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Isoperimetric and Poincaré Inequalities on Non-Self-Similar Sierpiński Sponges: the Borderline Case

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Abstract: In this paper we construct a large family of examples of subsets of Euclidean space that support a 1-Poincaré inequality yet have empty interior. These examples are formed from an iterative process that involves removing well-behaved domains, or more precisely, domains whose complements are uniform in the sense of Martio and Sarvas.
While existing arguments rely on explicit constructions of Semmes families of curves, we include a new way of obtaining Poincaré inequalities through the use of relative isoperimetric inequalities, after Korte and Lahti. To do so, we further introduce the notion of of isoperimetric inequalities at given density levels and a way to iterate such inequalities. These tools are presented and apply to general metric measure measures.
Our examples subsume the previous results of Mackay, Tyson, and Wildrick regarding non-self similar Sierpiński carpets, and extend them to many more general shapes as well as higher dimensions.

Keywords: Isoperimetry; Sierpiński carpet; Self-improvement; Poincaré

MSC: 26A45; 30L99 (28A75; 28A80; 31E05)

1 Introduction

A p-Poincaré inequality, in the sense of [11], captures the notion of possessing many (rectifiable) curves in a space that connect prescribed pairs of points – an idea made precise in [12, 18] for example. A smaller exponent p for a p-Poincaré inequality indicates a richer supply of curves, and our focus will be on the borderline case — that is, the 1-Poincaré inequality.

Alternatively, such inequalities are related to how easy it is to separate the space by "small" sets – i.e. the role of isoperimetry. Specifically, we consider the notion for *relative* isoperimetry, and how boundaries separate a set from its complement; for a precise formulation of these notions, see Section 3. Our context will mainly be Euclidean spaces in all dimensions $d \geq 2$, though many techniques on isoperimetry are completely general and apply to the metric space setting.

In passing from a given space to a subset of that space, the number of curves decreases and it (often) becomes easier to separate the subset (as a space, in its own right). By such reasoning, a subset $A \subset \mathbb{R}^d$ often will not support a Poincaré inequality. This does not always hold, however, since also the functions and sets become more restricted as well. If, however, one removes a collection of sufficiently "sparse" obstacles from the underlying space, then intuitively the Poincaré inequality could be preserved for the subset. It is a subtle issue, however, of "how sparse" these obstacles can be. Our main result, Theorem 1.3, gives a general sufficient condition for a 1-Poincaré inequality to hold for subsets of $\mathbb{R}^d$ arising from such a removal process.

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In the case of $\mathbb{R}^2$, it was shown in [14, Theorems 1.5–1.6] that a certain family of positive (Lebesgue) measured subsets satisfy p-Poincaré inequalities, despite having empty interior. Their results are remarkable, in that they give sharp characterizations of the range of exponents p for which the p-Poincaré inequality holds. Our work here employs substantially different techniques, and yields a more general class of examples for the exponent $p = 1$. See Appendix A for a more detailed discussion.

An earlier work by the authors [9] studied the case of exponents $p > 1$ separately and uses completely different techniques, both from here and from [14]. In a similar theme, however, our results here hinge on new sufficient conditions for a 1-Poincaré inequality. Together, they form a complete generalization of the main result of [14].

In our main theorem, we make a step towards understanding which removal processes are permitted, when forming these subsets. To that end, we adopt a new perspective on isoperimetric inequalities. Instead of testing isoperimetry for all sets, we first require the inequality to hold only for sufficiently "dense" sets, as measured by a density parameter τ. The new notion of a (τ, C)-isoperimetric inequality allows the flexibility of proving "sufficiently good" estimates at every scale, which when iterated, leads to isoperimetry for all sets and at all densities. Indeed, this added flexibility allows us to consider each scale independent of others, and is the crucial tool in our proof.

The (τ, C)-isoperimetric inequality can be further thought of as a scale-invariant weak estimate that improves itself. This idea of self-improvement is frequent in harmonic analysis and geometric analysis and has appeared, for example, in the following classical contexts. The results often involve mild topological assumptions and to obtain them, one iterates the relevant estimate in an appropriate case-dependent way.

1. Muckenhoupt (ϵ, δ)-conditions improve to A_p-type conditions [20, Proposition V.4].
2. Weak quasisymmetries are quasisymmetries [22, Lemma 6.5].
3. A "balled" Loewner condition improves to a Loewner condition [4, Proposition 3.1].
4. The Loewner condition improves to a more quantitative estimate [11, Theorem 3.6].
5. "Weak"-type Poincaré conditions at a given level, improve to true p-Poincaré inequalities for some $p > 1$ [8, Theorems 1.2 and 1.8]. See also [9, Theorem 2.19] for a more quantitative version.

1.1 Subsets arising from removing obstacles

The sets $S = S_{\mathbf{n}}$ we consider arise by removing "obstacles" R from a set Ω, and thus are of the form

$$S_{\mathbf{n}} := \Omega \setminus \bigcup_{k \in \mathbb{N}} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} R,$$

where Ω is a so-called "uniform" domain in $\mathbb{R}^d$ and each $\mathcal{R}_{\mathbf{n},k}$ is a collection of "co-uniform" domains R; that is, each $\mathbb{R}^d \setminus R$ is uniform and ∂R is connected.

For the precise notion of a uniform domain, see Definition 2.7; for the moment, however, we note that these include convex sets with bounded eccentricity, or regions without cusps. In particular, planar domains whose boundaries are quasicircles are also uniform, see [9, Remark 4.16] for the definitions of a quasicircle, some references on such examples. (As a technical point, in this paper we allow uniform domains to be closed sets.)

The notion of a uniformly sparse collection of co-uniform domains was introduced in [9, Definition 4.21] and forms the starting point for our analysis. Below, for sets K and K' we denote their "distance" by $d(K, K') := \inf\{d(x, x') : x \in K, x' \in K'\}$.

Definition 1.1. Let Ω be a non-empty compact subset of $\mathbb{R}^d$. Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers, and consider scales $s_0 = \operatorname{diam}(\Omega)$ and

$$s_k = \frac{1}{n_k} s_{k-1}$$

for $k \in \mathbb{N}$. A sequence of collections of domains $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ in $\Omega \subset \mathbb{R}^d$ forms a UNIFORMLY ($\mathbf{n}$-)SPARSE COLLECTION OF CO-UNIFORM DOMAINS IN Ω if there are constants $\delta \in (0, 1)$ and $L, A > 0$ so that for each $R \in \mathcal{R}_{\mathbf{n},k}$:

1. $R \subset \Omega$;
2. R is A-co-uniform and Ω is A-uniform;
3. $\operatorname{diam}(R) \leq Ls_k$;
4. $d(R, \Omega^c) \geq \delta s_{k-1}$;
5. for each $R' \in \mathcal{R}_{\mathbf{n},k'}$ with $k \geq k'$, then $d(R, R') \geq \delta s_{k-1}$;

Moreover, call $\{\mathcal{R}_{\mathbf{n},k}\}$ DENSE in Ω whenever $\bigcup_{k\in\mathbb{N}} \bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} R$ is a dense subset of Ω.

We note that versions of these conditions also appear in the context of uniformization of metric carpets, see e.g. [3].

Uniform sparseness by itself does not ensure a Poincaré inequality, and one needs to impose a condition on $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$, the sequence of ratios between scales. In [9], it suffices to assume that $\{n_k^{-1}\}_{k=1}^{\infty} \in \ell^d$ to obtain a p-Poincaré inequality for all $p > 1$.

In the $p = 1$ case, however, we will also consider the projections of such collections onto subspaces. Below, let $\pi_1, \dots, \pi_d$ be any collection of linearly independent projections of $\mathbb{R}^d$ onto subspaces of codimension one, that is, the collection of the normal vectors of hyperplanes $\pi_i(\mathbb{R}^d)$ form a linearly independent set in $\mathbb{R}^d$. Up to a coordinate change, we will often assume that the π_i are coordinate projections.

Definition 1.2. A uniformly ($\mathbf{n}$-)sparse collection of co-uniform domains $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ is said to HAVE SMALL PROJECTIONS if, with the same constant $L > 0$ as before,

(6) For $k \in \mathbb{N}$, if $r \geq s_{k-1}$ then for each $x \in \Omega$ and $i = 1, \dots, d$ it holds that

$$\mathcal{H}_{d-1}\left(\pi_i\left(B(x, r) \cap \bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} R\right)\right) \leq \frac{Lr^{d-1}}{n_k^{d-1}}.$$

With this notion, we now formulate our main result.

Theorem 1.3. *Fix constants $L, A \geq 1$, $\delta > 0$ as in Definition 1.1. Suppose $\Omega \subset \mathbb{R}^d$ is a compact uniform domain and that $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ is a uniformly* $\mathbf{n}$*-sparse collection of co-uniform domains in Ω that has small projections. The set*

$$S_{\mathbf{n}} := \Omega \setminus \bigcup_{k\in\mathbb{N}} \bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} R.$$

has positive Lebesgue measure and satisfies a 1*-Poincaré inequality (with respect to the restricted measure and metric from $\mathbb{R}^d$) if*

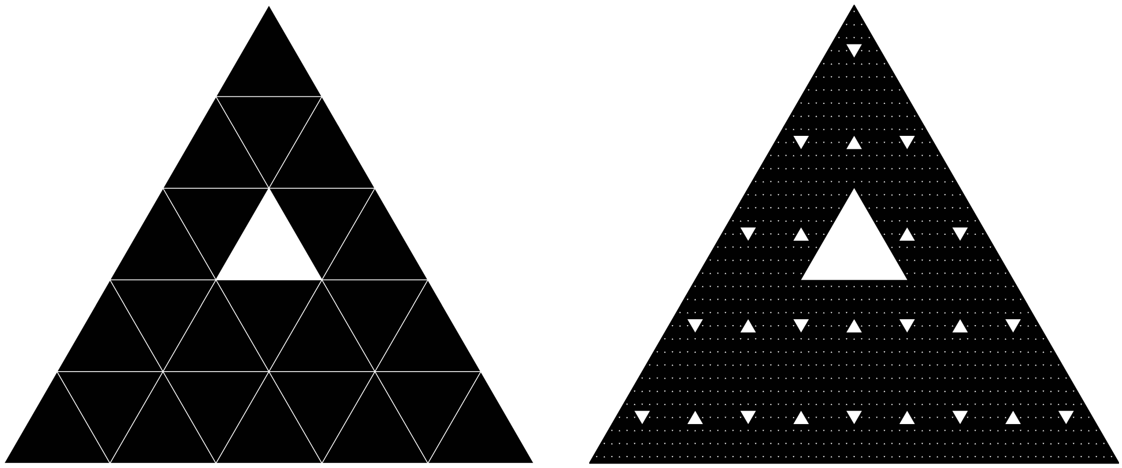
$$\sum_{k=1}^{\infty} \frac{1}{n_k^{d-1}} < \infty. \tag{1.4}$$

Note that the summability condition in equation (1.4) implies that the set $S_{\mathbf{n}}$ has positive measure. Indeed, this will be shown in Lemma 2.22.

Remark 1.5 (Previous results). As a special case of Theorem 1.3, we obtain immediately a new proof that certain "non-self similar Sierpiński carpets" satisfy a 1-Poincaré inequality, as first shown in [14]. In particular, the removed obstacles R there are coordinate squares and the uniformly sparse collections of squares have small coordinate projections. (For this specific construction and other similar ones, see Appendix A.)

In contrast to [14], whose results apply only to a construction involving d-dimensional cubes with $d = 2$, our result applies immediately in all dimensions simultaneously and allows for many variations. For instance, one could imagine sets $S_{\mathbf{n}}$ where the sets R, Ω in Definition 1.1 are all circles, or as in Figure 1, where they are all triangles. One could even go so far as to choose randomly-sided polyhedra with the number of sides, though uniformly bounded, varying with each scale s_k! This way, our result and proof give a flexible way to approach such results without overly restricting the geometry.

We refer to [9] for a more expansive discussion on the relevance of these results.



(a) Step one: Removal of the central white triangle (relative to height) with $n_1 = 5$.

(b) Three steps: Domain is in black, and removed triangles in white. We used $n_1 = 5, n_2 = 7, n_3 = 11$.

Figure 1: Construction of a non-self-similar triangular version of a Sierpiński carpet.

Remark 1.6 (Necessity of conditions). Note that the conditions given in Definitions 1.1 and 1.2 are close to necessary when $d = 2$. Indeed, in the planar case, versions of conditions (1)-(5) are necessary (see [9, Theorem 4.40]) while without (6) and the summability condition (1.4) one may construct counterexamples [14, Proposition 4.1].

In the case of specific constructions, such as [14] and the one in Figure 1, condition (6) is sharp. In cases lacking sufficient symmetry, however, there is a subtlety regarding the precise placement of obstacles. In a similar spirit as say, the quasiconformal Jacobian problem, it appears very difficult to formulate a completely sharp result; see e.g. [19] for further discussion on similar characterization problems.

In any event, some minimal assumption, say the weaker condition that $(n_k^{-1})_{k=1}^{\infty} \in \ell^d$, is needed for the set to have positive Lebesgue measure and guarantee the validity of some Poincaré inequality.

In order to obtain Theorem 3.5, we need a flexible way to prove such inequalities, and a condition which handles such inequalities at fixed scales. This involves a new notion of isoperimetry which applies to all metric measure spaces, not just Euclidean ones. Indeed, the only place where the Euclidean structure is used is in using projections and a projected isoperimetric inequality, see Lemma 3.11. It is conceivable that analogous structures exist in other settings. For example, the Heisenberg group has a natural collection of vertical projections [6, Definition 2.2], and one may push our main result to such settings. We leave this for future exploration.

1.2 Iterating Isoperimetry

The case of $p = 1$ is a borderline case for the Poincaré inequality and an inherently geometric one. Consider, for example, the well-known correspondence between the Sobolev embedding theorem and the isoperimetric inequality.

Related to this, we will employ the characterization of Lahti and Korte [13] which asserts that the validity of a 1-Poincaré inequality is equivalent to a so-called "relative isoperimetric inequality", see Theorem 3.5. Since, this fact and many of the following results hold true in general metric measure spaces, the forthcoming discussion will also be formulated in the context of metric measure spaces.

The isoperimetric inequality is easier to establish for "larger" sets in the sense of density, as defined below.

Definition 1.7. Let (X, μ) be a measure space and let E, F be measurable subsets of X with $\mu(F) > 0$. We define the DENSITY OF E RELATIVE TO F (or just RELATIVE DENSITY for short) as

$$\Theta_\mu(E, F) := \frac{\min\{\mu(E \cap F), \mu(F \setminus E)\}}{\mu(F)}.$$

Note that the relative density is symmetric for E and its complement E^c. We are interested in those sets E with $\Theta_\mu(E, F) \geq \tau$ for a given $\tau > 0$ – i.e. the ones that are neither empty nor full, quantitatively. In our proof of Theorem 1.3, for these sets we can throw away and control a junk-set coming from conditions (5) and (6) in Definitions 1.1 and 1.2. Interestingly enough, to prove isoperimetry it suffices to consider such sets. More precisely, we prove the following fact, which applies to all metric measure spaces and may be of independent interest. Here, as in Definition 3.7, a (τ, C)-isoperimetric inequality is one that only holds for sets $E \subset B = B(x, r)$ with $\Theta_\mu(E, B) \geq \tau$, where C is a multiplicative constant.

Theorem 1.8. *Let (X, d, μ) be D-doubling with $D \geq 2$, and let $\tau \in (0, \frac{1}{D^3}]$. If X satisfies a (τ, C)-isoperimetric inequality with inflation factor Λ, then X satisfies the relative isoperimetric inequality with constants $C_S = C_S(D, \Lambda) = D^{7+\log_2(\Lambda)}C$ and $\Lambda_S = \Lambda_S(\Lambda) = 2\Lambda$.*

The proof of this theorem involves "iterating" an estimate at appropriate scales. This method of relative isoperimetry via iteration is new. To the authors' knowledge, this is the first instance where it is used to verify a previously-conjectured Poincaré inequality. This method has the advantage that it allows to throw away small sets (such as those arising from condition (6) in Definition 1.2). It further allows to focus on a single scale at a time.

1.3 Outline

The remainder of the paper is organised as follows. In Section §2 we discuss preliminaries on measure and uniformity and state crucial lemmas. Section §3 is devoted to facts about isoperimetry and stating the isoperimetric inequality. In that section we also prove Theorem 1.8 and give the projected isoperimetric inequality in $\mathbb{R}^d$ in Lemma 3.11.

The proof of the main result (Theorem 1.3) is then left to Section §4. This rests on establishing the (τ, C)-isoperimetric inequality for subsets $E \subset B(x, r) \cap S_\mathbf{n}$. First, we reduce to the case of the sum in Equation (1.2) being small. This is done by localizing the argument. Following this, one replaces $B(x, r)$ by a better ball, which does not intersect too large obstacles. To this ball one applied first Lemma 3.11 to obtain some Euclidean boundary. The restriction on projections, and the large enough density of E and its complement, means that some of this boundary must lie in the original carpet. A precise quantitative bound yields the result.

In Appendix A we give the explicit example of a non-self similar Sierpiński sponge similar to [14], and show how the higher dimensional generalization of their result follows from Theorem 1.3.

2 Preliminaries

Notational convention: There are many constants to keep track of in our proof. In doing so, we shall use the notation $C = C(A, B, \dots)$ to indicate when a constant C in a statement depends on other constants A, B in the same statement.

2.1 Measure theoretic preliminaries

For the interest of generality, many of the statements in this preliminary section will be formulated for general metric measure spaces, while our main result (Theorem 1.3) is formulated only for $X = S_{\mathbf{n}} \subset \mathbb{R}^d$, where we employ the restricted measure $\mu = \lambda|_{S_{\mathbf{n}}}$, with λ the usual Lebesgue measure on $\mathbb{R}^d$.

To this end, open balls in a metric space are denoted $B = B(x, r)$, and their dilations by $CB = B(x, Cr)$, despite the ambiguity that balls may not be uniquely defined by their radii. Where necessary, we will include subscripts to indicate the ambient space in which the balls are located. Thus if $X \subset \mathbb{R}^d$, then with center $x \in X$ and radius $r > 0$, the ball in $\mathbb{R}^d$ is $B(x, r) = B_{\mathbb{R}^d}(x, r)$ while the ball in X is $B_X(x, r) = B(x, r) \cap X$. We also apply this subscript notation for *relative boundaries*, i.e. $\partial_X E$ refers to the boundary of E, when E is treated as a subset of X.

Throughout this paper we will consider only measures μ whose support is all of the underlying metric space X, i.e. $\operatorname{supp}(\mu) = X$ and that $\mu(B(x, r)) \in (0, \infty)$ for all balls in the metric space.

The volume of any unit ball $B(x, 1)$ in $\mathbb{R}^d$ is $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$, where $\Gamma(x)$ is the standard Gamma function.

Definition 2.1. A metric measure space (X, d, μ) with a Radon measure μ is said to be D-(MEASURE) DOUBLING if for all $r \in (0, \operatorname{diam}(X))$ and any $x \in X$ we have

$$0 < \mu(B(x, 2r)) \leq D\mu(B(x, r)) \tag{2.2}$$

and (X, d, μ) is said to be AHLFORS Q-REGULAR (with constant $C_{AR} > 0$) if for all $r \in (0, \operatorname{diam}(X))$ and any $x \in X$ we have

$$\frac{1}{C_{AR}} r^Q \leq \mu(B(x, r)) \leq C_{AR} r^Q. \tag{2.3}$$

It is easy to see that the doubling condition (2.2) forces $D \geq 1$ and that every Ahlfors Q-regular space is $2^Q C_{AR}^2$-doubling.

Further, every D-doubling space is metrically doubling, in the following sense: a metric space X is N-METRICALLY DOUBLING if there is a constant $N \in \mathbb{N}$ so that for every ball $B(x, r) \subset X$, there are centers $x_1, x_2, \ldots, x_N$ (possibly not distinct) so that

$$B(x, r) \subset \bigcup_{i=1}^{N} B(x_i, r/2).$$

The metric doubling constant N of a D-doubling space can be chosen as $N = N(D) = D^4$. This is really an upper bound, as often N can be chosen smaller in specific spaces. See [10] for further details about metrically and measure doubling spaces.

Occasionally, we will assume that X is geodesic, that is between any pair of points $x, y \in X$ there is a rectifiable curve $\gamma : I \to X$ that connects them and with length $d(x, y)$. This is automatically true for our main space of interest, $X = \mathbb{R}^d$. For metric measure spaces in general, it ensures that $\mu(\partial B(x, r)) = 0$, for any ball $B(x, r) \subset X$, in which case the map $(x, r) \to \mu(B(x, r))$ then becomes continuous. See, for example, [5, Corollary 2.2].

For any subset $E \subset X$ and $x \in X$, we denote the distance to the set E by

$$d(x, E) = \inf_{y \in E} d(x, y).$$

(For empty sets E, we interpret the infimum and hence the distance as infinite.)

If $E \subset X$ is a μ-measurable set, then $x \in X$ is called a point of density of E if

$$\lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 1. \tag{2.4}$$

The following lemma allows for choosing scales with density within a desired range, once an upper bound for the density is met.

Lemma 2.5. *Let $X = (X, d, \mu)$ be a D-doubling metric measure space and let E be a Borel subset of X. If x is a point of density of E and if $R > 0$ is such that*

$$\frac{\mu(E \cap B(x, R))}{\mu(B(x, R))} \leq b$$

for some $b \in (0, 1)$, then there exists $r' \in (0, R)$ such that

$$\frac{b}{D} \leq \frac{\mu(E \cap B(x, r'))}{\mu(B(x, r'))} \leq b.$$

Remark 2.6. Above, if X is such that the map $(x, r) \to \mu(B(x, r))$ is continuous, then the upper bound is attained for some r', i.e.

$$\frac{\mu(E \cap B(x, r'))}{\mu(B(x, r'))} = b.$$

This occurs, for instance, when X is a geodesic metric space, or when X is a subset of a geodesic metric space and equipped with the restricted measure and distance.

Proof of Lemma 2.5. Put $h(t) := \frac{\mu(E\cap B(x,t))}{\mu(B(x,t))}$, so $\lim_{t\to 0} h(t) = 1$, and put $R_k = 2^{-k}R$. Since x is a point of density of E, there is some $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have $h(R_{k+1}) > b$. Let K be the largest index such that $h(R_K) \leq b$.

From doubling we have $\frac{h(t/2)}{h(t)} \leq D$ for all $t > 0$. So if $h(R_K) \leq D^{-1}b$, then

$$h(R_{K+1}) = h\left(\frac{R_K}{2}\right) \leq D \cdot h(R_K) \leq b$$

which is a contradiction to K being the largest index with this property. The desired estimate for $r' = R_K$ follows. □

2.2 Preliminaries on Uniformity

Here, a curve is a continuous map $\gamma\colon I \to X$ and $\Omega \subset X$ will denote a closed set.

Definition 2.7. Given $x, y \in \Omega$ and $A \geq 1$, we say that $\gamma\colon [0, 1] \to X$ is an A-UNIFORM CURVE between x and y in Ω if $\gamma(0) = x$, $\gamma(1) = y$, $\operatorname{diam}(\gamma) \leq Ad(x, y)$, and

$$d(\gamma(t), \Omega^c) \geq \frac{1}{A} \min\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\}. \tag{2.8}$$

for all $t \in [0, 1]$.

A subset $\Omega \subset X$ is called A-UNIFORM if for every $x, y \in \Omega$ there is an A-uniform curve between x and y. If $\Omega^c = \emptyset$, we apply the convention $d(x, \emptyset) = \infty$, and the condition is vacuously satisfied for $\Omega = X$.

An open subset R of X is called $(A-)$CO-UNIFORM if ∂R is connected and $X \setminus R$ is A–uniform.

The notion of uniform domains Ω (in Euclidean spaces) has been introduced by Martio and Sarvas, [15] where it was initially required that such sets Ω be open. We remark, that if a closed subset Ω is uniform then its interior $\operatorname{int}(\Omega)$ is uniform in the classical sense. One can also show that $\partial\Omega$ is a porous subset, and thus has zero Lebesgue measure. This allows us to apply many of the calculations in classical literature, see [9, Section 4.2] for a more detailed discussion. We also refer to [2, 21] for further, fundamental results about such domains. Co-uniformity, which appears in [9], was introduced as a further, convenient context for multiply-connected domains.

Clearly every set is a uniform domain in itself, i.e. if $\Omega = X$ then the conditions in Definition 2.7 are automatically satisfied. Of the next two lemmas, the first relates uniformity of domains to the previous notion of doubling (Definition 2.1) and the second is a technical estimate to be used later. See [9, Lemma 4.24] and [9, Lemma A.3] for detailed arguments, respectively. The first of these also follows easily from [2, Lemma 4.2].

Lemma 2.9. *Suppose $\Omega \subset X$ is A-uniform and $A \geq 1$ and that X is Q-Ahlfors regular with constant C, then Ω is Q-Ahlfors regular with constant $C' = C'(C, Q)$ when equipped with the restricted measure.*

Lemma 2.10. *Let $\Omega \subset X$ be a closed subset and let $x, y \in \Omega$. If $\gamma\colon [0,1] \to \Omega$ is an A-uniform curve between x and y in Ω, then for every $t \in [0,1]$ it holds that*

$$d(\gamma(t), \Omega^c) \geq \frac{1}{4A} \min\{d(x, \Omega^c) + \operatorname{diam}(\gamma|_{[0,t]}), d(y, \Omega^c) + \operatorname{diam}(\gamma|_{[t,1]})\}.$$

We will also need the following result. It affirms the intuitive idea that nontrivial overlaps of uniform domains are also uniform. To formulate it, recall that a metric space X is C-quasiconvex, if for every pair of points $x, y \in X$ there exists a rectifiable curve $\gamma : [0,1] \to X$ so that $\gamma(0) = x, \gamma(1) = y$, and $\operatorname{Len}(\gamma) \leq Cd(x,y)$. We emphasise that we will only apply this result with $X = \mathbb{R}^d$, which is clearly 1-quasiconvex (and in particular, convex). The proof is somewhat technical, however, and can be found in [9, Theorem 4.22].

Theorem 2.11. *([9, Theorem 4.22]) Fix structural constants $A_1, A_2, C, D \geq 1$. Let X be a C-quasiconvex, D-metric doubling metric space, let Ω be an A_1-uniform subset of X, and let S be a bounded, A_2-co-uniform subset of X. If*

$$\overline{S} \subset \operatorname{int}(\Omega)$$

then $\Omega \setminus S$ is A'-uniform in X, with dependence $A' = A'(A_1, A_2, C, D, \frac{d(S,\partial\Omega)}{\operatorname{diam}(S)})$.

We will also need the following “collared” estimate for neighborhoods of uniform domains. In general, for non-empty subsets E of X we define their neighborhoods $N_r(E)$ for $r > 0$ by

$$N_r(E) = \bigcup_{e \in E} B(e, r). \tag{2.12}$$

Lemma 2.13. *Let $A, D \geq 1$ and let (X, d, μ) be a D-doubling space. There are constants $C_N = C_N(D, A) \geq 1$ and $b = b(D, A) > 0$ so that for every nonempty A-uniform subset U of X, every $x \in U$, every $r \in (0, \operatorname{diam}(U)]$, and every $\delta \in (0,1)$ it holds that*

$$\mu(B_U(x,r)) \cap N_{\delta r}(U^c)) \leq C_N \delta^b \mu(B_U(x,r)).$$

This follows directly from [2]. More precisely, as stated in [2, Lemma 4.2] uniform domains satisfy the “corkscrew” condition, so by [2, Theorem 2.8] this is equivalent to the “local b-shell” conditions for balls; this is precisely the estimate in our preceding Lemma. While the proofs in [2] are formulated for uniform domains that are open, they apply in our setting of closed uniform domains too, since $\mu(\partial R) = 0$; see the discussion above after Definition 2.7. The following lemma allows us to exchange balls in an appropriate way for ones which are strictly contained inside Ω.

Lemma 2.14. *Let $A, D \geq 1$, let X be a geodesic D-doubling space, and let $\Omega \subset X$ be A-uniform. For any $\eta \in (0,1)$ there is a $\sigma = \sigma(D, A, \eta) \in (0,1)$ so that for any choice of $x \in \Omega$, $r > 0$, and $E \subset \Omega$ with*

$$\Theta_\mu(E, B(x,r)) \geq \eta,$$

and any $s \in (0, \sigma)$ there exists $p \in B(x, 8Ar)$ so that $d(p, \Omega^c) \geq \sigma r$ and

$$\Theta_\mu(E, B(p, sr)) \geq \frac{1}{2D^2} \Theta_\mu(E, B(x,r)) \geq \frac{1}{2D^2}\eta.$$

Proof. Let $x \in \Omega$ and $r > 0$. With the same constants as above, let $b = b(D, A)$ and $C_N = C_N(D, A) \geq 1$ be as in Lemma 2.13. Now put

$$\sigma = \frac{1}{32A} \left(\frac{\eta}{2C_N}\right)^{1/b},$$

so $\sigma \in (0,1)$. Let $s \in (0,\sigma)$ be given, and in what follows, put $\delta := 8A\sigma < \frac{1}{2}(\frac{\eta}{2C_N})^{1/b}$ and let

$$F_i = \begin{cases} E \setminus N_{2\delta r}(\Omega^c), & \text{if } i = 1 \\ B(x,r) \setminus (E \cup N_{2\delta r}(\Omega^c)), & \text{if } i = 2. \end{cases}$$

It follows from our hypothesis that

$$\min\{\mu(E \cap B(x,r)), \mu(B(x,r) \setminus E)\} \geq \eta\mu(B(x,r)). \tag{2.15}$$

By our choices of σ and δ,

$$C_N(2\delta)^b < C_N \cdot \frac{\eta}{2C_N} \leq \frac{1}{2}\Theta_\mu(E, B(x,r)),$$

so for the case $i = 1$, using $U = \Omega$ in Lemma 2.13 we have

$$\begin{aligned} \mu((E \setminus N_{2\delta r}(\Omega^c)) \cap B(x,r)) &\geq \mu(E \cap B(x,r)) - \mu(E \cap N_{2\delta r}(\Omega^c) \cap B(x,r)) \\ &\geq \mu(E \cap B(x,r)) - \mu(\Omega \cap N_{2\delta r}(\Omega^c) \cap B(x,r)) \\ &\geq \mu(E \cap B(x,r)) - C_N(2\delta)^b \mu(B(x,r) \cap \Omega) \\ &> \Theta_\mu(E, B(x,r))\mu(B(x,r)) - \frac{\Theta_\mu(E, B(x,r))}{2}\mu(B(x,r)). \end{aligned}$$

From this (and replacing E with its complement, for $i = 2$) we conclude

$$\mu(F_i \cap B(x,r)) \geq \frac{\Theta_\mu(E, B(x,r))}{2}\mu(B(x,r)). \tag{2.16}$$

Now consider the sets

$$\begin{aligned} \mathcal{D} &:= \{(y,z) \in X \times X \colon y \in B(x,2r),\ d(y,\Omega^c) \geq \delta r,\ d(z,y) \leq sr\}, \\ \mathcal{D}_i' &:= \{(y,z) \in X \times X \colon z \in B(x,r) \cap F_i,\ d(z,y) \leq sr\}, \end{aligned}$$

where, clearly, $\mathcal{D}_i' \subset \mathcal{D}$. Using Fubini's Theorem, for $i = 1, 2$ it follows that

$$\begin{aligned} \int\limits_{B(x,2r)\setminus N_{\delta r}(\Omega^c)} \frac{\mu(F_i \cap B(y,sr))}{\mu(B(y,sr))}\, d\mu(y) &= \iint\limits_{\mathcal{D}} \frac{1_{F_i}(z)}{\mu(B(y,sr))}\, d\mu(z)\, d\mu(y) \\ &\geq \frac{1}{D} \iint\limits_{\mathcal{D}_i'} \frac{1_{F_i}(z)}{\mu(B(z,sr))}\, d\mu(z)\, d\mu(y) \\ &\geq \int\limits_{F_i \cap B(x,r)} \frac{1}{D\mu(B(z,sr))} \int\limits_{B(z,sr)} 1\, d\mu(y)\, d\mu(z) \\ &\geq \frac{1}{D}\mu(F_i \cap B(x,r)). \end{aligned}$$

As a result, for both $i = 1, 2$, there thus exists some $y_i \in B(x,2r) \setminus N_{\delta r}(\Omega^c)$ so that

$$\mu(B(x,2r) \setminus N_{\delta r}(\Omega^c)) \frac{\mu(F_i \cap B(y_i,sr))}{\mu(B(y_i,sr))} \geq \frac{1}{D}\mu(F_i \cap B(x,r)).$$

By Equation (2.16) we have for such $y_i \in B(x,2r) \setminus N_{\delta r}(\Omega^c)$ that

$$\frac{\mu(F_i \cap B(y_i,sr))}{\mu(B(y_i,sr))} \geq \Theta_\mu(E, B(x,r)) \frac{1}{2D} \frac{\mu(B(x,r))}{\mu(B(x,2r))} \geq \frac{1}{2D^2}\Theta_\mu(E, B(x,r)).$$

In summary, there must exist $y_i \in B(x,2r) \setminus N_{\delta r}(\Omega^c)$ so that

$$\mu(F_i \cap B(y_i,sr)) \geq \frac{1}{2D^2}\Theta_\mu(E, B(x,r))\mu(B(y_i,sr)). \tag{2.17}$$

We next consider three different cases. If

$$\mu(B(y_1, sr) \setminus E) > \frac{1}{2D^2}\mu(B(y_i, sr)). \tag{2.18}$$

then since $d(x, y_1) \le 2r \le 2Ar$, the claim would follow for $p := y_1$. Similarly, if equation (2.18) held with y_2 replacing y_1, then the claim would hold for $p := y_2$ instead. Note that in these first two cases, we obtain the weaker density estimate from the claim. In the final case, which we handle next, we obtain a point p so that $\Theta_\mu(E, B(p, sr)) = 1/2$.

We can therefore assume for $i = 1, 2$ that

$$\mu(B(y_i, sr) \setminus E) \le \frac{1}{2D^2}\mu(B(y_i, sr)). \tag{2.19}$$

Let $\gamma\colon I = [0, 1] \to X$ be a A-uniform curve joining y_1 and y_2, so by Lemma 2.10 we get

$$d(\gamma(t), \Omega^c) \ge \frac{1}{4A}\min\{d(y_1, \Omega^c), d(y_2, \Omega^c)\} \ge sr$$

for all $t \in I$. Since X is geodesic and doubling, the map

$$T(t) := \frac{\mu(B(\gamma(t), sr) \cap E)}{\mu(B(\gamma(t), sr))}$$

is continuous. From Equation (2.19) we get $T(0) \ge (1 - \frac{1}{2D^2}) \ge \frac{1}{2}$ and $T(1) \le \frac{1}{2D^2} \le \frac{1}{2}$, so by continuity, there is some t so that $T(t) = \frac{1}{2}$. Moreover,

$$d(\gamma(t), x) \le \mathrm{diam}(\gamma) + d(y_1, x) \le Ad(y_1, y_2) + 2r \le 4Ar + 2r \le 8Ar$$

follows from the definition of an A-uniform curve, in which case $p = \gamma(t)$ satisfies $p \in B(x, 8Ar)$ as well as

$$\Theta_\mu(E, B(y, sr)) = T(t) = \frac{1}{2} \ge \frac{1}{2D^2} \ge \frac{1}{2D^2}\Theta_\mu(E, B(x, r))$$

which is the desired conclusion. □

By a similar argument one also gets the following.

Corollary 2.20. *Fix $D \ge 1$. Suppose $X \subset \mathbb{R}^d$ is connected, suppose $\mu = \lambda|_X$ is D-doubling, and fix $r \in (0, \mathrm{diam}(X))$. There is a constant $L = L(D)$ so that the following holds: if $\eta \in (0, 1)$, $E \subset X$, and $B(x, r) \subset X$ satisfy*

$$\Theta_\mu(E, B(x, r)) \ge \eta$$

then for each $r_1 \in (0, r)$ there exists $x_1 \in X$ so that

$$\Theta_\mu(E, B(x_1, r_1)) \ge \frac{1}{L}\eta.$$

Remark 2.21. The proof of Corollary 2.20 proceeds exactly as in Lemma 2.14, but with the following substitutions:

- replace every instance of sr by r_1;
- replace $N_{\delta r}(\Omega^c)$ and $N_{2\delta r}(\Omega^c)$ by empty sets, and remove where appropriate;
- use the entire space X, so no complementary set Ω^c is needed. Instead, choose

$$F_i = \begin{cases} E, & \text{if } i = 1 \\ X \setminus E, & \text{if } i = 2 \end{cases}$$

$$\mathcal{D} = \{(y, z) \in X \times X\colon d(x, y) < 2r, d(z, y) \le r_1\}$$

$$\mathcal{D}_i' = \{(y, z) \in X \times X\colon z \in B(x, r) \cap F_i, d(z, y) \le r_1\};$$

- to finish the proof, use directly that X is connected and that the map

$$z \mapsto \frac{\mu(B(z, r_1) \cap E)}{\mu(B(z, r_1))}$$

is continuous, in which case no explicit curve γ is needed.

We need also a lemma on volumes for subsets of interest in Euclidean spaces.

Lemma 2.22. *Under the hypotheses of Theorem 1.3, then $S_{\mathbf{n}} \subset \mathbb{R}^d$ is d-Ahlfors regular with constant C_{AR} depending only on those constants from Definition 1.1 and the sequence* $\mathbf{n}$.

Proof. Let A, δ, L be the constants for the uniform sparseness condition. As μ is the restriction of Lebesgue measure, we clearly have $\mu(B_{S_{\mathbf{n}}}(x, r)) \le \omega_d r^d$, . so it suffices to show the lower bound in (2.3). Since $n_k \in \mathbb{N}$ with $n_k \ge 3$ it clearly holds that

$$\sum_{k=1}^{\infty} \frac{1}{n_k^d} < \sum_{k=1}^{\infty} \frac{1}{n_k^{d-1}} < \infty.$$

If we set $Y = \Omega$, then Lemma 2.9 further implies Y is d-Ahlfors regular with some constant $C(A, d)$. The same is true with a different constant, if we set $Y = \Omega \setminus R$ for any $R \in \mathcal{R}_{\mathbf{n},k}$, as Theorem 2.11 implies that Y is A'-uniform, for some A' depending solely on A (and the doubling constant of $\mathbb{R}^d$). Either way, such Y is Ahlfors d-regular with a constant $C = C(A', D)$.

In the following, we prove a lower bound only for some small scales depending on the sequence $\mathbf{n}$. In particular, choosing $K \in \mathbb{N}$ so that

$$\sum_{k=K}^{\infty} \frac{1}{n_k^d} \le \frac{\delta^d}{4^{d+1} C L^d \omega_d},$$

it suffices to prove a lower bound for $r \in (0, \delta s_{K-1}/2)$. Once this bound is established, with some constant C', it applies to all $r \le \delta s_{K-1}/4$; so if instead $r \in [\delta s_{K-1}/2, \operatorname{diam}(S_{\mathbf{n}})]$ then

$$\mu(B_{S_{\mathbf{n}}}(x, r)) \ge \mu(B_{S_{\mathbf{n}}}\left(x, \frac{\delta s_{K-1}}{4}\right)) \ge C'\left(\frac{\delta s_{K-1}}{4}\right)^d \ge C'\left(\frac{\delta s_{K-1}}{4}\right)^d \frac{r^d}{\operatorname{diam}(S_{\mathbf{n}})^d}$$

in which case $S_{\mathbf{n}}$ is C_{AR}-Ahlfors regular with $C_{AR} = C'(\frac{\delta s_{K-1}}{4 \operatorname{diam}(S_{\mathbf{n}})})^d$.

To this end, choose $J \ge K$ so that $\frac{1}{2}\delta s_J < r < \frac{1}{2}\delta s_{J-1}$. Let $x \in S_{\mathbf{n}}$. By condition (5) in Definition 1.1, the ball $B(x, r)$ intersects at most one $R \in \mathcal{R}_{\mathbf{n},k}$ for $k \le J$. For any such R, let $Y = \Omega \setminus R$; otherwise let $Y = \Omega$. In either case, Y is C-Ahlfors regular and satisfies

$$B(x, r) \cap S_{\mathbf{n}} = B(x, r) \cap Y \setminus \left(\bigcup_{k>J} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} R \right).$$

Now put $\mathcal{R}_{\mathbf{n},k}(x, r) := \{R \in \mathcal{R}_{\mathbf{n},k} : R \cap B(x, r) \ne \emptyset\}$. As a special case, assuming first that for all $k > J$ we have

$$\mu\left(\bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} B(x, r) \cap R \right) \le \omega_d \left(\frac{4Lr}{\delta n_k} \right)^d, \tag{2.23}$$

then summing over k gives

$$\mu\left(\bigcup_{k>J} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} B(x, r) \cap R \right) \le \sum_{k=J}^{\infty} \mu\left(\bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} B(x, r) \cap R \right) \le \left(\sum_{k=K}^{\infty} \frac{1}{n_k^d} \right) \omega_d \frac{4^d L^d r^d}{\delta^d} \le \frac{r^d}{4C}$$

and along with the previous equality of sets, the lower bound follows with $C' = 1/(2C)$:

$$\begin{aligned} \mu(B(x, r) \cap S_{\mathbf{n}}) &= \mu\left(B(x, r) \cap Y \setminus \bigcup_{k>J} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} R \right) \\ &= \mu(B(x, r) \cap Y) - \mu\left(\bigcup_{k>J} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} B(x, r) \cap R \right) \ge \frac{r^d}{2C}. \end{aligned}$$

To see how estimate (2.23) is valid in the general case, observe that each $R \in \mathcal{R}_{\mathbf{n},k}$ that intersects $B(x,r)$ can also be included in a ball $B(x_R, Ls_k)$ and that the rescaled balls $B(x_R, \delta s_{k-1}/2)$ are disjoint and are contained in $B(x, 2r)$. Thus there are at most $(\frac{4r}{\delta s_{k-1}})^d$ such balls and summing over all previous such R yields

$$\mu\Bigg(\bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} B(x,r)\cap R\Bigg) \le \sum_{R\in\mathcal{R}_{\mathbf{n},k}(x,r)} \omega_d L^d s_k^d \le \omega_d L^d s_k^d \frac{4^d r^d}{\delta^d s_{k-1}^d} = \frac{\omega_d 4^d L^d r^d}{\delta^d n_k^d},$$

which is (2.23) as desired. □

3 Isoperimetry for Sierpiński sponges

In this section, we develop tools regarding isoperimetry, which are used to prove Theorem 1.3. For completeness, we define Poincaré inequalities here, although, we shall quickly pivot to the equivalent notion of (relative) isoperimetry, which is what we actually prove and use.

Definition 3.1. Let $1 \le p < \infty$. A closed subset X of $\mathbb{R}^d$, with $\mu = \lambda|_X$, is said to satisfy a p-Poincaré inequality (with constants $C, \Lambda \ge 1$) if for all Lipschitz functions $f\colon X \to \mathbb{R}$ and all $x \in X$ and $r \in (0, \operatorname{diam}(X))$ we have for balls $B := B(x,r)$ that

$$\fint_B |f - f_B|\, d\mu \le Cr \Bigg(\fint_{\Lambda B} |\nabla f|^p\, d\mu\Bigg)^{1/p}. \tag{3.2}$$

Here, for any locally Lebesgue integrable $f\colon X \to \mathbb{R}$ its average value on a ball is

$$f_B := \fint_B f\, d\mu := \frac{1}{\mu(B)} \int f\, d\mu,$$

and by Rademacher's Theorem, the restriction of the gradient ∇f to X is well-defined almost-everywhere.

This inequality is essentially a local property, as the following version of [1, Theorem 1.3] shows. This quantitative version does not appear explicitly in the reference, but follows directly from their argument.

Lemma 3.3. *Suppose that (X, d, μ) is a bounded, connected, D-doubling metric measure space. If X satisfies Definition 3.1 with constant (C, Λ) for all $r \in (0, r_0)$, then X satisfies Definition 3.1 for all $r > 0$ with constants $C_1 = C_1(C, \Lambda, \frac{\operatorname{diam}(X)}{r_0})$ and $\Lambda_1 = \Lambda_1(C, \Lambda, \frac{\operatorname{diam}(X)}{r_0})$.*

3.1 Definition of isoperimetry and iteration

Lahti and Korte discuss in [13] various criteria that are equivalent to a 1-Poincaré inequality. To formulate them, we will require two additional notions. If μ is a doubling measure on X, then put

$$h(B(x,r)) := \frac{1}{r}\mu(B(x,r))$$

and define for any set $E \subset X$ the codimension-one Hausdorff content as

$$\mathcal{H}_{h,\delta}(E) := \inf\Bigg\{\sum_{i=1}^{\infty} h(B(x_i,r_i)) \;\Bigg|\; r_i \le \delta, E \subset \bigcup_{i=1}^{\infty} B(x_i,r_i)\Bigg\},$$

and the codimension-one Hausdorff measure as

$$\mathcal{H}_h(E) := \lim_{\delta\to 0} \mathcal{H}_{h,\delta}(E).$$

If we use $h(B(x,r)) = r^s$ instead, we obtain the s-dimensional (spherical) Hausdorff measure $\mathcal{H}_s(E)$ (up to a scalar multiple, depending on convention).

Remark 3.4. Note that if μ is Ahlfors Q-regular with constant C_{AR}, then $\mathcal{H}_h$ is comparable to $(Q-1)$-dimensional (spherical) Hausdorff measure $\mathcal{H}_{Q-1}$. Indeed, we have

$$\frac{1}{C_{AR}}\mathcal{H}_{Q-1}(E) \leq \mathcal{H}_h(E) \leq C_{AR}\mathcal{H}_{Q-1}(E),$$

as follows easily from the definition. We will use this fact, as we will be giving bounds for the $d-1$-dimensional Hausdorff measure instead of the codimension-one Hausdorff measure.

We are now ready to introduce the criterion [13, Theorem 1.1].

Theorem 3.5 (Korte-Lahti)**.** *A doubling metric measure space (X, d, μ) satisfies the 1-Poincaré inequality if and only if there are constants $C_S, \Lambda_S \in [1, \infty)$ such that for every ball $B = B(x, r)$ and any Borel set $E \subset X$ we have*

$$\Theta_\mu(E, B) \leq C_S r \frac{\mathcal{H}_h(\partial E \cap \Lambda_S B)}{\mu(\Lambda_S B)}. \tag{3.6}$$

Inequality (3.6) is known as a RELATIVE ISOPERIMETRIC INEQUALITY: once a ball is given, the measure of the boundary of E within that ball controls the density of E and its complement relative to that ball. To specify the dependence on parameters, we sometimes refer to (3.6) as a RELATIVE ISOPERIMETRIC INEQUALITY WITH CONSTANTS C_S AND Λ_S.

The relative isoperimetric inequality can be considered as a property of every subset E in X, relative to every ball in the space X. However, it will be helpful to introduce a "density level" in our proofs, i.e. to consider isoperimetric inequalities only for "large enough" sets.

Definition 3.7. Let $\tau, C > 0$. A metric measure space (X, d, μ) is said to satisfy a (τ, C)-ISOPERIMETRIC INEQUALITY if there is a $\Lambda \geq 1$ such that every Borel set $E \subset X$ and ball $B = B(x, r)$ satisfies the following: if $\Theta_\mu(E, B) \geq \tau$, then

$$\Theta_\mu(E, B) \leq Cr \frac{\mathcal{H}_h(\partial E \cap \Lambda B)}{\mu(\Lambda B)}. \tag{3.8}$$

To specify the dependence on parameters, we say that X satisfies a (τ, C)-ISOPERIMETRIC INEQUALITY WITH INFLATION FACTOR Λ.

Since the left hand side of (3.8) is bounded below by τ, it would really suffice to give just a constant lower bound. However, we wish the constants to match those in Equation (3.6) as closely as possible, and simply to weaken the condition by restricting the sets considered.

Proof of Theorem 1.8. Given parameters $D \geq 2$, $\tau \in (0, D^{-3}]$, $C > 0$, and $\Lambda \geq 1$, we will assume that X satisfies the (C, τ)-isoperimetric inequality with inflation factor Λ and prove the isoperimetric inequality (3.6) with $C_S = 2D^{7+\log_2(\Lambda)}C$ and $\Lambda_S = 2\Lambda$.

Let E be any Borel subset of X. Without loss of generality assume

$$\mu(E \cap B(x, r)) < \frac{1}{2}\mu(B(x, r)),$$

otherwise we prove the inequality by replacing E by E^c. Now if $x \in X$ and $r > 0$ satisfy

$$\frac{\mu(E \cap B(x, 2r))}{\mu(B(x, 2r))} \geq \frac{1}{D^3}$$

then by hypothesis, the inequality equation (3.8) is exactly what we want for (3.6) except for an extra factor of D arising when the quantity $\mu(\Lambda B)$ is adjusted for $\mu(2\Lambda B)$.

We can therefore assume that

$$\frac{\mu(E \cap B(x, 2r))}{\mu(B(x, 2r))} < \frac{1}{D^3}.$$

Consider the set of density points

$$S = \left\{ z \in B(x, r) \,\middle|\, \lim_{t \to 0} \frac{\mu(B(z, t) \cap E)}{\mu(B(z, t))} = 1 \right\}.$$

By Lebesgue differentiation, we have $\mu(S) = \mu(E \cap B(x, r))$ and for each $z \in S$ we have

$$\frac{\mu(E \cap B(z, r))}{\mu(B(z, r))} \le D^2 \frac{\mu(E \cap B(x, 2r))}{\mu(B(z, 4r))} \le D^2 \frac{\mu(E \cap B(x, 2r))}{\mu(B(x, 2r))} < \frac{1}{D}.$$

Thus, from Lemma 2.5 there exists $r_z \le r$ with

$$\frac{1}{D^2} < \frac{\mu(B(z, r_z) \cap E)}{\mu(B(z, r_z))} < \frac{1}{D}. \tag{3.9}$$

By the $5B$-covering lemma (see e.g. [16, Theorem 2.1]), there is a countable subset $\{z_i\}_{i\in I}$ of S and radii $s_i = r_{z_i}$ such that $\{B(z_i, \Lambda s_i)\}_{i\in I}$ is pairwise disjoint, that $B_i := B(z_i, s_i)$ satisfy (3.9), and that

$$S \subset \bigcup_{i\in I} B(z_i, 5\Lambda s_i).$$

By (3.9) and the hypotheses of the theorem, each $E \cap B_i$ satisfies $\Theta_\mu(E, B_i) \ge \frac{1}{D^2} > \tau$ and hence the (τ, C)-isoperimetric inequality as well:

$$\mathcal{H}_h(\partial E \cap B(z_i, \Lambda s_i)) \ge \frac{\mu(B(z_i, \Lambda s_i))}{Cs_i D^2} \ge \frac{\mu(B(z_i, 5\Lambda s_i))}{Cs_i D^5}. \tag{3.10}$$

From this and the inclusion $B(z_i, \Lambda s_i) \subseteq B(x, 2\Lambda r)$, for each $i \in I$, it follows from a repeated use of doubling of the measure that

$$\begin{aligned}
\mathcal{H}_h(\partial E \cap B(x, 2\Lambda r)) &\ge \sum_{i\in I} \mathcal{H}_h(\partial E \cap B(z_i, \Lambda s_i)) \\
&\overset{(3.10)}{\ge} \frac{1}{CD^5} \sum_{i\in I} \frac{\mu(B(z_i, 5\Lambda s_i))}{s_i} \\
&\ge \frac{1}{CD^5} \frac{\mu(S)}{r} \\
&\ge \frac{1}{2CD^{7+\log_2(\Lambda)}} \frac{\mu(B(x, 2\Lambda r))}{r} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \\
&\ge \frac{1}{C_S} \frac{\mu(B(x, 2\Lambda r))}{r} \Theta_\mu(E, B(x, r))
\end{aligned}$$

which implies the relative isoperimetric inequality, as desired. □

3.2 A Euclidean isoperimetric inequality

We will prove a "projected" isoperimetric inequality for Borel sets E relative to axis-aligned rectangles in Euclidean spaces $\mathbb{R}^d$. This guarantees that $\partial_{S_n} E$ has large projections, and when combined with condition (6) from Definition 1.2, gives lower bounds for $\mathcal{H}_{d-1}(\partial_{S_n} E)$. To this end we formulate an isoperimetric inequality in terms of a direction-wise Euclidean boundary.

For $x \in \mathbb{R}^d$, denote by $l_{i,x}$ the line containing x that is parallel to the i'th coordinate axis. If $E \subset \mathbb{R}^d$, we also put

$$\partial_{+,i} E = \{\, x \mid x \in \partial_{l_{i,x}}(l_{i,x} \cap E)\,\}.$$

In other words, the set $\partial_{+,i} E$ consists of those points x, where a sequence of points exists in the i'th coordinate direction within E, and outside of E, which converges to it. Next, denote by π_i the projection of $\mathbb{R}^d$ onto the hyperplane defined by $x_i = 0$.

We now relate the density of sets with respect to boxes with the size of the projections of their boundaries. The following lemma is likely classical, but we include its proof for completeness.

Lemma 3.11. *Let $Q = \prod_{i=1}^d (a_i, b_i)$ be a rectangle, and $E \subset \mathbb{R}^d$ a Borel set. Then,*

$$\Theta_\lambda(E, Q) \le d \sum_{i=1}^{d} \frac{\mathcal{H}_{d-1}(\pi_i(\partial_{+,i} E \cap Q))}{\mathcal{H}_{d-1}(\pi_i(Q))}$$

Proof. The statement is invariant under affine functions of x_i, so we can assume that $Q = (0,1)^d$. Also, the statement is clear for $d = 1$.

Towards a proof by induction, assume that the statement has been proven for dimension $d-1$ and that $d \geq 2$. Without loss of generality assume

$$\lambda_E = \lambda(E \cap Q) \leq \frac{1}{2}.$$

For $t \in (0,1)$ and $\mathbf{y} \in (0,1)^{d-1}$ define

$$\begin{aligned} H_t &:= \{(t, \mathbf{x}) \in (0,1)^{d-1} \mid \mathbf{x} \in (0,1)^{d-1}\} \\ l_{\mathbf{y}} &:= \{(s, \mathbf{y}) \in (0,1)^{d-1} \mid s \in (0,1)\} \end{aligned}$$

and consider the following images under π_1:

$$\begin{aligned} I_0 &:= \{\mathbf{x} \in (0,1)^{d-1} \mid l_{\mathbf{x}} \subset E \cap Q\}, \\ O_0 &:= \{\mathbf{x} \in (0,1)^{d-1} \mid l_{\mathbf{x}} \cap E \cap Q = \emptyset\}, \\ P_0 &:= (0,1)^{d-1} \setminus (I_0 \cup O_0), \end{aligned}$$

in which case it is clear that

$$\begin{aligned} P_0 &= \pi_1(\partial_{+,1} E \cap Q) \\ \pi_1(E \cap Q) &\subset P_0 \cup I_0 \\ \pi_1(Q \setminus E) &\subset P_0 \cup O_0. \end{aligned}$$

If $\mathcal{H}_{d-1}(P_0) \geq \frac{1}{d}\lambda_E$ then the statement of the lemma is trivially true, so assume $\mathcal{H}_{d-1}(P_0) \leq \frac{1}{d}\lambda_E$ which implies that

$$\lambda_E \leq \mathcal{H}_{d-1}(\pi_1(E \cap Q)) \leq \mathcal{H}_{d-1}(P_0 \cup I_0) \leq \frac{1}{d}\lambda_E + \mathcal{H}_{d-1}(I_0).$$

Then, $\mathcal{H}_{d-1}(I_0) \geq \frac{d-1}{d}\lambda_E$. Similarly, we assumed $\lambda_E \leq \frac{1}{2} \leq \lambda(Q \setminus E)$ so it follows that

$$\min\{\mathcal{H}_{d-1}(I_0), \mathcal{H}_{d-1}(O_0)\} \geq \frac{d-1}{d}\lambda_E.$$

Now if $t \in (0,1)$ then $E \cap H_t$ contains a translate of I_0, and its complement contains a translate of O_0. Then, we get from the $d-1$-dimensional statement for $E \cap H_t$, that

$$\frac{(d-1)\lambda_E}{d} \leq (d-1)\sum_{j=2}^{d} \mathcal{H}_{d-2}(\pi_j(\partial_{+,j} E \cap H_t)),$$

which when integrated over t and using Fubini's theorem gives

$$\lambda_E \leq d\sum_{j=2}^{d} \mathcal{H}_{n-1}(\pi_j(\partial_{+,j} E \cap Q)).$$

This gives the desired inequality. □

Remark 3.12. The above gives a fairly simple inductive proof of the isoperimetric inequality in $\mathbb{R}^d$, although with sub-optimal constants.

4 Proof of Theorem 1.3

To begin, recall from §2 that subscripts for balls indicate the space (and hence, the choice of metric) on which those balls are defined.

Remark 4.1 (Dependence on parameters). In the proof below, many choices will depend on the parameters from Definitions 1.1 and 1.2, as well as lemmas from earlier sections. Two parameters, β and ϵ_1, will be determined at the end of the proof, as they depend on many intermediate parameters. However, none of the other parameters will depend on the choice of β and ϵ_1.

Proof of Theorem 1.3. Definitions 1.1 and 1.2 are scale-invariant, so without loss of generality we may assume that $\operatorname{diam}(\Omega) = 1$. By a coordinate change we can take the maps π_i to be orthogonal projections onto hyperplanes given by $x_i = 0$, for $i = 1, \dots, d$.

Before fixing other parameters, we first claim that $S_{\mathbf{n}}$ is connected. Indeed, by Theorem 2.11 the domains obtained by removing finitely many co-uniform sets from Ω, i.e.

$$S_{j,\mathbf{n}} := \Omega \setminus \Big(\bigcup_{k=1}^{j} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} R \Big) \tag{4.2}$$

are a nested sequence of compact connected sets, so the intersection $S_{\mathbf{n}} = \bigcap_{j=1}^{\infty} S_{j,\mathbf{n}}$ is also connected.

It follows from Lemma 2.22 that $S_{\mathbf{n}}$ is d-Ahlfors regular with constant $C_{AR} \geq \omega_d$ depending on all parameters $A, L, \delta, d, \mathbf{n}$. Thus it is also D-doubling with fixed constant $D = 2^d C_{AR}^d$.

We now proceed in four steps, indicating reductions and strategies as needed.

Step I: Reduction to a small sum: Since $\mathbb{R}^d$ is D-doubling and 1-quasiconvex, Theorem 2.11 implies that for any $k \in \mathbb{N}$ and for any fixed obstacle $R \in \mathcal{R}_{\mathbf{n},k}$ the set $\Omega \setminus R$ is A'-uniform for some $A' \geq A$.

With $\epsilon_1 \in (0, \frac{1}{2})$ to be determined later, choose N so that

$$\sum_{k=N}^{\infty} \frac{1}{n_k^{d-1}} < \epsilon_1$$

and construct $S'_{\mathbf{n}}$ analogously to $S_{\mathbf{n}}$ by using the obstacle sets $\mathcal{R}'_{\mathbf{n},l} = \emptyset$, for $l < N$, and $\mathcal{R}'_{\mathbf{n},l} = \mathcal{R}_{\mathbf{n},l}$ for $l \geq N$, and with $\Omega' = \Omega \setminus R$ in place of Ω for any $R \in \mathcal{R}_{\mathbf{n},k}$. (In other words, the difference between Ω and Ω' is that in the latter we have re-inserted the obstacles contained in $\mathcal{R}_{\mathbf{n},l}$ for $l < N$ - except possibly for a given R. The set $S'_{\mathbf{n}}$ thus depends on a choice of R.) Clearly $S_{\mathbf{n}} \subset S'_{\mathbf{n}} \subset \mathbb{R}^d$, so the d-Ahlfors regularity of $S_{\mathbf{n}}$ implies the d-Ahlfors regularity of $S_{\mathbf{n}'}$, with the same constant C_{AR} or smaller, and hence the same fixed doubling constant D as above (or smaller).

So at this level N, if $r < \delta s_{N-1}/2$ then by condition (5) of Definition 1.1 the ball $B(x, r)$ can intersect only one set ρ in the collection $\{\Omega^c\} \cup \bigcup_{k=1}^{N-1} \mathcal{R}_{\mathbf{n},k}$. Then, with the above construction of $S'_{\mathbf{n}}$ with $R = \rho$, it holds that

$$B_{S_{\mathbf{n}}}(x, r) = S_{\mathbf{n}} \cap B(x, r) = S'_{\mathbf{n}} \cap B(x, r) = B_{S_{\mathbf{n}'}}(x, r). \tag{4.3}$$

It suffices to prove a Poincaré inequality for $S'_{\mathbf{n}}$, say with constants C_{PI}, Λ_{PI}. To see why, by applying (4.3) and considering only radii $r \in (0, \delta s_{N-1}/(2\Lambda_{PI}))$, this gives a local Poincaré inequality in $S_{\mathbf{n}}$ as both sides of inequality (3.1) coincide in $S_{\mathbf{n}}$ and $S'_{\mathbf{n}}$. The set $S_{\mathbf{n}}$ is bounded, D-doubling and connected, and so, by Lemma 3.3, a local Poincaré inequality further yields a global Poincaré inequality.

We now simplify notation by only considering $S'_{\mathbf{n}}$ and dropping the primes, that is by increasing A we assume now that $A = A'$ and that $S_{\mathbf{n}} = S'_{\mathbf{n}}$ is d-Ahlfors regular with constant C_{AR}. By the construction of $S'_{\mathbf{n}}$ we may also assume that $\mathcal{R}_{\mathbf{n},k} = \emptyset$ for $k < N$. We will also use a simplified notation for balls and for relative boundaries, that is: $B_{\mathbf{n}}(x, r) := B_{S_{\mathbf{n}}}(x, r)$ and $\partial_{\mathbf{n}} E := \partial_{S_{\mathbf{n}}} E$. Further, for each $i \in \mathbb{N}$ we re-index n_{i+N-1} as n_i. If necessary, we replace Ω by $\Omega \setminus \rho$ as before.

Step II: The strategy for small sums $\sum_{k=N}^{\infty} \frac{1}{n_k^{d-1}} < \epsilon_1$ while assuming $\mathcal{R}_{\mathbf{n},k} = \emptyset$ for $k < N$: By Theorems 1.8 and 3.5, it suffices to prove that there are constants C, Λ so that for sets $E \subset S_{\mathbf{n}}$ and balls $B_{\mathbf{n}}(x, r)$, if $x \in \Omega$ satisfies

$$\Theta_\mu(E, B_{\mathbf{n}}(x, r)) \geq \frac{1}{D^3}$$

then it would follow that

$$\frac{Cr\mathcal{H}_h(\partial_{\mathbf{n}}E \cap B_{\mathbf{n}}(x, \Lambda r))}{\mu(B_{\mathbf{n}}(x, \Lambda r))} \geq \Theta_\mu(E, B_{\mathbf{n}}(x, r)). \tag{4.4}$$

It suffices to take $r \in (0, \operatorname{diam}(S_{\mathbf{n}})]$. We fix such a ball, as well as a set E for the remainder. By Remark 3.4 it suffices to prove this estimate with $\mathcal{H}_{d-1}$ replacing $\mathcal{H}_h$, in which case the result follows with constant CC_{AR} in place of C.

The proof will proceed by finding a "good" ball in two steps. We construct, as necessary, balls $B_{\mathbf{n}}(x_1, r_1)$ and $B(x_2, r_2)$, with $x_1, x_2 \in \Omega$, so that $\Theta_\mu(E, B_{\mathbf{n}}(x_1, r_1)) \geq \eta_1$ and $\Theta_\lambda(E, B(x_2, r_2)) \geq \eta_2$; here the precise choices of η_i, to be made at the end of Step III below, will be quantitative in the previous parameters. In order to pass to Euclidean bounds and apply Lemma 3.11, the second ball in this process will be a Euclidean ball. Putting $x = x_0$ and $r = r_0$, for $i = 1, 2$, we will also ensure that $d(x_i, x_{i-1}) \leq S_i r_{i-1}$, and $r_i \in [\frac{1}{S_i} r_{i-1}, r_{i-1}]$ for some $S_i > 1$ which will depend quantitatively on the parameters.

As a result, we show that $x_2 \in \Omega$ satisfies both $d(x_2, x) \leq Sr$ and $r \geq r_2 \geq \frac{1}{S} r$ for some universal constant $S = \max\{S_1 S_2, S_1 + S_2\}$, as well as

$$\mathcal{H}_{d-1}(\partial_{\mathbf{n}}E \cap B(x_2, \sqrt{d} r_2)) \geq \Delta r_2^{d-1} \tag{4.5}$$

for some universal Δ depending on all of the constants in the statement.

Moreover by doubling, the fact that $\Theta_\mu(E, B_{\mathbf{n}}(x, r)) \geq \frac{1}{D^3}$, and the choice of the good balls, we can deduce estimate (4.4) from inequality (4.5). The constants C, Λ come directly from doubling and Ahlfors regularity.

With the fixed parameters A, D, δ, and L from before, let $A' = A'(A, D, \delta, L)$ be the uniformity constant from Theorem 2.11. This coincides with an upper bound for the uniformity constant of $\Omega \setminus R$ for any fixed obstacle R. We note for clarity, that this instance of A' is different from the previous A' in Step I – indeed, due to the abbreviation of notation, the new uniformity constant would arise from the removal of up to two obstacles from the domain we started from.

Since $S_{\mathbf{n}} \subset \Omega$, and $S_{\mathbf{n}} \subset \Omega \setminus R$ for any obstacle R, then both Ω and $\Omega \setminus R$ are Ahlfors regular with constants no larger than C_{AR}. Thus, they are also D-doubling.

Step III: Choosing a good ball: Put $S_1 = \frac{32A'\sqrt{d}}{\delta} > 1$ and $r_1 = \min\{r, 1/S_1\}$. If $r \leq 1/S_1$ then it suffices to choose $x_1 = x$ and $\eta_1 = \frac{1}{D^3}$. Otherwise $r \in (1/S_1, \operatorname{diam}(S_{\mathbf{n}})]$, so by Corollary 2.20 there exist $L_1 > 0$ and $x_1 \in S_{\mathbf{n}}$, depending quantitatively on the parameters, so that

$$\Theta_\mu(E, B_{\mathbf{n}}(x_1, r_1)) \geq \frac{1}{L_1 D^3}, \tag{4.6}$$

in which case choose instead $\eta_1 = (L_1 D^3)^{-1}$. Note in the first case that $d(x_1, x) = 0$ and in the second case that

$$d(x_1, x) \leq \operatorname{diam}(S_{\mathbf{n}}) \leq \operatorname{diam}(\Omega) S_1 r = S_1 r.$$

In either case, the distance is bounded by $S_1 r$ and we have $S_1^{-1} r \leq r_1 \leq r$. Now choose $k \in \mathbb{N}$ so that

$$\frac{\delta}{32A'\sqrt{d}} s_k \leq r_1 \leq \frac{\delta}{32A'\sqrt{d}} s_{k-1}.$$

We proceed in three separate cases for $i = 2$.

First, if $B(x_1, 16A'\sqrt{d} r_1)$ does not intersect Ω^c or any obstacles R in $\mathcal{R}_{\mathbf{n},l}$ for $l \leq k$, then choose $x_2 = x_1$ and $r_2 = r_1$. Second, with $\beta > 0$ to be determined later, if instead $B(x_1, 16A'\sqrt{d} r_1)$ does not intersect Ω^c but does intersect such obstacles and if the largest such obstacle R in $\mathcal{R}_{\mathbf{n},l}$ for $l \leq k$ satisfies $\operatorname{diam}(R) \leq \beta r_1$, then we also choose $x_2 = x_1$ and $r_2 = r_1$. In both cases, putting $\eta_2 = \omega_d^{-1} C_{AR}^{-1} \eta_1$, this yields

$$\Theta_\lambda(E, B(x_2, r_2)) \geq \eta_2 \tag{4.7}$$

where we use (4.6) and the Ahlfors regularity of $S_{\mathbf{n}}$, i.e. $\mu(B_{\mathbf{n}}(x_2, r_2)) \geq \frac{1}{\omega_d C_{AR}} \lambda(B(x_2, r_2))$.

If neither of these two cases occurs, then $B(x_1, 16A'\sqrt{d} r_1)$ intersects either Ω^c, or some obstacle R in $\mathcal{R}_{\mathbf{n},l}$ for $l \leq k$ with $\operatorname{diam}(R) \geq \beta r_1$. Both cannot occur at the same time, and there can be at most one such obstacle, due to conditions (3) and (4) in Definition 1.1 and by the above choice of k.

If there is such a obstacle R, then define $\Omega' = \Omega \setminus R$, and if there is none, then define $\Omega' = \Omega$; either way, Ω' is A'-uniform with A' fixed as above. As noted at the end of Step II, Ω' is also C_{AR}-Ahlfors regular and D-doubling. Denote by μ' the restricted Lebesgue measure on Ω' for which we have

$$\Theta_{\mu'}(E, B(x_1, r_1)) \geq C_{AR}^{-2}\eta_1.$$

Here, we similarly used estimate (4.6) and the Ahlfors regularity of Ω'. For this domain, applying Lemma 2.14 to Ω' instead of Ω and with $C_{AR}^{-2}\eta_1$ for η, there exist $\sigma = \sigma(D, A', \eta_1) \in (0, 1)$ and $x_2 = p \in B(x_1, 8A'r_1) \cap \Omega'$ so that $B(x_2, \sigma r_1) \subset \Omega' \subset \Omega$ as well as

$$\Theta_\lambda(E, B(x_2, \frac{\sigma}{4A'\sqrt{d}}r_1)) = \Theta_{\mu'}(E, B(x_2, \frac{\sigma}{4A'\sqrt{d}}r_1)) \geq \frac{1}{2(8\sqrt{d})^d C_{AR}^2 D^2}\eta_1. \tag{4.8}$$

Let $r_2 = \frac{\sigma}{4A'\sqrt{d}}r_1$ and put $\eta_2 = \frac{1}{2(8A'\sqrt{d})^d C_{AR}^2 D^2}\eta_1$, from which Equation (4.7) also follows; it now suffices to take $S_2 := \max\{8A', \frac{4A'\sqrt{d}}{\sigma}\}$. With this choice of ball, we will have

$$\frac{\sigma\delta}{128A'^2 d}s_k \leq r_2 \leq \frac{\delta}{32A'\sqrt{d}}s_{k-1}.$$

Step IV: An isoperimetric estimate for a good ball: By construction, $B(x_2, 2A'\sqrt{d}r_2)$ does not intersect Ω^c. From our choice of r_2 above, any obstacle R that intersects $B(x_2, 2A'\sqrt{d}r_2)$ will also intersect $B(x_1, 16A'\sqrt{d}r_1)$ and is contained in Ω' and therefore satisfies either $R \in \mathcal{R}_{\mathbf{n},l}$ for some $l > k$ or $R \in \mathcal{R}_{\mathbf{n},l}$ with $l \leq k$ with

$$\operatorname{diam}(R) \leq \beta r_1 \leq \frac{4A'\sqrt{d}}{\sigma}\beta r_2. \tag{4.9}$$

If there is an obstacle for which the second case applies, then call it R_0; otherwise let $R_0 = \emptyset$.

Now, let Q be the cube centered at x_2 of side length $2r_2$ and with faces parallel to the coordinate planes, so Q contains the ball $B(x_2, r_2)$ and $\lambda(Q) \leq 2^d\sqrt{d}^d\omega_d^{-1}\lambda(B(x_2, r_2))$.

Then for $\eta_3 = \omega_d(2^d C_{AR}\sqrt{d}^d)^{-1}\eta_2$, the above estimates (4.8) and (4.7) imply

$$\Theta_\lambda(E, Q) \geq (\omega_d^{-1}\sqrt{d}^d 2^d)^{-1}\Theta_\lambda(E, B(x_2, r_2)) \geq (\omega_d^{-1}\sqrt{d}^d 2^d)^{-1}\omega_d^{-1}C_{AR}^{-1}\eta_2 = \eta_3.$$

By Lemma 3.11, there is an i so that

$$\mathcal{H}_{d-1}(\pi_i(Q \cap \partial_{+,i}E)) \geq \frac{(2r_2)^{d-1}\eta_3}{d^2}. \tag{4.10}$$

Fix a choice of such an index i.

Let $\mathcal{S} = \pi_i(Q \cap \bigcup_l \bigcup_{R\in\mathcal{R}_{\mathbf{n},l}} R)$, i.e. the "shadow" of all of the obstacles intersecting Q and removed from Ω. Consider the portion of the boundary not shadowed by obstacles:

$$\partial^i E := Q \cap \partial_{+,i}E \setminus \pi_i^{-1}(\mathcal{S}).$$

If $z \in \partial^i E$, then since $z \in Q \cap \partial_{+,i}E$, there is a sequence of points $z_j^E \in E \cap l_{i,z}$ ($j \in \mathbb{N}$) and a sequence in its complement $z_j^{E^c} \in E^c \cap I_{i,z}$ converging to z. These sequences lie in $S_{\mathbf{n}}$, since they are not shadowed by obstacles in Q. Indeed, $z \in \partial_{\mathbf{n}}E$ and we have shown

$$\partial^i E \subset \partial_{\mathbf{n}}E \cap Q. \tag{4.11}$$

It thus suffices to prove that $\mathcal{H}_{d-1}(\partial^i E)$ is greater than a multiple of r_2^{d-1}. We will do this by estimating $\mathcal{H}_{d-1}(\pi_i(\partial^i E))$ from below. To do this we note

$$\pi_i(\partial^i E) = \pi_i(Q \cap \partial_{+,i}E) \setminus \mathcal{S}. \tag{4.12}$$

Now, $\mathcal{S}$ consists of two parts: $\bigcup_{l>k}\bigcup_{R\in\mathcal{R}_{\mathbf{n},l}} \pi_i(R \cap Q)$ and $\pi_i(R_0 \cap Q)$. Since R_0 is either empty or satisfies Equation (4.9), it follows that

$$\mathcal{H}_{d-1}(\pi_i(R_0)) \leq \beta^{d-1}\left(\frac{4A'\sqrt{d}}{\sigma}\right)^{d-1} r_2^{d-1}. \tag{4.13}$$

Let $\rho = \frac{128A'^2 d}{\sigma\delta} r_2 \geq s_k$, so $Q \subset B(x_2, \rho)$. For the other part, we apply condition (6) from Definition 1.2 and the assumption that $\mathcal{R}_{\mathbf{n},k} = \emptyset$ for $k < N$. Indeed, putting $M = \max(k+1, N)$ we have

$$\mathcal{H}_{d-1}\Bigg(\pi_i\Bigg(\bigcup_{l>k}\bigcup_{R\in\mathcal{R}_{\mathbf{n},l}} \pi_i(R\cap B(x_2,\rho))\Bigg)\Bigg) \leq \sum_{l=M}^{\infty} \frac{L\rho^{d-1}}{n_l^{d-1}} \leq \frac{\epsilon_1 L(128A'^2 d)^{d-1}}{\sigma^{d-1}\delta^{d-1}} r_2^{d-1}. \tag{4.14}$$

Then estimates (4.13) and (4.14), together with $R_0 \cap Q \subset R_0$ and $Q \subset B(x_2, \rho)$, give

$$\begin{aligned}\mathcal{H}_{d-1}(\mathcal{S}) &\leq \mathcal{H}_{d-1}(\pi_i(R_0)) + \mathcal{H}_{d-1}\Bigg(\pi_i\Bigg(\bigcup_{l>k}\bigcup_{R\in\mathcal{R}_{\mathbf{n},l}} \pi_i(R\cap B(x_2,\rho))\Bigg)\Bigg)\\ &\leq \Bigg(\frac{\epsilon_1 L(128A'^2 d)^{d-1}}{\sigma^{d-1}\delta^{d-1}} + \beta^{d-1}\Big(\frac{4\sqrt{d}A'}{\sigma}\Big)^{d-1}\Bigg) r_2^{d-1}. \end{aligned}\tag{4.15}$$

Now, we choose $\epsilon_1 = \frac{\sigma^{d-1}\delta^{d-1}\eta_3}{4d^2 L(128A'^2 d)^{d-1}}$ and $\beta = \frac{\sigma}{4A'\sqrt{d}}(\frac{\eta_3}{4d^2})^{\frac{1}{d-1}}$. These choices, together with estimates (4.15) and (4.10) together with the equality (4.12) and inclusion (4.11) give

$$\begin{aligned}\mathcal{H}_{d-1}(\partial_{\mathbf{n}}E \cap B(x_2, \sqrt{d}r_2)) \geq \mathcal{H}_{d-1}(\partial_{\mathbf{n}}E\cap Q) &\geq \mathcal{H}_{d-1}(\partial^i E)\\ &\geq \mathcal{H}_{d-1}(\pi_i(\partial^i E))\\ &\overset{(4.12)}{\geq} \mathcal{H}_{d-1}(\pi_i(Q\cap \partial_{+,i}E)) - \mathcal{H}_{d-1}(\mathcal{S})\\ &\overset{(4.15)(4.10)}{\geq} \frac{(2r_2)^{d-1}\eta_3}{d^2} - \frac{r_2^{d-1}\eta_3}{2d^2} \geq \frac{\eta_3}{2d^2} r_2^{d-1}.\end{aligned}\tag{4.16}$$

This is estimate (4.5), from which (4.4) follows after applying doubling, Ahlfors regularity as well as the estimates for $d(x_2, x) \leq Sr$ and $r_2 \geq \frac{1}{S}r$ obtained by following the previous steps. This concludes the proof of the isoperimetric inequality. □

A Explicit examples

Here, we give an explicit application of Theorem 1.3. This is a generalization of a construction of Mackay, Tyson, and Wildrick [14] to higher dimensions.

Let $\mathbf{n} = (n_i)_{i=1}^{\infty}$ be a sequence of odd positive integers with $n_i \geq 3$. Fix a dimension $d \geq 2$ and consider the following iterative construction:

1. At the first stage, put $S_{0,\mathbf{n}} = [0,1]^d$ and $T^1_{0,\mathbf{n}} = [0,1]^d$ and $\mathcal{T}_{0,\mathbf{n}} = \{T^1_{0,\mathbf{n}}\}$.
2. Assuming that we have defined $S_{k,\mathbf{n}}$, $T^j_{k,\mathbf{n}}$, $\mathcal{T}_{k,\mathbf{n}}$ at the kth stage, for $k \in \mathbb{N}$,
   - Subdivide each $T \in \mathcal{T}_{k,\mathbf{n}}$ into $(n_{k+1})^d$ equal subcubes and exclude the central one.
   - Index the remaining subcubes in any fashion as $T^j_{k+1,\mathbf{n}}$, and let $\mathcal{T}_{k+1,\mathbf{n}} = \{T^j_{k+1,\mathbf{n}}\}$ be the collection of all the remaining cubes.

   The $k+1$'TH ORDER PRE-SPONGE is defined, consistent with (4.2), as the set

$$S_{k+1,\mathbf{n}} = \bigcup_{T\in\mathcal{T}_{k+1,\mathbf{n}}} T.$$

3. Define $\mathcal{R}_{\mathbf{n},k}$ to be the set of central $1/n_{k+1}$ subcubes that were removed from each $T \in \mathcal{T}_{k,\mathbf{n}}$ which are removed at the k'th stage and put $\overline{\mathcal{R}}_{\mathbf{n},k} = \bigcup_{l=1}^{k} \mathcal{R}_{\mathbf{n},l}$. Further we note that for $k \in \mathbb{N}$,

$$s_k = \prod_{i=1}^{k} \frac{1}{n_i}$$

   is the side length of each cube $T \in \mathcal{T}_{k,\mathbf{n}}$. (For consistency, let $s_0 = 1$.)

The SIERPIŃSKI-SPONGE ASSOCIATED TO THE SEQUENCE $\mathbf{n}$ is then defined as

$$S_{\mathbf{n}} = \bigcap_{k\geq 0} S_{k,\mathbf{n}}. \tag{A.1}$$

When $d = 2$, we often call these Sierpiński-carpets due to the fact when $\mathbf{n} = (3, 3, 3 \dots)$ the construction yields the usual "middle-thirds" Sierpiński carpet. Indeed, in the plane, all $S_{\mathbf{n}}$ are homeomorphic to this space.

The main results by Mackay, Tyson, and Wildrick [14, Theorem 1.5–1.6], for dimension $d = 2$, imply a characterization when a Sierpiński carpet with the restricted measure satisfies a Poincaré inequality and has positive measure. (A reader may notice that they consider instead a renormalized limit measure, which in the relevant positive measure case is comparable to the restricted measure.) Specifically, the subset satisfies a 1-Poincaré inequality if and only if $\mathbf{n}^{-1} = (\frac{1}{n_i}) \in \ell^1$. Their result also states that the carpet satisfies a p-Poincaré inequality for some (or any) $p > 1$ if and only if $\mathbf{n}^{-2} = (\frac{1}{n_i^2}) \in \ell^1$. The $p > 1$ regime behaves quite differently, and the authors investigated this in more detail in a separate paper [9]. In that paper, there also appears a version of the following theorem for $p > 1$; see [9, Theorem 1.6]. These together fully extend the results [14] to all dimension, as well as to obstacles with different geometries.

The following result is a higher dimensional analogue of the Mackay-Tyson-Wildrick Theorem, for the $p = 1$ case.

Theorem A.2. *Let $\mathbf{n} = (n_i)$ be a sequence of odd integers with $n_i \geq 3$, and let $d \geq 2$. The space $(S_{\mathbf{n}}, |\cdot|, \lambda)$ satisfies a 1-Poincaré inequality if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{n_i^{d-1}} < \infty. \tag{A.3}$$

Proof. We check the various conditions of having a sparse collection of co-uniform domains in $[0, 1]^d$ with small projections and then the claim follows from Theorem 1.3. These can be directly verified with the choices $L = 4(\sqrt{d}r)^{d-1}$, $\delta = \frac{1}{3}$, $\Omega = T_0$ and $A = \frac{1}{6d}$.

Conditions (1), (2): The uniformity and co-uniformity of squares is easy. For example, for the unit cube T_0, if we take $x, y \in T_0 = \Omega$ with $d(x, y) = s$, then we can form γ by first choosing radial paths towards the center of the square c from x and y of length $\min\{s/2, d(x, c)\}$ and $\min\{s/2, d(y, c)\}$, respectively, and then concatenating by a straight line segment. This gives a path of length at most $6s$, which is $\frac{1}{6d}$-uniform.

Condition (3): Each $R \in \mathcal{R}_{\mathbf{n},k}$ has side length s_k, and so diameter at most $\sqrt{d}s_k$.

Conditions (4), (5): If $R \in \mathcal{R}_{\mathbf{n},k}$, then R is a central $1/n_k$ cube of some $T \in \mathcal{T}_{\mathbf{n},k-1}$. Thus the distance to the boundary, or any other higher level obstacle, is at least the distance to the boundary of T, that is at least $\frac{1}{3}s_{k-1}$, since $n_k \geq 3$.

Condition (6): Set $K = 1$. Include $\pi_i(B(x, r) \cap S_{0,\mathbf{n}})$ into a $d-1$ dimensional cube Q_0 of side length at most $4\sqrt{d}r$, which is a union of $d-1$ cubes $\mathcal{Q} = \{Q_i : i = 1, \dots, N\}$, for some N, in the grid of side length s_{k-1}. Each cube Q_i of side length s_{k-1} will include at most one projected cube $\pi_i(R)$ for some $R \in \mathcal{R}_{\mathbf{n},k}$. Such a cube is centered and accounts for at most $\frac{1}{n_k^{d-1}}$ of the volume $\lambda(Q_i)$. Thus, $\lambda\left(\pi_i\left(\bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} \cap Q_i\right)\right) \leq \frac{1}{n_k^{d-1}}\lambda(Q_i)$. The entire volume of the cube Q_0 is at most $(4\sqrt{d}r)^{d-1}$. Summing over all $i = 1, \dots, N$ gives the claim with $L = (4\sqrt{d}r)^{d-1}$.

□

Remark A.4. One can replace the square lattice with a triangular lattice, and perform the removal procedure on a central triangle instead of a central square. The only crucial property one must ensure is that the central triangle does not intersect the boundary of its parent triangle. The triangle lattice also comes with two natural (non-orthogonal) projections and one may verify the conditions of Theorem 1.3 in a similar fashion. This would give a triangular version, depicted in Figure 1, of Theorem A.2. The details are left to the reader.

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