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ON THE LAURICELLA PROBLEM FOR THE EQUATION $\Delta^2 u(X)=f(X,u(X))$ IN THE CIRCLE

1. In [1] the Lauricella problem for the equation $\Delta^2 u(X) = f(X)$ in the circle was solved.

In the present paper we shall study the following Lauricella problem, in the circle $K = \{X: |X| < R\}$,

$$\Delta^2 u(X) = f(X,u(X)) \quad \text{for} \quad X \in K, \tag{1}$$

$$u(X) = f_1(X) \text{ for } X \in \partial K, \tag{2}$$

$$D_{n_X} u(X) = f_2(X) \quad \text{for} \quad X \in \partial K, \tag{3}$$

where f, f_1, f_2 are given functions, n_X denotes the inward normal to ∂K in the point $X \in \partial K$.

2. Using the convenient Green function G, we shall replace the problem (1)-(3) by an integral equation which may be solved by the Banach method of the contracting mapping.

Let us denote:

$X = (x_1, x_2)$ is an arbitrary point of K,

$Y = (y_1, y_2)$ is an arbitrary point of the plane E_2,

$\bar{X} = (\bar{x}_1, \bar{x}_2)$ is the symmetric image of X with respect to ∂K,

$$r^2(X;Y) = (y_1-x_1)^2 + (y_2-x_2)^2,$$

$$A = -(2R^2)^{-1}, \qquad B(X) = R^2 - r^2(0;X),$$

and consider the function G of the form

$$G(X;Y) = r^2(X;Y)g(X;Y) + AB(X)B(Y) \quad \text{for} \quad X \in K,\ X \neq Y,$$

where

$$g(X;Y) = \ln\left[r(0;X)r(\overline{X};Y)(Rr(X;Y))^{-1}\right] \quad \text{for} \quad X \neq 0,$$

$$g(0;Y) = \ln r(0;Y) - \ln R \quad \text{for} \quad Y \neq 0.$$

The function G satisfies the following conditions:

$$G(X;Y) = 0 \quad \text{for} \quad X \in \partial K,$$

$$D_{n_X}G(X;Y) = r^2(X;Y)D_{n_X}g(X;Y) + g(X;Y)D_{n_X}r^2(X;Y) +$$

$$+ AB(Y)D_{n_X}B(X) = 0 \quad \text{for} \quad X \in \partial K,$$

because, by [2] Vol. I, p. 250, we have $g(X;Y) = 0$, $r^2(X;Y)D_{n_X}g(X;Y) = R^{-1}B(X)$ and $AB(Y)D_{n_X}B(X) = -R^{-1}B(Y)$ for $X \in \partial K$.

Moreover

(4) $$\Delta_Y G(X;Y) = 2AB^2(X)r^{-2}(X;Y) \quad \text{for} \quad Y \in \partial K,$$

(5) $$D_{n_Y}\Delta_Y G(X;Y) = H_1(X;Y) + H_2(X;Y) \quad \text{for} \quad Y \in \partial K,$$

where

$$H_1(X;Y) = 2(R^3r^2(X;Y))^{-1}B^2(X);\ H_2(X;Y) = 2(R^3r^4(X;Y))^{-1}B^3(X).$$

3. Now we shall give the theorem concerning the unicity of the problem (1) - (3). Let us introduce the following two definitions.

D e f i n i t i o n 1. We shall call the class (N) the set of all functions $u \in C(\overline{K})$ such that $|u| \leqslant r$ (r being a positive number).

D e f i n i t i o n 2. We shall call the class (F) the set of all functions $f(X,u)$ such that the functions $D_u^{(i)}f(X,u)$, $i=0,1$, are continuous and bounded for $(X,u) \in D_1 = \{(X,u): X \in \overline{K},\ u \in [-r,r]\}$.

T h e o r e m 1. If $f \in (F)$, $D_u^{(1)} f(X,u) \leqslant 0$ for $(X,u) \in D_1$, and the functions u_1, u_2 of class $C^4(K) \cap C^3(\bar{K})$ are solutions of the problem (1)-(3), then $u_1(X) = u_2(X)$ for $X \in K$.

P r o o f . By [2], Vol.II, p.179, we have

$$P_1 = \iint_K \left\{ \left[\Delta(u_1-u_2)\right]^2 - (u_1-u_2)\left[\Delta^2(u_1-u_2)\right]\right\} dY =$$

$$= - \int_{\partial K} \left[(D_{n_Y}(u_1-u_2))\Delta(u_1-u_2) - \right.$$

$$\left. - (u_1-u_2) D_{n_Y} \Delta(u_1-u_2)\right] ds_Y = P_2.$$

By the formula $\Delta^2 u_i = f(X,u_i)$ $(i=1,2)$ and by the mean value theorem, we obtain

$$P_1 = \iint_K \left\{ \left[\Delta(u_1-u_2)\right]^2 - (u_1-u_2)^2 D_u f(Y,\bar{u})\right\} dY \geqslant 0,$$

where $\bar{u} = u_1 + Q(u_2-u_1)$, $0 < Q < 1$, therefore $\bar{u} \in (N)$. Similarly by the boundary conditions (2), (3), we get $P_2 = 0$. Hence $P_1 = P_2 = 0$ implies $u_1 \equiv u_2$, for $X \in K$.

4. Let the function V denote the solution of the biharmonic equation

$$\Delta^2 u(X) = 0 \quad \text{for} \quad X \in K, \tag{1a}$$

satisfying the boundary conditions (2) and (3).

Consider the integral equation

$$u(X) = S(X;u) + V(X), \tag{6}$$

where

$$S(X;u) = I_1(X;u) + I_2(X;u); \; V(X) = V_1(X) + V_2(X)$$

and

$$I_1(X;u) = A_1 \iint_K f(Y;u(Y)) B(X) B(Y) dY,$$

$$I_2(X;u) = A_1 \iint_K f(Y;u(Y))r^2(X;Y)g(X;Y)dY,$$

$$V_1(X) = A_2 \iint_{\partial K} f_1(Y)D_{n_Y}\Delta_Y G(X;Y)dS_Y,$$

$$V_2(X) = A_2 \int_{\partial K} f_2(Y)\ \Delta_Y G(X;Y)dS_Y,$$

$A_1 = -(4\pi)^{-1}$, $A_2 = -(8R)^{-1}$, the functions $D_{n_X}\Delta_X G$, $\Delta_Y G$ being given by formulae (4), (5).

Let $\|W-Z\| = \sup_{X\in\bar{K}} |W(X)-Z(X)| = d(W,Z)$ for any functions $W,Z \in C(\bar{K})$. By [1], we have

$$|V_1(X) + V_2(X)| \leqslant B = \sup_{X\in\partial K} |f_1(X)| + 2R \sup_{X\in\partial K} |f_2(X)|.$$

Now we shall give the lemma concerning the estimate of the Green potentials.

L e m m a 1. If $f \in (F)$ and $M_1 = \sup_{(X;u)\in D_1} |f(X;u)|$, the following inequalities hold

(a) $|I_1(X;u)| < \frac{\pi}{2} |A_1| M_1 R^6$; (b) $|I_2(X;u)| \leqslant H(R)$,

where

$$H(R) = C_1R^2\left[C_2+C_3 |\ln R| + R^2C_4 |\ln R| + C_5R^2\right],$$

C_i $(i=1,\dots,5)$ being a convenient positive constants.

P r o o f . We omit the simple proof of the inequality (a) and we shall prove only (b). Let us write the function I_2 in the form

$$I_2(X,u) = J_1(X,u)+J_2(X,u)+J_3(X,u),$$

where

$$J_1(X,u) = A_1 \iint_K f(Y,u(Y))r^2(X;Y)\ \ln\left[r(0,X)r(\bar{X};Y)\right] dY,$$

$$J_2(X,u) = -A_1 \iint_K f(Y,u(Y))r^2(X;Y)\ln R\, dY,$$

$$J_3(X,u) = \iint_K f(Y,u(Y))r^2(X;Y)\ln r(X;Y)\, dY.$$

For the function J_2 we have the following estimation

$$|J_2| \leq 4\pi|A_1|M_1R^4|\ln R| = 2A_3R^2|\ln R|, \text{ where } A_3 = 2\pi|A_1|M_1R^2.$$

Now we shall estimate the function J_1. By [2], p.249, we have the following inequalities

$$(7)\qquad R(R-r(0,X)) \leq r(0,X)r(\bar{X},Y) \leq R(R+r(0,X)).$$

We shall consider two cases:

1^0 $\quad r(0,X) > \frac{R}{n}$,

2^0 $\quad r(0,X) = 0$ for $X = 0$.

By (7), in the case 1^0 we have

$$\left(1 - \frac{1}{n}\right)R^2 \leq r(0,X)r(\bar{X},Y) \leq \left(1 + \frac{1}{n}\right)R^2, \quad n = 1,2,\dots$$

and we shall consider two cases:

(a) $\left(1 + \frac{1}{n}\right)R^2 < 1$, $\qquad$ (b) $\left(1 + \frac{1}{n}\right)R^2 > 1$.

Ad (a). $\qquad |J_1(X,u)| \leq -2A_3R^2\ln\left[\left(1 - \frac{1}{n}\right)R^2\right]$.

Ad (b). Let $M(R) = \max\left(\left|\ln\left[\left(1 - \frac{1}{n}\right)R^2\right]\right|, \left|\ln\left[\left(1 + \frac{1}{n}\right)R^2\right]\right|\right)$ and we have $|\ln[r(0,X)r(\bar{X},Y)]| \leq M(R)$. Thus $|J_1(X,u| \leq$ $\leq 2A_3R^2M(R)$.

Ad 2^0. By continuity of the function $g(X;Y)$, from [2], Vol.I, p.250 we have

$$g(0;Y) = -\ln r(0;Y) + \ln R.$$

Hence we can write the function $J_1(0,u)$ in the form $J_1(0,u) = L_1(0,u) + L_2(0,u)$, where

$$L_1(0,u) = A_1 \iint_K f(Y,u(Y))r^2(0;Y)\ln R\, dY,$$

$$L_2(0,u) = -A_1 \iint_K f(Y,u(Y))r^2(0,Y)\ln r(0,Y)dY.$$

For the integrals L_1, L_2 we have the following estimations

$$|L_1(0,u)| \leqslant \frac{1}{2} A_3R^2 |\ln R|; \quad |L_2(0,u)| \leqslant A_1M_1R^2 \iint_K |\ln r(0;Y)|\, dY.$$

For the estimation of the integral L_2 we shall distinguish two cases:

(c) $R \leqslant 1$, (d) $R > 1$.

Ad (c). Using the polar coordinates we obtain

$$|L_2(0,u)| \leqslant -\frac{1}{2} A_3R^2\ln R + \frac{1}{4} A_3R^2.$$

Ad (d). $|L_2(0,u)| \leqslant \frac{1}{2} A_3(R^2\ln R - \frac{1}{2} R^2 + 1)$.

Hence

$$|J_1(0,u)| \leqslant |L_1(0,u)| + |L_2(0,u)| \leqslant \frac{1}{2} A_3R^2 |\ln R| - \frac{1}{2} A_3R^2\ln R + \frac{1}{4} A_3R^2$$

for $R \leqslant 1$,

$$|J_1(0,u)| \leqslant \frac{1}{2} A_3R^2 |\ln R| + \frac{1}{2} A_3(R^2\ln R - \frac{1}{2} R^2+1) \quad \text{for} \quad R > 1.$$

Now we shall estimate the function J_3; we have

$$|J_3(X,u)| \leqslant A_1M_1 4R^2 \iint_K |\ln r(X;Y)|\, dY.$$

Applying the polar coordinates

$$y_1-x_1 = p\cos\varphi, \quad y_2-x_2 = p\sin\varphi, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant p \leqslant p(\varphi) \leqslant 2R,$$

we obtain

$$|J_3(X,u)| \leqslant A_3R^2(1-2\ln 2R) \quad \text{for} \quad R \leqslant \tfrac{1}{2} ,$$

$$|J_3(X,u)| \leqslant A_3(\tfrac{1}{2} + 2R^2\ln 2R - (2\pi)^{-1}R^2) \quad \text{for} \quad R > \tfrac{1}{2} .$$

By the foregoing inequalities for the functions J_1, J_2, J_3, we obtain the inequality (b).

5. Denote $F_V(X;u) = S(X;u) + V(X)$,

$$q = A_1 \sup_{(X;u)\in D_1} \left| \iint_K D_u f(Y,u)G(X;Y)dY \right| \text{ and } M_2 = \sup_{(X;u)\in D_1} |D_u f(X;u)|.$$

We shall prove the following lemma.

Lemma 2. If the functions W,Z,u belong to the class (N) and $|V(X)| < (1-q)r$, where $q \in (0,1)$, then

1^0 $S(X;0) = 0$.

2^0 $d(S(W),S(Z)) \leqslant qd(W,Z)$.

3^0 $\|F_V(W)-F_V(Z)\| < qd(W,Z)$.

4^0 $\|F_V(u)\| < r$.

Proof. The assertion 1^0 is evident.

Ad 2^0. We have

$$d(S(W),S(Z)) = |A_1| \sup_{X\in K} \left| \iint_K \left[f(Y,W(Y))-f(Y,Z(Y))\right] G(X;Y)dY \right|.$$

By the mean value theorem, we obtain

$$d(S(W),S(Z)) = |A_1| \sup_{(X;u)\in D_1} \left| \iint_K (W-Z)D_u f(Y,\bar{u})G(X;Y)dY \right| \leqslant$$

$$\leqslant |A_1| d(W,Z) \sup_{(X;u)\in D_1} \left| \iint_K D_u f(Y,\bar{u})G(X;Y)dY \right| = qd(W,Z),$$

where $\bar{u} = W+Q(Z-W)$, $0 < Q < 1$, therefore $\bar{u} \in (N)$.

Remark. Indeed $q \in (0,1)$ for M_2 sufficiently small, because, by Lemma 1, we have

$$\sup_{(X,u)\in D_1}\left|\iint_K D_u f(Y,u)G(X;Y)dY\right| \leqslant \sup_{(X,u)\in D_1}|D_u f(X,u)| \sup_{X\in K}\iint_K |G(X;Y)|dY =$$

$$= M_2 \sup_{X\in K}\iint_K |G(X;Y)|\,dY \leqslant M_2\left[(2)^{-1}\pi|A_1|\,R^6 M_1 + H(R)\right].$$

The proof of the assertion 3° is similar to that of 2°.

Ad 4°. By 1°, we have

$$\|F_V(u)\| = \|S(X,u) + V(X)\| = \|S(X,u)-S(X,0)+V(X)\| \leqslant$$

$$\leqslant \|S(X,u)-S(X,0)\| + \|V(X)\| < qd(u,0) + (1-q)r <$$

$$< qr + (1-q)r = r.$$

By Lemma 2, the mapping F_V is contracting for the functions of the class (N) and transforms every function $u \in (N)$ into a function of the class (N).

Lemma 2 and Theorem VIII.2 in [3] imply the following lemma.

Lemma 3. There exists exactly one function $u \in (N)$ satisfying the integral equation (6).

6. Let $Z_1 = \{X: R_1 < |X| \leqslant R\}$, where $0 < R_1 < R$. We shall prove the following theorem.

Theorem 2. If $f_1 \in C^1(Z_1)$, $f_2 \in C(\partial K)$, $f \in (F)$, $u \in (N)$, $u \in C^4(K) \cap C^3(\bar{K})$ and $D_u^{(1)} f(X,u) \leqslant 0$ for $(X,u) \in D_1$, then

1° the function u being the solution of the integral equation (6) is the solution of the problem (1)-(3),

2° the function u is the unique solution of the problem (1)-(3).

Proof. Ad 1°. Let us consider the integrals

$$I^{j,k}(X;u) = A_1 \iint_K f(Y,u(Y))D^j_{x_k} G(X;Y)dY, \quad j = 0,1, \quad k = 1,2.$$

By [1], the integrals $I^{j,k}$, $j=0,1$, $k=1,2$, are locally uniformly convergent at every point $X \in K$. Hence there exist the derivatives

$$D^j_{x_k} \iint_K f(Y,u(Y))G(X;Y)dY, \quad k=1,2, \quad j=0,1,$$

and

$$A_1 D^j_{x_k} \iint_K f(Y,u(Y))G(X;Y)dY = I^{j,k}(X;u), \quad j=0,1, \quad k=1,2.$$

By the properties of the function G, we obtain

$$D^i_{n_X} I^{0,k}(X;u) \to 0 \quad \text{as} \quad X \to X_0 \in \partial K, \; k=1,2, \; i=0,1.$$

From [1] we have

$$\Delta^2 I^{0,k}(X;u) = A_1 \iint_K f(Y;u(Y))\Delta^2_X G[X;Y)dY =$$

$$= A_1 \iint_K f(Y,u(Y))J(X;Y)dY,$$

where $I^{0,k}(X;u) = I_1(X;u) + I_2(X;u)$ and

$$J(X;Y) = 4g(X;Y) + 2R^{-2}B(Y) -$$

$$- 4 + 4r^{-2}(X;Y)(R^2+y_1^2+y_2^2+x_1y_1-y_2x_2-\bar{x}_1y_1-\bar{x}_2y_2).$$

By [4], p.303, we obtain

$$\Delta^2 I^{0,k}(X;u) = f(X;u(X)) \quad \text{for} \quad X \in K.$$

From the last formula and the integral equation (6), by the properties of the function V, we obtain

$$\Delta^2 u(X) = \Delta^2 V(X) + \Delta^2 I^{0,k}(X;u) = f(X,u(X)) \quad \text{for} \quad X \in K,$$

$$u(X) = [V(X)+I(X;u)] \longrightarrow f_1(X_0) \text{ as } X \longrightarrow X_0 \in \partial K,$$

$$D_{n_X}u(X) = \left[D_{n_X}V(X)+D_{n_X}I(X;u)\right] \longrightarrow f_2(X_0) \quad \text{as} \quad X \longrightarrow X_0 \in \partial K.$$

The assertion 2° follows from Theorem 1.

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