Quantifying lawlessness in finitely generated groups

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Abstract

We introduce a quantitative notion of lawlessness for finitely generated groups, encoded by the *lawlessness growth function* $\mathcal{A}_\Gamma : \mathbb{N} \to \mathbb{N}$. We show that $\mathcal{A}_\Gamma$ is bounded iff Γ has a nonabelian free subgroup. By contrast we construct, for any nondecreasing unbounded function $f : \mathbb{N} \to \mathbb{N}$, an elementary amenable lawless groups for which $\mathcal{A}_\Gamma$ grows more slowly that f. We produce torsion lawless groups for which $\mathcal{A}_\Gamma$ is at least linear using Golod-Shafarevich theory, and give some upper bounds on $\mathcal{A}_\Gamma$ for Grigorchuk's group and Thompson's group $\mathbf{F}$. We note some connections between $\mathcal{A}_\Gamma$ and quantitative versions of residual finiteness. Finally, we also describe a function $\mathcal{M}_\Gamma$ quantifying the property of Γ having no mixed identities, and give bounds for nonabelian free groups. By contrast with $\mathcal{A}_\Gamma$, there are *no* groups for which $\mathcal{M}_\Gamma$ is bounded: we prove a universal lower bound on $\mathcal{M}_\Gamma(n)$ of the order of $\log(n)$.

1 Introduction

A *law* for a group Γ is a non-trivial word-map which vanishes identically on Γ. Γ is *lawless* if it has no laws. The goal of this article is to introduce and study a quantitative notion of lawlessness.

1.1 Statement of results

Throughout, Γ is a group generated by a finite set S. Given a nontrivial reduced word w which is not a law for Γ, the *complexity* of w in Γ is the minimal length of a tuple in Γ not evaluating to the identity under w (where the length of a tuple of elements of Γ is the sum of the lengths of those elements in the word-metric induced by S on Γ). The *lawlessness growth function* $\mathcal{A}_\Gamma^S : \mathbb{N} \to \mathbb{N}$ of Γ then sends $n \in \mathbb{N}$ to the maximal complexity occurring among the words of length at most n. It is easy to see that if Γ has a nonabelian free subgroup, then Γ is lawless and $\mathcal{A}_\Gamma^S$ is a bounded function. Our first observation is that this is the only way that bounded lawlessness growth can arise.

Theorem 1.1. *Suppose Γ is lawless. Then Γ has a nonabelian free subgroup iff $\mathcal{A}_\Gamma^S$ is bounded.*

Theorem 1.1 immediately begs the question of how slowly $\mathcal{A}_{\Gamma}^{S}$ can grow for a lawless finitely generated group which contains no nonabelian free subgroup. We give a satisfying answer to this question, constructing examples where the growth can be as slow as desired.

Theorem 1.2. *Let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded nondecreasing function with $f(1) \geq 2$. There exists an elementary amenable lawless group Γ, generated by a finite set S such that for all $n \in \mathbb{N}$, $\mathcal{A}_{\Gamma}^{S}(n) \leq f(n)$.*

The groups Γ can all be built out of a fixed lawless elementary amenable group, using a wreath product construction. Being amenable, Γ contains no nonabelian free subgroup, so that by Theorem 1.1 $\mathcal{A}_{\Gamma}^{S}$ is unbounded. Next we turn to groups of fast lawlessness growth.

Theorem 1.3. *There exists a lawless finitely generated group Γ with $\mathcal{A}_{\Gamma}^{S}(n) \gg n$.*

The group Γ is constructed using Golod-Shafarevich theory: it has long been known that G-S theory enables the construction of infinite (indeed lawless) finitely generated p-torsion groups. We run this argument in an effective manner: ensuring that the orders of elements grow as slowly as possible as a function of word-length.

We then turn to the problem of estimating $\mathcal{A}_{\Gamma}^{S}$ for specific well-known examples of lawless groups without nonabelian free subgroups. The first example we investigate is Grigorchuk's (first) group of automorphisms of the binary rooted tree. This is a torsion group (so has no nonabelian free subgroup) and has every finite 2-group as a subgroup (so has no laws).

Theorem 1.4. *Let Γ be the first Grigorchuk group. Then there exists $C > 0$ such that:*

$$n^{2/3} \ll \mathcal{A}_{\Gamma}^{S}(n) \ll \exp(Cn). \tag{1}$$

The upper bound in Theorem 1.4 is based on embedding iterated wreath products of C_2 into Γ, close to the identity, and bounds on the lengths of the shortest laws for these groups. The lower bound derives from a bound on the rate of growth of element-orders in Γ, due to Bartholdi and Šunić. Being as it is based on complexity-bounds for words of a very specific type (power words), we expect that our lower bound in Theorem 1.4 is far from best-possible; indeed, we conjecture that Grigorchuk's group has super-polynomial lawlessness growth.

Our second example is Thompson's group $\mathbf{F}$ of homeomorphisms of the interval. Brin and Squier [5] prove that $\mathbf{F}$ has no nonabelian free subgroup and satisfies no law. Making their arguments effective, we obtain the following conclusion.

Theorem 1.5. *Thompson's group $\mathbf{F}$ satisfies $\mathcal{A}_{\mathbf{F}}^{S}(n) \ll n$.*

If the word w does not vanish under evaluation at the tuple $\mathbf{g}$, then whenever $\pi : \Gamma \to Q$ is a homomorphism in whose kernel $w(\mathbf{g})$ does not lie, w is not a law for Q. For residually finite lawless groups Γ, this observation draws a link between the behaviour of $\mathcal{A}_{\Gamma}^{S}$, the lengths of the shortest laws holding in finite groups, and the *residual finiteness growth function* $\mathcal{F}_{\Gamma}^{S} : \mathbb{N} \to \mathbb{N}$ due to Bou-Rabee [2]. For instance, in [3] the author and A. Thom proved the following result.

Theorem 1.6. *Suppose Γ contains a nonabelian free subgroup and let $\delta > 0$. There exists $C > 0$ such that:*

$$\mathcal{F}_\Gamma^S(n) \geq Cn^{3/2}/\log(n)^{9/2+\delta}.$$

By the preceding paragraph, we deduce the following conclusion valid for all lawless groups, of which Theorem 1.6 is a special case (since by Theorem 1.1 groups satisfying the hypothesis of Theorem 1.6 are precisely those for which $\mathcal{A}_\Gamma^S$ is bounded).

Theorem 1.7. *Let Γ be a finitely generated lawless group and let $\delta > 0$. There exists $C > 0$ such that:*

$$\mathcal{F}_\Gamma^S\left(n\mathcal{A}_\Gamma^S(n)\right) \geq Cn^{3/2}/\log(n)^{9/2+\delta}.$$

We also have an analogous bound for the residual p-finiteness growth. Finally we note a connection between $\mathcal{A}_\Gamma^S$ and decision problems for recursively presented groups.

Theorem 1.8. *If Γ is a finitely generated recursively presented lawless group with decidable word problem, then $\mathcal{A}_\Gamma$ is computable.*

If, instead of word-maps, we consider word-maps with coefficients, then we have the class of MIF groups (instead of lawless groups) and we can define an *MIF growth function* $\mathcal{M}_\Gamma$. By contrast with Theorem 1.1, we have a nonconstant universal lower bound.

Theorem 1.9. *There are no groups of bounded MIF growth. Indeed, for any finitely generated group Γ, $\mathcal{M}_\Gamma(l) \gg \log(l)$.*

We also have an effective version of the fact (due to G. Baumslag) that nonabelian free groups are MIF.

Theorem 1.10. *Let Γ be a nonabelian free group of finite rank. Then $\mathcal{M}_\Gamma(n) \ll n\log(n)$.*

1.2 Background and structure of the paper

There are some important similarities between the class of groups satisfying laws and the class of amenable groups: both classes are closed under taking subgroups; quotients and extensions; both contain all finite and abelian groups, and neither contains F_2. An important difference between the two classes is that an ascending union of amenable groups is amenable, whereas an ascending union of groups with laws may be lawless. Our Theorem 1.2 shows how stark a distinction is opened up by this property: by taking (an extension of) an ascending union of finite groups, we may construct amenable groups which are nonetheless "very" lawless.

In the other direction, there are known examples of nonamenable groups satisfying laws: for instance Adian showed that free Burnside groups of sufficiently large odd exponent are nonamenable. Such examples are, however, quite exotic. It is, for example, an open question whether every nonamenable *residually finite* group is lawless [6]. Our Theorem 1.7 shows that there is a tension between residual finiteness and lawlessness, in a quantitative sense: a group cannot be simultaneously "very" residually finite and "very" lawless.

There has also been great interest in recent years in estimating the lengths of the shortest laws for *finite* groups. One may view the following problem as the "finitary" version of the problem of estimating $\mathcal{A}_\Gamma$ for a fixed lawless group: given a sequence (G_n) of finite groups which do not share a common law, can we estimate the asymptotic behaviour of the length of the shortest law $w_n \in F_2$ for G_n as $n \to \infty$? This latter problem has been explored with G_n taken to be the symmetric group $\mathrm{Sym}(n)$ [11]; a finite simple group of Lie type of bounded rank [4], or the direct product of all groups of order at most n (see Theorem 8.2 below). We exploit analogies between the finite and infinite settings several times in this paper, most notably in Theorems 1.2; 1.4 and 1.7.

The paper is structured as follows. In Section 2 we define $\mathcal{A}_\Gamma^S$ and establish its basic properties. We also prove Theorem 1.8 in this Section. Sections 3-7 are devoted, respectively, to the proofs of Theorems 1.1-1.5. In Section 8 we establish connections between $\mathcal{A}_\Gamma^S$ and quantitative versions of residual finiteness. In Section 9 we define $\mathcal{M}_\Gamma^S$ and prove Theorems 1.9 and 1.10. We conclude with a selection of open problems, speculations and directions for future research.

2 Preliminaries

Let $|\cdot|_S : \Gamma \to \mathbb{N}$ be the word-length function induced on Γ by S. Denote by $B_S(l)$ the set of elements of Γ of length at most l. For F_k the free group of rank k, we denote by $|\cdot|$ the word length on F_k induced by a fixed free basis $X_k = \{x_1, \dots x_k\}$, and may write $B(l)$ for $B_{X_k}(l)$. The word $w \in F_k$ is a *law* for Γ if it is non-trivial in F_k and lies in the kernel of every homomorphism $F_k \to \Gamma$. The set of *non-laws* for Γ in F_k is:

$$N_k(\Gamma) = F_k \setminus \bigcap_\pi \ker(\pi)$$

where the intersection is taken over all homomorphisms $\pi : F_k \to \Gamma$. For $w \in N_k(\Gamma)$, we define the *complexity* of w in Γ (with respect to S) to be:

$$\chi_\Gamma^S(w) = \min\Big\{ \sum_{i=1}^{k} |g_i|_S : \mathbf{g} \in \Gamma^k, w(\mathbf{g}) \neq 1 \Big\}.$$

For $W \subseteq N_k(\Gamma)$, the *W-lawlessness growth function* (or anarchy growth; pandemonium growth etc.) of Γ with respect to S is defined to be:

$$\mathcal{A}_{\Gamma,W}^S(l) = \max\big\{ \chi_\Gamma^S(w) : w \in W; |w| \leq l \big\}.$$

If $N_k(\Gamma) = F_k \setminus \{1\}$, we denote $\mathcal{A}_{\Gamma,F_k\setminus\{1\}}^S$ by $\mathcal{A}_\Gamma^S$ and refer to it simply as the *lawlessness growth of* Γ (with respect to S).

Lemma 2.1. *If $W' \subseteq W \subseteq N_k(\Gamma)$, then $\mathcal{A}_{\Gamma,W}^S(l) \geq \mathcal{A}_{\Gamma,W'}^S(l)$ for all $l \in \mathbb{N}$.*

Notation 2.2. *For nondecreasing functions $F_1, F_2 : \mathbb{N} \to \mathbb{N}$ we write $F_1 \preceq F_2$ if there exists $K \in \mathbb{N}$ such that $F_1(l) \leq K F_2(Kl)$ for all $l \in \mathbb{N}$, and $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$. It is clear that $\approx$ is an equivalence relation.*

Remark 2.3. In our notation $\mathcal{A}_\Gamma^S$ for the lawlessness growth function, we leave implicit the rank k of the free group F_k in which our words lie. We shall assume k to be an arbitrary (but fixed) integer at least 2. It turns out that very little is lost by making this assumption. First suppose that Γ has no law in F_k and $1 \leq k' \leq k$. Embed $F_{k'}$ into F_k by extending the set of free variables (with ι the inclusion map). Then for $w \in F_{k'}$ and $g_1, \dots, g_{k'} \in \Gamma$,

$$w(g_1, \dots, g_{k'}) = \iota(w)(g_1, \dots, g_{k'}, e, \dots, e), \text{ so } \chi_\Gamma^S(w) = \chi_\Gamma^S\big(\iota(w)\big). \tag{2}$$

We apply Lemma 2.1 with $W' = \iota(F_{k'}) \setminus \{1\}$ and $W = F_k \setminus \{1\}$ so that by (2),

$$\mathcal{A}^S_{\Gamma, F_k\setminus\{1\}}(l) \geq \mathcal{A}^S_{\Gamma, \iota(F_{k'})\setminus\{1\}}(l) = \mathcal{A}^S_{\Gamma, F_{k'}\setminus\{1\}}(l)$$

for all $l \in \mathbb{N}$. Conversely suppose that $2 \leq k' \leq k$ and that Γ has no laws in $F_{k'}$. Let $\phi : F_k \hookrightarrow F_{k'}$ be an embedding, and for $1 \leq i \leq k$ write $\phi(x_i) = v_i \in F_{k'}$. For $w \in F_k$ non-trivial, $\phi(w) = w(v_1, \dots, v_k)$ is not a law for Γ, so there are $\mathbf{g} \in \Gamma^{k'}$ with $\phi(w)(\mathbf{g}) = w(v_1(\mathbf{g}), \dots, v_k(\mathbf{g})) \neq e$. Hence the k-tuple $(v_1(\mathbf{g}), \dots, v_k(\mathbf{g}))$ witnesses that w is not a law for Γ, and that $\chi_\Gamma^S(w) \leq C\chi_\Gamma^S\big(\phi(w)\big)$, where $C = \max\{|v_i| : 1 \leq i \leq k\}$. Since $|\phi(w)| \leq C|w|$ and $|v_i(\mathbf{g})|_S \leq C(|g_1|_S + \dots + |g_k|_S)$,

$$\mathcal{A}^S_{\Gamma, F_k\setminus\{1\}}(l) \leq Ck\mathcal{A}^S_{\Gamma, \phi(F_k)\setminus\{1\}}(Cl) \leq Ck\mathcal{A}^S_{\Gamma, F_{k'}\setminus\{1\}}(Cl) \tag{3}$$

for all $l \in \mathbb{N}$. Thus $\mathcal{A}^S_{\Gamma, F_k\setminus\{1\}} \approx \mathcal{A}^S_{\Gamma, F_{k'}\setminus\{1\}}$.

Lemma 2.4. *Let $\Delta \leq \Gamma$ be a subgroup generated by a finite set T. Then there exists $C > 0$ such that for all $1 \neq w \in F_k$, if w is not a law for Δ, then:*

$$\chi_\Delta^T(w) \geq C\chi_\Gamma^S(w). \tag{4}$$

Thus, if $W \subseteq N_k(\Delta)$, then $\mathcal{A}^T_{\Delta,W}(l) \geq C\mathcal{A}^S_{\Gamma,W}(l)$ for all $l \in \mathbb{N}$.

Proof. There exists $C > 0$ such that for all $g \in \Gamma$, $|g|_T \geq C|g|_S$. (4) follows immediately. ☐

In view of Remark 2.3 and Lemma 2.4, we study $\mathcal{A}_\Gamma^S$ up to $\approx$.

Corollary 2.5. *Let S_1 and S_2 be finite generating sets for Γ. Then $\mathcal{A}_\Gamma^{S_1} \approx \mathcal{A}_\Gamma^{S_2}$.*

Lemma 2.6. *Let $\pi : \Gamma \twoheadrightarrow Q$ be an epimorphism of groups. Then $N_k(Q) \subseteq N_k(\Gamma)$ and, for any $w \in N_k(Q)$,*

$$\chi_\Gamma^S(w) \leq \chi_Q^{\pi(S)}(w). \tag{5}$$

Thus, if $W \subseteq N_k(Q)$, then $\mathcal{A}^S_{\Gamma,W}(l) \leq \mathcal{A}^{\pi(S)}_{Q,W}(l)$ for all l.

Example 2.7. *Suppose that Γ is a torsion group. The* torsion growth function *of Γ (with respect to S) is defined to be:*

$$\pi_\Gamma^S(n) = \max\{o(g) : |g|_S \leq n\}$$

and was introduced in [9]. In case Γ is a p-group of infinite exponent, there is an intimate connection between torsion growth and lawlessness growth: let $W = \{x^{p^k} : k \in \mathbb{N}\} \subset F_2$ be the set of p-power words. Then for $m, n \in \mathbb{N}$,

$$\mathcal{A}^S_{\Gamma,W}(p^m) \geq n + 1 \Leftrightarrow \pi^S_\Gamma(n) \leq p^m.$$

Known finitely generated torsion p-groups of infinite exponent include various branch groups*, for which the torsion growth has been estimated. We discuss this further in Section 6.*

Our proof of Theorems 1.2 and 1.9 uses the following Proposition, which is Lemma 2.2 from [11]. For $w \in F_k$ and Γ any group, let:

$$Z(w, \Gamma) = \{\mathbf{g} \in \Gamma^k : w(\mathbf{g}) = e\}$$

be the *vanishing set* of w in Γ.

Proposition 2.8. *Let $k \geq 2$ and let $w_1, \dots, w_m \in F_k$ be nontrivial. There exists $w \in F_k$ nontrivial such that, for any group Γ,*

$$Z(w, \Gamma) \supseteq Z(w_1, \Gamma) \cup \cdots \cup Z(w_m, \Gamma)$$

and $|w| \leq 16m^2 \max_i |w_i|$.

Corollary 2.9. *Let Γ be a lawless group and let $k \geq 2$. Then for all $l \geq 1$, there exist $g_1, \dots, g_k \in \Gamma$ such that, for all $v \in F_k$ nontrivial with $|v| \leq l$, $v(g_1, \dots, g_k) \neq e$.*

Proof. Suppose not. Let $w_1, \dots, w_m \in F_k$ be a list of all nontrivial reduced words of length at most l, and let w be as in the conclusion of Proposition 2.8. Then for every $g_1, \dots, g_k \in \Gamma$, there exists $1 \leq i \leq m$ such that $(g_1, \dots, g_k) \in Z(w_i, \Gamma) \subseteq Z(w, \Gamma)$, so $w(g_1, \dots, g_k)$, and therefore w is a law for Γ. $\square$

Proof of Theorem 1.8. Let S be a finite generating set for Γ. We are given an algorithm `WORDPROBLEM` which takes as input an element of $F(S)$ and determines whether or not it evaluates to the identity element of Γ. We describe an algorithm `COMPLEX` which takes as input a word $w \in F_k$ and returns $\chi^S_\Gamma(w)$. Applying `COMPLEX` to all w of length $\leq n$, we compute $\mathcal{A}^S_\Gamma(n)$.

At the mth step, `COMPLEX` either verifies that $\chi^S_\Gamma(w) \leq m$ and terminates, or verifies that $\chi^S_\Gamma(w) > m$ and proceeds to the $(m+1)$th step. We thus establish the exact value of $\chi^S_\Gamma(w)$.

The mth step of `COMPLEX` runs as follows. Let B be a list of all elements of $F(S)$ of length at most m (a finite set). For each ordered k-tuple $\mathbf{v}$ of (not necessarily distinct) elements of B, apply `WORDPROBLEM` to $w(\mathbf{v})$. Note that $\chi^S_\Gamma(w) \leq m$ iff for some $\mathbf{v}$, $w(\mathbf{v})$ is non-trivial in Γ. $\square$

3 Bounded lawlessness growth

We recall two basic facts about free groups. Let F be a free group of finite rank.

Lemma 3.1. *Let* $1 \neq N \lhd F$.

(i) If $|F : N| < \infty$*, then* $\mathrm{rk}(N) - 1 = |F : N| \cdot (\mathrm{rk}(F) - 1)$*;*

(ii) If $|F : N| = \infty$ *then* $\mathrm{rk}(N) = \infty$.

In particular, if F *is nonabelian then* N *is not cyclic.*

Lemma 3.2. *The centralizer of a non-trivial element of* F *is a cyclic subgroup.*

Proof of Theorem 1.1. Suppose $a, b \in \Gamma$ with $\langle a, b\rangle \cong F_2$. Let $C > 0$ with $|a|_S + |b|_S \leq C$. Then for all $1 \neq w \in F_2$, $w(a, b) \neq 1$, so $\chi_\Gamma^S(w) \leq C$. Thus $\mathcal{A}_\Gamma^S$ is bounded.

Conversely suppose (for a contradiction) that $\mathcal{G}_\Gamma^S$ is bounded but that Γ has no subgroup isomorphic to F_2. Let $r \in \mathbb{N}$ be minimal such that there exist $(g_1, h_1), \dots, (g_r, h_r) \in \Gamma \times \Gamma$ such that for all nontrivial $w(x, y) \in F_2$, there exists $1 \leq i \leq r$ such that $w(g_i, h_i) \neq 1$. Note that $r \geq 2$ (else $\langle g_1, h_1\rangle \cong F_2$). By minimality of r, there exists $1 \neq w_r \in F_2$ such that:

$$w_r(g_1, h_1), \dots, w_r(g_{r-1}, h_{r-1}) = 1.$$

Let $N = \{w \in F_2 : w(g_1, h_1) = 1\}$. Then $N \lhd F_2$ (it is the kernel of the homomorphism sending an ordered basis of F_2 to g_1, h_1) and $1 \neq N$ (for instance, $w_r \in N$). But $N \leq C_{F_2}(w_r)$, as otherwise any nontrivial word of the form $[w_r, v]$, for $v \in N$, would be a law for Γ (contradicting hypothesis). By Lemma 3.2 N is cyclic, contradicting Lemma 3.1 □

4 Amenable groups of slow lawlessness growth

In this Section we prove Theorem 1.2. We start with an elementary combinatorial fact.

Lemma 4.1. *There exist increasing functions* $p, q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ *such that* $p(1) = q(1) = 0$ *and, for all* $i, j, k, l \in \mathbb{N}$,

(i) For $r = p$ *or* q*, if* $r(j) - r(k) = r(l) - r(i)$ *then either (a)* $i = k$ *and* $j = l$*, or (b)* $j = k$ *and* $i = l$*;*

(ii) If $q(j) - p(k) = q(l) - p(i)$ *then* $i = k$ *and* $j = l$.

Proof. We construct p and q via a recursive process. We have $p(1) = q(1) = 0$ and set $p(2) = 1$, $q(2) = 2$, so that (i) and (ii) hold for $i, j, k, l \in \{1, 2\}$. Supposing we have constructed $p(1) \leq q(1) < p(2) \leq q(2) < \dots < p(n) \leq q(n)$ such that (i) and (ii) hold for all $1 \leq i, j, k, l \leq n$, we define $p(n + 1)$ such that:

$$p(n + 1) \geq p(n) + q(n) + 1 \tag{6}$$

and then define $q(n + 1)$ by:

$$q(n + 1) \geq p(n + 1) + q(n) + 1 \tag{7}$$

so that $p(n) < q(n) < p(n + 1) < q(n + 1)$. We check that (i) and (ii) hold for all $1 \leq i, j, k, l \leq n + 1$.

For (i), consider the equation $p(j) - p(k) = p(l) - p(i)$. We may assume that $l \geq j \geq k$. Since p is increasing, if $j = k$ then $i = l$ and if $l = j$ then $i = k$, so we may assume $l > j > k$. We claim that $p(j) - p(k) < p(l) - p(i)$. Subject to our assumptions, the maximal value of $p(j) - p(k)$ is attained for $j = l - 1$ and $k = 1$, while the minimal value of $p(l) - p(i)$ is attained for $i = l - 1$, and $p(l-1) - p(1) < p(l) - p(l-1)$ by construction. The argument for q is exactly the same.

For (ii), consider the equation $q(j) - p(k) = q(l) - p(i)$. We may assume $l > j$ (as before, WLOG $l \geq j$ and if $l = j$ then $i = k$ as p is increasing). We distinguish two cases. In the case that $l \geq i$, we have $j \geq k$ (since the quantity in the equation is non-negative) so that:

$$q(j) - p(k) \leq q(l-1) < q(l) - p(l) \leq q(l) - p(i).$$

In the second case, $l < i$ so $j < k < i$ and:

$$q(l) - q(j) \leq q(l) < p(i) - p(i-1) \leq p(i) - p(k)$$

so that $q(l) - p(i) < q(j) - p(k)$. □

Remark 4.2. *Though we shall not need it in the sequel, it is not difficult to see that a minimal solution to the inequalities (6) and (7) yields functions p and q growing at most exponentially.*

Fix a lawless elementary amenable group Δ (not necessarily finitely generated). For example, we may take Δ to be the direct sum of any sequence of finite groups which do not have a common law, such as the sequence of all finite symmetric groups [11]. By Corollary 2.9, for each $l \geq 1$ we have $g_l, h_l \in \Delta$ such that for all $v \in F_2$ nontrivial, if $|v| \leq l$ then $v(g_l, h_l) \neq e$. Let $L : \mathbb{N} \to \mathbb{N}$ be an increasing function to be determined. Let $\hat{g}, \hat{h} : \mathbb{Z} \to \Delta$ be defined as follows:

(i) For each $n \in \mathbb{N}$, $\hat{g}(p(n)) = g_{L(n)}$ and $\hat{h}(q(n)) = h_{L(n)}$;

(ii) $\hat{g}(m) = e$ for $m \notin \mathrm{im}(p)$;

(iii) $\hat{h}(m) = e$ for $m \notin \mathrm{im}(q)$,

where p and q are as in Lemma 4.1. Let $G = \Delta \operatorname{Wr} \mathbb{Z} = \Delta^{\mathbb{Z}} \rtimes \mathbb{Z}$ be the *unrestricted* wreath product of Δ and $\mathbb{Z}$, with the $\mathbb{Z}$-factor being generated by t. Let $S = S(L) = \{\hat{g}, \hat{h}, t\}$ and define $\Gamma = \Gamma(L) = \langle S(L) \rangle \leq G$. We claim that, for an appropriate choice of the function L, the group Γ satisfies the conclusion of Theorem 1.2, the proof of which we divide between the next two results.

Theorem 4.3. *For any L, Γ is elementary amenable.*

Proof. Let $N = \Gamma \cap \Delta^{\mathbb{Z}}$. Then $N \lhd \Gamma$ with $\Gamma/N \cong \mathbb{Z}$, so it suffices to check that N is elementary amenable. $N/[N,N]$ is a countable abelian group, so it suffices to check that $[N,N]$ is elementary amenable.

The group N is generated by $X = \{\hat{g}^{t^n}, \hat{h}^{t^n} : n \in \mathbb{Z}\}$, so $[N,N]$ is normally generated in N by $Y = \{[\hat{f}_1, \hat{f}_2] : \hat{f}_1, \hat{f}_2 \in X\}$. If $Y \subseteq \bigoplus_{\mathbb{Z}} \Delta$, then $[N,N] \leq \bigoplus_{\mathbb{Z}} \Delta$

(since $\bigoplus_{\mathbb{Z}} \Delta \lhd \Delta^{\mathbb{Z}}$), so that $[N, N]$ is elementary amenable (being a subgroup of the elementary amenable group $\bigoplus_{\mathbb{Z}} \Delta$).

We therefore claim that $Y \subseteq \bigoplus_{\mathbb{Z}} \Delta$. Recall that the *support* of $\hat{f} \in \Delta^{\mathbb{Z}}$ is $\operatorname{supp}(\hat{f}) = \{n \in \mathbb{Z} : \hat{f}(n) \neq e\}$, so that $\bigoplus_{\mathbb{Z}} \Delta$ is precisely the group of finite-support elements of $\Delta^{\mathbb{Z}}$. We assert that:

(a) for every $\hat{f}_1, \hat{f}_2 \in \Delta^{\mathbb{Z}}$, $\operatorname{supp}([\hat{f}_1, \hat{f}_2]) \subseteq \operatorname{supp}(\hat{f}_1) \cap \operatorname{supp}(\hat{f}_2)$;

(b) For all pairs of distinct elements $\hat{f}_1, \hat{f}_2 \in X$, $|\operatorname{supp}(\hat{f}_1) \cap \operatorname{supp}(\hat{f}_2)| \leq 1$,

whence the desired claim. Observation (a) is clear, and (b) follows from Lemma 4.1: $\operatorname{supp}(\hat{g}) \subseteq \operatorname{im}(p)$ and $\operatorname{supp}(\hat{h}) \subseteq \operatorname{im}(q)$, so a point in $\operatorname{supp}(\hat{g}^{t^m}) \cap \operatorname{supp}(\hat{h}^{t^n})$ is $p(i) + m = q(l) + n$ for some $i, l \in \mathbb{N}$. A second point in the intersection would yield a second pair $j, k \in \mathbb{N}$ satisfying $p(k) + m = q(j) + n$, so that $q(j) - p(k) = q(l) - p(i)$, contradicting Lemma 4.1 (ii). We argue similarly for $\operatorname{supp}(\hat{g}^{t^m}) \cap \operatorname{supp}(\hat{g}^{t^n})$ and $\operatorname{supp}(\hat{h}^{t^m}) \cap \operatorname{supp}(\hat{h}^{t^n})$ for $m \neq n$, using Lemma 4.1 (i). $\square$

Theorem 4.4. *For every unbounded nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ with $f(1) \geq 2$, there exists L such that for all $n \in \mathbb{N}$, $\mathcal{A}_{\Gamma(L)}^{S(L)}(n) \leq f(n)$.*

Proof. Let $m \in \mathbb{N}$ and let $w \in F_2$ be nontrivial with $|w| \leq L(m)$. Then:

$$w\big(\hat{g}^{t^{q(m)-p(m)}}, \hat{h}\big)\big(q(m)\big) = w(g_{L(m)}, h_{L(m)}) \neq e$$

so that w is not a law for $\Gamma(L)$ and:

$$\chi_{\Gamma(L)}^{S(L)}(w) \leq \left|\hat{g}^{t^{q(m)-p(m)}}\right|_{S(L)} + \left|\hat{h}\right|_{S(L)} \leq 2\big(q(m) - p(m) + 1\big). \tag{8}$$

We require $L(m) \in \mathbb{N}$ sufficiently large that $f(L(m)) \geq 2\big(q(m+1) - p(m+1) + 1\big)$ (possible since f is unbounded).

Now let $n \in \mathbb{N}$. Suppose first that $n \geq L(1)$. Let $m \in \mathbb{N}$ with $L(m) \leq n \leq L(m+1)$. Then:

$$\begin{aligned}\mathcal{A}_{\Gamma(L)}^{S(L)}(n) \leq \mathcal{A}_{\Gamma(L)}^{S(L)}(n) &\leq \mathcal{A}_{\Gamma(L)}^{S(L)}\big(L(m+1)\big)\\ &\leq 2\big(q(m+1) - p(m+1) + 1\big) \text{ (by (8))}\\ &\leq f\big(L(m)\big)\\ &\leq f(n).\end{aligned}$$

On the other hand, if $n \leq L(1)$ then for any $w \in F_2$ nontrivial with $|w| \leq n$, $w(\hat{g}, \hat{h}) \neq e$, so $\chi_{\Gamma(L)}^{S(L)}(w) \leq 2$ and $\mathcal{A}_{\Gamma(L)}^{S(L)}(n) \leq 2$. $\square$

5 Golod-Shafarevich groups of linear lawlessness growth

The goal of this Section is to prove Theorem 1.3. Throughout this Section p is an arbitrary (but fixed) prime number. We follow the treatment of Golod-Shafaverich groups from Chapter 3 of [8]. For Γ an abstract group, denote by $\hat{\Gamma}_{(p)}$ the *pro-p completion* of Γ. Let $\mathbb{F}_p\langle\langle U_k\rangle\rangle$ be the algebra of power-series in the non-commuting variables $U_k = \{u_1, \dots, u_k\}$ over $\mathbb{F}_p$. For $f \in \mathbb{F}_p\langle\langle U_k\rangle\rangle$, the *degree* $\deg(f)$ of f is the minimal length of a monomial occuring in f with nonzero coefficient. Let $\hat{F}_k$ be a free pro-p group on the finite set $X_k = \{x_1, \dots, x_k\}$. There is a continuous monomorphism (the *Magnus embedding*) $\mu : \hat{F}_k \hookrightarrow \mathbb{F}_p\langle\langle U_k\rangle\rangle^*$ extending $x_i \mapsto 1 + u_i$ [8].

Definition 5.1. *(i) A pro-p group G is* Golod-Shafarevich *if it admits a pro-p presentation $G = \langle X_k \mid R\rangle$ such that there exists $\tau \in (0,1)$ with:*

$$1 - k\tau + \sum_{r\in R} \tau^{D(r)} < 0, \tag{9}$$

where for $r \in \hat{F}$, $D(r) = \deg(\mu(r) - 1)$ is the degree *of r;*

(ii) An abstract group Γ is Golod-Shafarevich if $\hat{\Gamma}_{(p)}$ is a Golod-Shafarevich pro-p group.

All the relations in the presentations we construct will be p-powers, and estimating the degrees of these is easily done, direct from the definition of the function D.

Lemma 5.2. *For all $w \in \hat{F}_k$, $D(w^p) = pD(w)$.*

The next, basic Lemma immediately implies that if $G = \langle X_k \mid R\rangle$ is an *abstract* group presentation satisfying (9), then G is an abstract Golod-Shafarevich group.

Lemma 5.3. *Let Γ be an abstract group, and suppose $\langle S \mid R\rangle$ is an abstract group presentation for Γ. Then $\langle S \mid R\rangle$ is a pro-p presentation for $\hat{\Gamma}_{(p)}$.*

Finally, we need a guarantee that the groups we construct are indeed lawless (see [14] p.224).

Theorem 5.4. *Let G be a Golod-Shafarevich pro-p group. Then G has a non-abelian free subgroup.*

Corollary 5.5. *Suppose Γ is an (abstract) Golod-Shafarevich group. Then Γ is lawless.*

Proof. Let $1 \neq w \in F_2$. Let $\mathcal{N}$ be the inverse system of normal subgroups of p-power index in Γ, and let $G = \varprojlim_{N\in\mathcal{N}} \Gamma/N$ be the pro-p completion of Γ. By Theorem 5.4 there exist $g, h \in G$ freely generating a rank-2 free subgroup of G. Then there exists $N \in \mathcal{N}$ such that the image of $w(g,h)$ is non-trivial in Γ/N. In particular, w is not a law for Γ/N. But Γ/N is also a quotient of Γ, so w is not a law for Γ either. □

Theorem 1.3 is immediate from Corollary 5.5 and the next result. Recall that π_Γ^S is the *torsion growth function* of Γ (see Example 2.7 above).

Proposition 5.6. *For all $k \geq 2$ there is a torsion Golod-Shafarevich p-group $\Gamma = \langle X_k \mid R'\rangle$, and a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\pi_\Gamma^{\pi(X_k)}(n) \leq Cn. \tag{10}$$

Proof of Theorem 1.3. Let v be as in Proposition 5.6. Let $W = \{x^{p^k} : k \in \mathbb{N}\}$. Then for all $n \in \mathbb{N}$,

$$\mathcal{A}_{\Gamma}^{X_k}(Cn) \geq \mathcal{A}_{\Gamma,W}^{X_k}(Cn) \geq n+1$$

(by Lemma 2.1 and Example 2.7). □

Proof of Proposition 5.6. Let $w_1, w_2, w_3 \dots$ be an enumeration of the non-trivial elements of $F(X_k)$, ordered such that $|w_n|$ is non-decreasing. Recall that there exists $C > 0$ such that $|B_{X_k}(l)| \leq C(2k-1)^l$. Choose $q > 1$ and let $0 < c < \log(q)/\log(2k-1)$, so that:

$$a_m := Cq^{p^m} \geq \left|B_{X_k}(cp^m)\right|. \tag{11}$$

for $m \geq 1$ (with $a_0 := 0$). Choose $m_0 \in \mathbb{N}$ and set $r_n = w_n^{p^{m+m_0}}$ for $a_{m-1}+1 \leq n \leq a_m$ and $R' = \{r_n : n \in \mathbb{N}\}$. Let $\Gamma = \langle X_k | R' \rangle$ and let $g \in \Gamma$. If $m \in \mathbb{N}$ is such that $cp^{m-1} \leq |g|_{X_k} \leq cp^m$, then there exists $1 \leq n \leq a_m$ such that $g = w_n$ in Γ, so that the order of g in Γ divides $p^{m+m_0} \leq p^{m_0+1}|g|_{X_k}/c$, so $\tau_{\Gamma}^{X_k}$ grows at most linearly.

It therefore suffices to check that we can choose q and m_0 such that Γ is Golod-Shafarevich. By Lemma 5.2, $D(r_n) \geq p^{m+1}$ for $n > a_m$, hence for $\tau \in (0,1)$,

$$\sum_{r \in R'} \tau^{D(r)} \leq \sum_{m=1}^{\infty} a_m \tau^{p^{m+m_0}} = C \sum_{m=1}^{\infty} \left(q\tau^{p^{m_0}}\right)^{p^m} = C \sum_{m=1}^{\infty} h^{p^m} \tag{12}$$

where $h = q\tau^{p^{m_0}}$. It is clear that the right-hand side of (12) can be made arbitrarily small by making h arbitrarily small. If we take $q = 2$ and $\tau = 3/4$, then we have (9) provided h is sufficiently small that the right-hand side of (12) is $< (3k-4)/4$. This can be achieved for m_0 larger than an absolute constant. □

Remark 5.7. It is not difficult to strengthen the proof of Proposition 5.6 to show that if $\tilde{\Gamma}$ is Golod-Shafarevich, then there is a torsion Golod-Shafarevich p-group Γ, a surjective homomorphism $\pi : \tilde{\Gamma} \to \Gamma$ and a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\pi_{\Gamma}^{\pi(S)}(n) \leq Cn. \tag{13}$$

For, the pro-p completion of $\tilde{\Gamma}$ already has a presentation satisfying (9). Adding in the relations R' from the proof of Proposition 5.6 yields a presentation for the pro-p completion of Γ, which has at most linear torsion growth. But we can make the contribution of R' to the left-hand side of (9) arbitrarily small, so that the inequality (9) still holds, and Γ is still Golod-Shafarevich.

6 Grigorchuk's group

For the duration of this Section, Γ will denote the first Grigorchuk group. For background on this group and automorphisms of rooted trees more generally, we refer to [10]. The group Γ is defined as a group of automorphisms of the binary rooted tree $\mathcal{T}$: $V(\mathcal{T}) = \{0,1\}^*$ is the set of finite formal words in the alphabet $\{0,1\}$, and for each $v \in \{0,1\}^*$, $\epsilon \in \{0,1\}$, there is an edge joining v to $v\epsilon$. For each $v \in V(\mathcal{T})$, let $\mathcal{T}_v$ be the subtree rooted at v, that is the induced subgraph on $\{vw : w \in V(\mathcal{T})\}$. Note that $\mathcal{T}_v \cong \mathcal{T}$ via $vw \mapsto w$. Let:

$$\mathrm{top}(v) = \{g \in \mathrm{Aut}(\mathcal{T}) : w^g = w \text{ for all } w \in V(\mathcal{T}) \setminus V(\mathcal{T}_v)\}$$

be the *restricted stabilizer* at v. Then for every v, the above isomorphism $\mathcal{T}_v \cong \mathcal{T}$ induces an isomorphism $\mathrm{top}(v) \to \mathrm{Aut}(\mathcal{T})$; we write $g \mid_{\mathcal{T}_v} \in \mathrm{Aut}(\mathcal{T})$ for the image under this isomorphism of $g \in \mathrm{top}(v)$.

Let $V_n = \{0,1\}^n$ be the set of words of length n (geometrically, the set of vertices at distance n from the root of the tree). Let $\mathrm{Stab}(n) \leq \mathrm{Aut}(\mathcal{T})$ be the pointwise stabilizer of V_n. Then for $g \in \mathrm{Stab}(n)$, there exist unique $g_v \in \mathrm{top}(v)$, for $v \in V_n$, such that:

$$g = \prod_{v \in V_n} g_v \tag{14}$$

(note that the g_v, being disjointly supported, commute). Extending our notation above, we write $g \mid_{\mathcal{T}_v} = g_v \mid_{\mathcal{T}_v}$ for $g \in \mathrm{Stab}(n)$, $v \in V_n$ and $g_v \in \mathrm{top}(v)$ as in (14). The decomposition (14) yields an isomorphism $\mathrm{Aut}(\mathcal{T})^{V_n} \cong \mathrm{Stab}(n)$, and for $K \leq \mathrm{Aut}(\mathcal{T})$ we shall identify K^{V_n} with its image in $\mathrm{Aut}(\mathcal{T})$ under this isomorphism. Likewise, when g and g_v are as in (14) we shall write $g = (g_v)_{v \in V_n}$.

The automorphisms a, b, c and d of $\mathcal{T}$ are defined as follows:

$$\begin{array}{ll} (0w)^a = 1w; & (1w)^a = 0w; \\ (0w)^b = 0(w^a); & (1w)^b = 1(w^c); \\ (0w)^c = 0(w^a); & (1w)^c = 1(w^d); \\ (0w)^d = 0w; & (1w)^d = 1(w^b) \end{array}$$

for any $w \in \{0,1\}^*$. In other words, a swaps the two subtrees $\mathcal{T}_0$ and $\mathcal{T}_1$, and $b, c, d \in \mathrm{Stab}(1)$ are given by:

$$b = (a, c),\ c = (a, d),\ d = (1, b).$$

Grigorchuk's group Γ is defined to be the subgroup of $\mathrm{Aut}(\mathcal{T})$ generated by $S = \{a, b, c, d\}$. Let $x = abab \in \Gamma$ and let $K = \langle x \rangle^{\Gamma} \lhd \Gamma$ be the normal closure of x in Γ.

Proposition 6.1 ([10] Proposition 8)**.** *We have $K^{V_1} \leq K$ (and hence $K^{V_n} \leq K$ for all n).*

Proposition 6.2. *Let $y_n \in K^{V_n}$ be given by:*

$$y_n \mid_{\mathcal{T}_v} = \begin{cases} x & v = 0^n \\ e & \textit{otherwise} \end{cases}$$

for $v \in V_n$. Then there exists $C > 0$ such that for all n, $|y_n|_S \leq C(1+\sqrt{3})^n$.

Proof. We have $y_0 = x$ and $y_1 = y = (x, 1)$. As noted in the proof of [10, Proposition 9], $[x, y] = (x^{-1}, 1, 1, 1) = y_2^{-1}$, so $y_2 = [y_1, y_0]$. Since for $n \geq 1$ we have $y_n = (y_{n-1}, 1)$, so by induction, for $n \geq 3$ we have:

$$y_n = (y_{n-1}, 1) = \big([y_{n-2}, y_{n-3}], 1\big) = \big[(y_{n-2}, 1), (y_{n-3}, 1)\big] = [y_{n-1}, y_{n-2}]. \tag{15}$$

Now there exists $C_0 > 0$ such that $|y_0|_S, |y_1|_S \leq C_0$ (the latter since $y_1 \in K^{V_1} \leq K \leq \Gamma$), and by (15) we have $|y_n|_S \leq 2|y_{n-1}|_S + |y_{n-2}|_S$ for $n \geq 2$. Solving the corresponding recurrence yields the required bound. $\square$

Let W_n be the $(n+1)$-fold iterated regular wreath product of C_2; that is $W_0 = C_2$ and $W_{n+1} = W_n \wr C_2$. Alternatively we may view W_n as a subgroup of $\mathrm{Aut}(\mathcal{T})$, as follows: let $a \in \mathrm{Aut}(\mathcal{T})$ be as above, and identify $W_0 = C_2$ with $\langle a \rangle \leq \mathrm{Aut}(\mathcal{T})$. Having defined $W_n \leq \mathrm{Aut}(\mathcal{T})$, a general element of W_{n+1} is $(g_0, g_1)a^{\epsilon} \in W_n \wr C_2$ (with $g_0, g_1 \in W_n$ and $\epsilon \in \{0, 1\}$). We identify this with the unique $g \in \mathrm{Aut}(\mathcal{T})$ satisfying:

$$ga^{\epsilon} \in \mathrm{Stab}(1) \text{ and } (ga^{\epsilon})\,|_{\mathcal{T}_v} = g_v \text{ for } v \in V_1.$$

It is easily seen that this identification yields an embedding of W_{n+1} as a subgroup of $\mathrm{Aut}(\mathcal{T})$. Moreover, the action of $W_n \leq \mathrm{Aut}(\mathcal{T})$ on V_{n+1} is faithful and yields an isomorphism of permutation groups from W_{n+1} to the imprimitive permutational wreath product $\langle a \rangle \wr_{V_{n+1}} W_n$ (since $V_{n+2} = V_{n+1} \times V_1$ via the identification of $v\epsilon$ with (v, ϵ)).

Define, for $n \in \mathbb{N}$, $a_n \in \mathrm{Aut}(\mathcal{T})$ by: $a_0 = a$ and $a_{n+1} \in \mathrm{top}\left(0^{n+1}\right)$ by:

$$a_{n+1}|_{T_{0^{n+1}}} = a$$

so that $S_n = \{a_0, a_1, \dots, a_n\}$ generates W_n.

Lemma 6.3. *For any $v \in V_{n+1}$, there exists $h \in W_n$ with $|h|_{S_n} \leq n+1$ and $(0^{n+1})^h = v$.*

Proof. If $v = \epsilon_0\epsilon_1 \cdots \epsilon_n$, then $h = a_n^{\epsilon_n} a_{n-1}^{\epsilon_{n-1}} \cdots a_1^{\epsilon_1} a_0^{\epsilon_0}$ works. $\square$

Proposition 6.4. *W_n has no law in F_k of length at most $n+1$. In other words, $B(n+1) \setminus \{1\} \subseteq N_k(W_n)$. More precisely, for any $1 \neq w \in F_k$ with $|w| \leq n+1$, there exist $g_1, \dots, g_k \in W_n$ with:*

$$w(g_1, \dots, g_k) \neq e \textit{ and } \sum_{i=1}^{k} |g_i|_{S_n} \leq (n+1)^2. \tag{16}$$

Proof. We proceed by induction on n. In fact, we make a stronger claim. For $w \in F_k$ and $0 \leq m \leq |w|$ define $w^{(m)}$ to be the *m-prefix* of w, that is, $w^{(m)} \in F_k$ is the unique element satisfying (i) $|w^{(m)}| = m$ and (ii) there exists $u^{(m)} \in F_k$ such that $|u^{(m)}| = |w| - m$ and $w = w^{(m)}u^{(m)}$ (so that $w^{(0)}$ is the empty word and $w^{(|w|)} = w$). Our claim is that for all $w \in F_k$ with $|w| \leq n+1$, there exists $\mathbf{g} = (g_1, \dots, g_k) \in W_n^k$ with:

$$\sum_{i=1}^{k} |g_i|_{S_n} \leq (n+1)^2$$

such that the points:

$$v_0 = 0^{n+1}, v_1 = v_0^{w^{(1)}(\mathbf{g})}, \dots, v_0^{w^{(|w|)}(\mathbf{g})} = v_0^{w(\mathbf{g})}$$

are all distinct. In particular, $v_0 \neq v_0^{w(\mathbf{g})}$, so $w(\mathbf{g}) \neq e$, and (16) follows. The claim clearly holds for $n = 0$.

Let $n \geq 1$, let $1 \neq w \in F_k$ with $2 \leq |w| \leq n+1$, (if $|w| = 1$ the claim is trivial) and suppose the claim fails for w and W_{n+1}. Let $u = w^{(|w|-1)}$; WLOG (by permuting and inverting the variables x_i) we may assume that $w = ux_1$. Since $1 \neq u$, $|u| \leq n$, we may assume by induction that there exists $\mathbf{g}' = (g_1', \dots, g_k') \in W_{n-1}^k$ witnessing the truth of the claim for u and W_{n-1}. Let $v_i' = (0^n)^{u^{(i)}(\mathbf{g}')} \in V_n$ for $0 \leq i \leq |u|$ (a sequence of $|u|+1$ distinct points). Now consider the g_j' as elements of $W_n = \langle a \rangle \wr_{V_n} W_{n-1}$, acting naturally on V_{n+1}, and consider the points $\tilde{v}_i = (0^{n+1})^{w^{(i)}(\mathbf{g}')}$ for $0 \leq i \leq |w| = |u|+1$. For $i \leq |u|$ we have $\tilde{v}_i = v_i'0$; these points are distinct. On the other hand, by assumption the $\tilde{v}_i$ are not all distinct, so we must have $\tilde{v}_{|w|} = \tilde{v}_0 = 0^{n+1}$.

Let $h \in W_{n-1}$ with $|h|_{S_{n-1}} \leq n$ and $(0^n)^h = v_{|u|}'$; such exists by Lemma 6.3. Then:

$$h^{-1}a_n h \in \mathrm{top}(v_{|u|}'), \text{ with } (h^{-1}a_n h) \mid_{\mathcal{T}_{v_{|u|}'}} = a, \text{ and } |h^{-1}a_n h|_{S_n} \leq 2n+1.$$

Set $g_1 = h^{-1}a_n h g_1'$ and $g_j = g_j'$ for $2 \leq j \leq k$. Consider $v_i = (0^{n+1})^{w^{(i)}(\mathbf{g})}$. We have $v_i = \tilde{v}_i$ for $0 \leq i \leq |u|$ since $v_i' \neq v_{|u|}'$ for $i \leq |u| - 1$ (note that w is a reduced word so the final letter of u is not x_1^{-1}); these points are distinct. But then:

$$v_{|w|} = v_{|u|}^{g_1} = ((v_{|u|}'0)^{h^{-1}a_n h})^{g_1'} = (v_{|u|}'1)^{g_1'} = 0^n 1$$

differs in its final letter from all $v_i = v_i'0$ for $i \leq |u|$. Moreover,

$$\sum_{i=1}^{k} |g_i|_{S_n} \leq (2n+1) + \sum_{i=1}^{k} |g_i'|_{S_n}.$$

The claim follows. □

The next Proposition is essentially proved in the course of Proposition 10 of [10]. We include a proof for the reader's convenience.

Proposition 6.5. *For each $n \in \mathbb{N}$ there is an injective homomorphism $\Phi_n : W_n \to \Gamma$ sending a_i to $k_{i+1} = y_{5i}^4$ for $0 \leq i \leq n$.*

Proof. We follow the proof of Proposition 4 of [10], specialized to our setting. We have $k_{i+1} \in \mathrm{top}(0^{5i})$, with $k_{i+1} \mid_{\mathcal{T}_{0^{5i}}} = x^4$, and $x^4 \in \mathrm{Stab}(4)$ is given, for $v \in V_3$, by:

$$x^4 \mid_{\mathcal{T}_{v0}} = a \text{ and } x^4 \mid_{\mathcal{T}_{v1}} = c;$$

in particular, every k_{i+1} has order 2. We prove the conclusion by induction on n. Certainly $W_0 = C_2 \cong \langle k_1 \rangle \leq \Gamma$, so the conclusion holds for $n = 0$. Let $Q_n = \langle k_1, \dots, k_n \rangle \leq \Gamma$; we suppose by induction that there is an isomorphism $\Phi_n : W_{n-1} \to Q_n$ sending a_i to k_{i+1} for $0 \leq i \leq n-1$. Note that the natural isomorphism $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{top}(0^5)$ (induced by the isomorphism of trees $\mathcal{T} \cong \mathcal{T}_{0^5}$) sends k_i to k_{i+1}; the restriction of this map to Q_n yields an isomorphism from Q_n to $P_n = \langle k_2, \dots, k_{n+1} \rangle$. Similarly the natural isomorphism $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{top}(0)$ sends a_i to a_{i+1}, so restricts to an isomorphism from W_{n-1} to

$\langle a_1, \ldots, a_n \rangle$. Composing, we have a monomorphism $\Psi_n : P_n \to W_n$ sending k_{i+1} to a_i for $1 \le i \le n$. We shall extend Ψ_n from P_n to Q_{n+1}.

Since $P_n \le \mathrm{top}(0^5)$ and $k_1 \mid_{T_{0^4}} = a$, we have $P_n^{k_1} \le \mathrm{top}(0^4 1)$, so $P_n, P_n^{k_1} \le \Gamma$ generate their direct product, and k_1 (being of order 2) acts by conjugation by swapping the two factors. Thus $Q_{n+1} = \langle k_1, P_n \rangle \cong P_n \wr C_2$, with the C_2-factor generated by k_1. To finish, we compose with the isomorphism $P_n \wr C_2 \cong W_{n-1} \wr C_2 = W_{n+1}$ induced by $P_n \cong W_{n-1}$. □

One of the original motivations for introducing Γ was the following, now-famous result.

Theorem 6.6. *For all $g \in \Gamma$, there exists $k \in \mathbb{N}$ such that $g^{2^k} = 1$.*

As we have seen (Example 2.7 above) slow torsion growth yields fast lawlessness growth for p-groups. The following is a consequence of Theorem 7.7 of [1] (see also the bullet-points at the end of Section 1 of that paper).

Theorem 6.7. *There exists $C > 0$ such that for all $g \in \Gamma$, $o(g) \le C|g|_S^{3/2}$.*

Proof of Theorem 1.4. For the upper bound, let $n \in \mathbb{N}$ and let $1 \neq w \in F_k$ with $|w| \le n+1$. Let $g_1, \ldots, g_k \in W_n$ be as in Proposition 6.4 and let $\Phi_n : W_n \to \Gamma$ be as in Proposition 6.5. We have:

$$e \neq \Phi_n\big(w(g_1, \ldots, g_k)\big) = w\big(\Phi_n(g_1), \ldots, \Phi_n(g_k)\big)$$

so that:

$$\chi_\Gamma^S(w) \le \sum_{i=1}^{k} |\Phi_n(g_i)|_S \le 4C(1+\sqrt{3})^{5n} \sum_{i=1}^{k} |g_i|_{S_n} \le 4C(n+1)^2(1+\sqrt{3})^{5n}$$

(the second inequality being by Proposition 6.2). Thus:

$$\mathcal{A}_\Gamma^S(n) \le 4C(n+1)^2(1+\sqrt{3})^{5n} \preceq \exp(n).$$

For the lower bound, let $w_m(x,y) = x^{2^m} \in F_2$. Suppose that $g, h \in \Gamma$ satisfies $|g|_S, |h|_S \le l$. Then by Theorems 6.6 and 6.7, $w_m(g,h) = e$ for all $m \le \log_2 C + (3/2)\log_2 l$. Thus:

$$\mathcal{A}_\Gamma^S(2^m) \ge \chi_\Gamma^S(w_m) \ge l + 1 \ge 2^{2m/3}/C.$$

□

7 Thompson's group F

In this Section we prove Theorem 1.5. We adopt the following model for Thompson's group **F**.

Definition 7.1. $\mathbf{F}$ *is the subgroup of* $\mathrm{Homeo}(\mathbb{R})$ *generated by the homeomorphisms* A *and* B*, where:*

$$A(x) = x + 1 \text{ and } B(x) = \begin{cases} x & x \leq 0 \\ 2x & x \in (0, 1] \\ x + 1 & x > 1 \end{cases}$$

Proposition 7.2. $\mathbf{F}$ *is the group of orientation-preserving piecewise-linear homeomorphisms* H *of* $\mathbb{R}$ *which are differentiable except at finitely many dyadic rational numbers; all of whose slopes are powers of* 2*, and such that there exist integers* K *and* L *such that* $H(x) = x + K$ *for all* $x \in \mathbb{R}$ *sufficiently large and* $H(x) = x + L$ *for all* $x \in \mathbb{R}$ *sufficiently small.*

Our proof of Theorem 1.5 is based closely on [5], where the following was proved.

Theorem 7.3 (Brin–Squier)**.** *The group* $\mathbf{F}$ *is lawless.*

The argument presented in [5] is constructive, and we extract from it a linear upper bound on $\mathcal{A}_{\mathbf{F}}$, as follows. Define $T : \mathbb{R} \to \mathbb{R}$ on each interval $[8n, 8(n+1)]$ by:

$$T(x) = \begin{cases} 2(x - 4n) & x \in [8n, 8n+1] \\ x + 1 & x \in [8n+1, 8n+2] \\ (x + (8n+4))/2 & x \in [8n+2, 8n+6] \\ x - 1 & x \in [8n+6, 8n+7] \\ 2(x - 4(n+1)) & x \in [8n+7, 8(n+1)] \end{cases}$$

(so that T restricts to an orientation-preserving homeomorphism of $[8n, 8(n+1)]$) and set $U = T^2$. For $n \in \mathbb{N}$ define $U_n : \mathbb{R} \to \mathbb{R}$ by:

$$U_n(x) = \begin{cases} U(x) & 0 \leq x \leq 8(n+1) \\ x & \text{otherwise} \end{cases}$$

and $V_n = A^2 U_n A^{-2}$, so that U_n and $V_n \in \mathbf{F}$.

Lemma 7.4. *There exists* $C > 0$ *such that for all* $n \in \mathbb{N}$*,*

$$\left|U_n\right|_S, \left|V_n\right|_S \leq C(n+1).$$

Proof. By Proposition 7.2, $U_0 \in \mathbf{F}$. Let $M = |U_0|_S \in \mathbb{N}$. Then $U_{n+1} = A^8 U_n A^{-8} U_0$, so $|U_n|_S \leq (M+16)n + M$ and $|V_n|_S \leq (M+16)n + M + 4$. □

Proposition 7.5. *Let* $w \in F_2$ *be non-trivial, with* $|w| \leq n$*. Then* $w(U_n, V_n) \neq e$*.*

Proof. This is essentially the content of [5] Section 4; we give a sketch. In [5] there are defined homeomorphisms f_0, f_1 of S^1, such that f_0^2, f_1^2 freely generate a rank-2 free subgroup of $\mathrm{Homeo}(S^1)$. This is shown using a ping-pong argument. Identifying S^1 with $\mathbb{R}/8\mathbb{Z}$ and letting $\pi : \mathbb{R} \to S^1$ be the induced covering map, it follows that for any continuous lifts $\tilde{f}_0, \tilde{f}_1$ of f_0, f_1 to this cover, $\tilde{f}_0^2, \tilde{f}_1^2$ freely generate a rank-2 free subgroup of Homeo($\mathbb{R}$). As per the description of f_0 given in [5], our map T is such a lift of f_0 to $\mathbb{R}$ fixing 0.

It is then shown that similarly, if $\tilde{f}_0$ is a continuous lift of f_0 to $\mathbb{R}$ fixing 0; and $g_0 : \mathbb{R} \to \mathbb{R}$ is given by:

$$g_0(t) = \begin{cases} \tilde{f}_0^2(t) & t \in [-4(n+1), 4(n+1)] \\ t & \text{otherwise} \end{cases}$$

(so that $g_0 \in$ Homeo($\mathbb{R}$)), and $g_1 = A^2 g A^{-2}$, then $w(g_0, g_1) \neq e$ in Homeo($\mathbb{R}$) (T_2 is the notation of [5] is A in our notation and actions in [5] are on the right while ours are on the left). Taking $\tilde{f}_0 = T$ as above, we have $U_n = A^{4(n+1)} g_0 A^{-4(n+1)}$ and $V_n = A^{4(n+1)} g_1 A^{-4(n+1)}$, so $w(U_n, V_n) = A^{4(n+1)} w(g_0, g_1) A^{-4(n+1)} \neq e$ also. □

Proof of Theorem 1.5. By Lemma 7.4 and Proposition 7.5, any non-trivial $w \in F_2$, with $|w| \leq n$, satisfies $\chi_{\mathbf{F}}^{\{A,B\}}(w) \leq 2C(n+1)$. □

8 Residual finiteness growth

Let $\mathcal{C}$ be a class of finite groups. Recall that Γ is *residually* $\mathcal{C}$ if, for every $1 \neq g \in \Gamma$, there exists $Q \in \mathcal{C}$ and a surjective homomorphism $\pi : \Gamma \to Q$ such that $\pi(g) \neq 1$. In this case we denote by $D_{\Gamma,\mathcal{C}}(g)$ the minimal value of $|Q|$ among $Q \in \mathcal{C}$ admitting such a homomorphism π. If S is a finite generating set for Γ, then the *residual* $\mathcal{C}$*-finiteness growth function* of Γ (with respect to S) is:

$$\mathcal{F}_{\Gamma,\mathcal{C}}^{S}(n) = \max\{D_{\Gamma,\mathcal{C}}(g) : |g|_S \leq n\}.$$

Residual finiteness growth was introduced by Bou-Rabee [2] and has been extensively studied for a wide variety of residually finite groups. Two classes $\mathcal{C}$ which are of particular interest are the class of *all* finite groups, and the class of all *p-groups* (for p a fixed prime). In these cases we denote the function $\mathcal{F}_{\Gamma,\mathcal{C}}^{S}$ by $\mathcal{F}_{\Gamma}^{S}$ and $\mathcal{F}_{\Gamma,p}^{S}$, respectively.

Proposition 8.1. *Let $W \subseteq N_2(\Gamma)$. Let $f : \mathbb{N} \to \mathbb{N}$ and suppose that for all l, there exists $w_l \in W$ of length at most $f(l)$ which is a law for every member of $\mathcal{C}$ of order at most l. Then:*

$$\mathcal{F}_{\Gamma,\mathcal{C}}^{S}\big(f(l) \cdot (\mathcal{A}_{\Gamma,W}^{S} \circ f)(l)\big) > l. \tag{17}$$

Proof. Let $w_l \in W$ be as in the statement. There exist $g, h \in \Gamma$ such that:

$$\max\{|g|_S, |h|_S\} \leq |g|_S + |h|_S \leq \mathcal{A}_{\Gamma,W}^{S}\big(f(l)\big)$$

and $w_l(g,h) \neq e$, while by construction, $D_{\Gamma,\mathcal{C}}\big(w_l(g,h)\big) > l$ and:

$$|w_l(g,h)|_S \leq |w_l| \cdot \max\{|g|_S, |h|_S\} \leq f(l) \cdot \mathcal{A}_{\Gamma,W}^{S}\big(f(l)\big).$$

□

Proposition 8.1 is useful in the presence of good upper bounds on the function f. The best currently known bounds for the classes of finite groups and finite p-groups are as follows, taken from [3] and [7], respectively.

Theorem 8.2. *For each $n \in \mathbb{N}$ there exists $1 \neq w_n \in F_2$ such that:*

(i) For all finite groups G of order at most n, w_n is a law for G;

(ii) For all $\delta > 0$, $|w_n| = O_\delta(n^{2/3} \log(n)^{3+\delta})$.

Theorem 8.3. *For each $m \in \mathbb{N}$ there exists $1 \neq w'_m \in F_2$ such that:*

(i) For all nilpotent groups G of class at most m, w'_m is a law for G (in particular, w'_m is a law for every finite p-group of order at most p^m);

(ii) $|w'_m| = O(m^\alpha)$, where $\alpha = \log(2)/(\log(1+\sqrt{5}) - \log(2)) \approx 1.440$.

Corollary 8.4. *Let Γ be a finitely generated lawless group and let $\delta > 0$. Then:*

$$\mathcal{F}_\Gamma\big(l\mathcal{A}_\Gamma(l)\big) \succeq l^{3/2}/\log(l)^{9/2+\delta}.$$

Proof. Let w_n be as in Theorem 8.2 and set $W = \{w_n\}_{n\in\mathbb{N}}$. Apply Proposition 8.1 with C the class of all finite groups. □

Corollary 8.5. *Let Γ be a lawless group and let p be a prime. Then:*

$$\mathcal{F}_{\Gamma,p}\big(l\mathcal{A}_\Gamma(l)\big) \succeq \exp\big(\log(p) l^{1/\alpha}\big),$$

where α is as in Theorem 8.3 (so that $1/\alpha \approx 0.694$).

Proof. Let w'_m be as in Theorem 8.3. and set $W = \{w'_m\}_{m\in\mathbb{N}}$. Apply Proposition 8.1 with C the class of all finite p-groups. □

9 Equations with coefficients

Laws are generalized by *mixed identities*. Given a group Γ, a non-trivial element w of the free product $\Gamma * F_k$, lying in the kernel of every homomorphism $\Gamma * F_k \to \Gamma$ which restricts to the identity on Γ. A mixed identity is therefore a law for Γ precisely when it lies in F_k. Mixed-identity-free (MIF) groups are rather rarer than lawless groups: any group decomposing as a non-trivial direct product, or with a non-trivial finite conjugacy class satisfies a mixed identity, as does any group possessing a non-trivial normal subgroup with a mixed identity.

One may define an analogue of lawlessness growth for mixed identities. For Γ a finitely generated group, and $w \in \Gamma * F_k$ non-trivial, not a mixed identity for Γ, the complexity of w in Γ is define exactly as for elements of F_k: it is the minimal word-length of a k-tuple (g_i) in Γ such that w does not lie in the kernel of the homomorphism $\Gamma * F_k \to \Gamma$ restricting to the identity on Γ and sending x_i to g_i. To define the induced MIF growth function $\mathcal{M}_\Gamma$ of Γ, we take the maximal complexity over elements $w \in \Gamma * F_k$ of word-length at most l. As with lawlessness growth, it is easy to see that the equivalence class of the function $\mathcal{M}_\Gamma$ does not depend on a choice of finite generating set for Γ or $\Gamma * F_k$. Moreover, since $\Gamma * F_k$ embeds into $F_k * \mathbb{Z}$ for every k, we may assume $k = 1$. Henceforth we take $\mathbb{Z} = \langle x \rangle$ and fix a finite generating set S for Γ.

Remark 9.1. It might alternatively occur to one to assign length one to all coefficients from Γ appearing in w. The distinction between such a length function and the word metric on $\Gamma * F_k$ coming from a finite generating set is roughly the same as that between the degree and "height" of a polynomial with integer coefficients. Importantly though, if we were to assign length one to every element of Γ, then for Γ infinite, there would be infinitely many words of bounded length, so it would not be clear that the associated MIF growth function would take finite values, even for Γ MIF-free.

We recall the following basic fact about subgroups of free groups, the proof of which is an easy consequence of the uniqueness of reduced-word representatives for elements of free products.

Lemma 9.2. *Let Γ and Δ be nontrivial groups, and let $e \neq h \in \Delta$. Then $\{[g,h] : e \neq g \in \Gamma\}$ freely generate a free subgroup of $\Gamma * \Delta$.*

Proof of Theorem 1.9. Let $g_1, \dots, g_m$ be an enumeration of the nontrivial elements of $B_S(l)$, so that for some $C > 0$ (independent of l) we have $m \leq \exp(Cl)$. For $1 \leq i \leq m$ let $w_i = [g_i, x]$. By Lemma 9.2 the w_i freely generate a free subgroup F of $\Gamma * \mathbb{Z}$ of rank m. Applying the construction of Proposition 2.8 to the w_i, we obtain a nontrivial word w in the w_i of reduced length at most $16m^2$, with the property that, whenever $\pi : F \to \Gamma$ is a homomorphism whose kernel contains some w_i, $w \in \ker(\pi)$ also.

Since every y_i has word-length at most $2(l+1)$ in $S \cup \{x\}$, w has word-length at most $32(l+1)m^2 \leq \exp(l)$ when viewed as an element of $\Gamma * \mathbb{Z}$. Meanwhile, w has complexity at least $l+1$ in $\Gamma * \mathbb{Z}$. For every homomorphism $\Gamma * \mathbb{Z} \to \Gamma$ resticting to the identity on Γ restricts to a homomorphism $F \to \Gamma$. If such a homomorphism sends x to some g_i, then w_i lies in its kernel, and by Proposition 2.8, so does w. $\square$

Now we turn to the proof of Theorem 1.10. Henceforth Γ is a finite-rank nonabelian free group with free basis S.

Proposition 9.3. *Let:*

$$w(x) = a_1 x^{k_1} \cdots a_l x^{k_l} \in \Gamma * \mathbb{Z} \tag{18}$$

for some $e \neq a_i \in \Gamma$ and $k_i \in \mathbb{Z}$ with $k_i \neq 0$ for $i \leq l-1$. Suppose $u \in \Gamma$ is not a proper power in Γ, and that $[u, a_i] \neq e$ for all i. Then for all $m \in \mathbb{N}$ sufficiently large that:

$$|u^m|_S \geq |u^2|_S l + \sum_{i=1}^{l} |a_i|_S, \tag{19}$$

$w(u^m)$ does not commute with u (and in particular $w(u^m) \neq e$).

Proof. We proceed by induction on l. For $l = 1$ the claim is clear for any $m \geq 1$: $a_1 u^{k_1 m}$ commutes with u iff a_1 does. Let $l \geq 2$ and suppose the claim holds for smaller l. If $w(u^m)$ commutes with u, then they are powers of a common word. Since u is not a proper power, there exists $k_0 \in \mathbb{Z}$ such that $u^{k_0} a_1 u^{k_1 m} \cdots a_l u^{k_l m} = e$.

Consider the Cayley graph C of Γ with respect to the free basis S. Since C is a tree there is, for every $g, h \in \Gamma$, a unique reduced path $[g,h]$ from g to h in C (of length equal to $|g^{-1}h|_S$). Writing $g_0 = e$, $h_0 = u^{k_0}$ and $g_i = h_{i-1} a_i$, $h_i = g_i u^{k_i m}$ for $i \geq 1$ (so that $h_l = e$), the union of all the paths $[g_j, h_j]$ and $[h_j, g_{j+1}]$ is a closed loop in C

and (using again the fact that C is a tree), $[g_1, h_1]$ is contained in the union of the other intervals. Each $[h_{j-1}, g_j]$ has length $|a_j|_S$, so for m as in the statement, there exists $i \neq 1$ such that $[g_1, h_1]$ overlaps with $[g_i, h_i]$ in an interval of length at least $|u^2|_S$.

We may write u uniquely as $y^{-1}zy$, for $y, z \in \Gamma$ reduced words in S, with z cyclically reduced and $|u|_S = 2|y|_S + |z|_S$. There exist $M, N \in \mathbb{Z}$ with $g_1^{-1}h_1 = y^{-1}z^M y$ and $g_i^{-1}h_i = y^{-1}z^N y$. Further, $I_1 = [g_1 y^{-1}, h_1 y^{-1}] = [g_1 y^{-1}, g_1 y^{-1} z^M]$ overlaps with $I_2 = [g_i y^{-1}, h_i y^{-1}] = [g_i y^{-1}, g_i y^{-1} z^N]$ in an interval of length at least $|u^2|_S - 2|y|_S \geq 2|z|_S$. Let p and q be the shortest prefixes of z^M and z^N, respectively, such that $g_1 y^{-1} p = g_i y^{-1} q$ (that is, the starting-points of common subinterval of I_1 and I_2).

We claim that $|p|_S \equiv |q|_S \mod |z|_S$, so that the initial common subinverval of I_1 and I_2 of length $|z|_S$ contains a point $g_1 y^{-1} z^{k_1'} = g_i y^{-1} z^{k_2'}$ for some $k_1', k_2' \in \mathbb{Z}$, so that:

$$u^{k_1'-k_2'-k_1 m} = g_1^{-1} g_i = a_2 u^{k_2 m} \cdots u^{k_{i-1} m} a_i$$

commutes with u, contradicting the inductive hypothesis.

Suppose the claim fails. First suppose M and N have the same sign. Comparing the initial common subintervals of I_1 and I_2 of length $2|z|_S$, we see that there exist $v, w \in \Gamma$ nontrivial reduced words with $z = vw = wv$, and $|z|_S = |v|_S + |w|_S$ (specifically, $|v|_S \equiv |p|_S - |q|_S \mod |z|_S$). Since v and w commute, they are powers of a common word, contradicting the fact that u (and hence z) is not a proper power. Similarly if M and N have opposite signs, we obtain $z = vw$ and $z^{-1} = wv = wzw^{-1}$, but no nontrivial element of a free group is conjugate to its inverse. We therefore have the desired claim. ☐

Proof of Theorem 1.10. Let $e \neq w \in \Gamma * \mathbb{Z}$ with $|w| \leq n$. Conjugating, we may assume that $w(x)$ has the form given in (18). Moreover,

$$\sum_{i=1}^{l} |a_i| \leq n$$

so that by Proposition 9.3 w has complexity at most $2|u|_S n + |u|_S + n$ for any $u \in \Gamma$ satisfying the conditions of Proposition 9.3. It therefore suffices to find such u satisfying $|u| \ll \log n$.

There exists $C > 0$ such that $|B_S(k)| \geq \exp(Ck)$ for all $k \in \mathbb{N}$. Since centralizers in free groups are cyclic, the union of the centralizers of the a_i cover at most $(2k+1)l \leq (2k+1)n$ points in $B_S(k)$. Hence there exists $C' > 0$ such that for all $k \geq C' \log(n)$, $B_S(k)$ contains an element u such that $[u, a_i] \neq e$ for all i. Taking u to be of minimal length among such elements, we may assume u is not a proper power. ☐

Remark 9.4. *Unlike the class of lawless groups, the class of MIF groups is not closed under taking overgroups. For instance $\Gamma \times C_2$ is non-MIF, for any group Γ. There seems to be no straightforward relationship between the MIF growth of a group Δ and that of a finitely generated overgroup Γ (in the spirit of Lemma 2.4), even if Γ is itself MIF.*

10 Open questions

There are many interesting questions one can ask about the spectrum of possible lawlessness growth functions. For instance one may wonder whether there is a universal *upper* bound on $\mathcal{A}_\Gamma$ for lawless groups.

Question 10.1. *Does there exist, for all non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$, a finitely generated lawless group $\Gamma = \Gamma(f)$ satisfying $\mathcal{A}_\Gamma \npreceq f$? Does there exist such a group satisfying $\mathcal{A}_\Gamma \succeq f$?*

Our main source of strong lower bounds on $\mathcal{A}_\Gamma$ comes from *upper* bounds on torsion growth for infinite finitely generated p-groups. Under this approach, there are viable methods for constructing finitely generated lawless p-groups which are unrelated to Golod-Shafarevich theory. For instance, there are many examples of branch p-groups beyond Grigorchuk's group, and all (weakly) branch groups are lawless (by an argument similar to our proof of Theorem 1.4).

Question 10.2. *How slow can the torsion growth of a finitely generated (weakly) branch p-group be? Can it be sublinear?*

For example, by Theorem 7.8 of [1], Grigorchuk's group has torsion growth at least linear. In a completely different direction, a result of Druţu-Sapir shows that any non-virtually-cyclic group which admits an asymptotic cone with a cut-point is lawless. By [13][Theorem 1.12] such groups include some finitely generated torsion groups (indeed p-groups, as is made clear in the proof). It is unclear to us at present how effective the construction in [13] is able to be made, and whether it could give rise to groups of fast lawlessness growth.

Our proof of the upper bound in Theorem 1.4, and the possibility of improving upon it, naturally connects to the following question about finite groups.

Question 10.3. *Let W_n be the $(n+1)$-fold iterated wreath product of C_2, as in Section 6. What is the length of the shortest law in F_k for W_n?*

As far as the author is aware, there is no shorter law known for W_n than the obvious power-word $x^{2^{n+1}}$.

Turning to MIF growth, we may ask for examples of groups for which $\mathcal{M}_\Gamma$ grows slowly. Since MIF is a difficult proerty to satisfy, exhibiting such groups may be rather challenging. For instance, it is natural (if rather ambitious) to ask whether Theorem 1.9 is sharp.

Question 10.4. *Does there exist Γ with $\mathcal{M}_\Gamma(n) \ll \log(n)$?*

Gromov hyperbolic groups form an important class of MIF groups, properly containing the nonabelian finite-rank free groups. We may ask for a generalization of the bound from Theorem 1.10 to this class, and for insight into the features of the geometry of groups on which the bound depends.

Problem 10.5. *Let Γ be a torsion-free Gromov hyperbolic group. Give an upper bound on $\mathcal{M}_\Gamma$. Does there exist, for each $\delta > 0$, an increasing function $f_\delta : \mathbb{N} \to \mathbb{N}$ such that, if Γ admits a δ-hyperbolic Cayley graph, then $\mathcal{M}_\Gamma \preceq f_\delta$?*

Finally, since words in F_k of a given length are a small subset of the elements of $\Gamma * F_k$, it is clear that for any MIF group Γ, $\mathcal{A}_\Gamma \preceq \mathcal{M}_\Gamma$, and in general one would expect it would be a remarkable achievement to identify a group in which elements of $\Gamma * F_k$ of essentially maximal comlexity for their length already occur in F_k.

Question 10.6. *Does there exist a finitely generated MIF group Γ with $\mathcal{A}_\Gamma \approx \mathcal{M}_\Gamma$?*

By Theorems 1.1 and 1.9 a group providing a positive answer to Question 10.6 would needs must have no F_2-subgroups. In a forthcoming work, we shall give a finite group analogue of a positive answer to Question 10.6: a sequence (G_n) of finite groups having no common law, such that for each n, the length of the shortest law for G_n is comparable to the length of the shortest mixed identity.

Following circulation of a preliminary version of the present article, J.M. Petschick has shared with the author a construction of a group answering the final part of Question 10.2 in the affirmative (as yet unpublished). His result therefore also strengthens the conclusion of our Theorem 1.3.

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