$ defined as follows*:

$$M = \langle (\Phi(a), 0), (p, 0), (0, \Phi(b)), (0, p), (b - 1, 1 - a) \rangle.$$

Then the following assertions hold:

(1) *M is an indecomposable $K[C_p \times C_p]$-module and* $\dim_K(M) = 2p^2$;

(2) *there exists a cocycle $T : C_p \times C_p \to \hat{M} = FM^+/M^+$ defined by*

$$T(a) = (1, 0) + M, \quad T(b) = (0, 1) + M;$$

(3) *the group $\mathfrak{Crys}(C_p \times C_p; M; T)$ is torsion-free.*

Proof. (1) Let $\overline{Z_p} = K/pK$ and regard $\overline{Z_p}$ as a $K[C_p \times C_p]$-module with $C_p \times C_p$ acting trivially. Consider the projective resolution

$$\cdots \to (K[C_p \times C_p])^{(3)} \xrightarrow{\tau_1} (K[C_p \times C_p]) \xrightarrow{\tau_0} \overline{Z_p} \to 0 \tag{8}$$

of $\overline{Z_p}$. It is easy to see that $\ker(\tau_0) = \langle a - 1, b - 1, p \rangle$, and

$$\begin{aligned}\ker(\tau_1) = \langle &(\Phi(a), 0, 0), (0, \Phi(b), 0),\\ &(b - 1, 1 - a, 0), (p, 0, 1 - a), (0, p, 1 - b) \rangle.\end{aligned}$$

The $K[C_p \times C_p]$-modules $\ker(\tau_0)$ and $\ker(\tau_1)$ are indecomposable. Each $x \in \ker(\tau_1)$ has the form

$$x = (u_1\Phi(a) + u_3(b-1) + pu_4, u_2\Phi(b) + u_3(1-a) + pu_5, u_4(1-a) + u_5(1-b)), \quad (9)$$

with $u_i \in K[C_p \times C_p]$ for $i = 1, \dots, 5$. We map x to the element

$$(u_1\Phi(a) + u_3(b-1) + pu_4, u_2\Phi(b) + u_3(1-a) + pu_5)$$

of M. It is easy to check that this defines an isomorphism of the $K[C_p \times C_p]$-modules $\ker(\tau_1)$ and M. Thus M is an indecomposable $K[C_p \times C_p]$-module. Since $\dim_K(T_0) = p^2$, we have

$$\dim_K(M) = \dim_K(\ker(\tau_1)) = \dim_K(K[C_p \times C_p])^{(3)} - \dim_K(\ker(\tau_0)) = 2p^2.$$

(2) Define $T : C_p \times C_p \to \hat{M}$ as follows:

$$T(a^i) = (1 + a + \cdots + a^{i-1}, 0) + M,$$
$$T(b^j) = (0, 1 + b + \cdots + b^{j-1}) + M,$$
$$T(a^i b^j) = a^i T(b^j) + T(a^i) + M; \quad T(1) = M,$$

for $0 < i, j \leqslant p-2$. It is easy to see that $\Phi(a)T(a) \subset M$, $\Phi(b)T(b) \subset M$ and $(a-1)T(b) - (b-1)T(b) \subset M$. It follows that T is a cocycle of $C_p \times C_p$ with values in $\hat{M} = FM^+/M^+$.

(3) It is sufficient to show that T is not cohomologous to the zero cocycle at every non-trivial element g of $C_p \times C_p$. Let $g = a^i b^j$, where $0 < i, j \leqslant p-1$. Suppose that there exists $z \in FM$ such that

$$T(g) = (g-1)z + M. \quad (10)$$

From the definition of T and from (10) it follows that

$$(1 + a + \cdots + a^{i-1}, a^i(1 + b + \cdots + b^{j-1})) = (g-1)z + x,$$

for some $x \in M$. Multiplying this equation by $\Phi(a)\Phi(b)$ taking into account that

$$\Phi(a)\Phi(b)M = p\Phi(a)\Phi(b)(K, K), \quad \text{and} \quad \Phi(a)\Phi(b)(g-1) = 0$$

we conclude that $(i, j) \in (pK, pK)$, which is impossible since $0 < i, j \leqslant p-1$. This contradicts the assumption that T is cohomologous to the zero cocycle at g (see (10)).

Similarly, we may show that T is not cohomologous to the zero cocycle at the remaining non-trivial elements of $C_p \times C_p$. Thus $\mathfrak{Crys}(C_p \times C_p; M; f)$ is torsion-free.

Now let $p = 2$, let $G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$ and let K be one of the rings $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_{(2)}$. We will study those groups $\mathfrak{Crys}(G; M; T)$ which are torsion-free.

The group G has the following irreducible K-representations:

$$\chi_0 : a \mapsto 1,\ b \mapsto 1; \qquad \chi_1 : a \mapsto -1,\ b \mapsto 1;$$
$$\chi_2 : a \mapsto -1,\ b \mapsto -1; \qquad \chi_3 : a \mapsto 1,\ b \mapsto -1.$$

Let $H = \langle h \rangle$ be a subgroup of G of order 2. The indecomposable K-representations of H, up to equivalence, are the following:

$$\gamma_0 : h \mapsto 1; \quad \gamma_1 : h \mapsto -1; \quad \gamma_2 : h \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Let Γ be a K-representation of G and $\Gamma|_H$ its restriction to H. Let M be a KG-module corresponding to the K-representation Γ and $T : G \to \hat{M}$ be an arbitrary cocycle of G with values in $\hat{M} = FM^+/M^+$ (where F is a field containing K). The following lemma gives necessary conditions for $\mathfrak{Crys}(G; M; T)$ to be torsion-free.

Lemma 13. *If $\mathfrak{Crys}(G; M; T)$ is torsion-free then for each non-trivial subgroup H of order 2, the trivial representation γ_0 is contained in the decomposition of $\Gamma|_H$ as a direct sum of indecomposable K-representations of H.*

Indirect proof. Assume that H is a subgroup of order 2 in G such that $\Gamma|_H$ does not have γ_0 as a direct summand. Then it follows from Lemmas 2 and 3 that any cocycle $T : G \to \hat{M}$ will be cohomologous in H to the zero cocycle, and this implies that $\mathfrak{Crys}(G; M; T)$ has elements of order 2.

We make some remarks about the K-representations of $G \cong C_2 \times C_2$. Let G act trivially on K and consider the projective resolution

$$\cdots \to (KG)^{(n)} \xrightarrow{\nu_n} (KG)^{(n-1)} \to \cdots$$
$$\cdots \xrightarrow{\nu_3} (KG)^{(2)} \xrightarrow{\nu_2} (KG) \xrightarrow{\nu_1} K \to 0 \tag{12}$$

of K. Each ν_n is a homomorphism of the KG-modules and $\ker(\nu_n)$ is an indecomposable KG-module with

$$\dim_K(\ker(\nu_n)) = 2n + 1.$$

Let Γ_n be the K-representation of G corresponding to some K-basis in $\ker(\nu_n)$, and let Γ_n^* be the contragradient K-representation of Γ_n, that is, $\Gamma_n^*(g) = \Gamma^T(g^{-1})$ for all $g \in G$, where the superscript T denotes transposition of matrices.

Lemma 14. (see [16, 22]). *Each indecomposable K-representation of $G \cong C_2 \times C_2$*

of odd degree is equivalent to just one of the following: χ_i, $\Gamma_n \otimes_K \chi_i$ *or* $\Gamma_n^* \otimes_K \chi_i$, *for some* $i \in \{0,1,2,3\}$ *and* $n \geqslant 1$.

Let $p = 2$ in (8) and let us consider the projective resolution for $\ker(\tau_0) = \langle a-1, b-1, 2\rangle$:

$$\cdots \to (KG)^{(t_n)} \xrightarrow{\tau_n} (KG)^{(t_{n-1})} \to \cdots$$
$$\cdots \xrightarrow{\tau_3} (KG)^{(t_2)} \xrightarrow{\tau_2} (KG)^{(t_1)} \xrightarrow{\tau_1} \ker(\tau_0) \to 0. \tag{13}$$

It is easy to show that in (13) we have $t_n = 2n+1$ and

$$\dim_K(\ker(\tau_n)) = 4n+4,$$

where $n \geqslant 0$. Moreover all of the KG-modules $\ker(v_n)$ are indecomposable. If we take the tensor product over K of the exact sequence (12) and the KG-module $\ker(\tau_0)$ and compare the result with the sequence (13), then we obtain easily the isomorphism

$$\ker(\tau_0) \otimes_K \ker(v_n) \cong \ker(\tau_n) \oplus P_n,$$

where P_n is a projective KG-module.

Lemma 15. *Let* W_n *be the* K*-representation of* $G = \langle a\rangle \times \langle b\rangle \cong C_2 \times C_2$ *corresponding to the module* $\ker(\tau_n)$ *where* $n \geqslant 0$. *This representation has the following form*:

$$W_0 : a \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ & -1 & 0 & 0 \\ & & 1 & 0 \\ & & & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 1 & 1 & 0 \\ & -1 & 0 & 0 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix};$$

$$W_n : a \mapsto \begin{pmatrix} D & 0 & 0 & 0 & 0 \\ & E_n & 0 & 0 & V_n \\ & & -E_n & V_n & 0 \\ & & & E_{n+1} & 0 \\ & & & & -E_{n+1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} D & 0 & 0 & S & 0 \\ & E_n & 0 & V_n' & 0 \\ & & -E_n & 0 & V_n' \\ & & & -E_{n+1} & 0 \\ & & & & E_{n+1} \end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

and $V_n = (0\ E_n)$, $V_n' = (E_n\ 0)$ *are matrices with* n *rows and* $n+1$ *columns for* $n \geqslant 1$.

Proof. The proof reduces to the determination of a K-basis of $\ker(\tau_n)$, and this is not difficult to construct by induction on n.

Lemma 16. *Each faithful indecomposable K-representation of $G = \langle a \rangle \times \langle b \rangle$ which satisfies the necessary condition for the existence of a torsion-free group $\mathfrak{Crys}(G; M; f)$ is one of the following*:

$$\Delta_n \ (n \geqslant 1); \quad \Delta_n^* \ (n \geqslant 1); \quad W_n \ (n \geqslant 0); \quad W_n^* \ (n \geqslant 0).$$

Here

$$\Delta_n(a) = \begin{pmatrix} E_n & 0 & 0 & E_n & 0 \\ & 1 & 0 & 0 & 0 \\ & & -E_n & 0 & E_n \\ & & & -E_n & 0 \\ & & & & E_n \end{pmatrix}, \quad \Delta_n(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & E_n & 0 & 0 & E_n \\ & & -E_n & E_n & 0 \\ & & & E_n & 0 \\ & & & & -E_n \end{pmatrix},$$

and Δ_n^* and W_n^* are K-representations of G contragradient to Δ_n and W_n, so that $\Delta_n^*(g) = \Delta_n^T(g)$ and $W_n^*(g) = W_n^T(g)$ for all $g \in G$.

Proof. All K-representations listed above satisfy the necessary condition for the existence of a torsion-free group $\mathfrak{Crys}(G; M; f)$. The analysis of all representations of odd degree (see Lemma 14) shows that among the representations $\Gamma_n \otimes \chi_i$ the necessary condition is satisfied only by Δ^* which is equivalent to Γ_{2n} $(n = 1, 2, \ldots)$. Besides the representations W_n and W_n^*, the group G has a parameterized series of representations whose degrees are divisible by 4. In this series the following pairs of matrices correspond to the pair of generating elements of G:

$$\begin{pmatrix} E_n & 0 & 0 & E_n \\ & -E_n & E_n & 0 \\ & & E_n & 0 \\ & & & -E_n \end{pmatrix}, \quad \begin{pmatrix} E_n & 0 & \mathfrak{F} & 0 \\ & -E_n & 0 & E_n \\ & & -E_n & 0 \\ & & & E_n \end{pmatrix},$$

where the matrix $\mathfrak{F}$ over K has Frobenius (i.e. rational) canonical normal form indecomposable modulo $2K$. Clearly the representations in this series do not satisfy the necessary condition. Consider the following pair of matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & E_n & 0 & 0 & 0 & E_n \\ & & -E_n & E_n & 0 & 0 \\ & & & E_n & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & -E_n \end{pmatrix}, \quad \begin{pmatrix} E_{n+1} & 0 & E_{n+1} & 0 \\ & -E_n & 0 & E_n \\ & & -E_{n+1} & 0 \\ & & & E_n \end{pmatrix}.$$

These matrices define indecomposable K-representations of G of degree congruent to 2 modulo 4 and obviously these representations do not satisfy the necessary condition. We can obtain the remaining representations of degree $4n-2$ either by the described process of tensor multiplication by irreducible K-representations or by taking contragradient representations.

As a result we get representations which do not satisfy the necessary condition for the existence of a torsion-free group $\mathfrak{Crys}(G; M; f)$. Thus we have considered all indecomposable K-representations of G. The lemma is proved.

Proof of Theorem 3. We can take the module M of a K-representation Γ of G of degree m to be the K-module of m-dimensional columns with entries from K. Then FM is the space of m-dimensional columns over F and $\hat{M} = FM^+/M^+$ is the group of m-dimensional columns with entries from $\hat{F} = F^+/K^+$. Let $f : G \to \hat{M}$ be a cocycle. The value $f(g)$ of f at $g \in G$ is an m-dimensional column over $\hat{F}$. We note that if $g, h \in G$ then the product $g.f(h)$ is the ordinary product of the matrices $\Gamma(g)$ and $f(h)$.

If we consider the coordinates of the vector $f(g)$ as elements of F, then the elements of the ring K will be replaced by 0.

Let Γ be any of the representations of G listed in Lemma 16, let M be the module of this representation and let $H = \langle h \rangle$ be a non-trivial subgroup of G. There exists only one basis vector v in M such that M is the direct sum $M = Kv \oplus M'$ of the KH-module Kv and the KH-module M' generated by the rest of the basis vectors of M. In addition, $hv = v$ and a K-representation Γ' of H corresponding to the module M' is a sum of representations of type γ_1 and γ_2 (see (11)). This allows us to replace the cocycle f by a cohomologous cocycle f_1 in such a way that the projection $f_1|_{\widehat{M'}}$ will be equal to zero for the element h (see Lemmas 2, 3).

The coordinate x_v of the vector $f(h)$ corresponding to the basis vector v will be called the *special component* of the vector $f(h)$. From $(1+h)f(h) = 0$, it follows that $2x_v = 0$ (in the group $\hat{F}$). For any $z \in \hat{M}$ the special component of $(h-1)z + f(h)$ is always equal to x_v. If $x_v = \frac{1}{2}$, then the cocycle f is not cohomologous to the zero cocycle at h.

These remarks justify the following plan for the construction of cocycles of the representations Γ from Lemma 16. The form of the representation Γ defines the special components of the vectors $f(a)$ and $f(b)$ (where a and b are the generators of G). We choose $f(a)$ such that we can deduce that the special component is $\frac{1}{2}$ and all other components are zero. The possible forms of the components of the vector $f(b)$ follow from the following conditions:

$$(1+b)f(b) = 0; \tag{14}$$

$$(a-1)f(b) = (b-1)f(a). \tag{15}$$

We will carry out the following operations on the vector $f(b)$: replace $f(b)$ by the vector

$$(b-1)z + f(b), \tag{16}$$

where $z \in \hat{M}$ and $(a-1)z = 0$.

We discard all those forms of $f(b)$ with a zero special component. For a vector $f(b)$ whose special component equals $\frac{1}{2}$, we find that

$$f(ab) = af(b) + f(a) \tag{17}$$

and we examine the solvability of the following equation

$$(ab - 1)z + f(ab) = 0 \tag{18}$$

with $z \in \hat{M}$. The group $\mathfrak{Crys}(G; M; f)$ is torsion-free if and only if the equation (17) has no solution.

We consider the following seven cases:

Case 1. Let $\Gamma = \Delta_n$. The special components are the $(n+1)$th entry in $f(a)$ and the first one in $f(b)$. Set the $(n+1)$th coordinate of $f(a)$ to $\frac{1}{2}$ and let all the rest be 0. Let

$$f^T(b) = (y, Y_1, Y_2, Y_3, Y_4), \tag{19}$$

where $y \in \hat{F}$, $Y_i \in \hat{F}^{(n)}$ and $i = 1, 2, 3, 4$.

Using the operation (16) we can replace Y_2 by the zero vector. From (15) it follows that $Y_3 = Y_4 = 0$, and from (14), it follows that $2y = 0$ and $2Y_1 = 0$. Let $y = \frac{1}{2}$, $Y_1 = (v_1, v_2, \dots, v_n)$. Using (17), it is easy to transform (18) to a linear system of equations (over $\hat{F}$) with the $(n+1) \times n$-matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

and coefficients $\frac{1}{2}, v_1, \dots, v_{n-1}, v_n + \frac{1}{2}$. This system is solvable if and only if

$$v_1 + \cdots + v_{n-1} + v_n = 0.$$

Case 2. Let $\Gamma = \Delta_n^*$. The matrices of the K-representation are the transposes of the matrices of Δ_n. The special components of $f(a)$ and $f(b)$ are the same as in Case 1. Let us assume that $f(a)$ and $f(b)$ are chosen at first in the same way as in the case of $\Gamma = \Delta_n$ (see (19)). Condition (14) and operation (16) transform the vector $f(b)$ to the following form:

$$f^T(b) = (y, 0, -2Y_3, Y_3, 0).$$

Let $Y_3 = (v_1, \dots, v_{n-1}, v_n)$. It follows from (15) that if $n \geqslant 2$ then

$$y - 2v_1 = 0; \quad 2v_2 = \cdots = 2v_n = 0;$$
$$2v_1 = 0; \dots ; 2v_{n-1} = 0; \quad 2v_n = \tfrac{1}{2},$$

and, if $n = 1$, then $y - 2v_1 = 0$, $2v_1 = \frac{1}{2}$.

If $n > 1$ and $y = \frac{1}{2}$, then (19) leads to a contradiction. If $n = 1$ and $y = \frac{1}{2}$, then $v_1 = \frac{1}{4}$.

Thus if $n > 1$ and f is a cocycle then the special component of the vector $f(b)$ is equal to zero. Then the cocycle f is cohomologous to the zero cocycle in the element $b \in G$. This means that $\mathfrak{Crys}(G; M; f)$ cannot be torsion-free if M corresponds to the representation $\Gamma = \Delta_n^*$ where $n > 1$.

Let $n = 1$. Then

$$f(a) = (0, \tfrac{1}{2}, 0, 0, 0), \quad f(b) = (\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{4}, 0), \quad f(ab) = (\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}).$$

It is easy to check that (18) is unsolvable.

Case 3. Let $\Gamma = W_n^*$ $(n > 0)$. The special components are the $(2n + 3)$rd of $f(a)$ and the last of $f(b)$. Let the special component of $f(a)$ be equal to $\frac{1}{2}$, and let all the rest be zero.

Let $f^T(b) = (Y_0, Y_1, Y_2, Y_3, Y_4)$, where $Y_0 \in \hat{F}^{(2)}$, $Y_1, Y_2 \in \hat{F}^{(n)}$, $Y_3, Y_4 \in \hat{F}^{(n+1)}$. Operation (16) allows us to replace Y_3 by the zero vector. It follows from (14) that $Y_1 = 0$. Condition (15) shows that $Y_2 = 0$, $Y_0 = (0, y)$ $(y \in \hat{F}, 2y = 0)$ and $2Y_4 = 0$. Consequently

$$f^T(b) = (0, y, 0, \dots, 0, v_1, \dots, v_n, \tfrac{1}{2}).$$

The special component of $f(ab)$ is the second coordinate which, according to (17), equals y. Therefore $y = \frac{1}{2}$ and for any $v_1, \dots, v_n$ $(2v_1 = 2v_2 = \cdots = 2v_n = 0)$ the group $\mathfrak{Crys}(G; M; f)$ is torsion-free.

Case 4. Let $\Gamma = W_0^*$. In this case it is easy to see that the cocycle f with

$$f(a) = (0, 0, \tfrac{1}{2}, 0), \quad f(b) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2})$$

determines a torsion-free group $\mathfrak{Crys}(G; M; f)$.

Case 5. Let $\Gamma = W_n$ $(n > 1)$. We take the vectors $f(a)$ and $f(b)$ in the same fashion as in Case 3. Condition (14) shows that all components of the vector Y_4, except the last, are zero. Then condition (15) leads to a contradiction.

We obtain a contradiction by setting the special component in $f(a)$ equal to $\frac{1}{2}$. Consequently, for $\Gamma = W_n$ with $n > 1$, any cocycle f is cohomologous to the zero cocycle at the generator a of G, and so $\mathfrak{Crys}(G; M; T)$ is not torsion-free in this case.

Case 6. Let $\Gamma = W_1$. For the cocycle f with

$$f(a) = (0,0,0,0,\tfrac{1}{2},0,0,0), \quad f(b) = (0,\tfrac{1}{2},0,\tfrac{1}{4},0,\tfrac{1}{2},0,\tfrac{1}{2}),$$

the special components of the vector $f(a)$ (the fifth one) and $f(b)$ (the last one), and $f(ab)$ (the second one) are all equal to $\frac{1}{2}$. The cocycle f determines a torsion-free group $\mathfrak{Crys}(G; M; f)$ (see also Lemma 12).

Case 7. Let $\Gamma = W_0$. The special components are the third for $f(a)$ and the fourth for $f(b)$. Let us assume that they are equal to $\frac{1}{2}$. Then there exists only one cocycle

$$f(a) = (0,0,\tfrac{1}{2},0), \quad f(b) = (0,0,0,\tfrac{1}{2}).$$

Hence $f(ab) = (\frac{1}{2},0,\frac{1}{2},\frac{1}{2})$ and the special component (the second one) for $f(ab)$ is equal to zero. The cocycle f is cohomologous to zero at the element ab and $\mathfrak{Crys}(G; M; f)$ has elements of order 2.

It follows from Lemma 16 that all K-representations Γ of G for which there are torsion-free groups $\mathfrak{Crys}(G; M; f)$ have been enumerated. Consequently Theorem 3 is proved.

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