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SIX-DIMENSIONAL LANDAU INEQUALITIES

Abstract. Let X be a complex Banach space, and let $t \to T(t)(\|T(t)\| \leq 1,\ t \geq 0)$ be a strongly continuous contraction semigroup (on X) with infinitesimal generator A.In this paper we prove that the following five inequalities

$$\|A^i x\|^6 \leq R_i^1(6)\|x\|^{6-i}\|A^6 x\|^i$$

hold for every $x \in D(A^6)$ and for a fixed number $i = 1,2,3,4,5$, where $R_i^1(6) = \frac{(i!)^6}{(6!)^i}\left[\begin{smallmatrix}6\\i\end{smallmatrix}\right]\varepsilon_i$ such that our symbol $\left[\begin{smallmatrix}6\\i\end{smallmatrix}\right] = \frac{6^6}{i^i(6-i)^{6-i}}$ holds as well as $\varepsilon_1 = 2^1 3^6 5^5$, $\varepsilon_2 = 5^6 7^6$, $\varepsilon_3 = 2^{15}$, $\varepsilon_4 = 2^4 3^6 7^6$, $\varepsilon_5 = 2^5 5^5$. Analogous inequalities hold for strongly continuous contraction cosine functions. In this case of cosine functions constants $R_i^1(6)$ are replaced by $R_i^3(6) = \frac{((2i)!)^6}{((12)!)^i}\left[\begin{smallmatrix}6\\i\end{smallmatrix}\right]\varepsilon_i$. Inequalities are established also for uniformly bounded strongly continuous semigroups and cosine functions.

1. Introduction

Edmund Landau (1913) [6] initiated the following fundamental *extremum problem*: The sharp inequality between the supremum-norms of derivatives of twice differentiable functions f such that

$$\|f'\|^2 \leq 4\|f\|\,\|f''\| \tag{L}$$

holds with norm referring to the space $C[0,\infty]$.

Then R. R. Kallman and G.-C. Rota (1970) [3] found the more general result that inequality

$$\|Ax\|^2 \leq 4\|x\|\,\|A^2 x\| \tag{1}$$

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holds for every $x \in D(A^2)$, and A the infinitesimal generator (i.e., the strong right derivative of T at zero) of $t \to T(t)(t \geq 0)$: a semigroup of linear contractions on a complex Banach space X.

Z. Ditzian [1] achieved the better inequality for every $x \in D(A^2)$,

$$\|Ax\|^2 \leq 2\|x\| \, \|A^2x\| \tag{2}$$

where A is the infinitesimal generator of a group $t \to T(t)$ ($\|T(t)\| = 1$, $t \in R$) of linear isometries on X.

Moreover H. Kraljević and S. Kurepa [4] established the even sharper inequality for every $x \in D(A^2)$,

$$\|Ax\|^2 \leq \frac{4}{3}\|x\| \, \|A^2x\| \tag{3}$$

and A the infinitesimal generator (i.e., the strong right second derivative of T at zero) of $t \to T(t)$ ($t \geq 0$): a strongly continuous cosine function of linear contractions on X.

Therefore the best *Landau's type constant* is $\frac{4}{3}$ (for cosine functions).

The above-mentioned inequalities (1)–(3) were extended by H. Kraljević and J. Pečarić [5] so that new *Landau's type inequalities* hold. In particular, they proved that

$$\|Ax\|^3 \leq \frac{243}{8}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \leq 24\|x\| \, \|A^3x\|^2 \tag{1a}$$

hold for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous contraction semigroup on X. Besides they obtained the analogous but better inequalities

$$\|Ax\|^3 \leq \frac{9}{8}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \leq 3\|x\| \, \|A^3x\|^2 \tag{2a}$$

which hold for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous contraction group on X. Moreover they got the set of analogous inequalities

$$\|Ax\|^3 \leq \frac{81}{40}\|x\|^2\|A^3x\|, \qquad \|A^2x\|^3 \leq \frac{72}{25}\|x\| \, \|A^3x\|^2 \tag{3a}$$

for every $x \in D(A^3)$, where A is the infinitesimal generator of a strongly continuous cosine function on X.

Denote the afore-mentioned *constants* by

$$R_1^1(2) = 4; \; R_1^2(2) = 2; \; R_1^3(2) = \frac{4}{3}$$

for every $x \in D(A^2)$ (on X), as well as

$$R_1^1(3) = \frac{243}{8}, \; R_2^1(3) = 24; \; R_1^2(3) = \frac{9}{8}; \; R_2^2(3) = 3; \; R_1^3(3) = \frac{81}{40}, \; R_2^3(3) = \frac{72}{25}$$

for every $x \in D(A^3)$ (on X). These constants are employed for the establishment of the two-dimensional Landau inequalities ([1], [2], [3], [4], [6]) and the three-dimensional Landau inequalities [5], respectively.

Similarly we established the four-dimensional Landau inequalities [7] and the five-dimensional Landau inequalities [8] such that inequalities

$$(+) \qquad \|A^i x\|^n \leq R_i^k(n)\|x\|^{n-i}\|A^n x\|^i$$

hold for every $x \in D(A^n)$ (on X), a fixed number $n = 4$ and $n = 5$, and $i = 1, 2, \ldots, n-1$, where $k = 1$ corresponds to the case of *semigroups*, $k = 2$ to *groups*, and $k = 3$ to *cosine functions*, as well as *constants* $R_i^k(n)$ ($n = 4$ and $n = 5$) are of the following form:

$$R_1^1(4) = \frac{1024}{3}, \quad R_2^1(4) = \frac{10^4}{9}, \quad R_3^1(4) = 192;$$
$$R_1^2(4) = 10\left(\frac{5}{6}\right)^4, \quad R_2^2(4) = \frac{16}{9}, \quad R_3^2(4) = \frac{5^4(13)^3}{2^5 3^6};$$
$$R_1^3(4) = \frac{1024}{315}, \quad R_2^3(4) = \frac{400}{49}, \quad R_3^3(4) = \frac{2880}{343}$$

for every $x \in D(A^4)$ (on X) and

$$R_1^1(5) = \frac{5^9}{3(2^7)}, \quad R_2^1(5) = \frac{2(5^8)}{3^2}, \quad R_3^1(5) = \frac{3^2 5^2 7^5}{2^6}, \quad R_4^1(5) = 3(5)(2^7);$$
$$R_1^2(5) = \frac{(15)^4}{2^{11}}, \quad R_2^2(5) = \frac{5^3(22)^2}{3^8}, \quad R_3^2(5) = \frac{5^2(17)^3}{2^8 3^4}, \quad R_4^2(5) = 5\left(\frac{3}{2}\right)^4;$$
$$R_1^3(5) = \frac{5^9}{9!}, \quad R_2^3(5) = \frac{(4!)^5 5^8}{(9!)^2}, \quad R_3^3(5) = \frac{(35(6!))^5}{4(10!)^3}, \quad R_4^3(5) = \frac{(5(8!))^5}{(10!/2)^4}$$

for every $x \in D(A^5)$ (on X).

In this paper, we extend the above inequalities so that the six-dimensional Landau inequalities (+) hold for every $x \in D(A^6)$ (on X), where A is infinitesimal generator of a uniformly bounded continuous semigroup (resp. cosine functions).

2. Semigroups

Let $t \to T(t)$ be a uniformly bounded ($\|T(t)\| \leq M < \infty$, $t \geq 0$) strongly semigroup of linear operators on X with infinitesimal generator A, such that $T(0) = I$ (:= Identity) in $B(X)$:= the Banach algebra of bounded linear operators on X, $\lim_{t \downarrow 0} T(t)x = x$, for every x, and

$$(4) \qquad Ax = \lim_{t \downarrow 0} \frac{T(t) - I}{t} x (= T'(0)x)$$

for every x in a linear subspace $D(A)(:=$ Domain of $A)$ dense in X [2]. For every $x \in D(A)$, we have the formula

$$T(t)x = x + \int_0^t T(u)Ax\,du. \tag{5}$$

Using integration by parts, we get the formula

$$\int_0^t \Big(\int_0^u T(v)A^2x\,dv \Big)\,du = \int_0^t (t-u)T(u)A^2x\,du. \tag{6}$$

Employing (6) and iterating (5), we find for every $x \in D(A^2)$ that

$$T(t)x = x + tAx + \int_0^t (t-u)T(u)A^2x\,du \tag{5a}$$

Similarly iterating (5a), we obtain for every $x \in D(A^6)$ that

(5b) $T(t)x$

$$= x + \frac{t}{1!}Ax + \frac{t^2}{2!}A^2x + \frac{t^3}{3!}A^3x + \frac{t^4}{4!}A^4x + \frac{t^5}{5!}A^5x + \frac{1}{5!}\int_0^t (t-u)^5T(u)A^6x\,du.$$

Consider our auxiliary (6×5) coefficient matrix

$(a_{ij}) : i = 0,1,2,3,4,5;\ j = 1,2,3,4,5$, such that

$$a_{01} = a_{02} = a_{03} = a_{04} = a_{05} = \frac{1}{5}t_1t_2t_3t_4t_5;$$
$$a_{11} = t_2t_3t_4t_5, \quad a_{12} = t_1t_3t_4t_5, \quad a_{13} = t_1t_2t_4t_5,$$
$$a_{14} = t_1t_2t_3t_5, \quad a_{15} = t_1t_2t_3t_4;$$
$$a_{21} = t_2t_3t_4 + t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5,$$
$$a_{22} = t_1t_3t_4 + t_1t_3t_5 + t_1t_4t_5 + t_3t_4t_5,$$
$$a_{23} = t_1t_2t_4 + t_1t_2t_5 + t_1t_4t_5 + t_2t_4t_5,$$
$$a_{24} = t_1t_2t_3 + t_1t_2t_5 + t_1t_3t_5 + t_2t_3t_5,$$
$$a_{25} = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4;$$
$$a_{31} = t_2t_3 + t_2t_4 + t_2t_5 + t_3t_4 + t_3t_5 + t_4t_5,$$
$$a_{32} = t_1t_3 + t_1t_4 + t_1t_5 + t_3t_4 + t_3t_5 + t_4t_5,$$
$$a_{33} = t_1t_2 + t_1t_4 + t_1t_5 + t_2t_4 + t_2t_5 + t_4t_5,$$
$$a_{34} = t_1t_2 + t_1t_3 + t_1t_5 + t_2t_3 + t_2t_5 + t_3t_5,$$
$$a_{35} = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4;$$
$$a_{41} = t_2 + t_3 + t_4 + t_5, \quad a_{42} = t_1 + t_3 + t_4 + t_5,$$
$$a_{43} = t_1 + t_2 + t_4 + t_5, \quad a_{44} = t_1 + t_2 + t_3 + t_5,$$

$$a_{45} = t_1 + t_2 + t_3 + t_4;$$
$$a_{51} = a_{52} = a_{53} = a_{54} = a_{55} = 1.$$

Also consider

$$d_1 = t_2t_3t_4t_5(t_2 - t_3)(t_2 - t_4)(t_2 - t_5)(t_3 - t_4)(t_3 - t_5)(t_4 - t_5),$$
$$d_2 = t_1t_3t_4t_5(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)(t_3 - t_4)(t_3 - t_5)(t_4 - t_5),$$
$$d_3 = t_1t_2t_4t_5(t_1 - t_2)(t_1 - t_4)(t_1 - t_5)(t_2 - t_4)(t_2 - t_5)(t_4 - t_5),$$
$$d_4 = t_1t_2t_3t_5(t_1 - t_2)(t_1 - t_3)(t_1 - t_5)(t_2 - t_3)(t_2 - t_5)(t_3 - t_5),$$
$$d_5 = t_1t_2t_3t_4(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_2 - t_3)(t_2 - t_4)(t_3 - t_4).$$

Finally consider vectors:

$$\overline{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}) \ (i = 1, 2, 3, 4, 5) \quad \text{and} \quad \overline{d} = (d_1, d_2, d_3, d_4, d_5)$$

as well as vectors: $\overline{a}_0 = (a_{01}, a_{02}, a_{03}, a_{04}, a_{05})$ such that $a_{0j} = \frac{1}{5}\omega_5$ ($j = 1, 2, 3, 4, 5$) where $\omega_5 = t_1t_2t_3t_4t_5$ and $\overline{1} = (1, 1, 1, 1, 1)$.

Note that

$$\sum_{j=1}^{5} a_{ij}d_j = \overline{a}_i \bullet \overline{d}$$

holds for $i = 1, 2, 3, 4, 5$, where "$\bullet$" is the dot ("inner") product vector operation.

Also *note* that

$$t_1t_2t_3t_4t_5 = \omega_5 = \overline{a}_0 \bullet \overline{1},$$
$$t_1t_2t_3t_4 + t_1t_2t_3t_5 + t_1t_2t_4t_5 + t_1t_3t_4t_5 + t_2t_3t_4t_5 = \overline{a}_1 \bullet \overline{1},$$
$$t_1t_2t_3 + t_1t_2t_4 + t_1t_2t_5 + t_1t_3t_4 + t_1t_3t_5 + t_1t_4t_5 + t_2t_3t_4 + t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5 = (\overline{a}_2 \bullet \overline{1})/2,$$
$$t_1t_2 + t_1t_3 + t_1t_4 + t_1t_5 + t_2t_3 + t_2t_4 + t_2t_5 + t_3t_4 + t_3t_5 + t_4t_5 = (\overline{a}_3 \bullet \overline{1})/3,$$
$$t_1 + t_2 + t_3 + t_4 + t_5 = (\overline{a}_4 \bullet \overline{1})/4, \quad \text{and} \quad 1 = (\overline{a}_5 \bullet \overline{1})/5.$$

Denote vector

$$\overline{T} = (T_1, T_2, T_3, T_4, T_5) \text{ with } T_j = (-1)^{j-1}\frac{\sigma_5}{D}d_jT(t_j),\ j = 1, 2, 3, 4, 5,$$

where $\sigma_5 = \frac{(5!)^4}{1!\,2!\,3!\,4!}$ ($= 720000$) and "determinant":

$$D = \sigma_5[t_1t_2t_3t_4t_5(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)(t_2 - t_3)(t_2 - t_4) \times (t_2 - t_5)(t_3 - t_4)(t_3 - t_5)(t_4 - t_5)] \tag{11}$$

or

$$\overline{T} = \sigma_5\frac{1}{D}(d_1T(t_1), -d_2T(t_2), d_3T(t_3), -d_4T(t_4), d_5T(t_5)).$$

Also denote

$$r_i = \begin{cases} 1, & \text{for } i \in \{1,2\} \\ \frac{1}{i-1}, & \text{for } i \in \{3,4,5\}. \end{cases} \tag{9}$$

THEOREM 1. *Let $t \to T(t)$ be a uniformly bounded ($\|T(t)\| \leq M < \infty$, $t \geq 0$) strongly continuous semigroup of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^6x \neq 0$. Then the following five inequalities*

$$\|A^i x\| \leq (i!)\left\{\left[M\frac{\sigma_5}{D}(\overline{a}_i \bullet \overline{d}) + \frac{1}{i\omega_5}(\overline{a}_i \bullet \overline{1}\right]\|x\| + \left[\frac{1}{6!}M(r_i(\overline{a}_{i-1} \bullet \overline{1}))\right]\|A^6x\|\right\}, \tag{7}$$

hold for every $x \in D(A^6)$ and a fixed number $i = 1,2,3,4,5$, where σ_5, D, ω_5, r_i $(i = 1,2,3,4,5)$, $\overline{a}_i$ $(i = 1,2,3,4,5), \overline{d}, \overline{1}, \overline{a}_0$ are given above. Besides (7) holds for every $t_i \in \mathbb{R}^+ = (0,\infty)$ $(i = 1,2,3,4,5) : 0 < t_1 < t_2 < t_3 < t_4 < t_5$.

Proof of Theorem 1. In fact, formula (5b) yields system of five equations:

$$\begin{aligned} &\frac{5!}{1!}t_iAx + \frac{5!}{2!}t_i^2A^2x + \frac{5!}{3!}t_i^3A^3x + \frac{5!}{4!}t_i^4A^4x + t_i^5A^5x \\ &\quad = 5!T(t)x - 5!x - \int_0^{t_i}(t_i - u)^5T(u)A^6x\,du, \qquad \text{for } i = 1,2,3,4,5. \end{aligned} \tag{10}$$

The coefficient determinant with respect to A^ix $(i = 1,2,3,4,5)$ of *algebraic system* (10) is D given by (11).

It is clear that D is positive because: $0 < t_1 < t_2 < t_3 < t_4 < t_5$. Therefore there is a unique solution of system (10) of the following form:

$$\begin{aligned} Ax = (1!)\{&[(t_2t_3t_4t_5)d_1T(t_1)x - (t_1t_3t_4t_5)d_2T(t_2)x \\ &+ (t_1t_2t_4t_5)d_3T(t_3)x - (t_1t_2t_3t_5)d_4T(t_4)x \\ &+ (t_1t_2t_3t_4)d_5T(t_5)x]/\frac{D}{\sigma_5} \\ &- [(t_1t_2t_3t_4 + t_1t_2t_3t_5 + t_1t_2t_4t_5 + t_1t_3t_4t_5 + t_2t_3t_4t_5)/\omega_5]x\} \\ &- \int_0^{t_5} K_1(\omega;u)T(u)A^6xdu, \end{aligned} \tag{12}$$

where $\omega = (t_1, t_2, t_3, t_4, t_5)$,

$$(12a)\quad \begin{aligned} A^2x = &-(2!)\{[(t_2t_3t_4 + t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5)d_1T(t_1)x \\ &- (t_1t_3t_4 + t_1t_3t_5 + t_1t_4t_5 + t_3t_4t_5)d_2T(t_2)x \\ &+ (t_1t_2t_4 + t_1t_2t_5 + t_1t_4t_5 + t_2t_4t_5)d_3T(t_3)x \\ &- (t_1t_2t_3 + t_1t_2t_5 + t_1t_3t_5 + t_2t_3t_5)d_4T(t_4)x \\ &+ (t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)d_5T(t_5)x]/\frac{D}{\sigma_5} \\ &- [(t_1t_2t_3 + t_1t_2t_4 + t_1t_2t_5 + t_1t_3t_4 + t_1t_3t_5 + t_1t_4t_5 + t_2t_3t_4 \\ &+ t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5)/\omega_5]x\} + \int_0^{t_5} K_2(\omega; u)T(u)A^6x du, \end{aligned}$$

$$(12b)\quad \begin{aligned} A^3x = &(3!)\{[(t_2t_3 + t_2t_4 + t_2t_5 + t_3t_4 + t_3t_5 + t_4t_5)d_1T(t_1)x \\ &- (t_1t_3 + t_1t_4 + t_1t_5 + t_3t_4 + t_3t_5 + t_4t_5)d_2T(t_2)x \\ &+ (t_1t_2 + t_1t_4 + t_1t_5 + t_2t_4 + t_2t_5 + t_4t_5)d_3T(t_3)x \\ &- (t_1t_2 + t_1t_3 + t_1t_5 + t_2t_3 + t_2t_5 + t_3t_5)d_4T(t_4)x \\ &+ (t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)d_5T(t_5)x]/\frac{D}{\sigma_5} \\ &- (t_1t_2 + t_1t_3 + t_1t_4 + t_1t_5 + t_2t_3 + t_2t_4 + t_2t_5 \\ &+ t_3t_4 + t_3t_5 + t_4t_5)/\omega_5]x\} \\ &- \int_0^{t_5} K_3(\omega; u)T(u)A^6x\, du, \end{aligned}$$

$$(12c)\quad \begin{aligned} A^4x = &-(4!)\{[(t_2 + t_3 + t_4 + t_5)d_1T(t_1)x - (t_1 + t_3 + t_4 + t_5)d_2T(t_2)x \\ &+ (t_1 + t_2 + t_4 + t_5)d_3T(t_3)x - (t_1 + t_2 + t_3 + t_5)d_4T(t_4)x \\ &+ (t_1 + t_2 + t_3 + t_4)d_5T(t_5)x]/\frac{D}{\sigma_5} \\ &- [(t_1 + t_2 + t_3 + t_4 + t_5)/\omega_5]x\} + \int_0^{t_5} K_4(\omega; u)T(u)A^6x\, du, \end{aligned}$$

$$(12d)\quad \begin{aligned} A^5x = &(5!)\Big\{[d_1T(t_1)x - d_2T(t_2)x + d_3T(t_3)x - d_4T(t_4)x \\ &+ d_5T(t_5)x]\Big/\frac{D}{\sigma_5} - [1/\omega_5]x\Big\} - \int_0^{t_5} K_5(\omega; u)T(u)A^6x\, du, \end{aligned}$$

where $K_i = K_i(\omega; u)$ are given by the following formula (13). The above mentioned solution A^ix $(i = 1, 2, 3, 4, 5)$ of algebraic system (10) is of the form:

$$(*)\ \ A^i x = (-1)^{i-1}\left\{(i!)\left[(\overline{a}_i \bullet \overline{T}) - \frac{1}{i\omega_5}(\overline{a}_i \bullet \overline{1})\right]x - \int_0^{t_5} K_i(\omega;u)T(u)A^6 x\,du\right\},$$

where $K_i = K_i(\omega;u) = (-1)^{i-1}\frac{1}{D}\overline{K}_i(\omega;u)$ such that $\overline{K}_i = \sigma_i \widetilde{K}_i$, and

$$\sigma_i = \frac{(5!)^4}{\prod\limits_{\substack{j=1\\(j\neq i)}}^{5} j!}\left(\text{e.g. } \sigma_3 = \frac{(5!)^4}{1!\,2!\,4!\,5!}\right),$$

as well as

$$\widetilde{K}_i = \begin{cases} a_{i1}d_1(t_1-u)^5 - a_{i2}d_2(t_2-u)^5 + a_{i3}d_3(t_3-u)^5 \\ \qquad -a_{i4}d_4(t_4-u)^5 + a_{i5}d_5(t_5-u)^5, & 0 \le u \le t_1 \\ -a_{i2}d_2(t_2-u)^5 + a_{i3}d_3(t_3-u)^5 - a_{i4}d_4(t_4-u)^5 \\ \qquad +a_{i5}d_5(t_5-u)^5, & t_1 \le u \le t_2 \\ a_{i3}d_3(t_3-u)^5 - a_{i4}d_4(t_4-u)^5 + a_{i5}d_5(t_5-u)^5, & t_2 \le u \le t_3, \\ -a_{i4}d_4(t_4-u)^5 + a_{i5}d_5(t_5-u)^5, & t_3 \le u \le t_4 \\ a_{i5}d_5(t_5-u)^5, & t_4 \le u \le t_5 \end{cases}$$

for a fixed number $i = 1,2,3,4,5$, or equivalently

$$\widetilde{K}_i = \widetilde{K}_{ik} = \sum_{j=k}^{5}(-1)^{j-1}a_{ij}d_j(t_j-u)^5, \quad t_{k-1} \le u \le t_k,$$

for a fixed number $i = 1,2,3,4,5$ and all numbers $k = 1,2,3,4,5 : t_0 = 0 < t_1 < t_2 < t_3 < t_4 < t_5$. Thus we get the formula

$$(13) \qquad K_i = (-1)^{i-1}\sigma_i\frac{1}{D}\sum_{j=k}^{5}(-1)^{j-1}a_{ij}d_j(t_j-u)^5, \quad t_{k-1} \le u \le t_k,$$

for a fixed number $i = 1,2,3,4,5$ and all numbers $k = 1,2,3,4,5 : t_0 = 0 < t_1 < t_2 < t_3 < t_4 < t_5$.

It is obvious that $(-1)^{5-1}K_i(\omega;u) \ge 0$ ($i = 1,2,3,4,5$) for every $u \in [0,t_5]$ and *claim* that

$$\int_0^{t_5} K_1\,du = \frac{\omega_5}{6!} \quad \text{and} \quad \int_0^{t_5} K_i\,du = \frac{i!}{(i-1)(6!)}(\overline{a}_{i-1} \bullet \overline{1})$$

hold, or

$$(14) \qquad \int_0^{t_5} K_i\,du = (-1)^{5-1}\frac{i!}{6!}r_i(\overline{a}_{i-1} \bullet \overline{1}) = \frac{i!}{6!}r_i(\overline{a}_{i-1} \bullet \overline{1})$$

where r_i ($i = 1,2,3,4,5$) are given by (9). In fact,

$$(15)\qquad \sum_{j=1}^{5}(-1)^{j-1}d_jt_j^6 = \omega_5 \sum_{j=1}^{5}\left[(-1)^{j-1}\left(\frac{d_j}{\prod\limits_{\substack{k=1\\(k\neq j)}}^{5} t_k}\right)t_j^5\right]$$

$$= \omega_5[(-1)^{5-1}(D)(t_1+t_2+t_3+t_4+t_5)/\sigma_5\omega]$$

$$= \frac{D}{4\sigma_5}(\bar{a}_4 \bullet \bar{1}).$$

Therefore for $i = 5$

$$\int_0^{t_5} K_5 du = \sigma_5 \sum_{j=1}^{5}(-1)^{j-1}d_jt_j^6 \Big/ 6D = \frac{\sigma_5}{6D}\left[\frac{D}{4\sigma_5}(\bar{a}_4 \bullet \bar{1})\right]$$

$$= \frac{1}{24}(\bar{a}_4 \bullet \bar{1}) = \frac{5!}{(5-1)(6!)}(\bar{a}_4 \bullet \bar{1}) = \frac{1}{6!}r_5(\bar{a}_4 \bullet \bar{1}).$$

Similarly we prove the other cases ($i = 1, 2, 3, 4$) of (14) .Besides

$$\bar{a}_i \bullet \overline{T} = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}) \bullet (T_1, T_2, T_3, T_4, T_5) = \sum_{j=1}^{5}(a_{ij}T_j)$$

$$= \frac{\sigma_5}{D}\sum_{j=1}^{5}[(-1)^{j-1}a_{ij}d_jT(t_j)],$$

or

$$(16)\qquad |\bar{a}_i \bullet \overline{T}| \le M\frac{\sigma_5}{D}(\bar{a}_i \cdot \bar{d})$$

$i = 1, 2, 3, 4, 5$, holds by the triangle inequality.

Therefore from the formulas $(*)$–(14) and the triangle inequality as well as the inequalities (16) we establish the inequalities (7). This completes the proof of *Theorem* 1.

Note that if we set

$$t_1 = t, \quad t_2 = m_1t, \quad t_3 = m_2t, \quad t_4 = m_3t, \quad t_5 = m_4t,$$
$$\text{or } t_j = (m_{j-1})t \ (j = 1, 2, 3, 4, 5),$$

$$(17)\qquad m_0(=1) < m_1 < m_2 < m_3 < m_4, \quad t > 0,$$

in matrix $(a_{ij}) : i = 0, 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4, 5$, then we obtain the new (6×5) matrix $(\alpha_{ij}) : i = 0, 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4, 5$, such that

$a_{0j} = \alpha_{0j}t^5, \quad j = 1, 2, 3, 4, 5,$ where

$$\alpha_{01} = \alpha_{02} = \alpha_{03} = \alpha_{04} = \alpha_{05} = \tfrac{1}{5}m_1m_2m_3m_4;$$

$a_{1j} = \alpha_{1j}t^4,\ j = 1, 2, 3, 4, 5,$ where

$$\alpha_{11} = m_1m_2m_3m_4, \quad \alpha_{12} = m_2m_3m_4, \quad \alpha_{13} = m_1m_3m_4,$$

$$\alpha_{14} = m_1 m_2 m_4, \quad \alpha_{15} = m_1 m_2 m_3;$$

$a_{2j} = \alpha_{2j} t^3$, $j = 1,2,3,4,5$, where

$$\alpha_{21} = m_1 m_2 m_3 + m_1 m_2 m_4 + m_1 m_3 m_4 + m_2 m_3 m_4,$$

$$\alpha_{22} = m_2 m_3 + m_2 m_4 + m_3 m_4 + m_2 m_3 m_4,$$

$$\alpha_{23} = m_1 m_3 + m_1 m_4 + m_3 m_4 + m_1 m_3 m_4,$$

$$\alpha_{24} = m_1 m_2 + m_1 m_4 + m_2 m_4 + m_1 m_2 m_4,$$

$$\alpha_{25} = m_1 m_2 + m_1 m_3 + m_2 m_3 + m_1 m_2 m_3;$$

$a_{3j} = \alpha_{3j} t^2$, $j = 1,2,3,4,5$, where

$$\begin{aligned}
\alpha_{31} &= m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4,\\
\alpha_{32} &= m_2 + m_3 + m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4,\\
\alpha_{33} &= m_1 + m_3 + m_4 + m_1 m_3 + m_1 m_4 + m_3 m_4,\\
\alpha_{34} &= m_1 + m_2 + m_4 + m_1 m_2 + m_1 m_4 + m_2 m_4,\\
\alpha_{35} &= m_1 + m_2 + m_3 + m_1 m_2 + m_1 m_3 + m_2 m_3;
\end{aligned}$$

$a_{4j} = \alpha_{4j} t$, $j = 1,2,3,4,5$, where

$$\alpha_{41} = m_1+m_2+m_3+m_4, \quad \alpha_{42} = 1+m_2+m_3+m_4, \quad \alpha_{43} = 1+m_1+m_3+m_4,$$
$$\alpha_{44} = 1 + m_1 + m_2 + m_4, \quad \alpha_{45} = 1 + m_1 + m_2 + m_3;$$

$a_{5j} = \alpha_{5j}$, $j = 1,2,3,4,5$, where

$$\alpha_{51} = \alpha_{52} = \alpha_{53} = \alpha_{54} = \alpha_{55} = 1.$$

Therefore

(17a) $$a_{ij} = \alpha_{ij} t^{5-i}: \quad i = 0,1,2,3,4,5; \quad j = 1,2,3,4,5,$$

where a_{ij} are given above. Also setting (17) in d_i $(i = 1,2,3,4,5)$, we get

(17b) $$d_j = \delta_j t^{10}, \quad j = 1,2,3,4,5,$$

where $10 = 4 + \binom{4}{2}$ and

$$\begin{aligned}
\delta_1 &= m_1 m_2 m_3 m_4 (m_1 - m_2)(m_1 - m_3)(m_1 - m_4)\\
&\quad \times (m_2 - m_3)(m_2 - m_4)(m_3 - m_4),\\
\delta_2 &= m_2 m_3 m_4 (1 - m_2)(1 - m_3)(1 - m_4)(m_2 - m_3)(m_2 - m_4)(m_3 - m_4),\\
\delta_3 &= m_1 m_3 m_4 (1 - m_1)(1 - m_3)(1 - m_4)(m_1 - m_3)(m_1 - m_4)(m_3 - m_4),\\
\delta_4 &= m_1 m_2 m_4 (1 - m_1)(1 - m_2)(1 - m_4)(m_1 - m_2)(m_1 - m_4)(m_2 - m_4),\\
\delta_5 &= m_1 m_2 m_3 (1 - m_1)(1 - m_2)(1 - m_3)(m_1 - m_2)(m_1 - m_3)(m_2 - m_3).
\end{aligned}$$

Therefore from (17a)–(17b) , we find that

$$\text{(17c)}\quad \overline{a}_i \bullet \overline{d} = \sum_{j=1}^{5} a_{ij}d_j = \sum_{j=1}^{5}(a_{ij}t^{5-i})(\delta_j t^{10}) = t^{15-i}\sum_{j=1}^{5} a_{ij}\delta_j, \quad \text{or}$$

$$\overline{a}_i \bullet \overline{d} = (\overline{\alpha}_i \bullet \overline{\delta})t^{15-i}, \quad i = 1,2,3,4,5,$$

where

$$15 = 5 + \left[4 + \binom{4}{2}\right] \quad \text{and vectors}$$

$\overline{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}, \alpha_{i5})$ $(i = 1,2,3,4,5)$ and $\overline{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$, as well as vectors $\overline{\alpha}_0 = (\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{04}, \alpha_{05})$ such that $\alpha_{0j} = \frac{1}{5}\omega_5^*$ $(j = 1,2,3,4,5)$:

$$\text{(17d)}\qquad \omega_5 = \omega_5^* t^5,$$

where $\omega_5^* = m_1m_2m_3m_4$.

Also *note* that

$\overline{a}_0 \bullet \overline{1} = \omega_5 = \omega_5^* t^5$, where

$\omega_5^* = \overline{\alpha}_0 \bullet \overline{1} = m_1m_2m_3m_4$,

$\overline{a}_1 \bullet \overline{1} = (\overline{\alpha}_1 \bullet \overline{1})t^4$, where

$$\overline{\alpha}_1 \bullet \overline{1} = m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 + m_2m_3m_4 + m_1m_2m_3m_4,$$

$(\overline{a}_2 \bullet \overline{1})/2 = [(\overline{\alpha}_2 \bullet \overline{1})/2]t^3$, where

$$\begin{aligned}(\overline{\alpha}_2 \bullet \overline{1})/2 = {} & m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4 \\ & + m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 + m_2m_3m_4,\end{aligned}$$

$(\overline{a}_3 \bullet \overline{1})/3 = [(\overline{\alpha}_3 \bullet \overline{1})/3]t^2$, where

$$\begin{aligned}(\overline{\alpha}_3 \bullet \overline{1})/3 = {} & m_1 + m_2 + m_3 + m_4 + m_1m_2 + m_1m_3 + m_1m_4 \\ & + m_2m_3 + m_2m_4 + m_3m_4,\end{aligned}$$

$(\overline{a}_4 \bullet \overline{1})/4 = [(\overline{\alpha}_4 \bullet \overline{1})/4]t$, where

$$(\overline{\alpha}_4 \bullet \overline{1})/4 = 1 + m_1 + m_2 + m_3 + m_4, \quad \text{and}$$

$(\overline{a}_5 \bullet \overline{1})/5 = [(\overline{\alpha}_5 \bullet \overline{1})/5]$, where

$(\overline{\alpha}_5 \bullet \overline{1})/5 = 1.$

Therefore in the afore-mentioned inequalities (7) we have

$$\text{(17e)}\qquad \left.\begin{aligned} \overline{a}_i \bullet \overline{1} &= (\overline{\alpha}_i \bullet \overline{1})t^{5-i} \\ \text{and}\quad \overline{a}_{i-1} \bullet \overline{1} &= (\overline{\alpha}_{i-1} \bullet \overline{1})t^{6-i} \end{aligned}\right\},$$

for $i = 1,2,3,4,5$. Besides

$\overline{T} = (T_1, T_2, T_3, T_4, T_5)$ with

$$T_j = (-1)^{j-1}\frac{\sigma_5}{D}d_jT(t_j) = (-1)^{j-1}\frac{\sigma_5}{(D^*)t^{15}}(\delta_j t^{10})T(m_{j-1}t),$$

$j = 1, 2, 3, 4, 5$, where

(17f) $$D = D^* t^{15}$$

with

(11a) $$D^* = \sigma_5[(m_1 m_2 m_3 m_4)(1 - m_1)(1 - m_2)(1 - m_3)(1 - m_4) \times (m_1 - m_2)(m_1 - m_3)(m_1 - m_4)(m_2 - m_3) \times (m_2 - m_4)(m_3 - m_4)]$$

and

$$T_j = \left[(-1)^{j-1} \frac{\sigma_5}{D^*} \delta_j T(m_{j-1} t)\right] t^{-5}.$$

Therefore

(11b) $$\overline{T} = \overline{T}^* t^{-5}, \quad \text{or } T_j = T_j^* t^{-5}$$

where

$\overline{T}^* = (T_1^*, T_2^*, T_3^*, T_4^*, T_5^*)$ with

$T_j^* = (-1)^{j-1} \frac{\sigma_5}{D^*} \delta_j \, T(m_{j-1} t), \quad j = 1, 2, 3, 4, 5$ or

$\overline{T}^* = \sigma_5 \frac{1}{D^*}(\delta_1 T(t), -\delta_2 T(m_1 t), \delta_3 T(m_2 t), -\delta_4 T(m_3 t), \delta_5 T(m_4 t)).$

Thus from (17a) and (11b) , we obtain

$$\overline{a}_i \bullet \overline{T} = \sum_{j=1}^{5} a_{ij} T_j = \sum_{j=1}^{5} (a_{ij} t^{5-i})(T_j^* t^{-5}) = t^{-i} \sum_{j=1}^{5} a_{ij} T_j^*,$$

or

(11c) $$\overline{a}_i \bullet \overline{T} = (\overline{\alpha}_i \bullet \overline{T}^*) t^{-i}, i = 1, 2, 3, 4, 5.$$

Besides

$$\overline{\alpha}_i \bullet \overline{T}^* = \sum_{j=1}^{5} \alpha_{ij} T_j^* = \frac{\sigma_5}{D^*} \sum_{j=1}^{5} [(-1)^{j-1} \alpha_{ij} \delta_j T(m_{j-1} t)],$$

or

(11d) $$|\overline{\alpha}_i \bullet \overline{T}^*| \leq M \frac{\sigma_5}{D^*} (\overline{a}_i \bullet \overline{\delta}), \ i = 1, 2, 3, 4, 5$$

holds by triangle inequality.

Finally we *claim* that

(11e) $$K_i(\omega; u) = (K_i^*(\omega^*; u^*)) t^{5-i}, \mathrm{t} > 0, m_{k-2} \leq u^* \left(= \frac{u}{t} \right) \leq m_{k-1}$$

where

$$K_i^*(\omega^*; u^*) = (-1)^{i-1} \sigma_i \frac{1}{D^*} \sum_{j=k}^{5} (-1)^{j-1} \alpha_{ij} \delta_j (m_{j-1} - u^*)^5,$$

for a fixed number $i = 1, 2, 3, 4, 5$, and all numbers $k = 1, 2, 3, 4, 5$:

$$m_{-1}(=0) < m_0(=1) < m_1 < m_2 < m_3 < m_4.$$

In fact, from (13) and (17)–(17a)–(17b)–(17f), we obtain

$$K_i = (-1)^{i-1} \frac{\sigma_i}{(D^*)t^{15}} \sum_{j=k}^{5} (-1)^{j-1} (\alpha_{ij} t^{5-i})(\delta_j t^{10})(m_j - u^*)^5 t^5,$$

$$m_{k-2} \leq u^* \left(= \frac{u}{t} \right) \leq m_{k-1},$$

yielding (11e).

Note that from (11e) and (17) as well as $u = u^* t : du = t du^*$, we obtain

$$\int\limits_{u=0}^{u=t_5} K_i(\omega; u) T(u) A^6 x du = \int\limits_{u^*=0}^{u^*=m_4} (K_i^*(\omega^*; u^*)) t^{5-i} T(u^* t) A^6 x (t du^*), \text{ or}$$

$$\int\limits_0^{t_5} K_i(\omega; u) T(u) A^6 x du = \left[\int\limits_0^{m4} K_i^*(\omega^*; u^*) T(u^* t) A^6 x du^* \right] t^{6-i}, \tag{11f}$$

for a fixed number $i = 1, 2, 3, 4, 5$.

Thus from (17d)–(17e) and (11c)–(11f) as well as (*), we find

$$\begin{aligned} A^i x = (-1)^{i-1} \Bigg\{ & \left\{ i! \left[(\overline{\alpha}_i \bullet \overline{T}^*) - \frac{1}{i\omega_5^*} (\overline{\alpha}_i \bullet \overline{1}) \right] x \right\} \frac{1}{t^i} \\ & - \left\{ \left[\int\limits_0^{m_4} K_i^*(\omega^*; u^*) T(u^* t) du^* \right] A^6 x \right\} t^{6-i} \Bigg\} \end{aligned} \tag{[*]}$$

for a fixed number $i = 1, 2, 3, 4, 5$.

THEOREM 2. *Let $t \to T(t)$ be a uniformly bounded ($\|T(t)\| \leq M < \infty$, $t \geq 0$) strongly continuous semigroup of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^6 x \neq 0$. Then the following five inequalities*

$$\|A^i x\|^6 \leq \frac{(i!)^6}{(6!)^i} \begin{bmatrix} 6 \\ i \end{bmatrix} M^i g_i(\omega^*) \|x\|^{6-i} \|A^6 x\|^i \tag{7a}$$

hold for every $x \in D(A^6)$ (on X) and a fixed number $i = 1, 2, 3, 4, 5$, where our symbol

$$\begin{bmatrix} 6 \\ i \end{bmatrix} = \frac{6^6}{i^i (6-i)^{6-i}}$$

holds for a fixed number $i = 1, 2, 3, 4, 5$ and

$$g_i(\omega^*) = \left[M \frac{\sigma_5}{D^*} (\overline{\alpha}_i \bullet \overline{\delta}) + \frac{1}{i\omega_5^*} (\overline{\alpha}_i \bullet \overline{1}) \right]^{6-i} \left[r_i (\overline{\alpha}_{i-1} \bullet \overline{1}) \right]^i$$

with $\omega^* = (m_1, m_2, m_3, m_4)$ *for some* $m_j \in R^+ = (0, \infty)(j = 1,2,3,4)$: $1 < m_1 < m_2 < m_3 < m_4$ *as well as* $\sigma_5, D^*, \omega_5^*, r_i (i = 1,2,3,4,5), \overline{\alpha}_i (i = 1,2,3,4,5), \overline{\delta}, \overline{1}, \overline{\alpha}_0$ *are given above.*

Proof of theorem 2 . Employing (17)–(17c)–(17d)–(17e) and (17f) in (7), we obtain the following five inequalities

$$\|A^i x\| \leq b_i \frac{1}{t^i} + c_i t^{6-i}, \tag{18}$$

where

$$b_i = \left\{ i! \left[M \frac{\sigma_5}{D^*} (\overline{\alpha}_i \bullet \overline{\delta}) + \frac{1}{i\omega_5^*} (\overline{\alpha}_i \bullet \overline{1}) \right] \right\} \|x\|,$$

$$c_i = M \left[\frac{i!}{6!} r_i (\overline{\alpha}_{i-1} \bullet \overline{1} \right] \|A^6 x\|,$$

for any fixed number $i = 1,2,3,4,5$.

Note that (18) yields also from [*] and (11d) as well as from the fact

$$\int_0^{m_4} K_i^*(\omega^*; u^*) du^* = \frac{i!}{6!} r_i (\overline{\alpha}_{i-1} \bullet \overline{1}) \quad \text{for } i = 1,2,3,4,5.$$

Minimizing the right-hand side functions of t of (18), we get the *sharper* inequalities

$$\|A^i x\|^6 \leq \begin{bmatrix} 6 \\ i \end{bmatrix} b_i^{6-i} c_i^i, \tag{18a}$$

where

$$\begin{bmatrix} 6 \\ i \end{bmatrix} = \frac{6^6}{i^i (6-i)^{6-i}} \tag{18b}$$

for a fixed number $i = 1,2,3,4,5$ and

$$b_i^{6-i} c_i^i = \frac{(i!)^6}{(6!)^i} M^i g_i(\omega^*) \|x\|^{6-i} \|A^6 x\|^i, \tag{18c}$$

as well as

$$g_i(\omega^*) = \left[M \frac{\sigma_5}{D^*} (\overline{\alpha}_i \bullet \overline{\delta}) + \frac{1}{i\omega_5^*} (\overline{\alpha}_i \bullet \overline{1}) \right]^{6-i} \left[r_i (\overline{\alpha}_{i-1} \bullet \overline{1}) \right]^i, \tag{18d}$$

with $\omega^* = (m_1, m_2, m_3, m_4)$ for a fixed number $i = 1,2,3,4,5$.

Therefore from (18a)–(18b)–(18c)–(18d), we obtain the inequalities (7a). This *completes* the proof of *Theorem 2.*

Note that the point $\omega^* = \omega_0^* = (m_{10}, m_{20}, m_{30}, m_{40})$ such that $m_1 = m_{10} = 2 + \sqrt{3}$, $m_2 = m_{20} = 4 + 2\sqrt{3}$, $m_3 = m_{30} = 6 + 3\sqrt{3}$, $m_4 = m_{40} = 7 + 4\sqrt{3}$, where $m_{10} = m_{20} - m_{10} = m_{30} - m_{20} = m_{40} - m_{30} + 1$ is the *common* global minimum point for all functions $g_i = g_i(\omega^*)$ $(i = 1,2,3,4,5)$.

Setting

$$\varepsilon_i = \min g_i(\omega^*) = g_i(\omega_0^*) \ (i = 1,2,3,4,5)$$

we find that

(18e) $\varepsilon_1 = 2^1 3^6 5^5, \varepsilon_2 = 5^6 7^6, \varepsilon_3 = 2^{15}, \varepsilon_4 = 2^4 3^6 7^6, \varepsilon_5 = 2^5 5^5 \ (= 100000)$

Consider constants $R_i^1(6)$ for a fixed number $i = 1,2,3,4,5$ such that

(18f) $$R_i^1(6) = \frac{(i!)^6}{(6!)^i} \begin{bmatrix} 6 \\ i \end{bmatrix} \varepsilon_i .$$

Therefore from (7a) (theorem 2 with $M = 1$) and from (18e) - (18f) we prove

THEOREM 3. *Let $t \to T(t)$ be a strongly continuous contraction ($\|T(t)\| \leq 1, t \geq 0$) semigroup of linear operators on a complex Banach space X with infinitesimal generator A, such that $A^6 x \neq 0$. Then the following five inequalities*

(7b) $$\|A^i x\|^6 \leq R_i^1(6) \|x\|^{6-i} \|A^6 x\|^i ,$$

hold for every $x \in D(A^6)$ (on X) and for a fixed number $i = 1,2,3,4,5$, where constants $R_i^1(6) (i = 1,2,3,4,5)$ are given above.

3. Cosine functions

Let $t \to T(t)$ $(t \geq 0)$ be a uniformly bounded ($\|T(t)\| \leq M < \infty, t \geq 0$) strongly continuous cosine function with infinitesimal operator A, such that $T(0) = I$ (: = identity) in $B(X)$, $\lim_{t \downarrow 0} T(t)x = x, \forall x$, and A is defined as the strong second derivatives of T at zero,

(19) $$Ax = T''(0)x$$

for every x in a linear subspace $D(A)$ dense in X [5]. For every $x \in D(A)$, we have the formula

(20) $$T(t)x = x + \int_0^t (t-u)T(u)Axdu .$$

Using integration by parts, we get from (20) the formula

(21) $$\int_0^t (t-u)\left(\int_0^u (u-v)f(v)dv\right)du = \frac{1}{6}\int_0^t (t-v)^3 f(v)dv ,$$

where $f(v) = T(v)A^2x$. Note the *Leibniz formula*

(22) $$\frac{d}{du}\left(\int_0^u (u-v)^n f(v)dv\right) = n\left(\int_0^u (u-v)^{n-1} f(v)dv\right) .$$

Employing (21)–(22) and iterating (20) we find for every $x \in D(A^2)$ that

$$T(t)x = x + \frac{t^2}{2!}Ax + \frac{1}{3!}\int_0^t (t-u)^3 T(u)A^2 x dx \ . \tag{20a}$$

Similarly iterating (20a) we obtain for every $x \in D(A^6)$ that

$$\begin{aligned} T(t)x =& x + \frac{t^2}{2!}Ax + \frac{t^4}{4!}A^2x + \frac{t^6}{6!}A^3x + \frac{t^8}{8!}A^4x + \frac{t^{10}}{10!}A^5x + \\ & \frac{1}{11!}\int_0^t (t-u)^{11} T(u)A^6 x du \ . \end{aligned} \tag{20b}$$

THEOREM 4 . *Let $t \to T(t)$ be a uniformly bounded ($\|T(t)\| \le M < \infty, t \ge 0$) strongly continuous cosine function on a complex Banach space X with infinitesimal generator A, such that $A^6x \neq 0$. Then the following five inequalities*

$$\begin{aligned} \|A^ix\| \le ((2i)!)\Bigg\{ & \left[M\frac{\sigma_5^+}{D^+}(\overline{a}_i \bullet \overline{d}) + \frac{1}{i\omega_5}(\overline{a}_i \bullet \overline{1}) \right] \|x\| \\ & + \left[\frac{1}{(12)!} M(r_i(\overline{a}_{i-1} \bullet \overline{1})) \right] \|A^6x\| \Bigg\} \ , \end{aligned} \tag{8}$$

hold for every $x \in D(A^6)$ and a fixed number $i = 1,2,3,4,5$, where σ_5, D, ω_5 r_i $(i = 1,2,3,4,5), \overline{a}_i$ $(i = 1,2,3,4,5,)$, $\overline{d}$, $\overline{1}$, $\overline{a}_0$ are given above and $\frac{\sigma_5^+}{D^+} = \frac{\sigma_5}{D}$.

Proof of Theorem 4. In fact, setting $t(>0)$ instead of t^2 in (20b), we get

$$\begin{aligned} T(\sqrt{t}x) = & x + \frac{t}{2!}Ax + \frac{t^2}{4!}A^2x + \frac{t^3}{6!}A^3x + \frac{t^4}{8!}A^4x + \frac{t^5}{10!}A^5x \\ & + \frac{1}{11!}\int_0^{\sqrt{t}} (\sqrt{t}-u)^{11} T(u)A^6 x du \ . \end{aligned} \tag{20c}$$

Formula (20c) yields an *algebraic system* of five equations with respect to A^ix:

$$\begin{aligned} & \frac{11!}{2!}t_iAx + \frac{11!}{4!}t_i^2A^2x + \frac{11!}{6!}t_i^3A^3x + \frac{11!}{8!}t_i^4A^4x + \frac{11!}{10!}t_i^5A^5x \\ & = 11!T(\sqrt{t}x) - 11!x - \int_0^{\sqrt{t_i}} (\sqrt{t_i}-u)^{11} T(u)A^6 x du \ , \end{aligned} \tag{21}$$

$i = 1,2,3,4,5.$

The coefficient determinant D^+ of the system (21) with respect to unknowns $A^i x (i = 1, 2, 3, 4, 5)$ is

$$D^+ = \sigma_5^+ \frac{D}{\sigma_5} \tag{11a}$$

where

$$\sigma_5^+ = \frac{(11!)^5}{2!\ 4!\ 6!\ 8!\ 10!}\ , \qquad \sigma_5 = \frac{(5!)^4}{1!\ 2!\ 3!\ 4!} \tag{11b}$$

and D is given by (11). Also from (11) we find

$$\frac{D^+}{\sigma_5^+} = \frac{D}{\sigma_5} = t_1 t_2 t_3 t_4 t_5 (t_1 - t_2)(t_1 - t_3)(t_1 - t_4)(t_1 - t_5)(t_2 - t_3) \times (t_2 - t_4)(t_2 - t_5)(t_3 - t_4)(t_3 - t_5)(t_4 - t_5). \tag{11c}$$

The solution $A^i x (i = 1, 2, 3, 4, 5)$ of the system (21) is *unique* and of the form

(22)

$$A^i x = (-1)^{i-1} \left\{ (2i)! \left[(\overline{a}_i \bullet \overline{T}^+) - \frac{1}{i\omega_5} (\overline{a}_i \bullet \overline{1}) \right] x - \int_0^{\sqrt{t_5}} K_i^+(\omega; u) T(u) A^6 x du \right\} ,$$

where

$$\overline{T}^+ = \sigma_5^+ \frac{1}{D^+} (d_1 T(\sqrt{t_1}), -d_2 T(\sqrt{t_2}), d_3 T(\sqrt{t_3}), -d_4 T(\sqrt{t_4}), d_5 T(\sqrt{t_5}))$$

and $K_i^+ = K_i^+(\omega; u) (i = 1, 2, 3, 4, 5)$:

$$K_i^+ = (-1)^{i-1} \sigma_i^+ \frac{1}{D^+} \sum_{j=k}^{5} (-1)^{j-1} a_{ij} d_j (\sqrt{t_j} - u)^5, \ \sqrt{t_{k-1}} \le u \le \sqrt{t_k} \tag{13a}$$

for a fixed number $i = 1, 2, 3, 4, 5$ and all numbers $k = 1, 2, 3, 4, 5$ such that $\sqrt{t_0} = 0 < \sqrt{t_1} < \sqrt{t_2} < \sqrt{t_3} < \sqrt{t_4} < \sqrt{t_5}$ and

$$\sigma_i^+ = \frac{((11)!)^4}{\prod_{\substack{j=1 \\ (j \ne i)}}^{5} (2j)!} \left(\text{e.g. } \sigma_3^+ = \frac{((11)!)^4}{2!\ 4!\ 8!\ 10!} \right) .$$

Note that $K_i^+ = K_i^+(\omega; u) \ge 0 (i = 1, 2, 3, 4, 5)$ for every $u \in [0, \sqrt{t_5}]$ and

$$\int_0^{\sqrt{t_5}} K_i^+ du = \frac{(2i)!}{(12)!} r_i (\overline{a}_{i-1} \bullet \overline{1}) \ , \tag{14a}$$

where $r_i (i = 1, 2, 3, 4, 5)$ are given by (9).

The rest of the proof is similar to that of Theorem 1. Thus the proof of *Theorem 4* is *complete.*

Similarly from Theorems 2–3 we establish the following Theorems 5–6.

THEOREM 5. *Let* $t \to T(t)$ *be a uniformly bounded* ($\|T(t)\| \le M < \infty,\ t \ge 0$) *strongly continuous cosine function on a complex Banach space* X *with infinitesimal generator* A, *such that* $A^6x \neq 0$. *Then the following inequalities*

$$\|A^i x\|^6 \le \frac{((2i)!)^6}{((12)!)^i}\begin{bmatrix}6\\ i\end{bmatrix} M^i g_i(\omega^*)\|x\|^{6-i}\|A^6x\|^i\ , \tag{8a}$$

hold for every $x \in D(A^6)$ (*on* X) *and a fixed number* $i = 1, 2, 3, 4, 5$, *where* $g_i = g_i(\omega^*)$ *and* $\begin{bmatrix}6\\ i\end{bmatrix}$ are given as in Theorem 2.

THEOREM 6. *Let* $t \to T(t)$ *be a strongly continuous contraction* ($\|T(t)\| \le 1, t \ge 0$) *cosine function on a complex Banach space* X *with infinitesimal generator* A, *such that* $A^6x \neq 0$. *Then the following five inequalities*

$$\|A^i x\|^6 \le R_i^3(6)\|x\|^{6-i}\|A^6x\|^i\ , \tag{8b}$$

hold for every $x \in D(A^6)$ (*on* X) *and for a fixed number* $i = 1, 2, 3, 4, 5$, *where constants* $R_i^3(6)$ ($i = 1, 2, 3, 4, 5$) *are of the form*

$$R_i^3(6) = \frac{((2i)!)^6}{((12)!)^i}\begin{bmatrix}6\\ i\end{bmatrix}\varepsilon_i$$

with $\begin{bmatrix}6\\ i\end{bmatrix}$ *given as in Theorem 2 and* ε_i ($i = 1, 2, 3, 4, 5$) *by* (18e).

Query. The corresponding case to *groups* ($k = 2$) and the computation of the constants $R_i^2(6)$ ($i = 1, 2, 3, 4, 5$) is still *open.*

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