DEMONSTRATIO MATHEMATICA
Vol. XXXII No 1 1999

Paola Mannucci, Domingo Alberto Tarzia

THE SUPERCOOLED ONE-PHASE STEFAN PROBLEM IN SPHERICAL SYMMETRY

Abstract. The supercooled one-phase Stefan problem in spherical symmetry with a heat flux condition at the fixed face is considered. The relation between the heat flux and the initial temperature is analysed in order to characterize the cases with a global solution (possibility of continuing the solution for arbitrarily large time intervals), a finite time extinction and a blow-up at a finite time.

1. Introduction

We study a supercooled one-phase Stefan problem in spherical symmetry ($r \in [r_0, 1]$, $r_0 > 0$) corresponding to a positive heat flux condition at the fixed face and a negative initial temperature. Problems of this kind have been studied by other authors in connection with the freezing of a supercooled liquid. Several different boundary conditions were analysed in [3], [5], [6], [7], [10], [11], [13], for the one-dimensional case, in [1], [2] for cylindrical symmetry and in [9] for spherical symmetry.

In Section 1 we give the preliminaries corresponding to the description of the problem and in Section 2 we obtain conditions for data in order to characterize the cases with a global solution (possibility of continuing the solution for arbitrarily large time intervals), a finite time extinction and a blow-up at a finite time. In Section 3 we study the asymptotic behaviour of the solution and we give some results concerning the particular case $r_0 = 0$ with null heat flux, which are a sequel to those given in [9].

In this paper we study the following problem:

The first author is partially supported by the Italian M.U.R.S.T. National Project "Problemi non lineari..."

Key words:One-phase Stefan problem, phase-change problem, parabolic free boundary problem.

AMS Subject classifications: 35R35, 80A22, 35B40, 35K05.

PROBLEM I. Find a triple $(\theta(\rho,\tau),\sigma(\tau),T_1)$ such that:

i) $T_1>0$

ii) $\sigma(\tau)\in C^0([0,T_1])\cap C^1((0,T_1,))$, $\rho_0<\sigma(\tau)<b$, $0<\tau<T_1$, $\rho_0>0$.

iii) $\theta(\rho,\tau)$ is a bounded continuous function in $\rho_0\le\rho\le\sigma(\tau)$, $0\le\tau\le T_1$, $\theta_\rho(\rho,\tau)$ is continuous in $\rho_0\le\rho\le\sigma(\tau)$, $0<\tau<T_1$, $\theta_{\rho\rho},\theta_\tau$ are continuous in $\rho_0<\rho<\sigma(\tau)$, $0<\tau<T_1$.

iv) $\sigma(\tau)$ and $\theta(\rho,\tau)$ obey the conditions:

$$\begin{aligned}
&\theta_\tau=\alpha\Big(\theta_{\rho\rho}+\frac{2\theta_\rho}{\rho}\Big), && \rho_0<\rho<\sigma(\tau),\ 0<\tau<T_1,\\
&\sigma(0)=b,\\
&\theta(\rho,0)=\widetilde{h}(\rho), && \rho_0<\rho<b,\\
&\theta_\rho(\rho_0,\tau)=\widetilde{g}(\tau), && 0<\tau<T_1,\\
&\theta(\sigma(\tau),\tau)=0, && 0<\tau<T_1,\\
&\alpha c\theta_\rho(\sigma(\tau),\tau)=-\Lambda\dot{\sigma}(\tau), && 0<\tau<T_1.
\end{aligned}$$

The nomenclature is the following:

α material thermal diffusivity,
c specific heat,
Λ latent heat of melting,
θ temperature,
σ free boundary,
ρ radial coordinate variable,
τ time.

The adimensional problem corresponding to Problem I is obtained by the following transforms:

$$r=\frac{\rho}{b},\quad t=\frac{\alpha}{b^2}\tau, r_0=\frac{\rho_0}{b}<1,\quad T=\frac{T_1\alpha}{b^2},$$
$$u(r,t)=\frac{c}{\Lambda}\theta(\rho,\tau),\quad s(t)=\frac{\sigma(\tau)}{b},$$
$$h(r)=\frac{c}{\Lambda}\widetilde{h}(\rho),\quad g(t)=\frac{cb}{\Lambda}\widetilde{g}\Big(\frac{\alpha}{b^2}\tau\Big).$$

Then the variables (T,u,s) satisfy the problem

PROBLEM II. Find a triple (T,s,u) such that:

i) $T>0$.

ii) $s(t)\in C([0,T])$, $s\in C^1((0,T))$, $r_0<s(t)<1$, for $0<t<T$ and $r_0>0$.

iii) $u(r,t)$ is a bounded function, continuous in $r_0\le r\le s(t)$, $0\le t<T$, $u_r(r,t)$ is continuous in $r_0\le r\le s(t)$, $0<t<T$, u_{rr}, u_t are continuous in $r_0<r<s(t)$, $0<t<T$.

iv) The following conditions are satisfied:

$$u_t = u_{rr} + \frac{2}{r}u_r, \text{in } D_T = \{(r,t) : r_0 < r < s(t), 0 < t < T\}, \tag{1}$$

$$s(0) = 1, \tag{2}$$

$$u(r,0) = h(r), \qquad r_0 < r < 1, \tag{3}$$

$$u_r(r_0,t) = g(t), \qquad 0 < t < T, \tag{4}$$

$$u(s(t),t) = 0, \qquad 0 < t < T, \tag{5}$$

$$u_r(s(t),t) = -\dot{s}(t), \qquad 0 < t < T, \tag{6}$$

where we impose, from now on, the following assumptions:

(A_1) $h(r) \le 0,\ r_0 < r < 1,\ h(r)$ is a continuous function,

(A_2) $g(t) \ge 0,\ 0 < t < T,\ g(t)$ is a piecewise continuous function.

(Whenever we consider the derivatives of h and g we suppose further regularity of these functions).

Three cases can occur ([4], [5], [12]):

(A) The problem has a solution with arbitrarily large T (global solution),

(B) There exists a constant $T_B > 0$ such that $\lim_{t\to T_B^-} s(t) = r_0$ (exctintion time),

(C) There exists a constant $T_C > 0$ such that $\lim_{t\to T_C^-} s(t) > r_0$ and $\lim_{t\to T_C^-} \dot{s}(t) = -\infty$ (blow-up).

As we shall see, any of these cases can actually occur with an appropriate choice of the functions $h(r), g(t)$ in (3), (4).

2. Study of the three cases

In order to characterize the three cases we obtain some preliminary properties.

LEMMA 2.1. *If (T, s, u) solves problem (1)-(6) then*

$$\frac{s^2(t)}{2} = \frac{1}{2} + \int_{r_0}^{1} r h(r)\, dr - \int_{r_0}^{s(t)} r\, u(r,t)\, dr - \int_0^t (u(r_0,\tau) + r_0\, g(\tau))\, d\tau, \tag{7}$$

$$\frac{s^3(t)}{3} = \frac{1}{3} + \int_{r_0}^{1} r^2 h(r)\, dr - \int_{r_0}^{s(t)} r^2\, u(r,t)\, dr - r_0^2 \int_0^t g(\tau)\, d\tau, \tag{8}$$

$$\frac{s^2(t)}{4}(s^2(t) + 2r_0^2) = \frac{1}{4}(1 + 2r_0^2) + \int_{r_0}^{1} (r_0^2 + r^2) r h(r)\, dr - \tag{9}$$

$$-2r_0^3 \int_0^t g(\tau)\, d\tau + 2 \iint_{D_t} r\, u(r,\tau)\, dr\, d\tau - \int_{r_0}^{s(t)} (r^2 + r_0^2) r\, u(r,t)\, dr.$$

Proof. Consider Green's identity

$$\iint_{D_t}(vLz - zL^*v)\,dr\,d\tau = \int_{\partial D_t}(z_r v - z v_r)\,d\tau + vz\,dr,$$

where L denotes the heat operator and L^* its adjoint, with $z = ru(r,t)$. If $v = 1$ and $v = r$ we get respectively (7) and (8). (9) follows adding to (7) the integral representation obtained with $v = r^2$, taking into account the condition (4). ■

LEMMA 2.2. *If assumptions* (A_1) *and* (A_2) *hold, we have:*

i) $u(r,t) \le 0$, *in* D_T,

ii) $\dot{s}(t) < 0$, $\forall\, t \in (0,T)$,

iii) *If* $h'(r) \ge 0$, *then* $u_r(r,t) \ge 0$ *in* D_T,

iv) *If* $h(r) \ge -h_1(1-r)$, $h_1 > 0$ *and* $0 < g(t) \le h_1$, *then* $u(r,t) \ge -h_1(1-r)$ *in* D_T.

Proof. i), iii) iv) follow from the maximum principle applied respectively to u, u_r, and $u + h_1(1-r)$.

ii) is a consequence of i) and (6). ■

Now, we shall prove a result which gives us a bound from below for $\dot{s}$; that is we avoid the occurrence of case (C)(for the planar case see ([5])).

LEMMA 2.3. *Let* (T,s,u) *be a solution of* (1)–(6). *If* $h(r) \ge -h_1(1-r)$, $0 \le r \le 1$ *and if there exist two constants* $d \in (0, s_T - r_0)$, $z_0 \in (0,1)$ *with* $s_T = \inf_{t\in[0,T]} s(t) > r_0$ *such that* $u(s(t)-d,t) \ge -z_0$, $h_1 d < z_0$, $\forall\, t \in (0,T)$ *then*

$$\dot{s}(t) \ge \frac{1}{d}\log(1 - z_0), \qquad \forall\, t \in (0,T). \tag{10}$$

Proof. Fix $\epsilon > 0$ and define $a = a(\epsilon) = -\inf_{t\in(0,T-\epsilon)}\dot{s}(t) > 0$, $w(r,t) = \dfrac{-z_0}{1-e^{-ad}}(1 - e^{a(r-s(t))}) \le 0$.

From the maximum principle, $w(r,t) \le u(r,t)$ in $\Omega_\epsilon = \{(r,t) : s(t) - d \le r \le s(t),\ 0 \le t \le T - \epsilon\}$. It follows that $u_r(s(t),t) \le w_r(s(t),t)$, hence from the definition of a and assumptions on h_1, z_0 and d, we get (10). ■

Under assumptions (A_1) and (A_2), we proceed now to characterize, cases (A), (B) and (C) in dependence on the value $Q(t)$, where

$$Q(t) = 1 + 3\int_{r_0}^{1} r^2 h(r)\,dr - 3r_0^2\int_0^t g(\tau)d\tau. \tag{11}$$

We remark that $\dot{Q}(t) \le 0$, $\forall\, t \in [0,T]$.

PROPOSITION 2.4. *Under assumptions* (A_1), (A_2) *and* $h(r) \ge -h_1(1-r)$, *for* $r \in [0,1]$ *and* $0 < g(t) \le h_1$ *for* $t > 0$, *then*

$$\textit{case}\ (\mathrm{B}) \Longrightarrow Q(T_B) = r_0^3.$$

Proof. Performing the limit $t \to T_B$ in (11) and taking into account that $u(r,t)$ is bounded in D_T, from (8) we get the result. ∎

REMARK 2.5. We consider the case $g(t) \equiv g$, where g is a positive constant. From (11), we have

$$Q(T_1) = r_0^3 \iff T_1 = \frac{1 - r_0^3 + 3\int_{r_0}^1 r^2 h(r)\,dr}{3r_0^2 g}.$$

Therefore, we obtain

$$\text{If } h(r) \geq -h_1(1-r),\ 0 < h_1 < \frac{4(1-{r_0}^3)}{1 - r_0^3(4 - 3r_0)},\ \text{then } T_1 > 0.$$

PROPOSITION 2.6. *Let $h(r) \geq -h_1(1-r)$, $0 < g(t) \leq h_1$, $h_1 \leq 1$. If there exists a T_1 such that $Q(T_1) = r_0^3$ then case (B) occurs with $T_B = T_1$.*

Proof. From the assumptions and iv) of Lemma 1.2 we have that $u(r,t) \geq -h_1(1-r)$.

From (8) we have

$$\begin{aligned} s^3(T_1) &= r_0^3 - 3\int_{r_0}^{s(T_1)} r^2 u(r,T_1)\,dr \leq r_0^3 + 3h_1 \int_{r_0}^{s(T_1)} r^2(1-r)\,dr \\ &= r_0^3 + h_1(s(T_1)^3 - r_0^3) - \frac{3}{4}h_1(s(T_1)^4 - r_0^4). \end{aligned}$$

Hence

$$(s(T_1)^3 - r_0^3)(h_1 - 1) - \frac{3}{4}h_1(s(T_1)^4 - r_0^4) \geq 0,$$

and, if $h_1 \leq 1$ this implies $s(T_1) = r_0$. ∎

PROPOSITION 2.7. *If the assumptions of Lemma (2.3) hold and $h'(r) \geq 0$, then*

$$\textit{case (C)} \Longrightarrow Q(T_C) \leq r_0^3.$$

Proof. If case (C) occurs, from Lemma 2.3 the isotherm $u(r,t) = -1$ exists and reaches the free boundary at $t = T_C$.

Moreover, from the assumptions and from Lemma 2.2, we have $u_r(r,t) \geq 0$. Hence $u(r,T_C) \leq -1$, $\forall\, r \in [r_0, s(T_C)]$ and from (8) and (11) we get

$$\begin{aligned} s^3(T_C) &= Q(T_C) - 3\int_{r_0}^{s(T_C)} r^2 u(r,t)\,dr \\ &\geq Q(T_C) + 3\int_{r_0}^{s(T_C)} r^2\,dr = Q(T_C) + (s^3(T_C) - r_0^3). \end{aligned}$$ ∎

COROLLARY 2.8. *Let the assumptions of Proposition* 2.6 *and Proposition* 2.7 *hold.*
Case (C) $\Longrightarrow$ $Q(T_C) < r_0^3$.

Proof. Easily follows from Proposition 2.6 and Proposition 2.7. ∎

PROPOSITION 2.9. *If* $\|g\|_1 = \int_0^{+\infty} g(t)dt < +\infty$, *then*

$$\textit{case (A)} \Longrightarrow Q(t) \geq r_0^3, \ \forall\, t > 0.$$

Proof. Suppose that there exists a $T_0 > 0$ such that $Q(T_0) < r_0^3$. Since $\dot{Q}(t) < 0$, for all $t > 0$, we get

$$Q(t) \leq Q(T_0) < r_0^3, \qquad \forall\, t \geq T_0.$$

From (8) and (11), we have

$$\int_{r_0}^{s(t)} r^2 u(r,t)\, dr = \frac{Q(t) - s^3(t)}{3} < \frac{Q(T_0) - s^3(t)}{3} \leq -\frac{r_0^3 - Q(T_0)}{3} < 0, \qquad \forall\, t \geq T_0,$$

then

$$(12) \qquad r_0 \iint_{D_t} ru\, dr d\tau \leq r_0 \iint_{D_{T_0,t}} ru\, dr d\tau \leq \int_{T_0}^{t} d\tau \int_{r_0}^{s(\tau)} r^2 u(r,\tau)\, dr \leq -\frac{r_0^3 - Q(T_0)}{3}(t - T_0).$$

From (9),

$$(13) \qquad 2\iint_{D_t} ru(r,\tau)\, dr d\tau = \frac{s^2(t)}{4}(s^2(t) + 2r_0^2) - \frac{1}{4}(1 + 2r_0^2) - \int_{r_0}^{1} (r_0^2 + r^2) r h(r)\, dr + 2r_0^3 \int_0^t g(\tau)d\tau + \int_{r_0}^{s(t)} (r^2 + r_0^2) r u(r,t)\, dr$$
$$\geq \frac{r_0^2}{4}(3r_0^2) - \frac{1}{4}(1 + 2r_0^2) + 2r_0^3\|g\|_1 + (1 + r_0) \int_{r_0}^{s(t)} r^2 u(r,t)\, dr.$$

From (8)

$$\int_{r_0}^{s(t)} r^2 u(r,t)\,dr \geq \int_{r_0}^{1} r^2 h(r)\,dr - r_0^2 \|g\|_1 \geq C, \qquad \forall\, t > 0$$

where C is a suitable constant.

Hence (13) becomes

$$2 \iint_{D_t} r u(r,\tau)\,dr d\tau \geq \widetilde{C},\ \forall\, t > 0$$

where $\widetilde{C}$ is a suitable constant, on the contrary to (12). ∎

COROLLARY 2.10. *Let the assumptions of Proposition* 2.6 *and Proposition* 2.9 *hold. Then*

$$\textit{Case (A)} \implies Q(t) > r_0^3,\ \forall\, t > 0.$$

P r o o f. Easily follows from Proposition 2.6 and Proposition 2.9. ∎

COROLLARY 2.11. *Let the assumptions of Proposition* 2.4 *and Proposition* 2.7 *hold. Then*

$$Q(t) > r_0^3,\ \forall\, t > 0 \implies \textit{case (A)}.$$

P r o o f. It is obtained excluding the other cases. ∎

3. Asymptotic behaviour of the solution

PROPOSITION 3.1. *Let* (T, s, u) *be a solution of* (1)–(6) *of case* (A) *under assumptions of Corollary* 2.10 *and let there exist* $\lim_{t\to+\infty} g(t)$. *We denote by* $Q_\infty = \lim_{t\to+\infty} Q(t)$ *and* $s_\infty = \lim_{t\to+\infty} s(t)$. *Then*

$$s_\infty = (Q_\infty)^{1/3} \geq r_0 \tag{14}$$

where $Q_\infty = 1 + 3 \int_{r_0}^{b} r^2 h(r)\,dr - 3r_0^2 \|g\|_1$.

P r o o f. From the assumptions we obtain that

$$\lim_{t\to+\infty} g(t) = 0.$$

Moreover if $u_\infty(r) \equiv \lim_{t\to+\infty} u(r,t)$, $u_\infty(r)$ satisfies

$$\begin{cases} u_\infty'' + \dfrac{2}{r} u_\infty' = 0, & \text{in } (r_0, s_\infty), \\ u_\infty'(r_0) = 0, \\ u_\infty(s_\infty) = 0, \end{cases}$$

hence ([8]) $u_\infty(r) \equiv 0$ and, from (8) and (11), we get the result. ∎

REMARK 3.2.

$$s_\infty = r_0 \iff \int_0^{+\infty} g(\tau)d\tau = \frac{1 - r_0^3 + 3\int_{r_0}^1 r^2 h(r)\,dr}{3r_0^2}.$$

REMARK 3.3. Taking into account (8) and $u(r,t) \le 0$ we obtain the estimate

$$s(t) \ge [Q(t)]^{1/3}, \qquad \forall t \le T. \tag{15}$$

Now we shall consider briefly the particular case $r_0 = 0$ and since we are in spherical symmetry, the condition $g(t) \equiv 0$ is the most natural one.

The local in time existence and uniqueness of the classical solution for the case $r_0 = 0$ is given in [9]. The results obtained in this section are in sequel with those of [9] in order to clarify the behaviour of the solution as a function of the parameter R, defined below by 19.

We may obtain, as in the proof of Lemma 2.1, the integral representation (7)-(9) with $r_0 = 0$, $g(t) \equiv 0$,

$$\frac{s^2(t) - 1}{2} = \int_0^1 r h(r)\,dr - \int_0^{s(t)} r u(r,t)\,dr - \int_0^t u(0,\tau)d\tau, \tag{16}$$

$$\frac{s^3(t) - 1}{3} = \int_0^1 r^2 h(r)\,dr - \int_0^{s(t)} r^2 u(r,t)\,dr, \tag{17}$$

and

$$\frac{s^4(t) - 1}{4} = 2\int_0^t \int_0^{s(\tau)} r u(r,\tau)\,dr d\tau + \int_0^1 h(r) r^3\,dr - \int_0^{s(t)} r^3 u(r,t)\,dr. \tag{18}$$

Moreover the result of Lemma 2.3 can be proved in the same way as before. We may characterize cases (A), (B), (C) in dependence of the constant R defined by

$$R = 1 + 3\int_0^1 r^2 h(r)\,dr. \tag{19}$$

PROPOSITION 3.4. *Let $h(r) \ge -h_1(1-r)$ for $r \in [0,1]$. Then we have the following properties:*

(i) *Case (B)* $\Longrightarrow$ $R = 0$,
(ii) $R = 0$ *and case* (A) $\Longrightarrow \lim_{t\to\infty} s(t) = 0$,
(iii) *Case* (A) $\Longrightarrow R \ge 0$,
(iv) $R < 0 \Longrightarrow$ case (C).

Proof. We shall only prove (iii). Using (17) and the assumptions on $h(r)$ (thus $u(r,t) \geq -h_1(1-r)$), we get

$$-3\int_0^{s(t)} r^2 u(r,t)\,dr = s^3(t) - 1 - 3\int_0^1 r^2 h(r)\,dr \leq \frac{h_1}{4}. \tag{20}$$

Hence

$$-\int_0^{s(t)} r^3 u(r,t)\,dr \leq \frac{h_1}{12}. \tag{21}$$

Using (18) we have

$$2\int_0^t\int_0^{s(\tau)} ru(r,\tau)d\tau \geq -\frac{1}{4} - \frac{h_1}{12}. \tag{22}$$

If $R < 0$ from (17) we obtain

$$-\int_0^{s(t)} ru(r,t)\,dr \geq -\int_0^{s(t)} r^2 u(r,t)\,dr = \frac{s^3(t)}{3} - \frac{R}{3} > 0, \qquad \forall\, t > 0.$$

By integrating with respect to t the last inequality we get

$$\int_0^t\int_0^{s(\tau)} ru(r,\tau)\,drd\tau \leq \frac{R}{3}t,$$

hence a contradiction with (22). ∎

Acknowledgements. The authors wish to thank Prof. Pedro Marangunic for helpful discussions.

References

[1] D. Andreucci, *Continuation of the solution of a free boundary problem in cylindrical symmetry*, Meccanica 19 (1984), 91–97.

[2] D. Andreucci, A. Fasano and M. Primicerio, *On the occurrence of singularities in axisymmetrical problems of Hele-Shaw type*, Int. Series Numer. Math. 106 (1992), 23–38.

[3] E. Comparini, R. Ricci and D. A. Tarzia, *Remarks on a one-dimensional Stefan problem related to the diffusion-consumption model*, ZAMM 64 12 (1984), 543–550.

[4] A. Fasano and M. Primicerio, *General free-boundary problems for the heat equation*, J. Math. Anal. Appl. 57 (1977), 694–723.

[5] A. Fasano and M. Primicerio, *New results on some classical parabolic free-boundary problems*, Quart. Appl. Math. 38 (1980), 439–460.

[6] A. Fasano and M. Primicerio, *A critical case for the solvability of Stefan-like problems*, Math. Meth. Appl. Sci. 5 (1983), 84–96.

[7] A. Fasano, M. Primicerio, S. D. Howison and J. R. Ockendon, *Some remarks on the regularization of supercooled one-phase Stefan problems in one dimension*, Quart. Appl. Math. 47 (1990), 153–168.

[8] A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, Englewood Cliff (1964).
[9] A. M. Meirmanov, *The Stefan problem with surface tension in the three dimensional case with spherical symmetry: nonexistence of a classical solution*, European J. Appl. Math. 5 (1994), 1–19.
[10] A. G. Petrova, D. A. Tarzia and C. V. Turner, *The one-phase supercooled Stefan problem with temperature boundary condition*, Adv. Math. Sci. Appl. 4 (1994), 35–50.
[11] M. Primicerio, *The occurrence of pathologies in some Stefan-like problems*, in: Numerical Treatment of Free Boundary Value Problems, J. Albrecht-L. Collatz-K.H. Hoffmann (Eds.), ISNM 58, Birkhauser Verlag, Basel (1982), 233–244.
[12] B. Sherman, *A general one-phase Stefan problem*, Quart. Appl. Math. 28 (1970), 377–382.
[13] D. A. Tarzia and C. V. Turner, *The one-phase supercooled Stefan problem with a convective boundary condition*, Quart. Appl. Math., 55 (1997), 41–50.

Paola Mannucci
DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA
UNIVERSITÀ DEGLI STUDI DI PADOVA
Via Belzoni, 7
PADOVA, ITALY
E-mail: mannucci@math.unipd.it

Domingo Alberto Tarzia
DEPTO MATEMATICA
FCE UNIVERSIDAD AUSTRAL
Paraguay 1950, 2000-ROSARIO, ARGENTINA
E-mail: tarzia@uaufce.edu.ar

Received June 8, 1998.