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Research Article

On Integral Inequalities Involving Generalized Lipschitzian Functions

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A new class of mappings that includes the class of Lipschitzian mappings is introduced. For this kind of mappings, new integral inequalities of Hadamard's type are obtained. Our results are extensions of many previous contributions related to integral inequalities for Lipschitzian mappings.

1. Introduction

A function $\omega : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is said to be L-Lipschitzian, where $L > 0$, if

$$|\omega(\sigma) - \omega(\kappa)| \leq L|\sigma - \kappa|, \tag{1}$$

for all $\sigma, \kappa \in \mathbb{I}$. In [1, 2], some integral inequalities of Hadamard's type involving L-Lipschitzian functions were derived. In particular, it was shown that if ω is L-Lipschitzian in $\mathbb{I} = [\alpha, \beta]$, $\alpha < \beta$, then

$$\left|\frac{1}{\beta - \alpha}\int_\alpha^\beta \omega(\sigma)\, d\sigma - \omega\left(\frac{\alpha + \beta}{2}\right)\right| \leq \frac{L(\beta - \alpha)}{4}, \tag{2}$$

$$\left|\frac{1}{\beta - \alpha}\int_\alpha^\beta \omega(\sigma)\, d\sigma - \frac{\omega(\alpha) + \omega(\beta)}{2}\right| \leq \frac{L(\beta - \alpha)}{4}. \tag{3}$$

In [3], some Hadamard-type and Bullen-type inequalities were obtained for the same class of functions. We recall below the main results established in [3]. Let ω be L-Lipschitzian in $\mathbb{I} = [\alpha, \beta]$.

(i) Let $M, N \in \mathbb{I}$ with $M \leq N$. Let $\xi \in [0, 1]$, $\mathscr{W} = (1 - \xi)\alpha + \xi\beta$, and

$$\begin{aligned}\mathscr{W}_\xi(M, N) = {} & (M - \alpha)^2 + (\beta - N)^2 + \operatorname{sgn}\,(N - \mathscr{W})(N - \mathscr{W})^2 \\ & + \operatorname{sgn}\,(W - M)(\mathscr{W} - M)^2,\end{aligned} \tag{4}$$

where sgn denotes the signum function. Then,

$$\left|\xi\omega(M) + (1 - \xi)\omega(N) - \frac{1}{\beta - \alpha}\int_\alpha^\beta \omega(\sigma)\, d\sigma\right| \leq \frac{L\mathscr{W}_\xi(M, N)}{2(\beta - \alpha)} \tag{5}$$

Notice that if $\xi = 1/2$ and $M = N = (\alpha + \beta)/2$, then (5) reduces to (2). Moreover, if $\xi = 1/2$ and $(M, N) = (\alpha, \beta)$, then (5) reduces to (3).

(ii) Let $P, Q, R \in \mathbb{I}$, $P \leq Q \leq R$ and $\{\xi_i\}_{i=1}^3 \subset [0, 1]$, where $\xi_1 + \xi_2 + \xi_3 = 1$. Let $\mathscr{W}_1 = (1 - \xi_1)\alpha + \xi_1\beta$, $\mathscr{W}_2 = \xi_3\alpha + (\xi_1 + \xi_2)\beta$ and

$$\begin{aligned}\mathscr{W}_{\xi_1,\xi_2,\xi_3}(P, Q, R) = {} & (P - \alpha)^2 + \operatorname{sgn}\,(\mathscr{W}_1 - P)(\mathscr{W}_1 - P)^2 \\ & + \operatorname{sgn}\,(Q - \mathscr{W}_1)(Q - \mathscr{W}_1)^2 \\ & + \operatorname{sgn}\,(\mathscr{W}_2 - Q)(\mathscr{W}_2 - Q)^2 \\ & + \operatorname{sgn}\,(R - \mathscr{W}_2)(R - \mathscr{W}_2)^2 + (\beta - R)^2\end{aligned} \tag{6}$$

Then,

$$\left|\xi_1\omega(P)+\xi_2\omega(Q)+\xi_3\omega(R)-\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(\sigma)\,d\sigma\right|\le\frac{L\mathscr{W}_{\xi_1,\xi_2,\xi_3}(P,Q,R)}{2(\beta-\alpha)}. \tag{7}$$

Notice that if $(P,Q,R)=(\alpha,(\alpha+\beta)/2,\beta)$ and $(\xi_1,\xi_2,\xi_3)=(1/4,1/2,1/4)$, then (7) reduces to the Bullen-type inequality

$$\left|\frac{1}{2}\left[\omega\left(\frac{\alpha+\beta}{2}\right)+\frac{\omega(\alpha)+\omega(\beta)}{2}\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(\sigma)\,d\sigma\right|\le\frac{L(\beta-\alpha)}{8}. \tag{8}$$

Moreover, if $(P,Q,R)=(\alpha,(\alpha+\beta)/2,\beta)$ and $(\xi_1,\xi_2,\xi_3)=(1/6,2/3,1/6)$, then (7) reduces to the Bullen-type inequality (see [4])

$$\left|\frac{1}{6}\left[4\omega\left(\frac{\alpha+\beta}{2}\right)+\omega(\alpha)+\omega(\beta)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(\sigma)\,d\sigma\right|\le\frac{5L(\beta-\alpha)}{36}. \tag{9}$$

For other works related to integral inequalities for Lipschitzian functions, see, for example, [5–11] and the references therein.

Motivated by the above mentioned results, in this paper, we obtain some Hadamard-type integral inequalities for a new class of functions which includes the class of Lipschitzian functions.

2. The Class of Generalized Lipschitzian Functions

Let $\omega:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ be a given function and $\sigma_0\in\mathbb{I}$ be fixed.

$$|\omega(\sigma)-\omega(\kappa)|\le L(|\sigma-\sigma_0|+|\kappa-\sigma_0|), \tag{10}$$

for all $\sigma,\kappa\in\mathbb{I}$.

(i) ω is (L,σ_0)-Lipschitzian

(ii) For all $\sigma\in\mathbb{I}$,

$$|\omega(\sigma)-\omega(\sigma_0)|\le L|\sigma-\sigma_0| \tag{11}$$

$$\begin{aligned}|\omega(\sigma)-\omega(\kappa)|&\le|\omega(\sigma)-\omega(\sigma_0)|+|\omega(\kappa)-\omega(\sigma_0)|\\&\le L|\sigma-\sigma_0|+L|\kappa-\sigma_0|=L(|\sigma-\sigma_0|+|\kappa-\sigma_0|),\end{aligned} \tag{12}$$

which shows that ω is (L,σ_0)-Lipschitzian.

Definition 1. We say that ω *is* (L,σ_0) *-Lipschitzian, where* $L>0$ *, if*

Proposition 2. *Let $L>0$. The following statements are equivalent:*

Proof. Suppose that ω is (L,σ_0)-Lipschitzian. Taking $\kappa=\sigma_0$ in (10), (ii) follows. Suppose now that ω satisfies (11). Then, for all $\sigma,\kappa\in\mathbb{I}$,

Remark 3. It can be easily seen that if ω *is* L -Lipschitzian, then ω is (L,σ) -Lipschitzian, for all $\sigma\in\mathbb{I}$.

Let us denote by $\mathscr{L}(\mathbb{I})$ the set of functions $\omega:\mathbb{I}\to\mathbb{R}$ such that there exists $L>0$ for which ω is L-Lipschitzian. Moreover, we denote by $\widetilde{\mathscr{L}(\mathbb{I})}$ the set of functions $\omega:\mathbb{I}\to\mathbb{R}$ such that there exist $L>0$ and $\sigma_0\in\mathbb{I}$ for which ω is (L,σ_0) -Lipschitzian. The following example shows that

$$\mathscr{L}(\mathbb{I})\mathbb{O}\widetilde{\mathscr{L}(\mathbb{I})}. \tag{13}$$

$$\omega(\sigma)=\begin{cases}\sigma & \text{if } 0\le\sigma\le\dfrac{1}{4},\\ \dfrac{1}{2}-\sigma & \text{if } \dfrac{1}{4}\le\sigma\le\dfrac{1}{2},\\ \sigma-\dfrac{1}{2} & \text{if } \dfrac{1}{2}\le\sigma\le\dfrac{3}{4},\\ \sigma-\dfrac{3}{4} & \text{if } \dfrac{3}{4}<\sigma\le 1.\end{cases} \tag{14}$$

Example 1. Let $\omega:\mathbb{I}=[0,1]\to\mathbb{R}$ be the function defined by

One observes that ω is not continuous at $\sigma=3/4$, which shows that $\omega\in\mathscr{L}(\mathbb{I})$. On the other hand, one has

$$\left|\omega(\sigma)-\omega\left(\frac{1}{2}\right)\right|=|\omega(\sigma)|=\omega(\sigma)\le\left|\sigma-\frac{1}{2}\right|, \tag{15}$$

for all $\sigma\in\mathbb{I}$, which shows that ω is $(1,1/2)$-Lipschitzian. Hence, $\omega\in\widetilde{\mathscr{L}(\mathbb{I})}$.

3. Inequalities of Hadamard's Type

We first fix some notations that will be used throughout this section. Let $\omega:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ be (L,σ_0)-Lipschitzian, where $\mathbb{I}=[\alpha,\beta]$, $\alpha<\beta$, $L>0$, and $\sigma_0\in\mathbb{I}$. We denote by μ_{σ_0} the quantity defined by

$$\mu_{\sigma_0}=\left(\sigma_0-\frac{\alpha+\beta}{2}\right)^2+\left(\frac{\beta-\alpha}{2}\right)^2. \tag{16}$$

$$\left|\sum_{i=1}^n\xi_i\omega(M_i)-\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(\sigma)\,d\sigma\right|\le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\sum_{i=1}^n\xi_i\,|\,M_i-\sigma_0\,|\right). \tag{17}$$

$$\begin{aligned}\left|\sum_{i=1}^n\xi_i\omega(M_i)-\frac{1}{\beta-\alpha}\int_\alpha^\beta\omega(\sigma)\,d\sigma\right|&=\frac{1}{\beta-\alpha}\left|\sum_{i=1}^n\xi_i\int_\alpha^\beta[\omega(M_i)-\omega(\sigma)]\,d\sigma\right|\\&\le\frac{1}{\beta-\alpha}\sum_{i=1}^n\xi_i\int_\alpha^\beta|\omega(M_i)-\omega(\sigma)|d\sigma.\end{aligned} \tag{18}$$

Theorem 4. *Let $n \in \mathbb{N}$ (n is a positive integer), $\{M_i\}_{i=1}^n \subset \mathbb{I}$, and $\{\xi_i\}_{i=1}^n \subset [0,1]$ be such that $\sum_{i=1}^n \xi_i = 1$. Then,*

Proof. Since $\sum_{i=1}^n \xi_i = 1$, one has

Hence, by the assumption on ω, one obtains

$$\begin{aligned}\left|\sum_{i=1}^n \xi_i \omega(M_i) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| &\le \frac{L}{\beta-\alpha}\sum_{i=1}^n \xi_i \int_\alpha^\beta (|M_i-\sigma_0|+|\sigma-\sigma_0|)\,d\sigma \\ &= \frac{L}{\beta-\alpha}\sum_{i=1}^n \xi_i\left(|M_i-\sigma_0|(\beta-\alpha)+\int_\alpha^\beta |\sigma-\sigma_0|\,d\sigma\right),\end{aligned} \tag{19}$$

which yields

$$\begin{aligned}\left|\sum_{i=1}^n \xi_i \omega(M_i) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ \le L\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta |\sigma-\sigma_0|\,d\sigma + \sum_{i=1}^n \xi_i |M_i-\sigma_0|\right).\end{aligned} \tag{20}$$

On the other hand, an elementary calculation shows that

$$\int_\alpha^\beta |\sigma-\sigma_0| d\sigma = \mu_{\sigma_0} \tag{21}$$

Therefore, combining (20) with (21), (17) follows.

We investigate below some particular cases of Theorem 4.

$$\begin{aligned}\left|\xi\omega(M_1)+(1-\xi)\omega(M_2) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha} + \xi|M_1-\sigma_0|+(1-\xi)|M_2-\sigma_0|\right).\end{aligned} \tag{22}$$

$$\begin{aligned}\left|\xi\omega(\rho\alpha+(1-\rho)\beta)+(1-\xi)\omega(\rho\alpha+(1-\rho)\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha} + \xi|\rho\alpha+(1-\rho)\beta-\sigma_0|+(1-\xi)|(1-\rho)\alpha+\rho\beta-\sigma_0|\right).\end{aligned} \tag{23}$$

$$\left|\omega(M) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+|M-\sigma_0|\right). \tag{24}$$

$$\left|\omega(\rho\alpha+(1-\rho)\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+|\rho\alpha+(1-\rho)\beta-\sigma_0|\right). \tag{25}$$

Corollary 5. *Let $\xi \in [0,1]$ and $M_1, M_2 \in \mathbb{I}$. Then,*

Proof. In Theorem 4, let $n=2$, $\xi_1=\xi$ and $\xi_2 = 1-\xi$. Then, by (17), one obtains (22).

Corollary 6. *Let $\xi, \rho \in [0,1]$. Then,*

Proof. In Corollary 5, let $M_1 = \rho\alpha+(1-\rho)\beta$ and $M_2 = \rho\alpha + (1-\rho)\beta$. Then, by (22), one obtains (23).

Corollary 7. *Let $M \in \mathbb{I}$. Then,*

Proof. In Corollary 5, let $M_1 = M_2 = M$. Then, by (22), one obtains (24).

Corollary 8. *Let $\rho \in [0,1]$. Then,*

Proof. In Corollary 7, let $M = \rho\alpha + (1-\rho)\beta$. Then, by (24), one obtains (25).

Corollary 9. *We have*

$$\left|\omega\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\left|\frac{\alpha+\beta}{2}-\sigma_0\right|\right). \tag{26}$$

Proof. Taking $\rho = 1/2$ in (25), one obtains (26).

Corollary 10. *We have*

$$\left|\omega(\sigma_0) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le \frac{L\mu_{\sigma_0}}{\beta-\alpha}. \tag{27}$$

Proof. In Corollary 7, let $M = \sigma_0$. Then, by (24), one obtains (27).

Corollary 11. *Let $\sigma_0 = (\alpha+\beta)/2$, that is, ω is $(L, (\alpha+\beta)/2)$ -Lipschitzian. Then,*

$$\left|\omega\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le \frac{L(\beta-\alpha)}{4}. \tag{28}$$

Proof. By (16), for $\sigma_0 = (\alpha+\beta)/2$, one has

$\mu_{\sigma_0} = (\beta-\alpha/2)^2$.(14)(52)

Then, by (27), one obtains (28).

Remark 12. Corollary 11 shows that the inequality (2) which was obtained in [1, 2] for the class of L-Lipschitzian functions still holds for $(L, (\alpha+\beta)/2)$ -Lipschitzian functions.

Corollary 13. *Let $\xi \in [0,1]$. Then,*

$$\begin{aligned}\left|\xi\omega(\alpha)+(1-\xi)\omega(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha} + (2\xi-1)\sigma_0 - \xi(\alpha+\beta)+\beta\right).\end{aligned} \tag{29}$$

Proof. In Corollary 5, taking $M_1 = \alpha$ and $M_2 = \beta$, (29) follows.

Corollary 14. *We have*

$$\left|\frac{\omega(\alpha)+\omega(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\frac{\beta-\alpha}{2}\right). \tag{30}$$

Proof. In (29), taking $\xi = 1/2$, (30) follows.

Corollary 15. *Let $\sigma_0 = (\alpha+\beta)/2$, that is, ω is $(L,(\alpha+\beta)/2)$ -Lipschitzian. Then,*

$$\left|\frac{\omega(\alpha)+\omega(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le \frac{3L(\beta-\alpha)}{4}. \tag{31}$$

Proof. In (30), taking $\sigma_0 = (\alpha+\beta)/2$, (31) follows.

Corollary 16. *Let $M_1, M_2, M_3 \in \mathbb{I}$, and $\xi_1, \xi_2, \xi_2 \in [0,1]$ be such that $\xi_1+\xi_2+\xi_3 = 1$. Then,*

$$\begin{aligned}&\left|\xi_1\omega(M_1)+\xi_2\omega(M_2)+\xi_3\omega(M_3)-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ &\quad\le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\xi_1\,|\,M_1-\sigma_0|+\xi_2\,|\,M_2-\sigma_0|+\xi_3\,|\,M_3-\sigma_0\,|\right).\end{aligned} \tag{32}$$

Proof. In Theorem 4, let $n = 3$. Then, by (17), one obtains (32).

Corollary 17. *Let $\delta \in [0,1]$. Then,*

$$\begin{aligned}&\left|\delta\left(\frac{\omega(\alpha)+\omega(\beta)}{2}\right)+(1-\delta)\omega\left(\frac{\alpha+\beta}{2}\right)-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ &\quad\le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\delta\left(\frac{|\alpha-\sigma_0|+|\beta-\sigma_0\,|}{2}\right)+(1-\delta)\left|\frac{\alpha+\beta}{2}-\sigma_0\right|\right).\end{aligned} \tag{33}$$

Proof. In Corollary 16, let $(M_1, M_2, M_3) = (\alpha, (\alpha+\beta)/2, \beta)$ and $(\xi_1, \xi_2, \xi_3) = (\delta/2, 1-\delta, \delta/2)$. Then, using (32), (33) follows.

Corollary 18. *We have*

$$\begin{aligned}&\left|\frac{1}{4}\left[\omega(\alpha)+\omega(\beta)+2\omega\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ &\quad\le L\left[\frac{\mu_{\sigma_0}}{\beta-\alpha}+\frac{1}{4}\left(|\alpha-\sigma_0|+|\beta-\sigma_0|+2\left|\frac{\alpha+\beta}{2}-\sigma_0\right|\right)\right].\end{aligned} \tag{34}$$

Proof. In (33), taking $\delta = 1/2$, (34) follows.

Corollary 19. *Let $\sigma_0 = (\alpha+\beta)/2$, that is, ω is $(L,(\alpha+\beta)/2)$ -Lipschitzian. Then,*

$$\left|\frac{1}{4}\left[\omega(\alpha)+\omega(\beta)+2\omega\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le \frac{L(\beta-\alpha)}{2}. \tag{35}$$

Proof. In (34), taking $\sigma_0 = (\alpha+\beta)/2$, (35) follows.

Corollary 20. *We have*

$$\begin{aligned}&\left|\frac{1}{6}\left[\omega(\alpha)+\omega(\beta)+4\omega\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \\ &\quad\le L\left(\frac{\mu_{\sigma_0}}{\beta-\alpha}+\frac{|\alpha-\sigma_0|+|\beta-\sigma_0\,|}{6}+\frac{2}{3}\left|\frac{\alpha+\beta}{2}-\sigma_0\right|\right).\end{aligned} \tag{36}$$

Proof. In (33), taking $\delta = 1/3$, (36) follows.

Corollary 21. *Let $\sigma_0 = (\alpha+\beta)/2$, that is, ω is $(L,(\alpha+\beta)/2)$ -Lipschitzian. Then,*

$$\left|\frac{1}{6}\left[\omega(\alpha)+\omega(\beta)+4\omega\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{1}{\beta-\alpha}\int_\alpha^\beta \omega(\sigma)\,d\sigma\right| \le \frac{5L(\beta-\alpha)}{12}. \tag{37}$$

For the next result, let $\mathscr{V}$ be the set of functions $\vartheta : \mathbb{I} \to \mathbb{R}$ satisfying the following conditions:

(P) $\vartheta \in C^1(\mathbb{I})$.

(P) $\vartheta'(\sigma) > 0$, for all $\sigma \in \mathbb{I}$.

Notice that under the above conditions, the function $\vartheta : \mathbb{I} = [\alpha,\beta] \to [\vartheta(\alpha), \vartheta(\beta)]$ is invertible. We denote by ϑ^{-1} its inverse.

Theorem 22. *Let $\vartheta \in \mathscr{V}$. Then, for all $\sigma \in \mathbb{I}$,*

$$\begin{aligned}&\left|\omega(\sigma)-\frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_\alpha^\beta \omega(\kappa)\vartheta'(\kappa)\,d\kappa\right| \\ &\quad\le L\left(|\sigma-\sigma_0|+\frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\vartheta(\alpha)}^{\vartheta(\beta)}|\,\vartheta^{-1}(\tau)-\sigma_0\,|\,d\tau\right).\end{aligned} \tag{38}$$

Proof. Taking $\tau = \vartheta(\kappa)$, one obtains

$$\int_\alpha^\beta \omega(\kappa)\vartheta'(\kappa)\,d\kappa = \int_{\vartheta(\alpha)}^{\vartheta(\beta)} \omega\left(\vartheta^{-1}(\tau)\right)d\tau. \tag{39}$$

On the other hand, for $\sigma \in \mathbb{I}$, one has

$$\omega(\sigma) = \frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\vartheta(\alpha)}^{\vartheta(\beta)} \omega(\sigma)\,d\tau. \tag{40}$$

Combining (39) with (40), it holds that

$$\begin{aligned}&\left|\omega(\sigma)-\frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\alpha}^{\beta}\omega(\kappa)\vartheta'(\kappa)\,d\kappa\right|\\&\quad\leq\frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\vartheta(\alpha)}^{\vartheta(\beta)}\left|\omega(\sigma)-\omega\left(\vartheta^{-1}(\tau)\right)\right|d\tau.\end{aligned}\tag{41}$$

Next, using the assumption on ω, one obtains

$$\left|\omega(\sigma)-\omega\left(\vartheta^{-1}(\tau)\right)\right|\leq L\left(|\sigma-\sigma_0|+|\vartheta^{-1}(\tau)-\sigma_0|\right),\tag{42}$$

for all $\tau\in[\vartheta(\alpha),\vartheta(\beta)]$, which yields

$$\begin{aligned}&\int_{\vartheta(\alpha)}^{\vartheta(\beta)}\left|\omega(\sigma)-\omega\left(\vartheta^{-1}(\tau)\right)\right|d\tau\\&\quad\leq L\left((\vartheta(\beta)-\vartheta(\alpha))\,|\sigma-\sigma_0|+\int_{\vartheta(\alpha)}^{\vartheta(\beta)}|\vartheta^{-1}(\tau)-\sigma_0|\,d\tau\right).\end{aligned}\tag{43}$$

Finally, combining (41) and (43), (38) follows.
We investigate below some particular cases of Theorem 22.

Corollary 23. *Let $\vartheta\in\mathscr{V}$. Then,*

$$\begin{aligned}&\left|\omega(\sigma_0)-\frac{1}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\alpha}^{\beta}\omega(\kappa)\vartheta'(\kappa)\,d\kappa\right|\\&\quad\leq\frac{L}{\vartheta(\beta)-\vartheta(\alpha)}\int_{\vartheta(\alpha)}^{\vartheta(\beta)}|\vartheta^{-1}(\tau)-\sigma_0|d\tau.\end{aligned}\tag{44}$$

Proof. Taking $\sigma=\sigma_0$ in (38), (44) follows.

Corollary 24. *Let $\alpha>0$. Then, for all $\sigma\in\mathbb{I}$,*

$$\begin{aligned}&\left|\omega(\sigma)-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\omega(\kappa)}{\kappa^2}\,d\kappa\right|\\&\quad\leq L\left[|\sigma-\sigma_0|+\frac{\alpha\beta}{\beta-\alpha}\left(\ln\left(\frac{\alpha\beta}{\sigma_0^2}\right)+\left(\frac{\alpha+\beta}{\alpha\beta}-\frac{2}{\sigma_0}\right)\sigma_0\right)\right].\end{aligned}\tag{45}$$

Proof. Let

$$\vartheta(\kappa)=-\frac{1}{\kappa},\ \kappa\in\mathbb{I}.\tag{46}$$

It can be easily seen that $\vartheta\in\mathscr{V}$ and

$$\vartheta^{-1}(\tau)=-\frac{1}{\tau},\quad\tau\in\left[-\frac{1}{\alpha},-\frac{1}{\beta}\right].\tag{47}$$

Hence, by Theorem 22, one has

$$\left|\omega(\sigma)-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\omega(\kappa)}{\kappa^2}\,d\kappa\right|\leq L\left(|\sigma-\sigma_0|+\frac{\alpha\beta}{\beta-\alpha}\int_{-1/\alpha}^{-1/\beta}\left|\frac{1}{\tau}+\sigma_0\right|d\tau\right).\tag{48}$$

On the other hand,

$$\begin{aligned}\int_{-1/\alpha}^{-1/\beta}\left|\frac{1}{\tau}+\sigma_0\right|d\tau&=\int_{1/\beta}^{1/\alpha}\left|\sigma_0-\frac{1}{\tau}\right|d\tau=\int_{1/\beta}^{1/\sigma_0}\left(\frac{1}{\tau}-\sigma_0\right)d\tau\\&\quad+\int_{1/\sigma_0}^{1/\alpha}\left(\sigma_0-\frac{1}{\tau}\right)d\tau=-\ln\sigma_0\\&\quad+\ln\beta-\sigma_0\left(\frac{1}{\sigma_0}-\frac{1}{\beta}\right)+\sigma_0\left(\frac{1}{\alpha}-\frac{1}{\sigma_0}\right)\\&\quad+\ln\alpha-\ln\sigma_0=\ln\left(\frac{\alpha\beta}{\sigma_0^2}\right)+\left(\frac{\alpha+\beta}{\alpha\beta}-\frac{2}{\sigma_0}\right)\sigma_0.\end{aligned}\tag{49}$$

Finally, combining (48) with (49), (45) follows.

Corollary 25. *Let $\alpha>0$. Then,*

$$\left|\omega(\sigma_0)-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\omega(\kappa)}{\kappa^2}\,d\kappa\right|\leq\frac{\alpha\beta L}{\beta-\alpha}\left(\ln\left(\frac{\alpha\beta}{\sigma_0^2}\right)+\left(\frac{\alpha+\beta}{\alpha\beta}-\frac{2}{\sigma_0}\right)\sigma_0\right).\tag{50}$$

Proof. Taking $\sigma=\sigma_0$ in (45), (50) follows.

Corollary 26. *Let $\alpha>0$. Let $\sigma_0=2\alpha\beta/(\alpha+\beta)$, that is, ω is $(L,2\alpha\beta/(\alpha+\beta))$ -Lipschitzian. Then, for all $\sigma\in\mathbb{I}$,*

$$\left|\omega(\sigma)-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\omega(\kappa)}{\kappa^2}\,d\kappa\right|\leq L\left(\left|\sigma-\frac{2\alpha\beta}{\alpha+\beta}\right|+\frac{\alpha\beta}{\beta-\alpha}\ln\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right)\right).\tag{51}$$

Proof. Taking $\sigma_0=2\alpha\beta/(\alpha+\beta)$ in (45), (51) follows.

Corollary 27. *Let $\alpha>0$. If ω is $(L,2\alpha\beta/(\alpha+\beta))$ -Lipschitzian, then,*

$$\left|\omega\left(\frac{2\alpha\beta}{\alpha+\beta}\right)-\frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\omega(\kappa)}{\kappa^2}\,d\kappa\right|\leq\frac{\alpha\beta L}{\beta-\alpha}\ln\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right).\tag{52}$$

Proof. Taking $\sigma=2\alpha\beta/(\alpha+\beta)$ in (51), (52) follows.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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