Research Article

On the Generalized Mass Transport Equation to the Concept of Variable Fractional Derivative

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The hydrodynamic dispersion equation was generalized using the concept of variational order derivative. The modified equation was numerically solved via the Crank-Nicholson scheme. The stability and convergence of the scheme in this case were presented. The numerical simulations showed that, the modified equation is more reliable in predicting the movement of pollution in the deformable aquifers, than the constant fractional and integer derivatives.

1. Introduction

Anomalous dispersion phenomena are systematically pragmatic in physics, chemistry, and biology fields [1–4]. To distinguish irregular diffusion phenomena, constant-order fractional diffusion equations were initiated and have received breathtaking success. Nonetheless, it has been recognized that the constant order fractional dispersion equations are not competent of typifying some multifaceted dispersion processes, for example, dispersion process in inhomogeneous or heterogeneous medium [5]. In addition, when we think about dispersion process in porous medium, if the medium structure or external field changes with time, in these circumstances, the constant-order fractional diffusion equation model cannot be used to well illustrate such phenomenon [6, 7]. This is the case of the groundwater pollution problem; the medium through which the pollution is dispersing is heterogeneous and change with time. Motionless in some environmental science dispersion processes, the concentration of particles will determine the dispersion pattern [8, 9]. To solve the above problems, the variable-order (VO) fractional dispersion equation models have been suggested for use [10].

The leading edge work of VO operator can be traced to Samko et al. by introducing the variable order integration and Riemann-Liouville derivative in [10]. It has been acknowledged as a prevailing modelling approach in the fields of viscoelasticity [10], viscoelastic deformation [11], viscous fluid [12], and anomalous transmission [13]. The problem encounter in groundwater is that the geometry of the aquifer in which the flow or the pollution takes place is not well known. In addition, the geological formation through which the contamination takes place changes in time and space. These phenomena cannot therefore accurately be described neither via the classical hydrodynamic dispersion equation nor via the fractional version of hydrodynamic dispersion equation. In this paper, we generalize the hydrodynamic dispersion equation using the concept of the variable order derivative.

2. Definition and Problem Formulation

The fractional operators (fractional derivatives and integrals) refer to the differential and integral operators of arbitrary order, and fractional differential equations refer to those containing fractional derivatives. The former are the generalization of integer-order differential and integral operators and the latter are the generalization of differential equations of integer order. The operators of variable-order, which fall into

a more complex category, are the derivatives and integrals whose orders are the functions of certain variables.

2.1. Variational Order Operator

Definition 1. Left and right Riemann-Liouville integrals of variable order: let $0 < \alpha(x, t) < 1$ for all $(x, t) \in [a, b]$ and $f \in L_1[a, b]$; then,

$$aI_t^{\alpha(\cdot,\cdot)} f(t) = \int_a^t \frac{1}{\Gamma[\alpha(x,t)]} (x - t)^{\alpha(x,t)-1} f(x) \, dx, \quad (t > a) \tag{1}$$

is called the left Riemann-Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$, while

$$tI_b^{\alpha(\cdot,\cdot)} f(t) = \int_t^b \frac{1}{\Gamma[\alpha(x,t)]} (t - x)^{\alpha(x,t)-1} f(x) \, dx, \quad (t < b) \tag{2}$$

is referred to as the right Riemann-Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$.

Definition 2. Left and right Riemann-Liouville derivatives of variable fractional order: let $0 < \alpha(x, t) < 1$ for all $(x, t) \in [a, b]$. If $aI_t^{1-\alpha(\cdot,\cdot)} f \in AC[a, b]$, then, the left Riemann-Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ which is given as

$$aD_t^{\alpha(\cdot,\cdot)} f(t) = \frac{d}{dt} \int_a^t \frac{1}{\Gamma[1-\alpha(x,t)]} (t - x)^{-\alpha(x,t)} f(x) \, dx, \quad (t > a) \tag{3}$$

is called the left Riemann-Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$, while in the same line of idea we have the following expression:

$$tD_b^{\alpha(\cdot,\cdot)} f(t) = \frac{d}{dt} \int_t^b \frac{1}{\Gamma[1-\alpha(x,t)]} (x - t)^{-\alpha(x,t)} f(x) \, dx, \quad (t < b) \tag{4}$$

which is referred to as the right Riemann-Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$.

Definition 3. Left and right Caputo derivatives of variable fractional order: let $0 < \alpha(x, t) < 1$ for all $(x, t) \in [a, b]$. If $aI_t^{1-\alpha(\cdot,\cdot)} f \in AC[a, b]$, then, the left Riemann-Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ is given as

$$aD_t^{\alpha(\cdot,\cdot)} f(t) = \int_a^t \frac{1}{\Gamma[1-\alpha(x,t)]} (t - x)^{-\alpha(x,t)} \frac{d}{dx} f(x) \, dx, \quad (t > a) \tag{5}$$

which is called the left Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$, while in the same line of idea we have the following expression:

$$tD_b^{\alpha(\cdot,\cdot)} f(t) = \int_t^b \frac{1}{\Gamma[1-\alpha(x,t)]} (x - t)^{-\alpha(x,t)} \frac{d}{dx} f(x) \, dx, \quad (t < b) \tag{6}$$

which is referred to as the right Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$.

However, this work we will throughout use the following definition.

2.2. Problem Formulation.

The concern here is the generalization of advection dispersion equation by including a possible effect of heterogeneity or variability of the aquifer into the mathematical formulation. A one-dimensional model consisting of an infinitely long homogeneous isotropic porous medium with a steady state uniform flow with a seepage velocity v is considered here. A particular chemical is injected from one end of the model for a period of time t_0 such that the input concentration varies as an exponential function of time. The value of that chemical concentration at any time t and at a distance x from the injection boundary, allowing for the decay and adsorption, may be obtained from the solution of the following set of equations [14]; more details for this model can be found in [15–18]. Consider

$$D \frac{\partial^2 C}{\partial x^2} - v \frac{\partial C}{\partial x} - \lambda RC = R \frac{\partial C}{\partial t} + f(x, t). \tag{7}$$

Subject to the initial and boundary conditions:

$$C(x, 0) = 0, \qquad C(0, t) = c_0 \exp(-\alpha t), \qquad C_x(\infty, t) = 0, \tag{8}$$

where D is the dispersion coefficient, v is the seepage velocity, R is the retardation factor, λ is the radioactive decay constant, c_0 is the initial concentration, α is a positive constant, and $f(x, t)$ is any source and sink in the system. However, in the case of the groundwater pollution, the function $f(x, t)$ is always neglected because we assume that there is no source and sink in the system under investigation. Therefore, in our case we will set the function to be zero. In order to include explicitly the medium structure or external field changes with time into the mathematical formulation, we replace the standard derivative operator by the variational order derivative operator as follows:

$$f(x, t) + D \frac{\partial^2 C}{\partial x^2} - v \frac{\partial C}{\partial x} - \lambda RC = R D_0^{\alpha(t,x)} (C). \tag{9}$$

Subject to the initial and boundary conditions:

$$C(x, 0) = 0, \qquad C(0, t) = c_0 \exp(-\gamma t), \qquad C_x(\infty, t) = 0, \tag{10}$$

where $\alpha(t, x)$ is a continuous function in $(0, 1]$ that can be approximated from the field observation. Note that there is no analytical solution available for this new equation; therefore, our next concern is to establish the existence and uniqueness of such solution. This will be done in the next section.

3. Numerical Solution

Numerical methods acquiesce estimated solutions to the overriding equation all the way through the discretization of space and time. Inside the discredited problem domain, the variable internal properties, boundaries, and stresses of the system are approximated. Deterministic, distributed-parameter, and numerical models can relax the rigid idealized conditions of analytical models or lumped-parameter models, and they can therefore be more realistic and flexible for simulating fields' conditions [19–29]. The finite difference schemes for constant-order time or space fractional diffusion equations have been widely studied [19–29]. Before performing the numerical methods, we assume that (9) has a unique and sufficiently smooth solution. To establish the numerical schemes for the above equation, we let $x_l = lh$, $0 \leq l \leq M$, $Mh = L$, $t_k = k\tau$, $0 \leq k \leq N$, and $N\tau = T$, and we let h be the step and τ be the time size, and M and N be grid points.

3.1. Crank-Nicolson Scheme [30]. We bring in the Crank-Nicolson idea as follows. At the outset, the discretization of first- and second-order space derivative is stated as

$$\frac{\partial C}{\partial x} = \frac{1}{2}\left(\left(\frac{C(x_{l+1}, t_{k+1}) - C(x_{l-1}, t_{k+1})}{2(h)}\right)\right.$$
$$\left. + \left(\frac{C(x_{l+1}, t_k) - C(x_{l-1}, t_k)}{2(h)}\right)\right) + O(h), \tag{11}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{2}\left(\left(\frac{C(x_{l+1}, t_{k+1}) - 2C(x_l, t_{k+1}) + C(x_{l-1}, t_{k+1})}{(h)^2}\right)\right.$$
$$\left. + \left(\frac{C(x_{l+1}, t_k) - 2C(x_l, t_k) + C(x_{l-1}, t_k)}{(h)^2}\right)\right)$$
$$+ O(h^2), \tag{12}$$

$$C = \frac{1}{2}(C(x_l, t_{k+1}) + C(x_l, t_k)). \tag{13}$$

The Crank-Nicolson scheme for the VO time fractional diffusion model can be stated as follows:

$$\frac{\partial^{\alpha_l^{k+1}} C(x_l, t_{k+1})}{\partial t^{\alpha_l^{k+1}}}$$
$$= \frac{\tau^{-\alpha_l^{k+1}}}{\Gamma(2 - \alpha_l^{k+1})}\left(C(x_l, t_{k+1}) - C(x_l, t_k)\right.$$
$$+ \sum_{j=1}^{k}\left[C(x_l, t_{k+1-j}) - C(x_l, t_{k-j})\right]$$
$$\left. \times \left[(j+1)^{1-\alpha_l^{k+1}} - (j)^{1-\alpha_l^{k+1}}\right]\right) + O(\tau). \tag{14}$$

Now replacing (11), (12), (13), and (14) in (9) we obtain the following:

$$b_l^{k+1}\left[\left(C_{l+1}^{k+1} - 2C_l^{k+1} + C_{l-1}^{k+1}\right) + \left(C_{l+1}^k - 2C_l^k + C_{l-1}^k\right)\right]$$
$$+ c_l^{k+1}\left[\left(C_{l+1}^{k+1} - C_{l-1}^{k+1}\right) + \left(C_{l+1}^k - C_{l-1}^k\right)\right]$$
$$+ d_l^{k+1}\left(C_l^{k+1} - C_l^k\right)$$
$$= C_l^{k+1} - C_l^k + \sum_{i=1}^{k}\left(C_l^{k+1-i} - C_l^{k-i}\right)e_j^{l,k+1}$$
$$+ \tau^{\alpha_l^{k+1}}\Gamma\left(2 - \alpha_l^{k+1}\right)f_l^{k+1}. \tag{15}$$

Here

$$b_l^{k+1} = \frac{D\tau^{\alpha_l^{k+1}}\Gamma\left(2 - \alpha_l^{k+1}\right)}{2R(h)^2},$$
$$c_l^{k+1} = -\frac{v\tau^{\alpha_l^{k+1}}\Gamma\left(2 - \alpha_l^{k+1}\right)}{2hR}, \tag{16}$$
$$d_l^{k+1} = \frac{\lambda\tau^{\alpha_l^{k+1}}\Gamma\left(2 - \alpha_l^{k+1}\right)}{2}$$

$e_j^{l,k+1} = (j+1)^{1-\alpha_l^{k+1}} - (j)^{1-\alpha_l^{k+1}}$, $p_j^{l,k+1} = e_{j-1}^{l,k+1} - e_j^{l,k+1}$ $j = 1, 2$, $\ldots, N$, and $l = 1, 2, \ldots, M - 1$. It is important to point out that the sum term on the right-hand side of (15) automatically vanishes when $k = 0$. Then, (15) can be reformulated as:

$$b_l^1\left[\left(C_{l+1}^1 - 2C_{l-1}^1 + C_{l-1}^0\right) + \left(C_{l+1}^0 - 2C_{l-1}^0 + C_{l-1}^0\right)\right]$$
$$+ c_l^1\left[\left(C_{l+1}^1 - C_{l-1}^1\right) + \left(C_{l+1}^0 - C_{l-1}^0\right)\right]$$
$$+ d_l^1\left(C_l^1 - C_l^0\right) = \tau^{\alpha_l^1}\Gamma\left(2 - \alpha_l^1\right)f_l^1,$$
$$\text{for } k = 0, \tag{17}$$

$$b_l^{k+1}\left[\left(C_{l+1}^{k+1} - 2C_l^{k+1} + C_{l-1}^{k+1}\right) + \left(C_{l+1}^k - 2C_l^k + C_{l-1}^k\right)\right]$$
$$+ c_l^{k+1}\left[\left(C_{l+1}^{k+1} - C_{l-1}^{k+1}\right) + \left(C_{l+1}^k - C_{l-1}^k\right)\right]$$
$$+ d_l^{k+1}\left(C_l^{k+1} - C_l^k\right)$$
$$= C_l^{k+1} - C_l^k + \sum_{i=1}^{k}\left(C_l^{k+1-i} - C_l^{k-i}\right)e_j^{l,k+1}$$
$$+ \tau^{\alpha_l^{k+1}}\Gamma\left(2 - \alpha_l^{k+1}\right)f_l^{k+1}, \quad \text{for } k \geq 1. \tag{18}$$

The above discretization can be rewritten in the following matrix form.

3.2. Stability Analysis of the Crank-Nicolson Scheme. In this section, we will analyze the stability conditions of the Crank-Nicolson scheme for the generalized advection dispersion equation.

Let $\zeta_l^k = c_l^k - C_l^k$; here C_l^k is the approximate solution at the point (x_l, t_k), $(k = 1, 2 \ldots, N,\ l = 1, 2, \ldots, M - 1)$ and in addition $\zeta^k = [\zeta_1^k, \zeta_2^k, \ldots, \zeta_{M-1}^k]^T$ and the function $\zeta^k(x)$ is chosen to be

$$\zeta^k(x) = \begin{cases} \zeta_l^k & \text{if } x_l - \dfrac{h}{2} < x \le x_l + \dfrac{h}{2},\ l = 1, 2, \ldots, M - 1 \\ 0 & \text{if } L - \dfrac{h}{2} < x \le L. \end{cases} \tag{19}$$

Then, the function $\zeta^k(x)$ can be expressed in Fourier series as follows:

$$\zeta^k(x) = \sum_{m=-\infty}^{m=\infty} \delta_m(m) \exp\left[\frac{2i\pi mk}{L}\right],$$

$$\delta_k(x) = \frac{1}{L} \int_0^L \rho^k(x) \exp\left[\frac{2i\pi mx}{L}\right] dx. \tag{20}$$

It was established by [25] that

$$\left\|\zeta^2\right\|_2^2 = \sum_{m=-\infty}^{m=\infty} \left\|\delta_k(m)\right\|^2. \tag{21}$$

Observe that, for all $k, l \ge 1$, $0 \le 1 - \alpha_l^{k+1} < 1$; in addition, according to the problem in point, the velocity seepage v, the dispersion coefficient D, the retardation factor R, and the radioactive decay constant λ are positive constants. Then the following properties of the coefficients c_l^{k+1}, d_l^{k+1}, $p_j^{l,k+1}$, and b_l^{k+1} can be established as follows.

(1) d_l^{k+1}, b_l^{k+1} are positive for all $l = 1, 2 \ldots, M - 1$.

(2) c_l^{k+1} is negative for all $l = 1, 2 \ldots, M - 1$.

(3) $0 < e_j^{l,k} \le e_{j-1}^{l,k} \le 1$ for all $l = 1, 2 \ldots, M - 1$.

(4) $0 \le p_j^{l,k} \le 1$, $\sum_{j=0}^{k-1} p_{j+1}^{l,k+1} = 1 - e_k^{l,k+1}$ for all $l = 1, 2 \ldots, M - 1$.

The error committed while approximating the solution of the generalized advection dispersion equation with Crank-Nicolson scheme can be presented as follows:

$$b_l^{k+1}\left[\left(\zeta_{l+1}^{k+1} - 2\zeta_l^{k+1} + \zeta_{l-1}^{k+1}\right) + \left(\zeta_{l+1}^k - 2\zeta_l^k + \zeta_{l-1}^k\right)\right]$$

$$+ c_l^{k+1}\left[\left(\zeta_{l+1}^{k+1} - \zeta_{l-1}^{k+1}\right) + \left(\zeta_{l+1}^k - \zeta_{l-1}^k\right)\right]$$

$$+ d_l^{k+1}\left(\zeta_l^{k+1} - \zeta_l^k\right)$$

$$= \zeta_l^{k+1} - \zeta_l^k + \sum_{i=1}^k \left(\zeta_l^{k+1-i} - \zeta_l^{k-i}\right) e_j^{l,k+1}$$

$$+ \tau^{\alpha_l^{k+1}} \Gamma\left(2 - \alpha_l^{k+1}\right) f_l^{k+1}. \tag{22}$$

If we assume that ζ_l^k in (22) can be put in the delta-exponential form as follows:

$$\zeta_l^k = \delta_k \exp\left[i\varphi lk\right], \tag{23}$$

where φ is a real spatial wave number, replacing (23) in (22) we obtain

$$\left[1 + 4b_l^1 \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^1 \sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_1$$

$$= \left[1 - 4b_l^1 \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^1 \sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_0, \quad \text{for } k = 0,$$

$$\left[1 + 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right)\right]\delta_{k+1}$$

$$= \left[1 - 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right]\delta_k$$

$$+ \sum_{j=0}^{k-1} p_{j+1}^{l,k+1} \delta_{k-j} + e_k^{l,k+1}\delta_0, \quad \text{for } k = 1, 2 \ldots, N - 1. \tag{24}$$

Equation (24) can be written in the following form:

$$\delta_1 = \frac{\left[1 - 4b_l^1 \sin^2\left(\varphi h/2\right) - 2d_l^1 \sin^2\left(\varphi h/2\right)\right]\delta_0}{\left[1 + 4b_l^1 \sin^2\left(\varphi h/2\right) - 2d_l^1 \sin^2\left(\varphi h/2\right)\right]},$$

$$\delta_{k+1} = \left(\left[1 - 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right]\delta_k\right.$$

$$\left. + \sum_{j=0}^{k-1} p_{j+1}^{l,k+1} \delta_{k-j} + e_k^{l,k+1}\delta_0\right)$$

$$\times \left[1 + 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right)\right]^{-1}. \tag{25}$$

Our next concern here is to show that for all $k = 1, 2, \ldots, N-1$ the solution of (25) satisfies the following condition:

$$\left|\delta_k\right| < \left|\delta_0\right|. \tag{26}$$

To achieve this, we make use of the recurrence technique on the natural number k.

For $k = 1$ and remembering that d_l^{k+1}, b_l^{k+1} are positive for all $l = 1, 2 \ldots, M - 1$, then we obtain

$$\frac{\left|\delta_1\right|}{\left|\delta_0\right|} = \left|\frac{\left[1 - 4b_l^1 \sin^2\left(\varphi h/2\right) - 2d_l^1 \sin^2\left(\varphi h/2\right)\right]}{\left[1 + 4b_l^1 \sin^2\left(\varphi h/2\right) - 2d_l^1 \sin^2\left(\varphi h/2\right)\right]}\right| < 1. \tag{27}$$

Assuming that for $m = 2, 3, \ldots, k$ the property is verified, then

$$\left|\delta_{k+1}\right| = \left|\left(\left[1 - 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right]\delta_k\right.\right.$$

$$\left. + \sum_{j=0}^{k-1} p_{j+1}^{l,k+1} \delta_{k-j} + e_k^{l,k+1}\delta_0\right)$$

$$\left.\times \left[1 + 4b_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k} \sin^2\left(\frac{\varphi h}{2}\right)\right]^{-1}\right|. \tag{28}$$

Making use of the triangular inequality, we obtain

$$
\begin{aligned}
\left|\delta_{k+1}\right| \leq & \left(\left|1 - 4b_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right) - e_1^{l,k+1}\right|\left|\delta_k\right| \right. \\
& + \left|\sum_{j=0}^{k-1} p_{j+1}^{l,k+1}\delta_{k-j}\right| + \left|e_k^{l,k+1}\delta_0\right| \Bigg) \\
& \times \left|1 + 4b_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|^{-1}.
\end{aligned}
\tag{29}
$$

Using the recurrence hypothesis, we have

$$
\begin{aligned}
& \left|\delta_{k+1}\right| \\
& \leq \left(\left(\left|1 - 4b_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right| + \left|\sum_{j=0}^{k-1} p_{j+1}^{l,k+1}\right|\right) \right. \\
& \quad \times \left|1 + 4b_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right) - 2d_l^{1+k}\sin^2\left(\frac{\varphi h}{2}\right)\right|^{-1}\Bigg) \left|\delta_0\right|, \\
& \left|\delta_{k+1}\right| \leq \left(\frac{\left|1 + 4b_l^{1+k}\sin^2(\varphi h/2) - 2d_l^{1+k}\sin^2(\varphi h/2)\right|}{\left|1 + 4b_l^{1+k}\sin^2(\varphi h/2) - 2d_l^{1+k}\sin^2(\varphi h/2)\right|}\right)\left|\delta_0\right|, \\
& \left|\delta_{k+1}\right| \leq \left|\delta_0\right|
\end{aligned}
\tag{30}
$$

and this completes the proof.

3.3. Convergence Analysis of the Crank-Nicolson Scheme. If we assume that $c(x_l, t_k)$ $(l = 1, 2, \ldots, M, \ k = 1, 2\ldots, N-1)$ is the exact solution of our problem at the point (x_l, t_k), by letting $w_l^k = c(x_l, t_k) - C_l^k$ and $w^k = (0, w_1^k, w_2^k, \ldots, w_{M-1}^k)$ and substituting this in (18), we obtain

$$
b_l^1\left[\left(w_{l+1}^1 - 2w_{l-1}^1\right)\right] + c_l^1\left[\left(w_{l+1}^1 - wC_{l-1}^1\right)\right] + d_l^1\left(w_l^1\right) = F_l^1,
$$

$$
\text{for } k = 0,
$$

$$
\begin{aligned}
& b_l^{k+1}\left[\left(w_{l+1}^{k+1} - 2w_l^{k+1} + w_{l-1}^{k+1}\right)\right] \\
& \quad + c_l^{k+1}\left[\left(w_{l+1}^{k+1} - w_{l-1}^{k+1}\right)\right] + d_l^{k+1}w_l^{k+1} \\
& = F_l^{k+1} + \sum_{i=1}^{k}\left(w_l^{k-i}\right)e_j^{l,k+1}, \quad \text{for } k \geq 1.
\end{aligned}
\tag{31}
$$

Here

$$
\begin{aligned}
F_l^{k+1} = & \, d_l^{k+1}\left(-C\left(x_l, t_{k+1}\right) + C\left(x_l, t_k\right)\right) \\
& + \sum_{i=1}^{k}\left(C\left(x_l, t_{k-i+1}\right)\right)e_j^{l,k+1} \\
& + b_l^{k+1}\left(-C\left(x_{l+1}, t_k\right) + 2C\left(x_l, t_k\right) - C\left(x_{l-1}, t_k\right)\right) \\
& - c_l^{k+1}\left(C\left(x_{l+1}, t_k\right) - C\left(x_{l-1}, t_k\right)\right).
\end{aligned}
\tag{32}
$$

From (11), (12), and (14),

$$
\begin{aligned}
& \frac{\partial C\left(x_l, t_{k+1}\right)}{\partial x} + V_1 \\
& = \frac{1}{2}\left(\left(\frac{C\left(x_{l+1}, t_{k+1}\right) - C\left(x_{l-1}, t_{k+1}\right)}{2\left(h\right)}\right)\right. \\
& \quad \left. + \left(\frac{C\left(x_{l+1}, t_k\right) - C\left(x_{l-1}, t_k\right)}{2\left(h\right)}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2 C\left(x_l, t_{k+1}\right)}{\partial x^2} + h^2 V_2 \\
& = \frac{1}{2}\left(\frac{\left(C\left(x_{l+1}, t_{k+1}\right) - 2C\left(x_l, t_{k+1}\right) + C\left(x_{l-1}, t_{k+1}\right)\right)}{h^2}\right. \\
& \quad \left. + \frac{\left(C\left(x_{l+1}, t_k\right) - 2C\left(x_l, t_k\right) + C\left(x_{l-1}, t_k\right)\right)}{h^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^{\alpha_l^{k+1}} C\left(x_l, t_{k+1}\right)}{\partial t^{\alpha_l^{k+1}}} + \tau V_3 \\
& = \frac{\tau^{-\alpha_l^{k+1}}}{\Gamma\left(2 - \alpha_l^{k+1}\right)}\left(C\left(x_l, t_{k+1}\right) - C\left(x_l, t_k\right)\right. \\
& \quad + \sum_{j=1}^{k}\left[C\left(x_l, t_{k+1-j}\right) - C\left(x_l, t_{k-j}\right)\right] \\
& \quad \left. \times \left[(j+1)^{1-\alpha_l^{k+1}} - (j)^{1-\alpha_l^{k+1}}\right]\right).
\end{aligned}
\tag{33}
$$

From the above we have that

$$
F_l^{k+1} \leq V\left(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k} + h\tau^{\alpha_l^k}\right),
\tag{34}
$$

where V_1, V_2, V_3, and V are constants. Taking into account Caputo type fractional derivative, the detailed error analysis on the above schemes can refer to the work by Diethelm et al. [31] and further work by Li and Tao [32].

Lemma 4. $\|w^{k+1}\|_\infty \leq V(\tau^{1+\alpha_l^{k+1}} + h^2\tau^{\alpha_l^k} + h\tau^{\alpha_l^k})(e_j^{l,k+1})^{-1}$ *is true for* $(k = 0, 1, 2\ldots, N-1)$ *where* $\|w^k\|_\infty = \max_{1\leq l\leq M-1}(w^k)$ *and* V *is a constant. In addition,*

$$
\alpha^{k+1} = \begin{cases} \min\limits_{1\leq l\leq M-1}\alpha_l^{k+1}, & \text{if } \tau < 1 \\ \max\limits_{1\leq l\leq M-1}\alpha_l^{k+1}, & \text{if } \tau > 1. \end{cases}
\tag{35}
$$

This can be achieved via the recurrence technique on the natural number k.

When $k = 0$, one has the following:

$$
\begin{aligned}
\left| w_l^1 \right| &\le \left(c_l^1 + b_l^1 \right) \left| w_{l+1}^1 \right| \\
&\quad + \left(c_l^1 - 2b_l^1 \right) \left| w_{l-1}^1 \right| + d_l^1 \left| w_l^1 \right| \\
&= \left| F_l^1 \right| \\
&\le V \left(\tau^{1+\alpha_l^{k+1}} + h^2 \tau^{\alpha_l^k} + h\tau^{\alpha_l^k} \right) \left(e_j^{l,k+1} \right)^{-1}.
\end{aligned}
\tag{36}
$$

Now suppose that $\|w^{i+1}\|_\infty \le V(\tau^{1+\alpha_l^{i+1}} + h^2\tau^{\alpha_l^i} + h\tau^{\alpha_l^i})(e_j^{l,i+1})^{-1}$, $i = 1, \ldots, N-2$. *Then*

$$
\begin{aligned}
\left| w_l^{1+k} \right| &\le \left| b_l^{k+1} \left[\left(w_{l+1}^{k+1} - 2w_l^{k+1} + w_{l-1}^{k+1} \right) \right] \right. \\
&\quad \left. + c_l^{k+1} \left[\left(w_{l+1}^{k+1} - w_{l-1}^{k+1} \right) \right] + d_l^{k+1} w_l^{k+1} \right| \\
&\le \left(b_l^{k+1} + c_l^{k+1} \right) \left| w_{l+1}^{k+1} \right| + \left(b_l^{k+1} - c_l^{k+1} \right) \left| w_{l-1}^{k+1} \right| \\
&\quad + \left(d_l^{k+1} - 2b_l^{k+1} \right) \left| w_l^{k+1} \right| \\
&= \left| F_l^{k+1} + \sum_{i=1}^{k} \left(w_l^{k-i} \right) e_j^{l,k+1} \right| \\
&\le \left| F_l^{k+1} \right| + \sum_{i=1}^{k} \left| w_l^{k-i} \right| e_j^{l,k+1} \\
&\le V \left(\tau^{1+\alpha_l^{k+1}} + h^2 \tau^{\alpha_l^k} + h\tau^{\alpha_l^k} \right) + \sum_{i=1}^{k} \left\| w_l^{k-i} \right\|_\infty e_j^{l,k+1} \\
&\le V \left(\tau^{1+\alpha_l^{k+1}} + h^2 \tau^{\alpha_l^k} + h\tau^{\alpha_l^k} \right) \\
&\quad \times \left(e_j^{l,k+1} + e_0^{l,k+1} - e_j^{l,k+1} \right) \left(e_j^{l,k+1} \right)^{-1} \\
&\le V \left(\tau^{1+\alpha_l^{k+1}} + h^2 \tau^{\alpha_l^k} + h\tau^{\alpha_l^k} \right) \left(e_0^{l,k+1} \right) \left(e_j^{l,k+1} \right)^{-1}, \\
\left| w_l^{1+k} \right| &\le V \left(\tau^{1+\alpha_l^{k+1}} + h^2 \tau^{\alpha_l^k} + h\tau^{\alpha_l^k} \right) \left(e_j^{l,k+1} \right)^{-1}.
\end{aligned}
\tag{37}
$$

This completes the proof.

Theorem 5. *The Crank-Nicolson scheme is convergent, and there exists a positive constant V such that*

$$
\begin{aligned}
\left| C_l^k - c\left(x_l, t_k \right) \right| &\le V \left(\tau + h + h^2 \right), \\
l &= 1, 2, \ldots, M-1, \quad k = 1, 2, \ldots, N.
\end{aligned}
\tag{38}
$$

An interested reader can find the solvability of the Crank-Nicholson scheme in the work done by [33]. Therefore, the details of the proof will not be presented in this paper.

4. Numerical Simulations

In this section, we present the numerical solutions of the following three cases.

Table 1: Theoretical parameters.

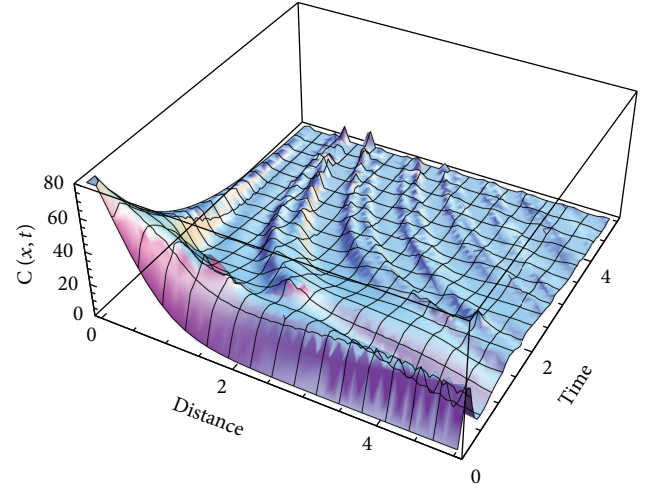| Parameters	D	v	R	γ	c_0	λ
Values	2	2	1	0.86	100	1

Figure 1: Approximate solution of the modified equation as function of time and space.

The case where we assume that the variation of the geological formation in which the contamination is migrating can be modelled as function of time and space as $\alpha(x,t)$. To include explicitly this variability in the mathematical formulation, we reformulate the hydrodynamic dispersion equation as follows:

$$
\begin{aligned}
D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} - \lambda RC &= RD_0^{\alpha(t,x)}(C), \\
\alpha(x,t) &= 1 - \sin(xt), \\
C(x,0) = 0, \qquad C(0,t) &= c_0 \exp(-\gamma t).
\end{aligned}
\tag{39}
$$

The theoretical parameters used in this simulation are given in Table 1.

Figure 1 shows the approximate solution of the main problem as function of space and time. Figure 2 shows the density plot of the approximate solution of the main problem. Figure 3 shows the variation of the concentration as function of time for a fixed space. Figures 1 and 3 show the solution of the variation order advection and advection equation, respectively, and Figure 2 shows the contour map of variational order advection equation.

5. Discussion and Conclusions

Natural geological deposits with highly contrasting permeability may form mobile and relatively immobile zones, where the potential mass exchange between mobile and immobile zones results in a wide time distribution for solute "trapping." The transport process, groundwater, is, by its very nature, always in contact with the matrix of an aquifer. There is thus a possibility that the solutes may interact with the rock
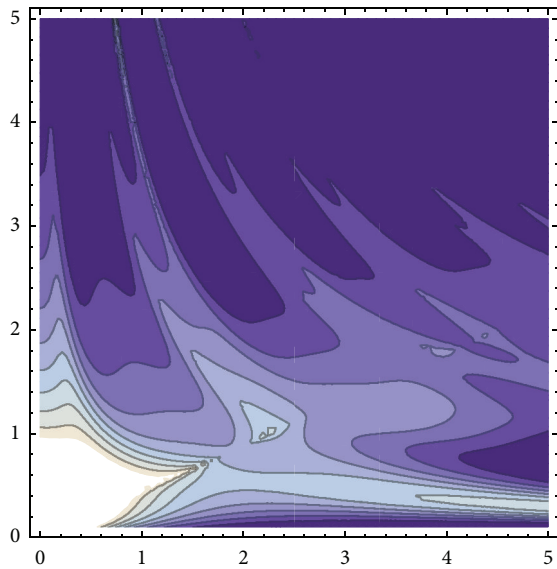
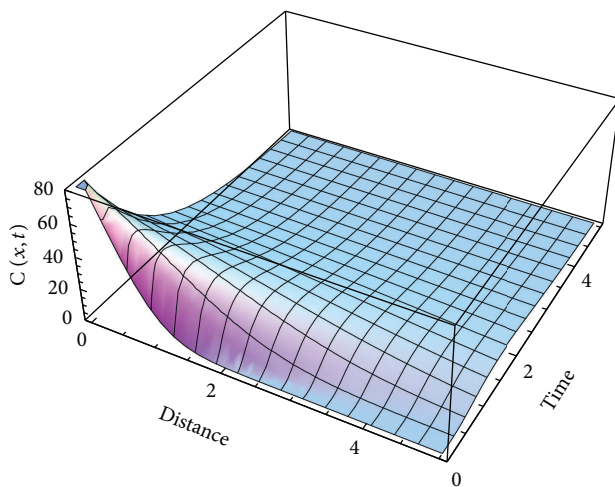FIGURE 2: Solution of the modified equation seing as contour map.

FIGURE 3: Solution of the standard equation.

the solution. These situations (a), (b), and (c) can be characterized efficiently by the time-nonlocal model, including the time variational order advection dispersion equation. If the high-permeable material tends to form preferential flow paths, such as the interconnected paleochannels observed in alluvial depositional systems, then the solute transport may show a heavy leading edge, which can be described by the Variational Order Advection Dispersion Equation (VOADE). Development of partial differential equations such as the advection-dispersion equation (ADE) begins with assumptions about the random behaviour of a single particle: possible velocities it may experience in a flow field and the length of time it may be immobilized. When assumptions underlying the ADE are relaxed, a fractional ADE (FADE) can arise, with a non-integer-order derivative on time or space terms. Fractional ADEs are nonlocal; they describe transport affected by hydraulic conditions at a distance. Space fractional ADEs arise when velocity variations are heavy tailed and describe particle motion that accounts for variation in the flow field over the entire system [34]. But when the geological formation, under which this pollution moves, is changing in time and space, the VOADE can be used. Many other uses of the variable-order derivatives can be found in the following works [35–40].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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matrix and one another. A true mathematical model for groundwater pollution must therefore be able to account for interactions between the dissolved solids and matrix of the aquifer. It will thus be advantageous to look at the nature of the interactions between dissolved solids and a porous medium that may be expected in groundwater pollution. Experimental evidence indicates that when a dissolved solid comes in contact with the matrix of a porous medium it may (a) pass through the medium with no apparent effect, (b) be absorbed by the porous matrix, and (c) react with the porous matrix and other substances dissolved in the fluid. The dissolved solids encountered in porous flow are, for this reason, often classified as conservative, nonconservative, and reactive tracers. This behaviour implies that the quantity of dissolved solids in a porous medium depends not only on the flow pattern but also on the nature of the porous matrix and

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