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Research Article

Strongly Reciprocally p-Convex Functions and Some Inequalities

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In this paper, we generalize the concept of strong and reciprocal convexity. Some basic properties and results will be presented for the new class of strongly reciprocally p-convex functions. Furthermore, we will discuss the Hermite–Hadamard-type, Jensen-type, and Fejér-type inequalities for the strongly reciprocally p-convex functions.

1. Introduction

The importance of convex functions and convex sets cannot be ignored, especially in nonlinear programing [1–5] and optimization theory [6], see, for instance, [7–14]. Generalization in the convexity is always appreciable. Also, many generalizations and extensions have been made in the theory of inequalities as well as in convexity. Several inequalities have been studied and established for the convexity of functions, and many generalizations, applications, and refinements take place, see [7, 9, 13, 15–18], for further study.

In the theory of inequalities, the famous inequality, Hermite–Hadamard inequality was established by Jaques Hadamard [19]. If σ: $L \longrightarrow R$ is a convex function, then

$$\sigma\left(\frac{c_1+c_2}{2}\right) \leq \left(\frac{1}{c_2-c_1}\right)\int_{c_1}^{c_2}\sigma(x)\mathrm{d}x \leq \left(\frac{\sigma(c_1)+\sigma(c_2)}{2}\right) \tag{1}$$

holds for all $c_1, c_2 \in L$ with $c_1 \leq c_2$.

In [10], Lipot Fejér established the weighted version of the Hermite–Hadamard inequality.

If σ: $L \longrightarrow R$ is a convex function, then the inequality

$$\begin{aligned}\sigma\left(\frac{c_1+c_2}{2}\right)\int_{c_1}^{c_2} w(x)\mathrm{d}x &\leq \left(\frac{1}{c_2-c_1}\right)\int_{c_1}^{c_2}\sigma(x)w(x)\mathrm{d}x\\ &\leq \left(\frac{\sigma(c_1)+\sigma(c_2)}{2}\right)\int_{c_1}^{c_2} w(x)\mathrm{d}x\end{aligned} \tag{2}$$

holds for all $c_1, c_2 \in L$ with $c_1 \leq c_2$ and w: $L \longrightarrow R$ is integrable, nonnegative, and symmetric about $((c_1+c_2)/2)$.

For more details on the Fejér inequality, see [8, 9, 11, 20–22]. The main motivation of this article is based on [18].

Mathematically, Jensen-type inequality is stated as if σ is a convex function defined on $L \subset R$, then

$$\sigma\left(\sum_{i=1}^{n}\mu_i x_i\right) \leq \sum_{i=1}^{n}\mu_i \sum_{i=1}^{n}\sigma(x_i) \tag{3}$$

holds for all $n \in R$, $x_1, x_2, \ldots, x_n \in L$ and $\mu_1, \mu_2, \ldots, \mu_n \geq 0$ with $\mu_1 + \mu_2 + \cdots + \mu_n = 1$.

This inequality has applications in probability and statistics.

The article is organized as follows: Section 2 is devoted to preliminaries and basic results, whereas in the last section, we will develop the main results for strongly reciprocally p-convex functions.

2. Preliminaries

This section concerns preliminaries and basic results for the strongly reciprocally p-convex functions.

Definition 1 (p-convex set; see [23]). An interval L is called the p-convex set if $[(rc_1^p + (1-r)c_2^p)^{(1/p)}] \in L$ for all $c_1, c_2 \in L$ and $r \in [0,1]$, where $p = 2u+1$ or $p = (d/c)$, $d = (vr+1)$, $c = 2w+1$, and $u, v, w \in N$.

Definition 2 (p-convex function; see [24]). A function $\sigma: L \longrightarrow R$ is called p-convex function if

$$\sigma\left[\left(rc_1^p + (1-r)c_2^p\right)^{(1/p)}\right] \leq r\sigma(c_1) + (1-r)\sigma(c_2), \quad (4)$$

for all $c_1, c_2 \in L$ and $r \in [0,1]$, where L is the p-convex set.

Definition 3 (strongly convex function; see [14]). Let μ be a positive number. A function $\sigma: L \longrightarrow R$ is called a strongly convex function if

$$\sigma(rc_1 + (1-r)c_2) \leq r\sigma(c_1) + (1-r)\sigma(c_2) - \mu r(1-r)(c_2 - c_1)^2, \quad (5)$$

for all $c_1, c_2 \in L$ and $r \in [0,1]$.

Definition 4 (strongly p-convex function; see [25]). Let μ be a positive number. A function $\sigma: L \longrightarrow R$ is called strongly p-convex function if

$$\sigma\left[\left(rc_1^p + (1-r)c_2^p\right)^{(1/p)}\right] \leq r\sigma(c_1) + (1-r)\sigma(c_2) - \mu r \cdot (1-r)\left(c_2^p - c_1^p\right)^2, \quad (6)$$

for all $c_1, c_2 \in L$ and $r \in [0,1]$.

Definition 5 (harmonic convex function; see [22]). Let $L = [c_1, c_2] \subset R$ be an interval. A function $\sigma: L \longrightarrow R$ is harmonic convex if

$$\sigma\left(\frac{(c_1 c_2)}{rc_1 + (1-r)c_2}\right) \leq r\sigma(c_1) + (1-r)\sigma(c_2), \quad (7)$$

for all $c_1, c_2 \in L$ and $r \in [0,1]$.

Definition 6 (harmonic p-convex function; see [26]). A function $\sigma: L \longrightarrow R$ is called a harmonic p-convex function if

$$\sigma\left[\left(\frac{\left(c_1^p c_2^p\right)}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right] \leq r\sigma(c_1) + (1-r)\sigma(c_2), \quad (8)$$

for all $c_1, c_2 \in L$ and $r \in [0,1]$.

Definition 7 (strongly reciprocally convex function; see [18]). Let $L \subseteq R$ and $\mu \in (0, \infty)$. A function $\sigma: L \longrightarrow R$ is said to be strongly reciprocally convex with modulus μ on L if the inequality

$$\sigma\left(\frac{c_1 c_2}{rc_1 + (1-r)c_2}\right) \leq r\sigma(c_1) + (1-r)\sigma(c_2) - \mu r(1-r) \cdot \left(\left(\frac{1}{c_1}\right) - \left(\frac{1}{c_2}\right)\right)^2, \quad (9)$$

holds for all $c_1, c_2 \in L$ and $r \in [0,1]$.

Now, we are ready to introduce a new class of convexity named as strongly reciprocally p-convex function.

Definition 8 (strongly reciprocally p-convex function). A function $\sigma: L \longrightarrow$ R is called strongly reciprocally p-convex with modulus μ on L if the inequality

$$\sigma\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] \leq r\sigma(x) + (1-r)\sigma(y) - \mu r \cdot (1-r)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2 \quad (10)$$

holds, for all $x, y \in L = [c_1, c_2]$ and $r \in [0,1]$.

Remark 1

(1) If we insert $p = 1$ in inequality (10), then we retrace the strong and reciprocal convexity [18]
(2) If we insert $\mu = 0$ in inequality (10), then we retrace the harmonic p-convexity [26]
(3) If we insert $p = 1$ and $\mu = 0$ in inequality (10), then we retrace the harmonic convexity [22]

The following proposition expresses the algebraic property of strongly reciprocally p-convex functions.

Proposition 1. *Let $\sigma, \varphi: L \longrightarrow R$ be two strongly reciprocally p-convex functions; then, the following statements hold:*

(i) $\sigma + \varphi: L \longrightarrow R$ is strongly reciprocally p-convex
(ii) For any $\lambda \geq 0$, $\lambda\sigma: L \longrightarrow R$ is strongly reciprocally p-convex corresponding to $\lambda\mu = \mu^*$

Proof

(i) Choose $v = [(x^p y^p/(rx^p + (1-r)y^p))^{(1/p)}]$; then, by the definition of σ and φ, we obtain

$$\begin{aligned}
(\sigma+\varphi)&\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] \\
&= \sigma\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] + \varphi\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] \\
&\leq r\sigma(x) + (1-r)\sigma(y) - \mu r(1-r)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2 \\
&\quad + r\varphi(x) + (1-r)\varphi(y) - \mu r(1-r)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2, \\
&= r(\sigma+\varphi)(x) + (1-r)(\sigma+\varphi)(y) - 2\mu r(1-r) \\
&\quad \cdot \left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2 \\
&\leq r(\sigma+\varphi)(x) + (1-r)(\sigma+\varphi)(y) - \mu r(1-r) \\
&\quad \cdot \left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2,
\end{aligned} \quad (11)$$

where $\mu \geq 0$.

(ii) Let $\lambda \geq 0$; then, by definition, we obtain

$$\begin{aligned}\lambda\sigma\left(\frac{x^p y^p}{rx^p + (1-r)y^p}\right) &\leq \lambda\left[r\sigma(x) + (1-r)\sigma(y) - \mu r(1-r)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2\right] \\ &= r\lambda\sigma(x) + (1-r)\lambda\sigma(y) - \mu^* r(1-r)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2,\end{aligned} \tag{12}$$

where $\mu^* = \lambda\mu$ and $\mu \geq 0$.

The next lemma establishes the connection between the strong and reciprocal p-convexity and harmonic p-convexity. □

Lemma 1. *Let σ: $L \longrightarrow R$ be a function; σ is strongly reciprocally p-convex iff the function φ: $L \longrightarrow R$, defined by $\varphi(x) = \sigma(x) - (\mu/x^{2p})$, is harmonically p-convex.*

Proof. Let σ be strongly reciprocally p-convex; then, we have

$$\begin{aligned}\varphi\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] &= \sigma\left[\left(\frac{(x^p y^p)}{rx^p + (1-r)y^p}\right)^{(1/p)}\right] - \mu\left(\frac{rx^p + (1-r)y^p}{(x^p y^p)}\right)^2 \\ &\leq r\sigma(y) + (1-r)\sigma(x) - \mu r(1-r)\left(\left(\frac{1}{x^p}\right) - \left(\frac{1}{y^p}\right)\right)^2 - \mu\left(\frac{rx^p + (1-r)y^p}{(x^p y^p)}\right)^2 \\ &= r\sigma(y) + (1-r)\sigma(x) - \mu r(1-r)\left(\left(\frac{1}{x^p}\right) - \left(\frac{1}{y^p}\right)\right)^2 - \mu\left(\frac{r^2}{y^{2p}} + \frac{(1-r)^2}{x^{2p}} + \frac{2r(1-r)}{(x^p y^p)}\right) \\ &= r\sigma(y) + (1-r)\sigma(x) - \mu\left(\left(\frac{r}{y^{2p}}\right) - \left(\frac{2r}{x^p y^p}\right) + \left(\frac{r}{x^{2p}}\right) - \left(\frac{r^2}{y^{2p}}\right) + \left(\frac{2r}{x^p y^p}\right) - \left(\frac{r^2}{x^{2p}}\right) + \left(\frac{r^2}{y^{2p}}\right)\right. \\ &\quad \left. + \left(\frac{2r}{x^p y^p}\right) - \left(\frac{2r^2}{x^p y^p}\right) + \left(\frac{1}{x^{2p}}\right) - \left(\frac{2r}{x^{2p}}\right) + \left(\frac{r^2}{x^{2p}}\right)\right) \\ &= r\sigma(y) + (1-r)\sigma(x) - \mu\left(\left(\frac{r}{y^{2p}}\right) + \left(\frac{1}{x^{2p}}\right) - \left(\frac{r}{x^{2p}}\right)\right) \\ &= r\sigma(y) + (1-r)\sigma(x) - \mu\left(\left(\frac{r}{y^{2p}}\right) + \left(\frac{1-r}{x^{2p}}\right)\right) \\ &= r\left(\sigma(y) - \left(\frac{\mu}{y^{2p}}\right)\right) + (1-r)\left(\sigma(x) - \left(\frac{\mu}{x^{2p}}\right)\right) \\ &= r\varphi(y) + (1-r)\varphi(x).\end{aligned} \tag{13}$$

This shows that φ is a harmonic p-convex function.

Conversely, if φ is harmonically p-convex, then

$$\begin{aligned}
\sigma\left[\left(\frac{(x^p y^p)}{rx^p+(1-r)y^p}\right)^{(1/p)}\right] &= \varphi\left[\left(\frac{x^p y^p}{rx^p+(1-r)y^p}\right)^{(1/p)}\right]+\mu\left(\frac{rx^p+(1-r)y^p}{(x^p y^p)}\right)^2\\
&\leq r\varphi(y)+(1-r)\varphi(x)+\mu\left(\frac{rx^p+(1-r)y^p}{(x^p y^p)}\right)^2\\
&= r\varphi(y)+(1-r)\varphi(x)+\mu\left(\left(\frac{r^2}{y^{2p}}\right)+\left(\frac{2r(1-r)}{x^p y^p}\right)+\frac{(1-r)^2}{x^{2p}}\right)\\
&\leq r\varphi(y)+(1-r)\varphi(x)+\mu\left(\frac{r(1-1+r)}{y^{2p}}+\frac{2r(1-r)}{x^p y^p}\right)+\mu\left(\frac{(1-r)(1-r)}{x^{2p}}\right)\\
&= r\varphi(y)+(1-r)\varphi(x)+\mu\left(\frac{r}{y^{2p}}-\frac{r(1-r)}{y^{2p}}\right)+\mu\left(\frac{2r(1-r)}{x^p y^p}+\frac{(1-r)}{x^{2p}}-\frac{r(1-r)}{x^{2p}}\right)\\
&= r\left(\varphi(y)+\mu\left(\frac{1}{y^{2p}}\right)\right)+(1-r)\left(\varphi(x)+\mu\left(\frac{1}{x^{2p}}\right)\right)-\mu r(1-r)\left(\left(\frac{1}{y^{2p}}\right)-\left(\frac{2}{x^p y^p}\right)+\left(\frac{1}{x^{2p}}\right)\right)\\
&= r\sigma(y)+(1-r)\sigma(x)-\mu r(1-r)\left(\left(\frac{1}{y^p}\right)-\left(\frac{1}{x^p}\right)\right)^2.
\end{aligned}\tag{14}$$

This implies that σ is a strongly reciprocally p-convex function for all $x, y \in L$ and $r \in [0,1]$. □

3. Main Results

In this section, Hermite–Hadamard-, Fejér-, and Jensen-type inequalities are investigated. The next theorem gives the generalization of the Hermite–Hadamard inequality for strongly reciprocally p-convex functions.

Theorem 1 *(Hermite–Hadamard-type inequality). Let $L \subset R/\{0\}$ be an interval on the real line. If σ: $L \longrightarrow R$ is a strongly reciprocally p-convex function with modulus $\mu \geq 0$ and $x \in L = [c_1, c_2]$, then*

$$\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p+c_2^p}\right)^{(1/p)}\right]+\frac{\mu}{12}\left(\frac{c_2^p-c_1^p}{c_1^p c_2^p}\right)^2 \leq \frac{pc_1^p c_2^p}{c_2^p-c_1^p}\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{(p+1)}}\mathrm{d}x \leq \frac{\sigma(c_1)+\sigma(c_2)}{2}-\left(\frac{\mu}{6}\right)\left(\frac{c_2^p-c_1^p}{c_1^p c_2^p}\right)^2,\tag{15}$$

for all $c_1, c_2 \in L$ with $c_1 \leq c_2$.

Proof. We start by the definition; set $r=(1/2)$ in inequality (10), and we have

$$\sigma\left[\left(\frac{2x^p y^p}{x^p+y^p}\right)^{(1/p)}\right]\leq\left(\frac{1}{2}\right)\sigma(x)+\left(\frac{1}{2}\right)\sigma(y)-\left(\frac{\mu}{4}\right)\left(\left(\frac{1}{y^p}\right)-\left(\frac{1}{x^p}\right)\right)^2.\tag{16}$$

Let $x=[(c_1^p c_2^p/(rc_1^p+(1-r)c_2^p))^{(1/p)}]$ and $y=[(c_1^p c_2^p/(rc_2^p+(1-r)c_1^p))^{1/p}]$, and by integrating w.r.t r over [0, 1], the above inequality yields

$$\begin{aligned}
&\int_0^1\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p+c_2^p}\right)^{(1/p)}\right]\mathrm{d}r\\
&\leq\frac{1}{2}\int_0^1\sigma\left[\left(\frac{c_1^p c_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\mathrm{d}r+\int_0^1\sigma\left[\left(\frac{c_1^p c_2^p}{rc_2^p+(1-r)c_1^p}\right)^{(1/p)}\right]\mathrm{d}r-\frac{\mu}{4}\left(\frac{c_2^p-c_1^p}{c_1^p c_2^p}\right)^2\int_0^1(1-2r)^2\mathrm{d}r
\end{aligned}\tag{17}$$

$$\int_0^1\sigma\left[\left(\frac{c_1^p c_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\mathrm{d}r=\left(\frac{pc_1^p c_2^p}{(c_2^p)-(c_1^p)}\right)=\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{(p+1)}}\mathrm{d}x=\int_0^1\sigma\left[\left(\frac{c_1^p c_2^p}{rc_2^p+(1-r)c_1^p}\right)^{(1/p)}\right]\mathrm{d}r,$$

and then inequality (17) is reduced to

$$\begin{aligned}&\sigma\left[\left(\frac{2c_1^pc_2^p}{c_1^p+c_2^p}\right)^{(1/p)}\right]\leq\left(\frac{pc_1^pc_2^p}{\left(c_2^p\right)-\left(c_1^p\right)}\right)\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{(p+1)}}\mathrm{d}x-\left(\frac{\mu}{12}\right)\left(\frac{\left(c_2^p\right)-\left(c_1^p\right)}{c_1^pc_2^p}\right)^2\\&\sigma\left[\left(\frac{2c_1^pc_2^p}{c_1^p+c_2^p}\right)^{(1/p)}\right]+\left(\frac{\mu}{12}\right)\left(\frac{c_2^p-c_1^p}{c_1^pc_2^p}\right)^2\leq\left(\frac{pc_1^pc_2^p}{\left(c_2^p\right)-\left(c_1^p\right)}\right)\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{(p+1)}}\mathrm{d}x,\end{aligned}\tag{18}$$

which is the left side of the inequality.

For the right side of inequality (15), set $x=c_1$ and $y=c_2$ in (10); we have

$$\sigma\left[\left(\frac{c_1^pc_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\leq r\sigma(c_1)+(1-r)\sigma(c_2)-\mu r\cdot(1-r)\left(\left(\frac{1}{c_1^p}\right)-\left(\frac{1}{c_2^p}\right)\right)^2.\tag{19}$$

Integrating w.r.t r over $[0,1]$, the above inequality yields

$$\begin{aligned}&\int_0^1\sigma\left[\left(\frac{c_1^pc_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\mathrm{d}r\leq\left(\frac{1}{2}\right)\sigma(c_1)+\left(\frac{1}{2}\right)\sigma(c_2)-\mu\left(\frac{c_2^p-c_1^p}{c_1^pc_2^p}\right)^2\int_0^1 r(1-r)\mathrm{d}r\\&\int_0^1\sigma\left[\left(\frac{c_1^pc_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\mathrm{d}r\leq\left(\frac{\sigma(c_1)+\sigma(c_2)}{2}\right)-\left(\frac{\mu}{6}\right)\left(\frac{c_2^p-c_1^p}{c_1^pc_2^p}\right)^2.\end{aligned}\tag{20}$$

Since

$$\int_0^1\sigma\left[\left(\frac{2c_1^pc_2^p}{rc_1^p+(1-r)c_2^p}\right)^{(1/p)}\right]\mathrm{d}r=\frac{pc_1^pc_2^p}{\left(c_2^p\right)-\left(c_1^p\right)}\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{(p+1)}}\mathrm{d}x,\tag{21}$$

then we obtain

$$\frac{pc_1^pc_2^p}{c_2^p-c_1^p}\int_{c_1}^{c_2}\frac{\sigma(x)}{x^{p+1}}\mathrm{d}x\leq\left(\frac{\sigma(c_1)+\sigma(c_2)}{2}\right)-\left(\frac{\mu}{6}\right)\left(\frac{c_2^p-c_1^p}{c_1^pc_2^p}\right)^2.\tag{22}$$

From (18) and (22), we get (15). □

Remark 2

(1) For $p=1$ in (15), Hermite–Hadamard inequality for strongly reciprocally convex functions is obtained [18].

(2) If we allow $\mu\longrightarrow 0^+$ in inequalities (15), we obtain the Hermite–Hadamard-type inequalities for harmonically convex functions [22].

For further details on Hermite–Hadamard inequities, see [27–30].

Theorem 2 *(Fejér-type inequality). Assume σ: $L\longrightarrow R$ is a strongly reciprocally p-convex function with modulus μ on L; then,*

$$\begin{aligned}&\sigma\left[\left(\frac{2c_1^pc_2^p}{c_1^p+c_2^p}\right)^{(1/p)}\right]\int_{c_1}^{c_2}\mathrm{d}x+\frac{\mu}{\left(2c_1^pc_2^p\right)^2}\int_{c_1}^{c_2}\frac{\left(2c_1^pc_2^p-\left(c_1^p+c_2^p\right)x^p\right)^2w(x)}{x^{3p+1}}\mathrm{d}x\\&\leq\int_{c_1}^{c_2}\frac{\sigma(x)w(x)}{x^{(p+1)}}\mathrm{d}x\\&\leq\frac{c_1^p}{c_2^p-c_1^p}\left[\sigma(c_1)+\sigma(c_2)\right]\int_{c_1}^{c_2}\frac{\left(c_2^p-x^p\right)w(x)}{x^{2p+1}}\mathrm{d}x-\frac{\mu}{c_1^pc_2^p}\int_{c_1}^{c_2}\frac{w(x)}{x^{(p+1)}}\mathrm{d}x+\frac{\mu}{\left(2c_1^pc_2^p\right)^2}\int_{c_1}^{c_2}\frac{\left(2c_1^pc_2^p-\left(c_1^p+c_2^p\right)x^p\right)^2w(x)}{x^{3p+1}}\mathrm{d}x\end{aligned}\tag{23}$$

holds for $c_1, c_2 \in L$ with $c_1 \le c_2$ and $x \in L = [c_1, c_2]$, where $w\colon L \longrightarrow R$ is a nonnegative integrable function that satisfies

$$w\left[\left(\frac{c_1^p c_2^p}{x^p}\right)^{(1/p)}\right] = w\left[\left(\frac{c_1^p c_2^p}{c_1^p + c_2^p - x^p}\right)^{(1/p)}\right]. \tag{24}$$

Proof. Since $\sigma\colon L \longrightarrow R$ is a strongly reciprocally p-convex function, then by definition for $r = (1/2)$ in (10), we have

$$\sigma\left[\left(\frac{2x^p y^p}{x^p + y^p}\right)^{(1/p)}\right] \le \left(\frac{\sigma(x) + \sigma(y)}{2}\right) - \left(\frac{\mu}{4}\right)\left(\left(\frac{1}{y^p}\right) - \left(\frac{1}{x^p}\right)\right)^2, \tag{25}$$

for all $x, y \in L$; suppose $x = [(c_1^p c_2^p/(rc_1^p + (1-r)c_2^p))^{(1/p)}]$ and $y = [(c_1^p c_2^p/(rc_2^p + (1-)c_1^p))^{(1/p)}]$ in the above inequality; then, we obtain

$$\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p + c_2^p}\right)^{(1/p)}\right] \le \left(\frac{1}{2}\right)\left[\sigma\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right] + \sigma\left[\left(\frac{c_1^p c_2^p}{rc_2^p + (1-r)c_1^p}\right)^{(1/p)}\right]\right] - \left(\frac{\mu}{4}\right)\left(\left(\frac{rc_2^p + (1-r)c_1^p}{c_1^p c_2^p}\right) - \left(\frac{rc_1^p + (1-r)c_2^p}{c_1^p c_2^p}\right)\right)^2. \tag{26}$$

Since w is nonnegative and symmetric, we have

$$\begin{aligned}
&\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p + c_2^p}\right)^{(1/p)}\right] w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right]\\
&\le \frac{1}{2}\left[\sigma\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right] + \sigma\left[\left(\frac{c_1^p c_2^p}{rc_2^p + (1-r)c_1^p}\right)^{(1/p)}\right]\right] w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right]\\
&\quad - \frac{\mu}{4}\left(\left(\frac{rc_2^p + (1-r)c_1^p}{c_1^p c_2^p}\right) - \left(\frac{rc_1^p + (1-r)c_2^p}{c_1^p c_2^p}\right)\right)^2 w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right].
\end{aligned} \tag{27}$$

The above inequality is integrated with respect to r over $[0, 1]$, and then putting $x = [(c_1^p c_2^p/(rc_1^p + (1-r)c_2^p))^{(1/p)}]$, we obtain

$$\begin{aligned}
&\frac{pc_1^p c_2^p}{c_2^p - c_1^p}\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p + c_2^p}\right)^{(1/p)}\right]\int_{c_1}^{c_2}\frac{w(x)}{x^{p+1}}\mathrm{d}x\\
&\le \frac{pc_1^p c_2^p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\sigma(x)w(x)}{x^{p+1}}dx - \frac{\mu p}{4\left(c_2^p - c_1^p\right)\left(c_1^p c_2^p\right)}\int_{c_1}^{c_2}\frac{\left(2c_1^p c_2^p - \left(c_1^p + c_2^p\right)x^p\right)^2}{x^{3p+1}}w(x)\mathrm{d}x.
\end{aligned} \tag{28}$$

After simplification, the above inequality becomes

$$\sigma\left[\left(\frac{2c_1^p c_2^p}{c_1^p + c_2^p}\right)^{(1/p)}\right]\int_{c_1}^{c_2}\frac{w(x)}{x^{p+1}}dx + \frac{\mu}{\left(2c_1^p c_2^p\right)^2}\int_{c_1}^{c_2}\frac{\left(2c_1^p c_2^p - \left(c_1^p + c_2^p\right)x^p\right)^2}{x^{3p+1}}w(x)\mathrm{d}x \le \int_{c_1}^{c_2}\frac{\sigma(x)w(x)}{x^{p+1}}\mathrm{d}x. \tag{29}$$

For the right-hand side of (23), set $x = c_1$ and $y = c_2$ in (10); we have

$$\begin{aligned}&\sigma\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right] w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right],\\ &\quad\leq \left[r\sigma(c_1) + (1-r)\sigma(c_2)\right] w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right] - \mu r(1-r)\left(\left(\frac{1}{c_1^p}\right) - \left(\frac{1}{c_2^p}\right)\right)^2 w\left[\left(\frac{c_1^p c_2^p}{rc_1^p + (1-r)c_2^p}\right)^{(1/p)}\right].\end{aligned} \tag{30}$$

Integrating with respect to r over $[0, 1]$ and then putting $x = [(c_1^p c_2^p/(rc_1^p + (1-r)c_2^p))^{(1/p)}]$, we obtain

$$\frac{pc_1^p c_2^p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\sigma(x)w(x)}{x^{p+1}}\mathrm{d}x \leq \frac{pc_1^p c_1^p c_2^p}{\left(c_2^p - c_1^p\right)^2}\left[\sigma(c_1) + \sigma(c_2)\right]\int_{c_1}^{c_2}\frac{\left(c_2^p - x^p\right)w(x)}{x^{2p+1}}\mathrm{d}x - \frac{\mu p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\left(c_2^p - x^p\right)\left(x^p - c_1^p\right)w(x)}{x^{3p+1}}\mathrm{d}x. \tag{31}$$

After simplification, we have

$$\int_{c_1}^{c_2}\frac{\sigma(x)w(x)}{x^{p+1}}\mathrm{d}x \leq \frac{c_1^p}{\left(c_2^p - c_1^p\right)}\left[\sigma(c_1) + \sigma(c_2)\right]\int_{c_1}^{c_2}\frac{\left(c_2^p - x^p\right)w(x)}{x^{2p+1}}\mathrm{d}x - \frac{\mu}{c_1^p c_2^p}\int_{c_1}^{c_2}\frac{\left(c_2^p - x^p\right)\left(x^p - c_1^p\right)w(x)}{x^{3p+1}}\mathrm{d}x. \tag{32}$$

From (32) and (27), we get (23). □

Remark 3. If we set $p = 1$ in (23), the Fejér-type inequality for strongly reciprocally convex functions is obtained.

Jensen-type inequality for the aforementioned inequality is described in the next theorem.

Theorem 3. *(Jensen-type inequality). If $\sigma\colon L \longrightarrow R$ is a reciprocally strongly p-convex function with modulus μ, then*

$$\sigma\left(\sum_{i=1}^n r\frac{1}{x_i^p}{}_i\right)^{(1/p)} \leq \sum_{i=1}^n r_i\sigma\left(\frac{1}{x_i}\right) - \mu\sum_{i=1}^n r_i\left(\left(\frac{1}{x_i^p}\right) - \left(\frac{1}{\overline{x}^p}\right)\right)^2 \tag{33}$$

holds for all $(1/x_1^p), (1/x_2^p), \ldots, (1/x_n^p) \in L$, $r_1, r_2, \ldots, r_n \geq 0$ with $r_1 + r_2 + \ldots + r_n = 1$ and $(1/\overline{x}^p) = (r_1(1/x_1^p) + r_2(1/x_2^p) + \ldots + r_n(1/x_n^p))$.

Proof. Fix $(1/x_1^p), (1/x_2^p), \ldots, (1/x_n^p) \in L$ and $r_1, r_2, \ldots, r_n > 0$ such that $r_1 + r_2 + \ldots + r_n = 1$.

Put $(1/\overline{x}^p) = r_1(1/x_1^p) + r_2(1/x_2^p) + \cdots + r_n(1/x_n^p)$, and suppose a function $w\colon L \longrightarrow R$ of the form

$$w\left(\frac{1}{x}\right) = \mu\left(\left(\frac{1}{x^p}\right) - \left(\frac{1}{\overline{x}^p}\right)\right)^2 + a\left(\left(\frac{1}{x^p}\right) - \left(\frac{1}{\overline{x}^p}\right)\right) + \sigma(\overline{x}), \tag{34}$$

supporting at $\overline{x}$, satisfying $w(1/\overline{x}) = \sigma(1/\overline{x})$ and $w(x) \leq \sigma(x)$, $x \in L$. Then, for every $i = \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}\sigma\left(\frac{1}{x_i}\right) \geq w\left(\frac{1}{x_i}\right) &= \mu\left(\left(\frac{1}{x_i^p}\right) - \left(\frac{1}{\overline{x}^p}\right)\right)^2 + a\left(\left(\frac{1}{x_i}\right)^p - \left(\frac{1}{\overline{x}^p}\right)\right)\\ &\quad + \sigma\left(\frac{1}{\overline{x}}\right).\end{aligned} \tag{35}$$

Multiplying both sides by r_i and summing up to n, we have

$$\sum_{i=1}^n r_i\sigma\left(\frac{1}{x_i}\right) \geq \mu\sum_{i=1}^n r_i\left(\frac{1}{x_i^p} - \frac{1}{\overline{x}^p}\right)^2 + a\sum_{i=1}^n r_i\left(\frac{1}{x_i^p} - \frac{1}{\overline{x}^p}\right) + \sigma\left(\frac{1}{\overline{x}}\right). \tag{36}$$

Since $\sum_{i=1}^n r_i((1/x_i^p) - (1/\overline{x}^p)) = 0$, we have

$$\sigma\left(\frac{1}{\overline{x}}\right)^{(1/p)} \leq \sum_{i=1}^n r\sigma\left(\frac{1}{x_i}\right)_i - \mu\sum_{i=1}^n r_i\left(\frac{1}{x_i^p} - \frac{1}{\overline{x}^p}\right)^2, \tag{37}$$

which completes the proof. □

Remark 4. In inequality (34), fixing $p = 1$ and $\mu = 0$ yields the Jensen-type inequality for the harmonic convex function

[22]. See [31–34] for more details on Jensen-type inequalities.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Hao Li analyzed all results and proofread and revised the paper, Muhammad Shoaib Saleem proposed the problem and supervised the work, Ijaz Hussain proved the results, and Muhammad Imran wrote the whole paper.

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