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Research Article

Basins of Attraction for Various Steffensen-Type Methods

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The dynamical behavior of different Steffensen-type methods is analyzed. We check the chaotic behaviors alongside the convergence radii (understood as the wideness of the basin of attraction) needed by Steffensen-type methods, that is, derivative-free iteration functions, to converge to a root and compare the results using different numerical tests. We will conclude that the convergence radii (and the stability) of Steffensen-type methods are improved by increasing the convergence order. The computer programming package Mathematica provides a powerful but easy environment for all aspects of numerics. This paper puts on show one of the application of this computer algebra system in finding fixed points of iteration functions.

1. Introduction

Suppose that we wish to find a solution ξ of a nonlinear equation $f(z) = 0$ numerically, where $z \in \mathbb{C}$ and the function $f : \psi \subseteq \mathbb{C} \to \mathbb{C}$ is an analytical complex function. Starting from some z_0, the Steffensen's method [1] uses the iteration:

$$z_{n+1} = z_n - \frac{f(z_n)^2}{f(z_n + f(z_n)) - f(z_n)}, \quad n = 0, 1, 2, \ldots. \tag{1}$$

If the initial value z_0 is close enough to a simple root of $f(z)$, this iteration converges quadratically (see [2, 3] for more information about Steffensen-type methods). If the root is not simple, the convergence becomes linear. Since then, a tremendous amount of effort has been made in the direction of improving the convergence and/or the simplicity of the method resulting in modified Steffensen method in the divided difference form [4] as follows (m is the multiplicity):

$$z_{n+1} = z_n - m\frac{f(z_n)}{f[z_n, w_n]}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

wherein $w_n = z_n + f(z_n)$ and $f[z_n, w_n]$ is the two-point divided difference. Notice that the modification for multiple roots given by (2) is the same as that given by Schröder for Newton's method. We remark that the notation of divided differences will be used throughout this paper.

Note that the Steffensen method could be written in a more generalized form with one free nonzero parameter [5] as follows:

$$z_{n+1} = z_n - \frac{f(z_n)}{f[z_n, w_n]}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

wherein $w_n = z_n + \beta f(z_n)$ and $\beta \in \mathbb{C} \setminus \{0\}$. It is known that this method has second order of convergence, for every nonzero value of the parameter β. However, as we will see in this paper, the parameter β plays an important role for choosing the initial estimation in the stability of the method, and so forth.

Usually, efficiency indices are used to compare the behavior of iterative methods. Traub in [5] used the operational efficiency index $p^{1/op}$, where p is the convergence rate and op is the number of products/quotients per iteration. Ostrowski in [6] defined the classical efficiency index $p^{1/d}$, wherein d is the number of functional evaluations per iteration. Similarly, one may use the informational index defined as p/d. In this work, we are going to introduce another criterion for comparing the iterative methods used: the basins of attraction, (see, e.g., [7–13]).

The dynamical analysis of the rational function associated with an iterative method gives us important information about its stability and reliability. There exists a huge number of publications related to the numerical and dynamical properties of iteration functions (see, e.g., [14–18]).

It is known that the basin of attraction of an iterative method on a particular function is an interesting tool to visually get to know how an iteration function behaves as a function of different initial points [19].

The remaining sections of this paper are organized as follows. Section 2 provides some basic concepts in order to deal with dynamics associated with iterative functions. It is followed by Section 3, wherein a discussion over the basins of attraction for the uniparametric Steffensen scheme (3) on higher degree polynomials, using the software Mathematica 8, is given. Section 4 gives the list of methods that we are going to compare and their dynamics on quadratic polynomials is analyzed. Next in Section 5, we present the basins of attraction for various high-order Steffensen-type methods. Some conclusions and discussions are illustrated in Section 6 to end the paper.

2. Basic Concepts

We need some basic definitions and notions before going to Section 3. Thus, now we remind them briefly.

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map on the Riemann sphere. Then, a periodic point z_0 of period m is such that $R^m(z_0) = z_0$, where m is the smallest such integer. If $m = 1$, z_0 is called fixed point of R and also point z_0 is called attracting if $|R'(z_0)| < 1$, repelling if $|R'(z_0)| > 1$, and neutral if $|R'(z_0)| = 1$. If the derivative is zero then the point is called super-attracting [20]. If R is the rational function associated with an iterative method on a function f, the fixed points of R different from the roots of $f(z) = 0$, are called strange fixed points.

Note that the Julia set of a nonlinear map $R(z)$, denoted $\mathcal{J}(R)$, is the closure of the set of its repelling periodic points. The complementary of $\mathcal{J}(R)$ in the Riemann sphere is the Fatou set $\mathcal{F}(R)$.

A point z_0 is in the Julia set if and only if dynamics in a neighborhood of z_0 shows strong dependence on the starting conditions, so that nearby initial conditions yield to wildly different behavior after a number of full iterations.

Definition 1 (basin of attraction [21]). If a fixed point p of R is attracting, then all nearby points of p are attracted toward p under the action of R, in the sense that their iterates under R converge to p. The collection of all points whose iterates under R converge to p is called the basin of attraction of p, denoted by $B_p = \{x \in \mathbb{C} : \lim_{k\to\infty} R^k(x) = p\}$ with $R^k(x) = R(R(\cdots R(x)\cdots))$ as the k-fold composite map of x under R.

Lemma 2 (see [20]). *Every attracting periodic orbit is contained in the Fatou set of R. In fact, the entire basin of attraction B_p, which is an open set, for an attracting periodic orbit is contained in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

Kalantari in [22] coined the term "polynomiography" to be the art and science of visualization in the approximation of roots of polynomial using iteration functions. Note that a polynomiograph may or may not result in a fractal image. Even when a polynomiograph is a fractal image it does not diminish its uniqueness.

As we endeavor to solve increasingly complex problems, computer algebra systems (CAS) are becoming more and more important to our work. We use CAS to help with calculations too time consuming. Mathematica 8 is one of the most popular CASs available today [23]. Due to its wide applicability and power along with easiness [24], we apply this programming package in finding the chaotic behaviors of Steffensen-type methods.

3. Basins of Attractions for the Uniparametric Steffensen Scheme

A wide dynamical study of Steffensen's method (1) on quadratic and cubic polynomials has been developed in [12]. In this paper, it was showed that no Scaling Theorem is possible for this derivative-free scheme and, therefore, the analysis was made in particular polynomials. In general, it can be concluded that the stability of derivative-free schemes is worse than the one of methods with derivatives. In fact, in the dynamical planes associated with some of these methods, frequently the basin of attraction of the infinity appears. In fact, the infinity can be a strange fixed point: if $R(z)$ is the rational function associated to the method, then we prove that the infinity is a fixed point if $Q(0) = 0$, where $Q(z) = 1/R(1/z)$. If this happens, the infinity (as any other strange fixed point) can be attractive, repulsive, or neutral. This character is obtained analyzing the value of $Q'(0)$.

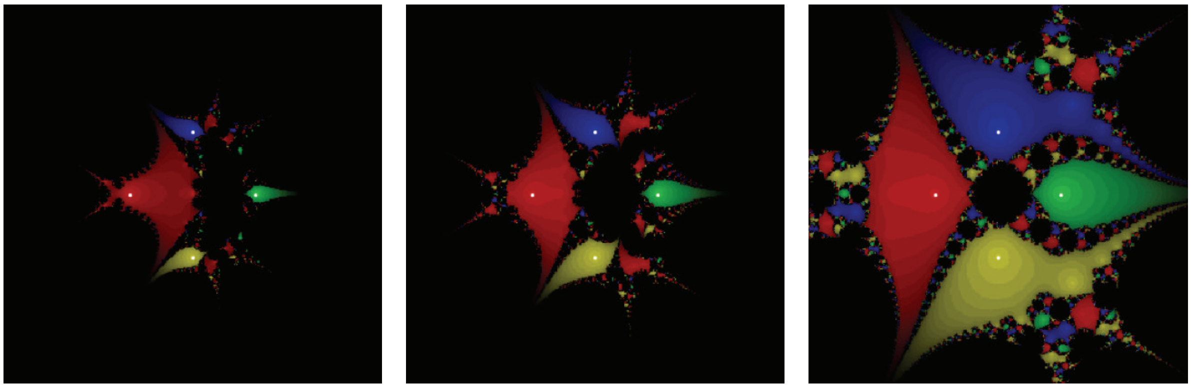
Before providing the chaotic behaviors of multipoint Steffensen-type methods, we conclude some points on the uniparametric Steffensen scheme (3).

Lemma 3. *The associated operator of the uniparametric Steffensen scheme on $z^2 - 1$ is $O_{St}(x) = (1 - \beta x + x^2 + \beta x^3)/(-\beta + 2x + \beta x^2)$. The only strange fixed point of $O_{St}(z)$ is the infinity, and its character is neutral.*

This result can be understood as the behavior of the infinity as fixed point can be attractive or repulsive, and it can be observed also when this method is applied on other functions, as we will see in Figures 5 to 10.

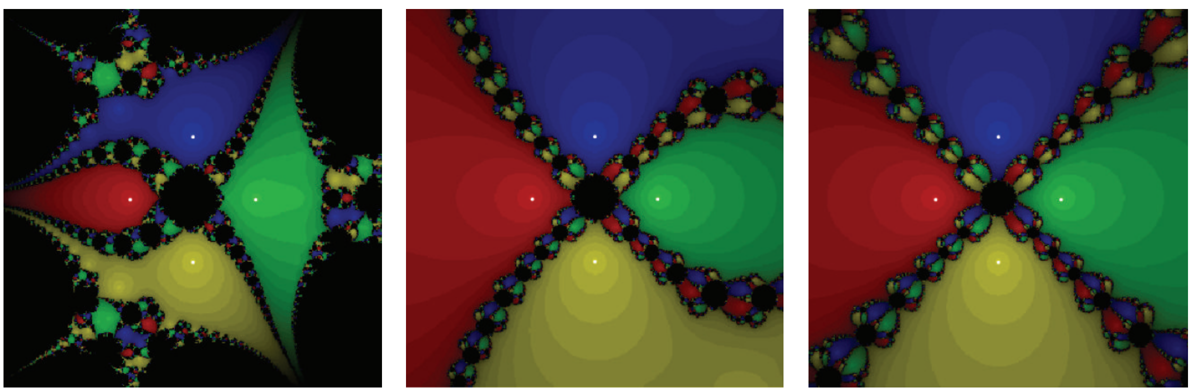
Toward this end and heretofore, we take a rectangle $D = [-3, 3] \times [-3, 3] \in \mathbb{C}$ and we assign a color to each point $z_0 \in D$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge after the maximum number of iterations. In this way, we distinguish the attraction basins by their colors for different methods.

The criteria we have used in our Mathematica 8 codes [25–27] are that the maximum number of iterations is 100.



(a) $\beta = 1$ (b) $\beta = 0.5$ (c) $\beta = 0.1$

Figure 1: Scheme (3) for the test problem 1.



(a) $\beta = -0.1$ (b) $\beta = 0.01$ (c) $\beta = 0.001$

Figure 2: Scheme (3) for the test problem 1.

That is to say, if the method does not reach the considered accuracy after 100 of its full iteration steps, we allocate the black color and also considered it as Not Convergent. The considered accuracy is the obtained residual of the function to be less than 10^{-4}. Note that the small white points will be shown in the exact location of the simple zeros in our fractal patterns.

In what follows, we define the test problems in this paper. For the first test, we have taken the following function with roots $\{-1., -i, i, 1.\}$:

$$p_1(z) = z^4 - 1. \tag{4}$$

The second test problem is a polynomial as follows when the simple zeros are $\{-2.20663, -0.45318 - 0.78493i, -0.45318 + 0.78493i, 0.906359, 1.10332 - 1.911i, 1.10332 + 1.911i\}$:

$$p_2(z) = z^6 + 10z^3 - 8. \tag{5}$$

The third test problem is chosen as follows:

$$p_3(z) = z^4 - \frac{1}{z}. \tag{6}$$

The roots are $\{0.309017 + 0.951057i, 0.309017 - 0.951057I, 1., -0.809017 + 0.587785i, -0.809017 - 0.587785i\}$.

The fourth test problem is taken into account as

$$p_4(z) = z^4 - z + i. \tag{7}$$

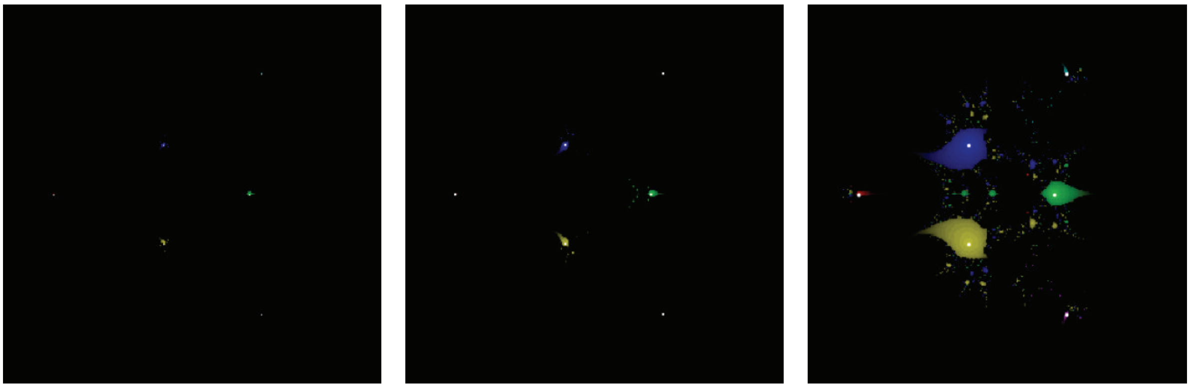
The roots are $\{-0.759845 + 0.592595i, -0.532605 - 1.08829i, 0.181924 + 0.732098I, 1.11052 - 0.236405i\}$.

The last test problem we have chosen is as follows when the roots are $\{-1.09112 + 0.629961i, 0. - 1.25992i, 1.09112 + 0.629961i\}$:

$$p_5(z) = z^3 - 2i. \tag{8}$$

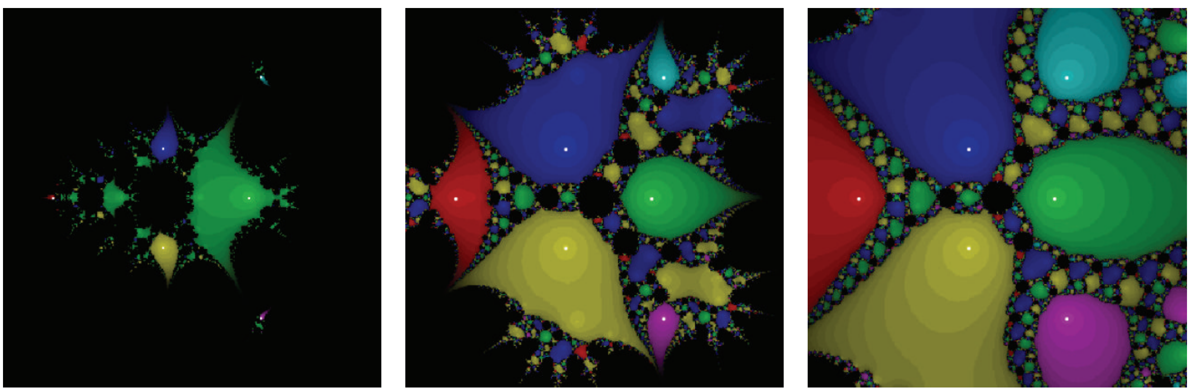
We can observe on Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 that scheme (3) for the choice $\beta = 0.001$ is very robust in contrast to the other cases. Method (3) has different radii of convergence according to the free nonzero parameter β. This is also true for all the test functions for Steffensen's scheme ($\beta = 1$).

From the graphical comparisons in this section, it is obvious that in the Steffensen uniparametric scheme (3), the basins of attractions can be widely improved by choosing a small value for the nonzero free parameter β. In fact, from the Taylor series expansion of the divided difference and the



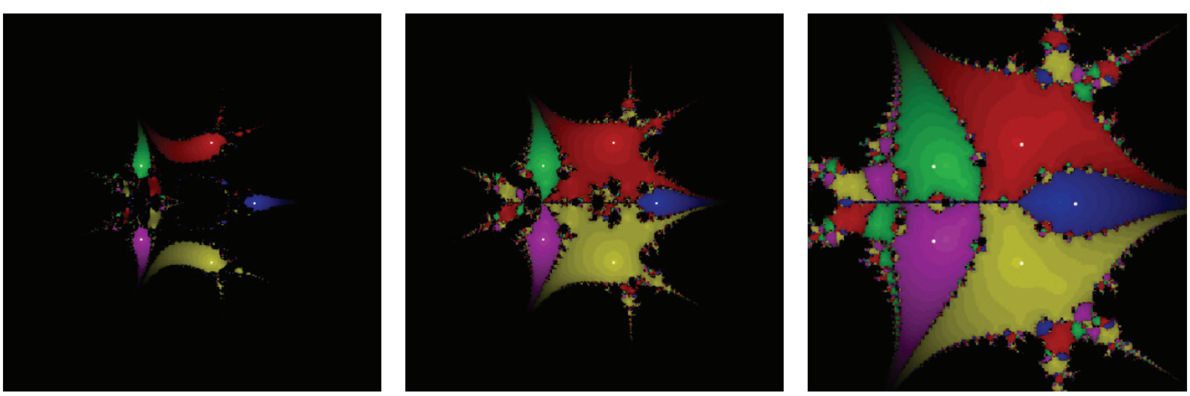
(a) $\beta = 1$ (b) $\beta = 0.5$ (c) $\beta = 0.1$

Figure 3: Scheme (3) for the test problem 2.



(a) $\beta = -0.1$ (b) $\beta = 0.01$ (c) $\beta = 0.001$

Figure 4: Scheme (3) for the test problem 2.

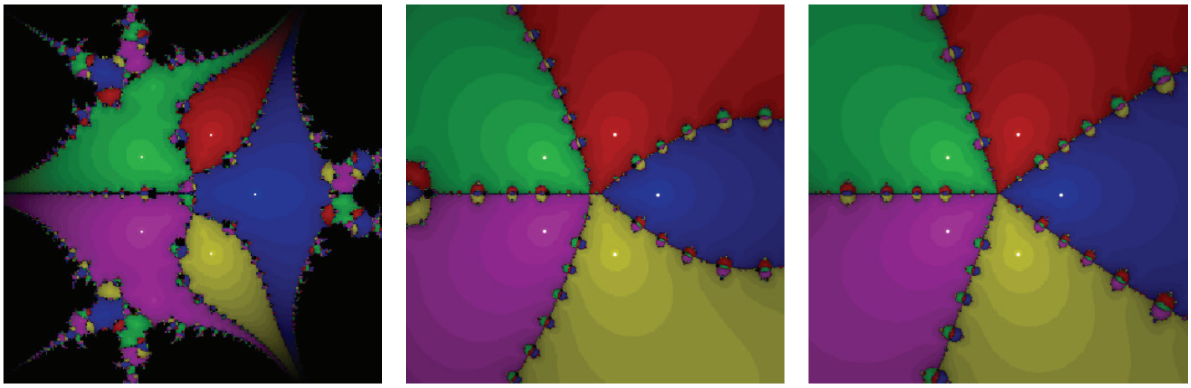


(a) $\beta = 1$ (b) $\beta = 0.5$ (c) $\beta = 0.1$

Figure 5: Scheme (3) for the test problem 3.

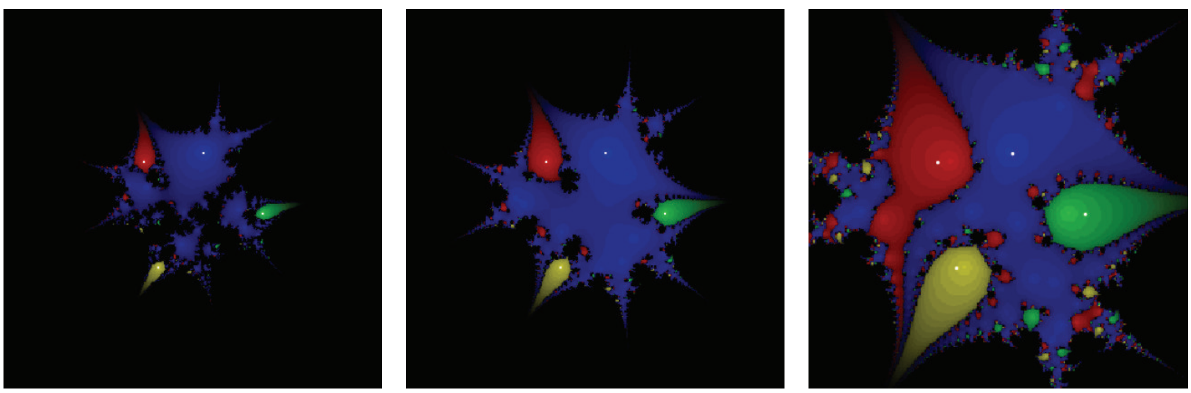
derivative around the solution ξ of the nonlinear equation, we have

$$\begin{aligned} f[x_n, w_n] &= \frac{f(x_n + \beta f(x_n)) - f(x_n)}{\beta f(x_n)} \\ &= f'(\xi)\left[1 + \left(2 + \beta f'(\xi)\right) c_2 e_n \right. \\ &\quad + \left(\beta f'(\xi) c_2^2 \right. \\ &\quad \left.\left. + \left(3 + 3\beta f'(\xi) + \beta^2 f'(\xi)^2\right) c_3\right) e_k^2\right] \\ &\quad + O\left(e_n^3\right), \\ f'(x_n) &= f'(\xi)\left[1 + 2f'(\xi) c_2 e_n + 3f'(\xi) c_3 e_n^2\right] + O\left(e_k^3\right). \end{aligned} \tag{9}$$



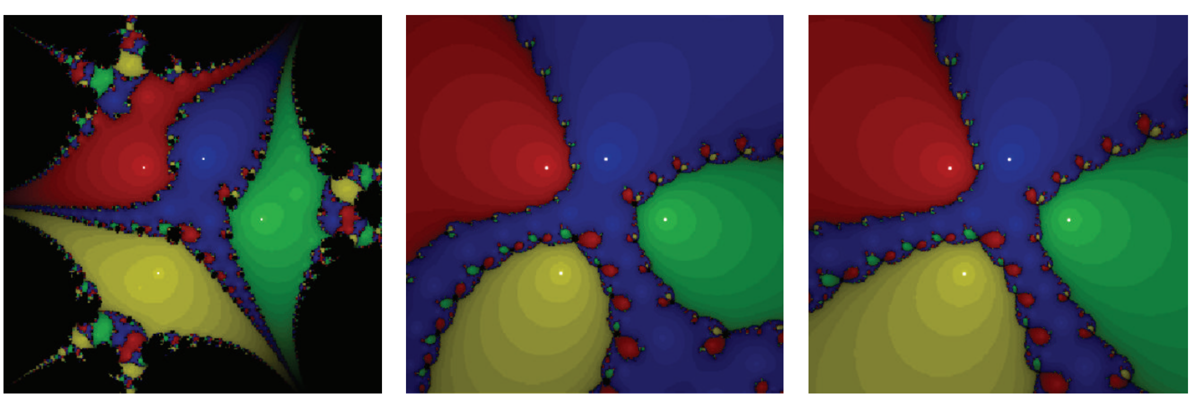
(a) $\beta = -0.1$ (b) $\beta = 0.01$ (c) $\beta = 0.001$

Figure 6: Scheme (3) for the test problem 3.



(a) $\beta = 1$ (b) $\beta = 0.5$ (c) $\beta = 0.1$

Figure 7: Scheme (3) for the test problem 4.

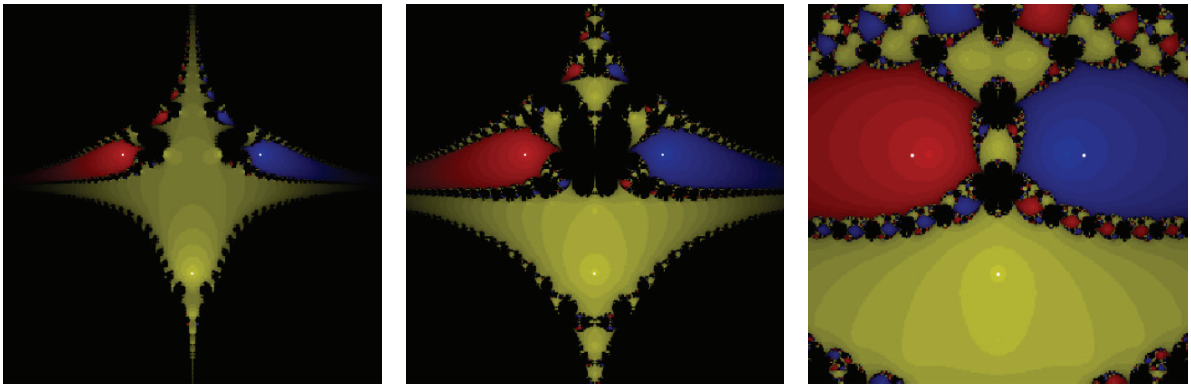


(a) $\beta = -0.1$ (b) $\beta = 0.01$ (c) $\beta = 0.001$

Figure 8: Scheme (3) for the test problem 4.

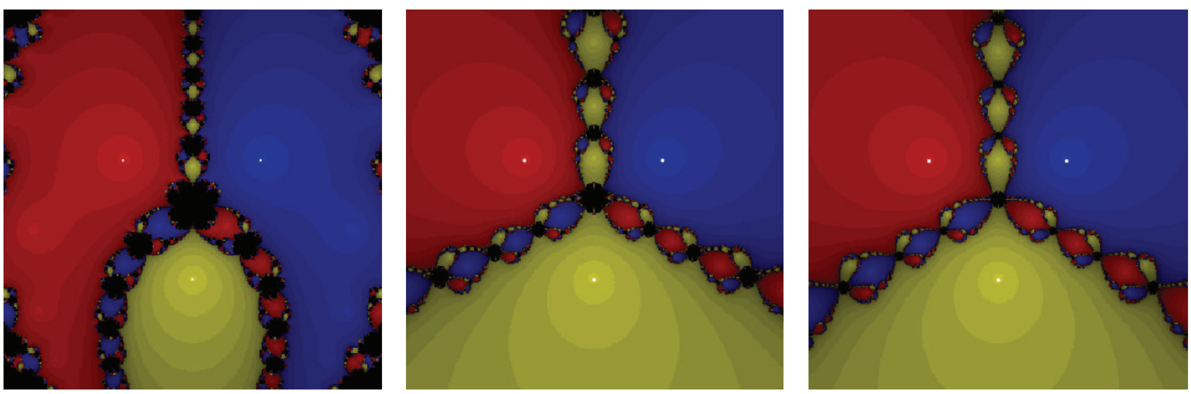
Let us note that, when $\beta \to 0$, both Taylor expansions coincide. Note that it is experimentally observable that (almost always) when the forward finite difference is used at the denominator of Steffensen's scheme, that is, $f[x_n, w_n] = (f(x_n + \beta f(x_n)) - f(x_n))/\beta f(x_n)$, then very small *negative* values for β results in larger basins of attractions and higher speed, while when the backward finite difference is used at the denominator, that is, $f[x_n, w_n] = (f(x_n) - f(x_n - \beta f(x_n)))/\beta f(x_n)$, then very small *positive* values for β yield in larger basins of attractions and higher speed.

This finding is very useful when we extend Steffensen's scheme for solving systems of nonlinear equations by defining the first order divided difference operator properly. In fact, by choosing very small values for the free nonzero



(a) $\beta = 1$ (b) $\beta = 0.5$ (c) $\beta = 0.1$

Figure 9: Scheme (3) for the test problem 5.



(a) $\beta = -0.1$ (b) $\beta = 0.01$ (c) $\beta = 0.001$

Figure 10: Scheme (3) for the test problem 5.

parameter the convergence radii will be similar to Newton's method for solving systems of nonlinear equations, while there is no need to compute the Jacobian matrix which is costly.

Even by applying an iteration on β and make the process *with memory*, we can obtain the cubical order of convergence. Another interesting point which should be included is the computational time required for obtaining the fractal behaviors of such schemes. In fact, in our numerical experiments, the computational CPU time has a dramatically fall when β is chosen as a very small value while choosing $\beta = 1$ takes more time to find the fixed points of the iteration functions.

Remark 4. The chaotic behavior of Steffensen's method can be simply improved by choosing very small entries for the free nonzero parameter (see Figures 6(c), 8(c), and 10(c)). In this case, the fractal behavior of the scheme tends to Newton's fractal. In fact, we want the methods to have very few black points on all examples not just one.

4. Behavior of Several Steffensen-Like Methods on Quadratic Polynomials

Many iterative methods have been improved by using various techniques [28, 29]. A drawback of Newton's method is that for many particular choices of the function f, especially in hard problems, the calculations of the derivatives take some deal of time. That is why higher order derivative-free methods are better root solvers and are in focus recently. The most important merit of Steffensen's method is that it has quadratic convergence like Newton's method. For this reason, in this section, we list some of the multipoint derivative-free methods we consider for comparisons and a brief dynamical analysis is made for these methods applied on quadratic polynomials.

Kung and Traub in the pioneer paper [30] provided the following two- and three-step derivative-free families ($\beta \in \mathbb{C} \setminus \{0\}$) of methods with orders four and eight, respectively,

$$y_n = x_n + \beta f(x_n),$$

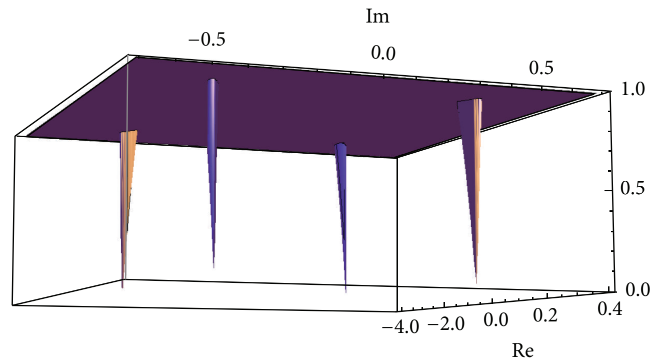


FIGURE 11: Stability function of attractive strange fixed points of KT4.

$$z_n = y_n - \beta \frac{f(x_n) f(y_n)}{f(y_n) - f(x_n)},$$
$$x_{n+1} = z_n - \frac{f(x_n) f(y_n)}{f(z_n) - f(x_n)} \left[\frac{1}{f[y_n, x_n]} - \frac{1}{f[z_n, y_n]} \right], \quad (10)$$

$$y_n = x_n + \beta f(x_n),$$
$$z_n = y_n - \beta \frac{f(x_n) f(y_n)}{f(y_n) - f(x_n)},$$
$$w_n = z_n - \frac{f(x_n) f(y_n)}{f(z_n) - f(x_n)} \left[\frac{1}{f[y_n, x_n]} - \frac{1}{f[z_n, y_n]} \right],$$
$$x_{n+1} = w_n - \frac{f(x_n) f(y_n) f(z_n)}{f(w_n) - f(x_n)}$$
$$\times \left[\frac{1}{f(w_n) - f(y_n)} \left\{ \frac{1}{f[w_n, z_n]} - \frac{1}{f[z_n, y_n]} \right\} - \frac{1}{f(z_n) - f(x_n)} \left\{ \frac{1}{f[z_n, y_n]} - \frac{1}{f[y_n, x_n]} \right\} \right]. \quad (11)$$

We would like to study the general convergence of methods (10) and (11) for quadratic polynomials. To be more precise (see [31, 32]), a given method is generally convergent if the scheme converges to a root for almost every starting point and for almost every polynomial of a given degree. The main problem is that no Scaling Theorem can be stated for derivative-free methods (see, e.g., [33]). So, only the behavior on specific polynomials can be analyzed. In this paper, the polynomial $p(z) = z^2 - 1$ will be used for this purpose. Therefore, let us denote by $O_{\mathrm{KT4}}(z)$ ($O_{\mathrm{KT8}}(z)$) the operator associated with the fourth-order (resp., eighth-order) scheme by Kung and Traub on $p(z)$.

The fixed points of $O_{\mathrm{KT4}}(z)$ are the roots of the equation $O_{\mathrm{KT4}}(z) = z$, that is, $z = -1$, $z = 1$, and the strange fixed points that are the zeros of the polynomial $1 + \beta^2 + \beta^4 - 4\beta(1 + 2\beta^2)z + (2 + 21\beta^2 - 4\beta^4)z^2 - 24\beta(1 - \beta^2)z^3 + (13 - 45\beta^2 + 6\beta^4)z^4 + (28\beta - 24\beta^3)z^5 + (23\beta^2 - 4\beta^4)z^6 + 8\beta^3 z^7 + \beta^4 z^8$.

The stability of the strange fixed points can be deduced from graphical analysis of the respective stability function, that is, representing graphically the regions of the complex plane in which the absolute value of the derivative of the operator evaluated at the strange fixed point is lower than one.

Lemma 5. *The number of simple strange fixed points of operator $O_{\mathrm{KT4}}(z)$ is eight, and their stability is described in the following cases.*

(i) *Four of them are always repulsive, so they remain in the Julia set.*

(ii) *Two of them can be attractive (but not super attractive) in a small complex neighborhood of the origin.*

(iii) *Finally, two of the strange fixed points are attractive in complex region around the origin and are super attractive for the following values of the parameter: $\beta \approx -0.406175 - 1.63893i$, $\beta \approx -0.406175 + 1.63893i$, $\beta \approx 0.276582 - 0.409103i$, and $\beta \approx 0.276582 + 0.409103i$.*

The stability region of the complex plane where these four fixed points are attractive is represented in Figure 11.

In order to determine the critical points, we calculate the first derivative of $O_{\mathrm{KT4}}(z)$. A classical result establishes that there is at least one critical point associated with each invariant Fatou component. It is clear that $z = -1$ and $z = 1$ are critical points and give rise to their respective Fatou components, but there exist in the family, some *free critical points*, that is, critical points different from the roots, some of them depending on the value of the parameter. If any of these free critical points is near a stable strange fixed point, then the last one would have its own basin of attraction.

Lemma 6. *Analyzing the equation $O'_{\mathrm{KT4}}(z) = 0$, we obtain fourteen free critical points, roots of the polynomial:*

$$2 + 7\beta^2 - 3\beta^4 - \beta^6 - \beta^8 + \left(-28\beta + 26\beta^3 + 10\beta^5 + 16\beta^7\right) z$$
$$+ \left(26 - 96\beta^2 - 25\beta^4 - 102\beta^6 + 7\beta^8\right) z^2$$
$$+ \left(140\beta - 24\beta^3 + 350\beta^5 - 96\beta^7\right) z^3$$

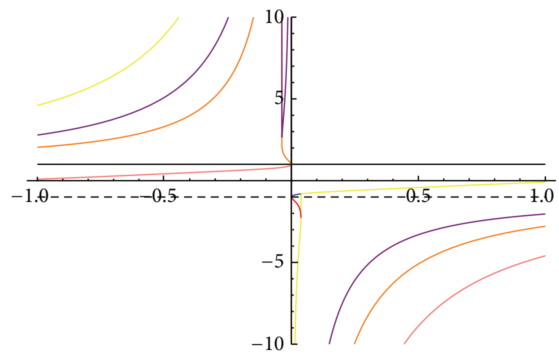


Figure 12: Real values of free critical points of $O_{KT4}(z)$.

$$\begin{aligned}
&+\left(-58+166\beta^2-750\beta^4+525\beta^6-21\beta^8\right)z^4\\
&+\left(-164\beta+1068\beta^3-1500\beta^5+240\beta^7\right)z^5\\
&+\left(30-904\beta^2+2430\beta^4-1060\beta^6+35\beta^8\right)z^6\\
&+\left(308\beta-2168\beta^3+2300\beta^5-320\beta^7\right)z^7\\
&+\left(827\beta^2-2495\beta^4+1065\beta^6-35\beta^8\right)z^8\\
&+\left(1098\beta^3-1550\beta^5+240\beta^7\right)z^9\\
&+\left(843\beta^4-534\beta^6+21\beta^8\right)z^{10}+\left(390\beta^5-96\beta^7\right)z^{11}\\
&+\left(107\beta^6-7\beta^8\right)z^{12}+16\beta^7z^{13}+\beta^8z^{14}.
\end{aligned}\tag{12}$$

Moreover, we can state the following facts.

(a) *There is no value of β that makes coincide a free critical point with a strange fixed point.*

(b) *For $\beta \approx 0.05$ one free critical point coincides with the root $z=-1$, so in this case the number of free critical points is reduced to 13.*

The real values of the critical points for a real range of values of the parameter are showed in Figure 12.

Let us state that the strange fixed points of $O_{KT8}(z)$ are the zeros of a polynomial of degree 39, whose coefficients depend on β.

The stability of the strange fixed points is described in the following lemma.

Lemma 7. *The number of simple strange fixed points of operator $O_{KT8}(z)$ is thirty-nine, and their stability is described in the following cases.*

(i) *Three of them are always repulsive, so they remain in the Julia set.*

(ii) *The other strange fixed points can be attractive or super attractive, in different complex regions.*

The stability region of some of these fixed points is represented in Figure 13.

By analyzing these stability functions, it is deduced that there are wide regions of the complex plane where it is easy to find two or more attractive strange fixed points (as, e.g., $\beta=-5i$ and $\beta=5i$). Moreover, there exist also wide regions of stability near the origin. In fact, no attractive strange fixed points can be found for $|\beta|<3/2$.

Analyzing the equation $O'_{KT8}(z)=0$, we obtain sixty free critical points that will coincide with some of the strange fixed points in case these are super attracting.

An optimal fourth-order derivative-free method [34] was introduced by Liu et al. in the following form:

$$\begin{aligned}
y_n &= x_n-\frac{f(x_n)^2}{f(x_n+f(x_n))-f(x_n)},\\
x_{n+1} &= y_n-\frac{f[x_n,y_n]-f[y_n,z_n]+f[x_n,z_n]}{f[x_n,y_n]^2}f(y_n),
\end{aligned}\tag{13}$$

where $z_n = x_n+f(x_n)$. It is denoted by L4. The rational function associated with this method on $p(z)$ is the operator $O_{L4}(z)$,

$$\begin{aligned}
O_{L4}(z) = {}&\left(1-5z+19z^2-27z^3-z^4+23z^5\right.\\
&\left.+5z^6+7z^7+8z^8+2z^9\right)\\
&\times\left(\left(-1+2z+z^2\right)\left(1-2z+3z^2+2z^3\right)^2\right)^{-1}.
\end{aligned}\tag{14}$$

Lemma 8. *The quantity of simple strange fixed points of operator $O_{L4}(z)$ is seven, $\{-3.4251, -1.98276-0.105743i, -1.98276+0.105743i, 0.109953-0.282276i, 0.109953+0.282276i, 0.585354-0.246666i, 0.585354+0.246666i\}$, and their stability is described in the following cases.*

(i) *All the complex strange fixed points are repulsive, so they remain in the Julia set.*

(ii) *The real strange fixed point is an attractor, but not super attractor.*

Lemma 9. *The equation $O'_{L4}(z)=0$ yields the poly-nomial $1+5x-31x^2+23x^3+64x^4+30x^5+4x^6$ whose roots are the free critical points, $\{-3.94434, -2.25291, -1.85186, -0.114261, 0.331686-0.151469i, 0.331686+0.151469i\}$.*

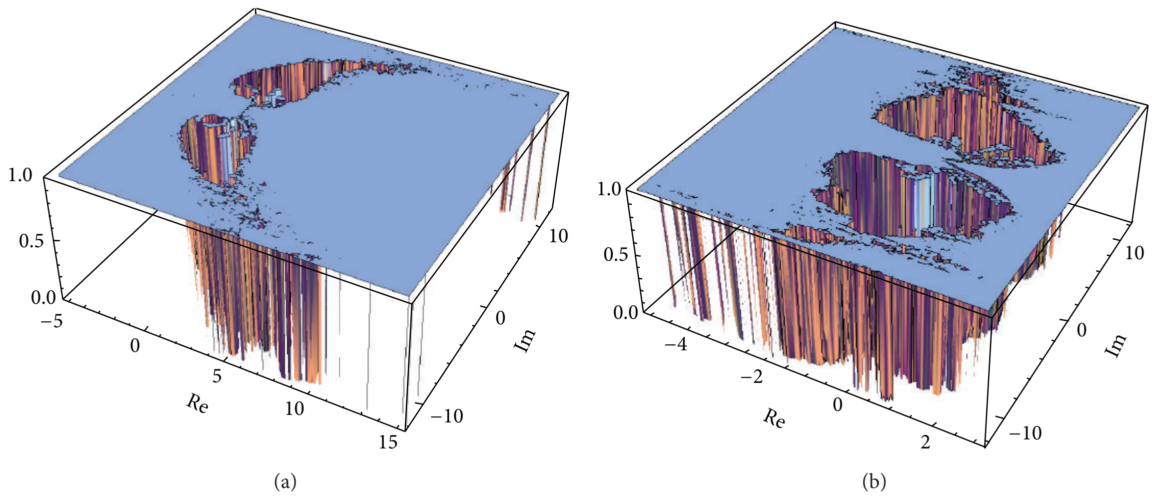


Figure 13: Stability regions of strange fixed points from KT8.

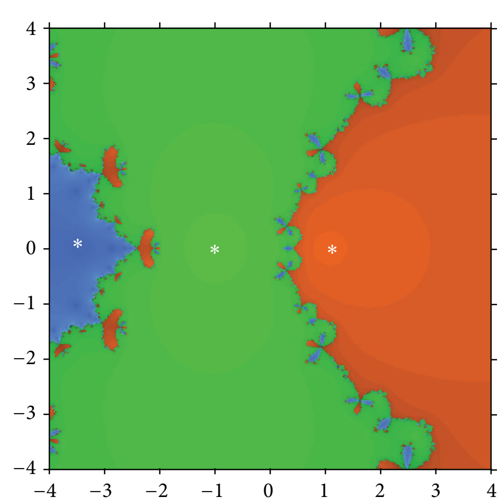


Figure 14: Dynamical plane of $O_{L4}(z)$ with three basins of attraction.

Let us remark that the existence of a free critical point near the real attracting fixed point will derive its own basin of attraction, as can be seen in Figure 14.

Cordero et al. in [35] proposed a seventh-order method, which is free from derivative:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)^2}{f(w_n) - f(x_n)}, \\ z_n &= y_n - f(y_n) \\ &\quad \times \left(\frac{(f(y_n) - f(w_n))}{(y_n - w_n)} - \frac{f(y_n)}{(y_n - w_n)} \right)^{-1}, \\ x_{n+1} &= z_n - f(z_n) \\ &\quad \times \left(\frac{(f(z_n) - f(y_n))}{(z_n - y_n)} - \frac{f(z_n)}{(y_n - z_n)} \right. \\ &\quad \left. - \frac{(f(y_n) - f(w_n))}{(y_n - w_n)} \right)^{-1} \end{aligned} \tag{15}$$

wherein $w_n = x_n + f(x_n)$. We will denote it by C7.

The rational function associated C7 on $p(z)$ is $O_{C7}(z)$,

$$\begin{aligned} O_{C7}(z) &= \left(1 - 9z + 13z^2 + 65z^3 - 112z^4 - 64z^5 + 119z^6 \right. \\ &\quad - 22z^7 - 98z^8 + 71z^9 + 152z^{10} + 85z^{11} + 21z^{12} \\ &\quad \left. + 2z^{13} \right) \\ &\quad \times \left(\left(-1 + 2z + z^2\right)^3 \right. \\ &\quad \left. \times \left(1 - 4z + 14z^2 + 8z^3 + 23z^4 + 12z^5 + 6z^6\right) \right)^{-1}. \end{aligned} \tag{16}$$

Lemma 10. *The simple strange fixed points of operator $O_{C7}(z)$ are*

(i) $\{-2.30882 - 0.732518i, -2.30882 + 0.732518i, -2.05354 - 0.276071i, -2.05354 + 0.276071i, -1.87555, -0.411291, 0.200847 - 0.0392285i, 0.200847 + 0.0392285i, 0.638264 - 0.0474315i, 0.638264 + 0.0474315i\}$, *and all of them are finite and repulsive as the derivable operator evaluated in each of them is greater than one, in absolute value;*

(ii) *the infinity is also a strange fixed point and its character is neutral.*

Lemma 11. *The free critical points of $O_{C7}(z)$ are the roots of the polynomial* $1 - 9z + 32z^2 + 96z^3 - 399z^4 - 2323z^5 + 1676z^6 + 9274z^7 + 2233z^8 - 13025z^9 - 12426z^{10} + 1194z^{11} + 8849z^{12} + 6817z^{13} + 2694z^{14} + 612z^{15} + 76z^{16} + 4z^{17}$.

As all the strange fixed points are repulsive and, therefore, they lay on the Julia set, the position of the free critical points in the complex plane has no interest.

Zheng et al. in [36] presented the following eighth-order derivative-free family without memory:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \qquad w_n = x_n + \beta f(x_n), \\ &\qquad\qquad\qquad \beta \in \mathbb{C} \setminus \{0\}, \\ z_n &= y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} &= z_n - f(z_n) \\ &\quad \times \left(f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) \right. \\ &\quad \left. + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n)\right)^{-1}, \end{aligned} \tag{17}$$

which will be denoted by Z8.

Lemma 12. *Operator $O_{Z8}(z)$ has ten simple strange fixed points, roots of the equation* $1 + 6\beta^2 + \beta^4 + (-24\beta - 16\beta^3)z + (21 + 48\beta^2 - \beta^4)z^2 + (-56\beta + 32\beta^3)z^3 + (35 - 84\beta^2 - 6\beta^4)z^4 + 56\beta z^5 + (7 + 14\beta^4)z^6 + (24\beta - 32\beta^3)z^7 + (30\beta^2 - 11\beta^4)z^8 + 16\beta^3 z^9 + 3\beta^4 z^{10}$. *Nine of these fixed points are repulsive for all values of parameter β; however, one of them can be attractive (even super attractive) in a small complex region around the origin. In Figure 15, the stability function of these specific fixed point is shown.*

Again, the existence of values of the parameter that yields attracting strange fixed points forces us to analyze the possibility of free critical points. As we know, if both elements coexist, basins of attraction of fixed points different from the roots appear.

Lemma 13. *Analyzing the equation $O'_{Z8}(z) = 0$, we obtain the free critical points:*

$$\begin{aligned} cr_1(\beta) &= \frac{-1 - \beta}{\beta}, \qquad cr_2(\beta) = \frac{-1 + \beta}{\beta}, \\ cr_3(\beta) &= \frac{-2\beta - \sqrt{2\beta^2 + \beta^4}}{\beta^2}, \\ cr_4(\beta) &= \frac{-2\beta + \sqrt{2\beta^2 + \beta^4}}{\beta^2}. \end{aligned} \tag{18}$$

Moreover,

(i) *if $\beta = 1/2$, then $cr_2(1/2) = cr_4(1/2) = -1$ and there are only two free critical points;*

(ii) *also when $\beta = -1/2$, then $cr_1(1/2) = cr_3(1/2) = 1$ and there are only two free critical points;*

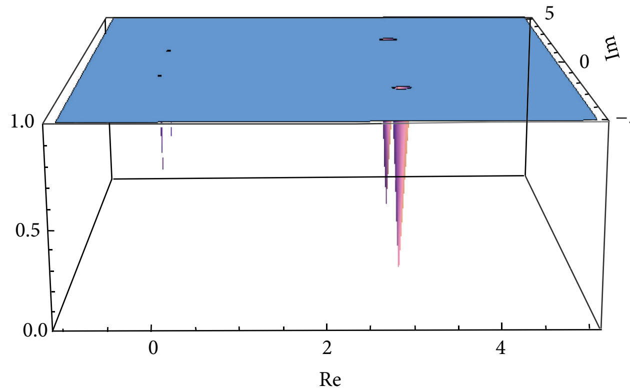
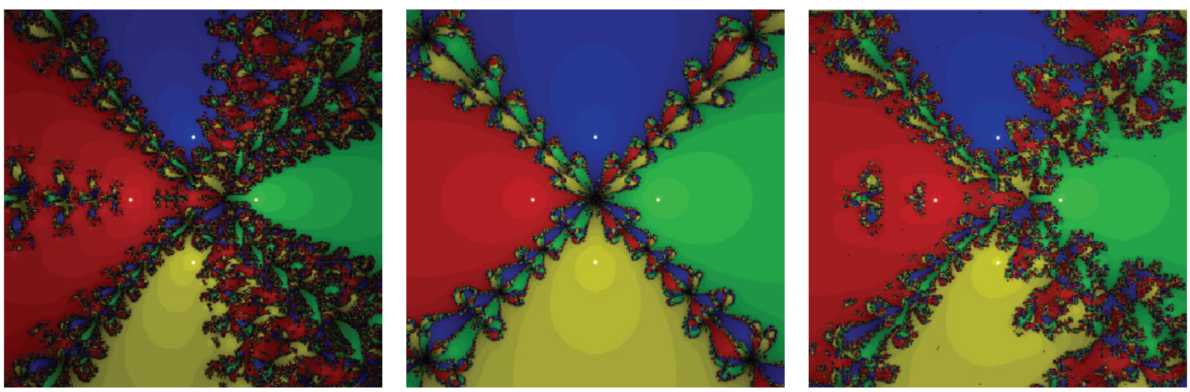


FIGURE 15: Stability function of the only attractive strange fixed point of Z8.



(a) Method (10), $\beta = 1$ (b) Method (10), $\beta = -0.001$ (c) Method (11), $\beta = 1$

FIGURE 16: Basins of attraction for the test problem 1.

(iii) *finally, if* $\beta = -\sqrt{2}i$ *or* $\beta = \sqrt{2}i$, *then* $cr_3(\pm\sqrt{2}i) = cr_4(\pm\sqrt{2}i) = \pm\sqrt{2}i$ *and the number of free critical points is three.*

Soleymani et al. in [37] proposed the following optimal three-step iteration family, including four function evaluations, just like (11) and (17):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \qquad w_n = x_n + \beta f(x_n), \\ &\qquad \beta \in \mathbb{C} \setminus \{0\}, \\ z_n &= y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} &= z_n - (1 + a_3(z_n - x_n))^2 f(z_n) \\ &\quad \times \left(a_1 - f(x_n) a_3 + 2a_2(z_n - x_n) \right. \\ &\quad \left. + a_2 a_3 (z_n - x_n)^2\right)^{-1}, \end{aligned} \tag{19}$$

where $a_1 = f[w_n, x_n] + f(w_n)a_3 - (w_n - x_n)a_2$, $a_2 = f[w_n, x_n, y_n] + a_3 f[w_n, y_n]$, and

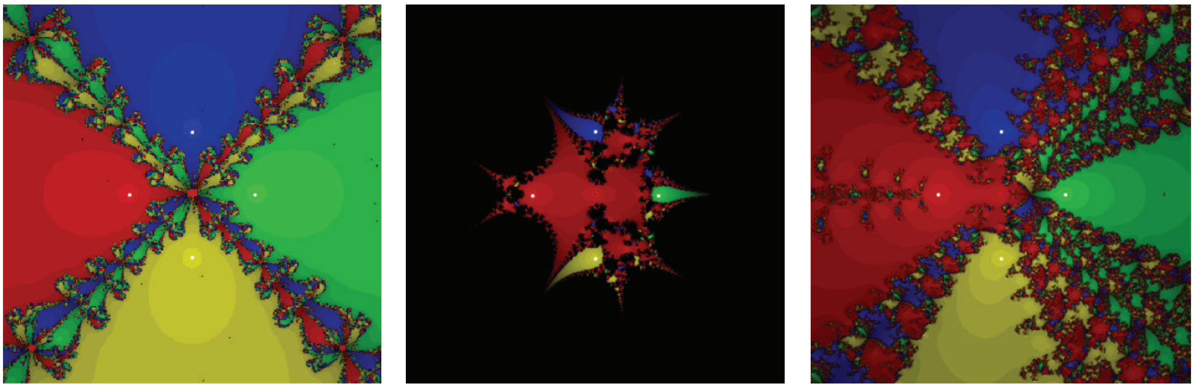
$$\begin{aligned} a_3 &= (w_n (f[z_n, x_n] - f[y_n, x_n]) - f[z_n, x_n] y_n \\ &\quad + f[w_n, x_n](y_n - z_n) + f[y_n, x_n] z_n) \\ &\quad \times ((z_n - y_n) f(w_n) + (w_n - z_n) f(y_n) \\ &\quad + (y_n - w_n) f(z_n))^{-1}. \end{aligned} \tag{20}$$

Let us denote this method by S8; the rational function associated with S8 when it is applied on the quadratic polynomial $p(z)$, $O_{S8}(z)$, is the same as the one of Z8. Then, their dynamics is the same (for this polynomial). Nevertheless, it is very different for other functions, as we will see in the following sections.

5. Attraction Basins for Various Steffensen-Type Methods

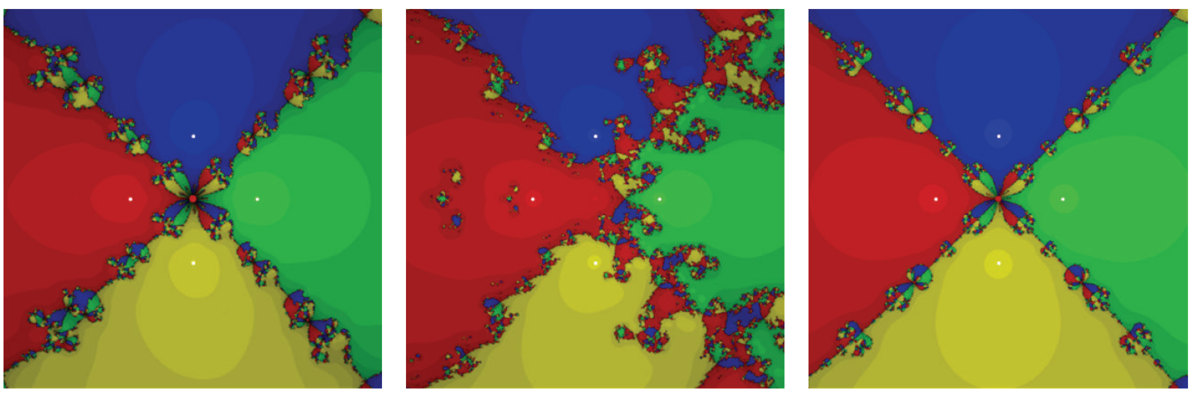
The aim herein is to use the basin of attraction as another tool for comparing the iteration algorithms given in Section 4.

We have used methods (10) with $\beta = 1$, (10) with $\beta = -0.001$, (11) with $\beta = 1$, (11) with $\beta = -0.001$, (13), (15), and



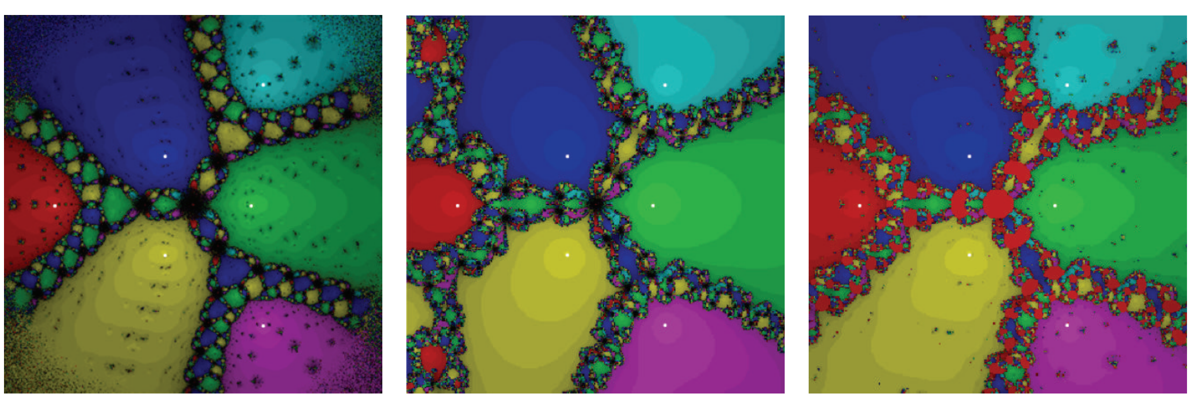
(a) Method (11), $\beta = -0.001$ (b) Method (13), $\beta = -0.001$ (c) Method (17), $\beta = 1$

Figure 17: Basins of attraction of (11), (13), and (15) for the test problem 1.



(a) Method (17), $\beta = -0.001$ (b) Method (19), $\beta = 1$ (c) Method (19), $\beta = -0.001$

Figure 18: Basins of attraction for the test problem 1.



(a) Method (10), $\beta = 1$ (b) Method (10), $\beta = -0.001$ (c) Method (11), $\beta = 1$

Figure 19: Basins of attraction for the test problem 2.

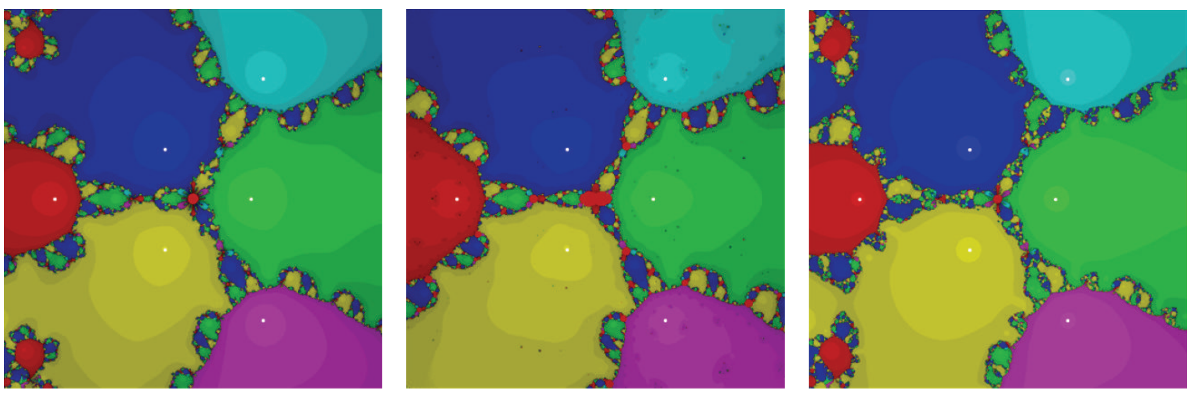
(17) with $\beta = 1$, (17) with $\beta = -0.001$, (19) with $\beta = 1$, and (19) with $\beta = -0.001$, for the test problems of Section 4. The fractal behavior of these comparisons is furnished in Figures 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, and 27. Let us remark that the dynamics of Test 3 is similar to the other test functions, so it is not included.

Remark 14. According to the discussion at the end of Section 4, the Steffensen-type methods, in which there is no nonzero free parameter in their structures, are not competitive and we expect to have small basins of attraction for them.

It is clear for the compared tests that even the higher order Steffensen-type methods (13) and (15) without

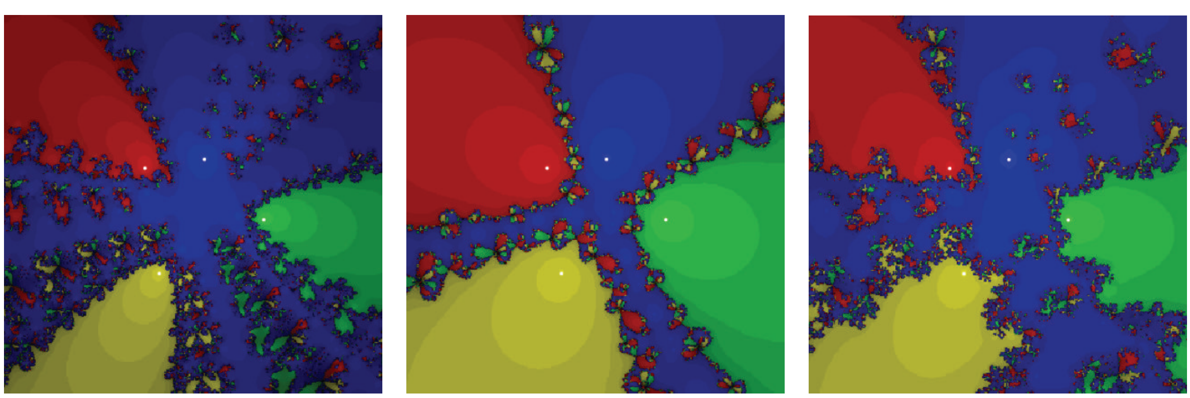
(a) Method (11), $\beta = -0.001$ (b) Method (13), $\beta = -0.001$ (c) Method (17), $\beta = 1$

Figure 20: Basins of attraction of (11), (13), and (15) for the test problem 2.



(a) Method (17), $\beta = -0.001$ (b) Method (19), $\beta = 1$ (c) Method (19), $\beta = -0.001$

Figure 21: Basins of attraction for the test problem 2.

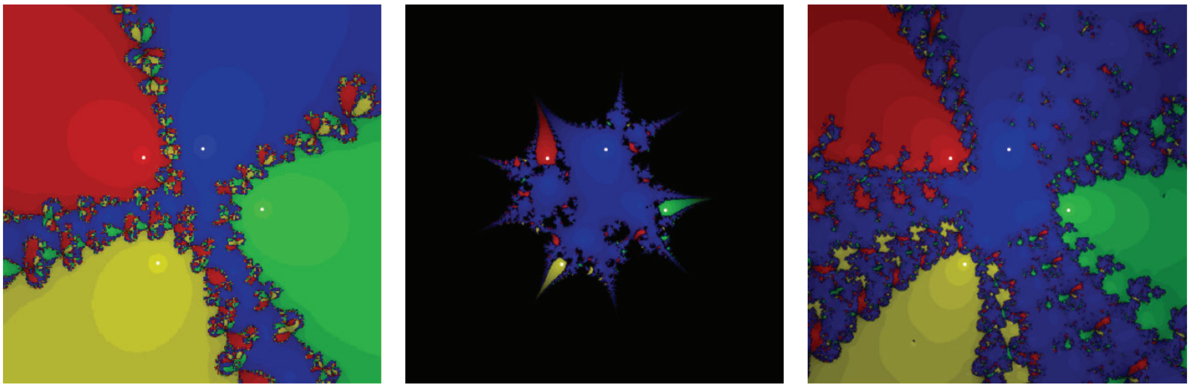


(a) Method (10), $\beta = 1$ (b) Method (10), $\beta = -0.001$ (c) Method (11), $\beta = 1$

Figure 22: Basins of attraction for the test problem 4.

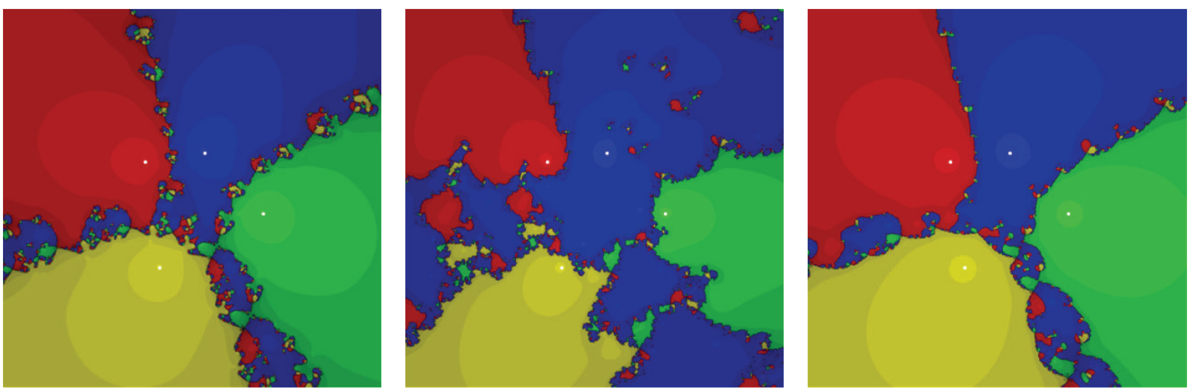
the free nonzero parameter have improved basins in contrast to Steffensen's scheme ($\beta = 1$). Furthermore, it should be noted that among all the methods compared in this section, Soleymani et al. optimal eighth-order method (19) with $\beta = -0.001$ has the best performance, followed by (17) with $\beta = -0.001$. In this work, the computer specifications are Microsoft Windows XP Intel(R), Pentium(R) 4 CPU, 3.20 GHz with 4 GB of RAM.

Remark 15. Unlike the Newton-type methods in which whatever the order is higher, the convergence radius is smaller, in the multipoint high-order efficient Steffensen-type methods



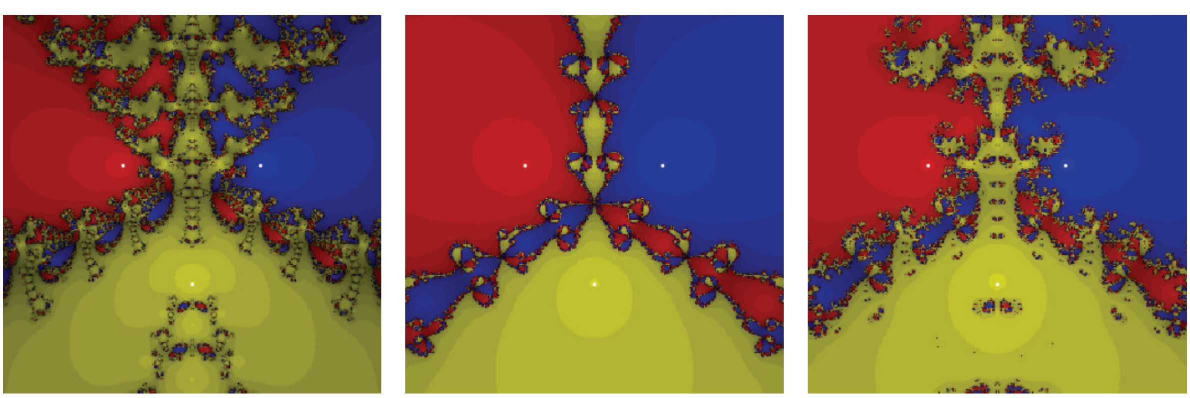
(a) Method (11), $\beta = -0.001$ (b) Method (13), $\beta = -0.001$ (c) Method (17), $\beta = 1$

Figure 23: Basins of attraction of (11), (13), and (15) for the test problem 4.



(a) Method (17), $\beta = -0.001$ (b) Method (19), $\beta = 1$ (c) Method (19), $\beta = -0.001$

Figure 24: Basins of attraction for the test problem 4.



(a) Method (10), $\beta = 1$ (b) Method (10), $\beta = -0.001$ (c) Method (11), $\beta = 1$

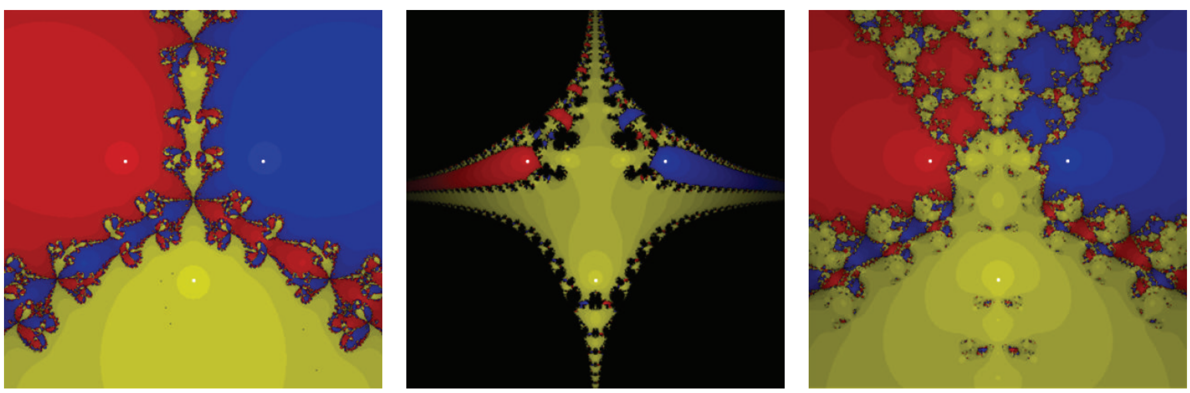
Figure 25: Basins of attraction for the test problem 5.

the increase of convergence order will automatically improve the convergence radius, though the chaotic behavior of the schemes for unappropriate values of the free nonzero parameter is too much. To avoid this chaotic behavior, one may follow Remark 4.

In order to summarize these results, we have attached a weight to the quality of the results obtained by each method. The weight of 1 is for the smallest Julia set and a weight of 4 for scheme with chaotic behavior alongside the convergence behavior. We then averaged those results to come up with

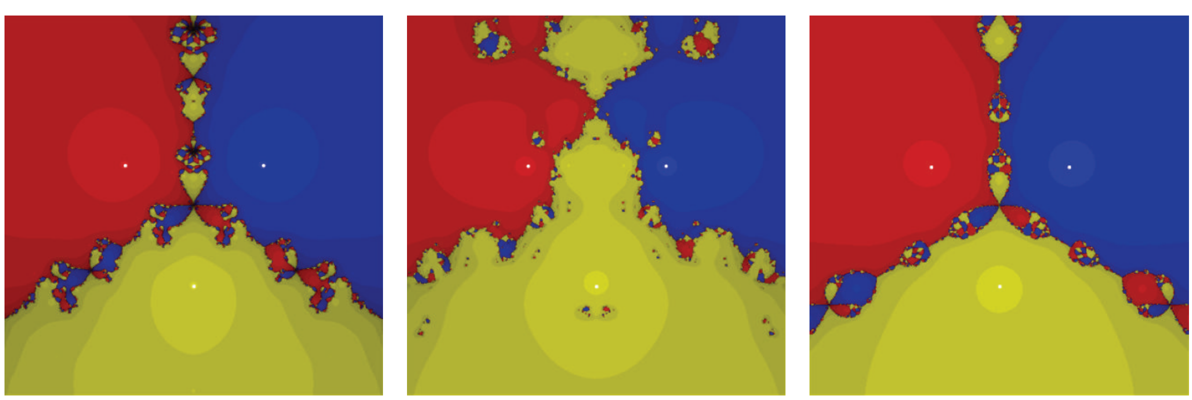
Table 1: Results of chaotic comparisons for different derivative-free methods.

Method	Test 1	Test 2	Test 4	Test 5	Average
(10) with $\beta = 1$	3	4	3	3	13/4
(10) with $\beta = -0.001$	2	3	2	2	9/4
(11) with $\beta = 1$	3	4	3	3	13/4
(11) with $\beta = -0.001$	2	3	2	2	9/4
(13)	4	4	4	3	15/4
(15)	4	4	4	3	15/4
(17) with $\beta = 1$	3	3	3	3	12/4
(17) with $\beta = -0.001$	2	2	1	2	7/4
(19) with $\beta = 1$	2	2	3	3	10/4
(19) with $\beta = -0.001$	1	1	1	1	4/4



(a) Method (11), $\beta = -0.001$ (b) Method (13), $\beta = -0.001$ (c) Method (17), $\beta = 1$

Figure 26: Basins of attraction of (11), (13), and (15) for the test problem 5.



(a) Method (17), $\beta = -0.001$ (b) Method (19), $\beta = 1$ (c) Method (19), $\beta = -0.001$

Figure 27: Basins of attraction for the test problem 5.

the smallest value for the best method overall and the highest for the worst. These data are presented in Table 1.

Notice again that the figures show how fast the method converges to a root based on shading to indicate speed of convergence.

6. Conclusions

In this paper, we have analyzed the dynamics of different Steffensen-type methods, firstly on quadratic polynomials and afterwards on other functions. We have concluded that in Steffensen-type methods whatever the order is higher, the convergence radius will be bigger. We have used Mathematica 8 for finding the fixed and critical points of the rational functions associated with the iterations functions and for drawing the basins of attraction. Besides, if the free nonzero parameter for the families analyzed tends to 0, then its fractal tends to be the same as Newton's fractal. Choosing very small magnitudes for the free nonzero parameter gives us the ability to avoid computation of the Jacobian matrix when dealing with systems of nonlinear equations and have an acceptable convergence radius. Although we have discussed simple zeros of nonlinear functions, such remarks are valid for Steffensen-type methods when finding multiple roots as well.

Conflict of Interests

The authors declare that they do not have any conflict of interests in their submitted paper.

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