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EQUICONTINUITY OF ITERATES OF CIRCLE MAPS

ANTONIOS VALARISTOS
Department of Mathematics
University of Florida
Gainesville, FL 32611 USA
(antonis@math.ufl.edu)

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ABSTRACT. Let f be a continuous map of the circle to itself. Necessary and sufficient conditions are given for the family of iterates $\{f^n\}_{n=1}^{\infty}$ to be equicontinuous.

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1. INTRODUCTION.

Let $C^0(X,Y)$ denote the set of continuous maps from X to Y, I a closed unit interval and S^1 the circle. Let $f \in C^0(I,I)$ and suppose that the family of iterates of f, i.e. $\{f^n\}_{n=1}^{\infty}$, is equicontinuous. Let F_1 and F_2 denote the fixed point set of f and f^2 respectively. A. M. Bruckner and T. Hu [4] have shown that $\{f^n\}$ is equicontinuous if and only if $F_2 = \cap_{n=1}^{\infty} f^n(I)$. We show that for maps of the circle the following result holds:

THEOREM. Let $f \in C^0(S^1,S^1)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if one of the following holds:

(1) f is conjugate to a rotation.
(2) F_1 consists of exactly two distinct points and every other point on S^1 has period two.
(3) F_1 consists of single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$.
(4) $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$.

2. PRELIMINARIES.

Let $f \in C^0(S^1,S^1)$. We think of the circle S^1 as R/Z and for $x, y \in S^1$ with $x \neq y$ we denote by $[x,y]$ the closed interval from x counterclockwise to y. Let $d(x,y)$ denote the $min\{|[x,y]|, |[y,x]|\}$ where $|[x,y]|$ is the length of the interval $[x,y]$. For any nonnegative integer n define f^n inductively by $f^n = f \circ f^{n-1}$, where f^0 is the identity map on S^1. A point $x \in S^1$ is a periodic point of f if there is a positive integer n such that $f^n(x) = x$. The least such n is called the period of x. A point of period one is called a fixed point. Let F_n denote the fixed point set of $f^n, \forall n \geq 1$ and $P(f)$ the set of periodic points of f.

If $x \in S^1$ then the trajectory of x is the sequence $\gamma(x,f) = \{f^n(x)\}_{n \geq 0}$ and the ω-limit set of x, $\omega(x,f) = \cap_{m \geq 0} \overline{\cup_{n \geq m} f^n(x)}$. Equivalently, $y \in \omega(x,f)$ if and only if y is a limit point of the trajectory $\gamma(x,f)$, i.e. $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$. Let $\mathcal{F} = \{f, f^2, f^3, ...\}$. The family of functions $\mathcal{F}$ is said to be equicontinuous if given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f^i(x), f^i(y)) < \epsilon$ whenever $d(x,y) < \delta$ for all $x, y \in S^1$ and all $i \geq 1$.

The following theorem is proved by J. Cano [5]:

THEOREM A. Let $f \in C^0(I,I)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then F_1 is connected and if it is non-degenerate then $F_1 = P(f)$.

The next theorem which is given in [4] and is due to A. M. Bruckner and Thakyin Hu (only if) and W. Boyce (if):

THEOREM B. Let $f \in C^0(I,I)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\cap_{n=1}^{\infty} f^n(I) = F_2$.

Combining these two theorems we get the following corollary:

COROLLARY. Let $f \in C^0(I,I)$. If f has a periodic point of period $n > 2$, then $\{f^n\}_{n=1}^{\infty}$ cannot be equicontinuous.

3. RESULTS

Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. We consider three cases:

(I) f has a fixed point on S^1.

(II) the smallest period of the periodic points of f on S^1 is $n \geq 2$.

(III) f has no periodic points on S^1.

We start with case (I). The basic result of this case is Theorem 1. We first show the following four lemmas:

LEMMA 1. Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1, and let J be the component of F_2 containing p. If J is either $\{p\}$ or a proper closed interval containing p then there exists an open interval K containing J such that $\omega(x,f) \subseteq J$, for every x in K.

PROOF. First suppose that $J = \{p\}$. Let $\epsilon = |S^1|/4 > 0$. By equicontinuity of $\{f^n\}_{n=1}^{\infty}$ there is an open interval K containing p such that for every x in K and for every $n \geq 1$, $d(f^n(x),p) < \epsilon$. Define $L = \overline{\cap_{j=0}^{\infty} f^j(K)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorems A and B), the fixed point set of $f|L$ and $f^2|L$ is connected and therefore, it is $\{p\}$. Moreover, by the above corollary all periodic points of $f|_L$ have period 1 or 2. But the fixed point p is the only periodic point of f in L. Therefore $P(f) = F_1$ and by [2] the ω-limit points coincide with the fixed points. Hence p is the only ω-limit point of f in L and thus $\omega(x,f) = \{p\} = J$, for every x in L. Since $K \subset L$, $\omega(x,f) = \{p\} = J$, for every x in K.

Now suppose that J is a proper closed interval containing p. Let q_1 and q_2 be the endpoints of J, which are fixed points under f^2. Let $\epsilon = |S^1 - J|/4 > 0$. By equicontinuity of $\{f^n\}_{n=1}^{\infty}$, there is an open interval K_1 around q_1 such that for every x in K_1 and for every $n \geq 1$, $d(f^n(x), f^n(q_1)) < \epsilon$. Similarly there is an open interval K_2 around q_2 such that for every x in K_2 and for every $n \geq 1$, $d(f^n(x), f^n(q_2)) < \epsilon$. Define $L = \overline{\cup_{j=0}^{\infty} f^j(K_1 \cup J \cup K_2)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorems A and B), the fixed point set of $f|L$ and $f^2|L$ is connected and therefore it is J. Moreover, by the Corollary all periodic points of $f|_L$ have period 1 or 2, which we know that lie in J. Since $P(f)$ is closed, by [2], it coincides with the set of ω-limit points. Therefore $\omega(x,f) \subset J$, for every x in L. Let $K = K_1 \cup J \cup K_2$. Then $K \subset L$ and $\omega(x,f) \subset J$, for every x in K. □

LEMMA 2. Let $f \in C^0(S^1,S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1, and let J be the component of F_2 containing p. Define $S = \{x \in S^1 : \omega(x,f) \subseteq J\}$. Then $S = S^1$.

PROOF. There are three cases:

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$.

If (iii) holds then obviously $S = J = S^1$.

Therefore assume (i) and (ii) hold. Then by Lemma 1, there exists an open interval K containing J such that $\omega(x,f) \subseteq J$, for every x in K. Note that S is nonempty since $S \supseteq K$. First we show that S is open: Let $x \in S$. Then $\omega(x, f) \subseteq J$, by definition of S. Choose N large enough such that $f^N(x) \in K$. By continuity of f^N there is a neighborhood U of x such that if $y \in U$ then $f^N(y) \in K$. But then $\omega(f^n(y), f) = \omega(y, f) \subseteq J$ and $y \in S$. Therefore S is open.

Let T be the component of S containing J and therefore K, as well. Then T is open and connected. We will show that $T = S^1$. Suppose $T \neq S^1$. Then $S^1 - T$ is a closed interval or a point. Let $J = [q_1, q_2]$ where possibly $q_1 = q_2 = p$.

Suppose first that $S^1 - T$ is a closed interval. Let z_1 and z_2 be the endpoints of this closed interval such that $[z_2, z_1] \cap J = \emptyset$. Let $\epsilon = \frac{1}{2} min\{d(q_1, z_1), d(q_2, z_2)\}$. By equicontinuity of $\{f^n\}$ at z_1, there is an open interval V_1 around z_1 such that for every x in V_1 and for every $n \geq 1$, $d(f^n(x), f^n(z_1)) < \epsilon$. Let $x \in T$ such that $d(x, z_1) < \epsilon$. Since $\omega(x, f) \subseteq J$ and the orbit of z_1 stays by definition out of T, there exists a positive integer k such that $d(f^k(x), f^k(z_1)) > \epsilon$, which is a contradiction.

Now suppose that $S^1 - T = \{z\}$. Let $\epsilon = \frac{1}{2} min\{d(z, q_1), d(q_2, z)\}$. By equicontinuity of $\{f^n\}$ at z there is an open interval V around z such that for every x in V and for every $n \geq 1$, $d(f^n(x), f^n(z)) < \epsilon$. Since $\omega(x, f) \subseteq J$ for every $x \in T$ and $f(z) = z$ is a fixed point of f, we get a contradiction.

Hence $T = S^1$. Thus $S = \{x \in S^1 : \omega(x, f) \subset J\} = S^1$. □

LEMMA 3. Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$ then F_1 cannot consist of exactly one point.

PROOF. Suppose that there is an $f \in C^0(S^1, S^1)$ such that $F_2 = S^1$ and $F_1 = \{p\}$. Let z be a point on $S^1 - \{p\}$ of period two. Let K be the closed interval with endpoints z and $f(z)$ which contains p and let L be the closed interval with the same endpoints that does not contain p. Since f is a homeomorphism, we have two cases:

(i) $f(K) = K$ and $f(L) = L$ or (ii) $f(K) = L$ and $f(L) = K$.

If (i) holds then, since $f(L) = L$, there would be another fixed point of f in L, which is a contradiction since $F_1 \subset K$.

If (ii) holds then $f(K) = L$ implies that p cannot be a fixed point which is again a contradiction. □

LEMMA 4. Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$ and F_1 consists of more than two distinct points then f is the identity on S^1.

PROOF. Assume that F_1 consists of exactly $k > 2$ distinct fixed points $p_1, p_2, ..., p_k$. Let $L_i = [p_i, p_{i+1}]$ for $i = 1, 2, ..., k-1$ and $L_k = [p_k, p_1]$ so that the interior of each L_i does not contain any fixed points. Then we have two cases: (i) $f(L_i) = L_i$ and (ii) $f(L_i) = S^1 - L_i$.

If (i) holds then pick x in the interior of L_i. Note that $f(x)$ is a point in the interior of L_i and denote by M_x the closed interval with endpoints x and $f(x)$ which is free of fixed points. If $f(M_x) = M_x$ then there would be another fixed point in M_x contradicting that the interior of L_i contains no fixed points. Thus the only choice is $x = f(x)$ for every $x \in L_i$ and $f|L_i$ is the identity map. The same argument applied to every L_i shows that f is the identity map on S^1.

If (ii) holds then there are points in L_i which map onto the other fixed points contradicting that $F_2 = S^1$. □

THEOREM 1. Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that there is a fixed point p on S^1. Then f has periodic points of period at most two and F_2 is connected. Furthermore F_1 is either connected or it consists of exactly two distinct points and every other point on S^1 has period two. Moreover if F_1 is a nondegenerate interval then $F_1 = P(f)$.

PROOF. Let J be the component of F_2 containing p. There are three cases:

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$.

Assume that (i) holds. Then, by Lemma 2, $\omega(x, f) - \{p\}$ for every $x \in S^1$. Thus the fixed point p is the only periodic point of f on S^1 and hence $P(f) = F_1 = F_2 = \{p\}$ is connected.

Assume that (ii) holds. Then, by Lemma 2, $\omega(x, f) \subseteq J$ for every $x \in S^1$ and the periodic points of f on S^1 lie in J. By results on the interval applied to $f|J$, either p is the unique fixed point of f on S^1 or

F_1 is a nondegenerate interval and the fixed points are the only periodic points of f on S^1. In particular, both F_1 and F_2 are connected.

Assume that (iii) holds. Then all of the points of S^1 are periodic with period 1 or 2 and F_2 is connected. By Lemma 3, F_1 cannot consist of one point and by Lemma 4 if F_1 consists of more than two points then f is the identity map. Otherwise F_1 consists of exactly two distinct points and every other point on S^1 has period two. □

We now investigate case (II) where the smallest period of the periodic points of f on S^1 is $n \geq 2$. The main result here is Theorem 2. We use Lemma 5 in the proof of the main theorem.

THEOREM 2. Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Suppose that the smallest period of the periodic points of f on S^1 is $n \geq 2$. Then every point on S^1 is periodic with period n.

PROOF. Let p be a periodic point of period n on S^1. Then $f^n(p) = p$ and therefore p is a fixed point of f^n. Applying Theorem 1 to f^n, we conclude that F_n is either connected or it consists of exactly two distinct points and every other point on S^1 has period $2n$.

We claim that there is no continuous map of the circle having two points of period two and every other point periodic of period four. Otherwise, if g is such a map, let $p, g(p)$ be the two points of period two and let $K = [p, g(p)]$ and $L = [g(p), p]$. Since g is a homeomorphism, we have two cases: (i) $g(K) = K$ and $g(L) = L$ or (ii) $g(K) = L$ and $g(L) = K$. In both cases $g^2(K) = K$. Hence if $x \in K$ is a point of period four then $g^2(x) \neq x$ and $g^2(x) \in K$. If M is the closed interval with endpoints x and $g^2(x)$ lying in K then $g^2(M) = M$. Therefore M contains a periodic point of period two, contradicting the assumption that p and $g(p)$ are the only points of period two and every other point has period four.

Hence F_n is connected. Suppose that $F_n \neq S^1$. Then F_n is a proper closed interval containing the orbit of p under f. Moreover $f(F_n) \subset F_n$. This implies that f has a fixed point on S^1, contradicting the hypothesis that the smallest possible period of the periodic points is $n > 1$. Hence $F_n = S^1$. □

For a proof of the following see [7].

LEMMA 5. Let $f \in C^0(S^1, S^1)$. Suppose that there exists a positive integer $n \geq 2$ such that every point on S^1 is periodic with period n. Then f is conjugate to a rational rotation.

Now we consider case (III) where $f \in C^0(S^1, S^1)$ has no periodic points and $\{f^n\}$ is equicontinuous. The main result here is listed in Theorem 3.

Note that f must be onto, since otherwise $f(S^1) = I$ is homeomorphic to a closed interval and $f(I) \subset I$, so f has a fixed point. We shall adapt the techniques and use results due to J. Auslander and Y. Katznelson [1].

Let $x \in S^1$. In [1], J_x is defined to be the largest interval containing x such that $f^m(x) \notin J_x, \forall m \geq 1$. Denote by z_1 and z_2 the endpoints of J_x, where possibly $z_1 = z_2 = x$. The following are showed in [1]: J_x is closed and $z_1, z_2 \neq f^k(x)$ for $k \geq 1$. If $x, y \in S^1$ then $y \in \omega(x, f)$ if and only if y is an endpoint of J_y. If z_1 and z_2 are the endpoints of J_x, then $f(z_1)$ and $f(z_2)$ are the endpoints of $f(J_x)$. Also $f(J_x) \cap J_x = \emptyset$ and $f^m(J_x) = J_{f^m(x)}$, $\forall m \geq 1$. The intervals $J_{f^m(x)}$ $(m = 0, 1, 2, \ldots)$ are pairwise disjoint and if $f(x) = f(x')$, then $J_x = J_{x'}$. The sets $\{J_x\}$ form a partition of S^1 (that is, if $x, y \in S^1$ then $J_x = J_y$ or $J_x \cap J_y = \emptyset$ and $\cup_{x \in S^1} J_x = S^1$). Finally, at most countably many of the sets J_x are non-degenerate $(J_x \neq \{x\})$.

Before we show our result, we state the following theorem proved in [6] which concerns homeomorphisms.

THEOREM C. Let f be an orientation preserving homeomorphism of S^1 to itself. For $x \in S^1$, let $R_\alpha(x) = x + \alpha \ (mod\ 1)$ denote irrational rotation by α. Then f is conjugate to some R_α if and only if some (all) orbits of f are dense on S^1.

We are now ready to show the following:

THEOREM 3. Let $f \in C^0(S^1, S^1)$ without periodic points and such that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then f is conjugate to an irrational rotation R_α.

PROOF. We first show that $\omega(x, f) = S^1$ for all $x \in S^1$. Since $y \in \omega(x, f)$ if and only if y is an endpoint of J_y, it suffices to show that $\forall y \in S^1$, $J_y = \{y\}$. By the way of contradiction assume that J_{y0} is a non-degenerate interval. Since f is onto, there exists $y_1 \in S^1$ such that $f(J_{y1}) = J_{y0}$. Continuing in this way, we obtain a sequence of intervals $\{J_{yn}\}$ such that $f(J_{yn}) = J_{yn-1}\, \forall n \geq 1$. Since there are countably many non-degenerate such intervals on S^1, $lim_{k\to\infty}|f^{-k}(J_{y0})| = 0$. Hence f^k maps arbitrarily small intervals onto J_{y0} (as $k \to \infty$) which contradicts equicontinuity. Therefore $\omega(x, f) = S^1$ for all $x \in S^1$.

This is equivalent to saying that all orbits of f are dense in S^1. If $f(y) = f(y')$, then $J_y = J_{y'}$ and hence f is a homeomorphism. By Theorem C it follows immediately that f is conjugate to an irrational rotation R_α.

4. PROOF OF THEOREM

We first state the following three lemmas which can be shown to hold on any compact metric space.

LEMMA 6. Let $f, g \in C^0(X, X)$, where X is a compact metric space. Suppose that f is conjugate in X to g and that $\{g^n\}_{n=1}^{\infty}$ is equicontinuous. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

LEMMA 7. Let $f \in C^0(X, X)$, where X is a compact metric space. Let k be a positive integer and $g = f^k$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\{g^n\}_{n=1}^{\infty}$ is equicontinuous.

LEMMA 8. Let $f \in C^0(X, X)$, where X is a compact metric space. If $\{(f|_{f(X)})^n\}_{n=1}^{\infty}$ is equicontinuous then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Finally, we summarize the results to the following theorem.

THEOREM. Let $f \in C^0(S^1, S^1)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if one of the following holds:

(1) f is conjugate to a rotation.

(2) F_1 consists of exactly two distinct points and every other point on S^1 has period two.

(3) F_1 consists of a single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$.

(4) $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$.

PROOF. We suppose that $\{f^n\}$ is equicontinuous. First assume that $F_1 = \emptyset$. If f has no periodic points on S^1, then by Theorem 3, f is conjugate to an irrational rotation, so that (1) holds.

If the smallest period of the periodic points of f on S^1 is $n \geq 2$, then By Theorem 2, every point on S^1 is periodic with period n. It follows by Lemma 5 that f is conjugate to a rational rotation, so that (1) holds again.

Now assume that $F_1 \neq \emptyset$. Then by Theorem 1, f has periodic points of period at most two. If F_1 is not connected, then by Theorem 1 it consists of exactly two distinct points and every other point of S^1 has period two, so that (2) holds.

If F_1 is connected then it consists of (i) a single point, (ii) a proper interval or (iii) the whole circle.

(i) First assume that F_1 consists of a single point p. Note that by Theorem1, F_2 is a connected proper interval of S^1. Moreover by Lemma 2, we have that for every $x \in S^1$, $\omega(x, f) \subseteq F_2$. As in the proof of Lemma 1, there exists an open interval K containing F_2 such that if $L = \overline{\cup_{j=0}^{\infty} f^j(K)}$ then L is a proper interval. Of course L is also closed and invariant. For $x \in S^1$, since $\omega(x, f) \subseteq F_2 \subset K \subset L$, there exists a positive integer N such that $f^N(x) \in K$. Then $f^m(x) \in L$ for every $m \geq N$. By continuity of f^N there exists an open neighborhood V_x of x such that $f^N(V_x) \in K$ and hence $f^m(V_x) \in L$ for every $m \geq N$. Note that for each $x \in S^1$ the collection $\{V_x\}_{x\in S^1}$ forms an open cover of S^1. By compactness of S^1 there exists a finite subcover, which we denote by $\{V_i\}_{i=1,..l}$. Consequently, for every V_i there exists a positive integer N_i such that $f^{N_i}(V_i) \subset K$, for $i = 1, 2, ...l$

and $f^{m_i}(V_i) \subset L$, for every $m_i \geq N_i$ and for $i = 1, 2, ...l$. Choose $N = max\{N_1, ..., N_l\}$. Then $f^m(V_i) \subset L$, for every $m \geq N$ and $i = 1, 2, ...l$. Thus $f^m(S^1) \subset L$ for every $m \geq N$. By Theorem B, $\cap_{n=1}^{\infty} f^n(L) = F_2$. Since $f^m(S^1) \subset L$, for every $m \geq N$, it follows that $\cap_{n=1}^{\infty} f^n(L) = \cap_{n=1}^{\infty} f^n(S^1) = F_2$. Hence (3) holds.

(ii) Now assume that F_1 is a proper interval of S^1. We know by Theorem 1, that F_1 coincides with the set of periodic points of f. By an argument similar to the above applied to F_1, we can see that $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$ and hence (4) holds.

(iii) If $F_1 = S^1$ then obviously (4) holds again.

This concludes one direction of the proof, namely that if $\{f^n\}_{n=1}^{\infty}$ is equicontinuous then one of (1), (2), (3) or (4) holds. Now we will show that all of these four cases imply that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Suppose that (1) holds i.e. f is conjugate to a rotation R. Then R^n is an isometry for every $n \geq 1$ and therefore $\{R^n\}$ is equicontinuous. It follows by Lemma 6 that $\{f^n\}$ is equicontinuous as well.

Suppose that (2) holds i.e. F_1 consists of exactly two distinct points and every other point on S^1 has period two. Then f^2 is the identity on S^1. Therefore $\{f^{2n}\}$ is equicontinuous. It follows by Lemma 7 that $\{f^n\}$ is equicontinuous as well.

Suppose that (3) holds i.e. F_1 consists of a single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$. Then $f(S^1) \neq S^1$, since otherwise $F_2 = S^1$ and we have seen in Lemma 3 that there is no continuous map of the circle with one fixed point and every other point of period two. Hence $f(S^1)$ is a proper interval of S^1 and $f|_{f(S^1)} : f(S^1) \rightarrow f(S^1)$ is a continuous map of the interval with fixed point set of $(f|_{f(S^1)})^2$ equal to F_2. Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n (f(S^1)) = F_2$, it follows by Theorem B, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 8 we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Finally suppose that (4) holds i.e. $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$. If $S^1 = f(S^1)$ then $F_1 = S^1$ and the identity map is equicontinuous. If $f(S^1)$ is a proper interval of S^1 then F_1 is a point or a proper interval of S^1. It follows that $f|_{f(S^1)} : f(S^1) \rightarrow f(S^1)$ is a continuous map of the interval such that its fixed point set equals the fixed point set of f on S^1. Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n (f(S^1)) = F_1$, it follows by Theorem B, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 8 we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. □

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