Research Article

Stability of SIRS Epidemic Model with Stochastic Perturbation and Distributed Delays

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A class of SIRS epidemic model with stochastic perturbation and distributed delays is proposed and discussed. Some sufficient conditions on the stability of the zero solution are established. Finally, concluding that, the white noise is favorable for the stability of zero solution and the distributed time delays have no impact on the stability of zero solution.

1. Introduction

As an important class of mathematical models, stochastic differential equations have been widely used in automatic control, biological, chemical reaction engineering, medicine, economics, demography, and many other scientific fields. For better application, people have done a lot of work in the stochastic differential equations and various particular equations proposed for the study of stochastic differential equations. Subsequently, a lot of the literature related to this topic was published and extensive research results were obtained. However, many of the problems have not been fully investigated and deserve further study. As it is well known, in recent years, the stochastic epidemic systems are extensively studied. Many important and influential results have been established and can be found in many articles and books. Particularly, the stability of zero solutions for various type stochastic epidemic systems has been extensively studied in [1–7] and the references are cited therein.

In [8], the authors studied the following classical SIRS epidemiological model with distributed time delay:

$$\dot{S}(t) = -\beta S(t) \int_0^h f(s) I(t-s)\, ds - \mu_1 S(t) + b + kR(t),$$

$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s)\, ds - (\mu_2 + \lambda) I(t),$$

$$\dot{R}(t) = \lambda I(t) - (\mu_3 + k) R(t). \tag{1}$$

Individual are susceptible, then infectious, recovered with temporary immunity, and then susceptible again when the immunity is lost. Then parameter $b > 0$ is the rate of susceptible individuals recruited into the population (either by birth or immigration) and $\mu_1 > 0$ is the natural death rate. The parameter $\lambda > 0$ describes the rate at which the infectious population becomes recovered, and $k > 0$ denotes the rate at which the recovered population loses immunity. The positive constant $\beta > 0$ is the average number of contacts per infective per day. The nonnegative constant h is the time delay. $f(s)$ is the fraction of vector population for which the time taken to become infectious, and it is assumed to be a nonnegative function on $[0, h]$ as density function and satisfies $\int_0^h f(s)\, ds = 1$. For detailed biological background, we refer to [8, 9].

In [8], the authors derived the disease-free equilibrium $P^0(S^0, 0, 0)$ and the basic reproductive number $R_0 = (\beta b / \mu_1 (\lambda + \mu_2))$ of system (1), where $S^0 = (b/\mu_1)$, β is the transmission coefficient, and $1/(\lambda + \mu_2)$ is the average residence time in the infectious individuals class. By straightforward computations, it can be seen that, for $R_0 > 1$, system (1) has a unique endemic equilibrium state $P^*(S^*, I^*, R^*)$ in the interior of $\mathfrak{R}_+^3$, where

$$S^* = \frac{S^0}{R_0} = \frac{\lambda + \mu_2}{\beta},$$

$$I^* = \frac{k + \mu_3}{\mu_1 (k + \lambda + \mu_2)} (b - \mu_1 S^*),$$

$$R^* = \lambda \frac{I^*}{k + \mu_3}.$$

$$(2)$$

On the other hand, white noise stochastic perturbations around the positive endemic equilibrium of epidemic models were considered in [1, 3]. In [1], the authors studied the following stochastic differential equation model:

$$\dot{S}(t) = -\beta S(t) \int_0^h f(s) I(t-s)\, ds - \mu_1 S(t)$$
$$+ b + \sigma_1 \left(S(t) - S^*\right) \dot{w}_1(t),$$

$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s)\, ds - \left(\mu_2 + \lambda\right) I(t)$$

$$(3)$$

$$+ \sigma_2 \left(I(t) - I^*\right) \dot{w}_2(t),$$

$$\dot{R}(t) = \lambda I(t) - \mu_3 R(t) + \sigma_3 \left(R(t) - R^*\right) \dot{w}_3(t),$$

where S^*, I^*, and R^* are the positive points of equilibrium for the corresponding deterministic system (1), constants σ_i ($i = 1, 2$) are the intensity of white noise, and $w_i(t)$ ($i = 1, 2, 3$) are the standard Wiener processes. In [1], the authors used the Lyapunov functionals method and the sufficient conditions for the stability of the zero solution are obtained for system (3).

Likewise, in [3], the authors studied the same type of stochastic perturbations and extended the deterministic model for the epidemics induced by virulent phages on bacteria in marine environment, allowing random fluctuations around the positive equilibrium, and concluded that the solution of the corresponding stochastic model for phage bacteria interaction was asymptotically mean square stable both analytically and numerically.

Motivated by the aforementioned works, in this paper, we consider the following SIRS epidemic model with stochastic perturbation and distributed delays:

$$\dot{S}(t) = -\beta S(t) \int_0^h f(s) I(t-s)\, ds - \mu_1 S(t) + b + kR(t)$$
$$+ \sigma_1 \left(S(t) - S^*\right) \dot{w}_1(t),$$

$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s)\, ds - \left(\mu_2 + \lambda\right) I(t)$$

$$+ \sigma_2 \left(I(t) - I^*\right) \dot{w}_2(t),$$

$$\dot{R}(t) = \lambda I(t) - \left(\mu_3 + k\right) R(t) + \sigma_3 \left(R(t) - R^*\right) \dot{w}_3(t),$$

$$(4)$$

where the constants σ_1, σ_2, and σ_3 are the intensity of white noise $\dot{w}_1(t)$, $\dot{w}_2(t)$, $\dot{w}_3(t)$ and $w_1(t)$, $w_2(t)$, $w_3(t)$ are standard Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, P)$. The detailed definition of Wiener process can be found in [2].

Let $u_1 = S - S^*$, $u_2 = I - I^*$, $u_3 = R - R^*$, where $N^* = S^* + I^* + R^*$. Then, system (4) is rewritten in the following form:

$$\dot{u}_1 = -\left(\beta I^* + \mu_1\right) u_1 - \beta S^* \int_0^h f(s) u_2(t-s)\, ds - \beta u_1$$

$$\times \int_0^h f(s) u_2(t-s)\, ds + ku_3 + \sigma_1 u_1 \dot{w}_1,$$

$$\dot{u}_2 = \beta I^* u_1 - \beta S^* u_2 + \beta S^* \int_0^h f(s) u_2(t-s)\, ds + \beta u_1$$

$$(5)$$

$$\times \int_0^h f(s) u_2(t-s)\, ds + \sigma_2 u_2 \dot{w}_2,$$

$$\dot{u}_3 = \lambda u_2 - \left(\mu_3 + k\right) u_3 + \sigma_3 u_3 \dot{w}_3.$$

It is easy to see that the stability of zero solution in system (4) is equivalent to the stability of zero solution in system (5). In order to obtain the sufficient conditions for the stability of zero solution in system (5), we will consider the following linear part of system (5):

$$\dot{z}_1 = -\left(\beta I^* + \mu_1\right) z_1 - \beta S^* \int_0^h f(s) z_2(t-s)\, ds$$

$$+ kz_3 + \sigma_1 z_1 \dot{w}_1,$$

$$\dot{z}_2 = \beta I^* z_1 - \beta S^* z_2 + \beta S^* \int_0^h f(s) z_2(t-s)\, ds + \sigma_2 z_2 \dot{w}_2,$$

$$\dot{z}_3 = \lambda z_2 - \left(\mu_3 + k\right) z_3 + \sigma_3 z_3 \dot{w}_3,$$

$$(6)$$

and the auxiliary system without delays

$$\dot{y}_1 = -\left(\beta I^* + \mu_1\right) y_1 + ky_3 + \sigma_1 y_1 \dot{w}_1,$$

$$\dot{y}_2 = \beta I^* y_1 - \beta S^* y_2 + \sigma_2 y_2 \dot{w}_2,$$

$$(7)$$

$$\dot{y}_3 = \lambda y_2 - \left(\mu_3 + k\right) y_3 + \sigma_3 y_3 \dot{w}_3.$$

2. Preliminaries

Consider the following stochastic differential equation [2]:

$$dx(t) = a(t, x_t)\, dt + b(t, x_t)\, dw(t), \quad x_0 = \varphi \in H, \quad (8)$$

where function $a(t, x_t)$ defined in $[t_0, \infty) \times R^n$ and $b(t, x_t)$ is an $n \times m$ matrix, $x_t(s) = x(t+s)$ with $s \leq 0$, $w(t)$ is m-dimensional Wiener process. We assume that $x = 0$ is a trivial solution of system (8), that is, $a(t, 0) = 0, b(t, 0) = 0$ for all $t \geq t_0$. Denote with H the space of f_0-adapted random variables φ, with $\varphi(s) \in \mathfrak{R}^n$, $s \leq 0$ and $\|\varphi\|_0 = \sup_{s \leq 0}|\varphi(s)|$, $\|\varphi\|_1^2 = \sup_{s \leq 0}\mathbb{E}|\varphi(s)|^2$ ($\mathbb{E}$ is the mathematical expectation). Let $V : [0, \infty) \times H \to R$ be a functional defined for $t \geq 0$, $\varphi \in H$.

Generating operator L of (8) is defined [2] by formula

$$LV(t, \varphi) = \lim_{\Delta \to 0} \frac{\mathbb{E}_{t,\varphi}V\left(t + \Delta, y_{t+\Delta}\right) - V(t, \varphi)}{\Delta}, \quad (9)$$

where a scalar functional $V(t, \varphi)$ is defined by $t \geq 0$, $\varphi \in H$, and $x(s)$ is the solution of (8) by $s \geq t$ with initial function $x_t = \varphi \in H$. Let us describe one class of functionals $V(t, \varphi)$ for which the operator L can be calculated in final form. We reduce the arbitrary functional $V(t, \varphi)$, $t \geq 0$, $\varphi \in H$, to the form $V(t, \varphi) = V(t, \varphi(0), \varphi(s))$, $s < 0$ and define the function

$$V_{\varphi}(t, x) = V(t, \varphi) = V(t, x, x(t + s)),$$

$$s < 0, \quad \varphi = x_t, \quad x = \varphi(0) = x(t). \tag{10}$$

Let D be the class of functionals $V(t, \varphi)$ for which functional $V_{\varphi}(t, x)$ has two continuous derivations with respect to x and one bounded derivative with respect to t for almost all $t \geq 0$. For functionals from D the generating operator L of (8) is defined and is equal to the following:

$$LV(t, x_t) = \frac{\partial V_{\varphi}(t, x)}{\partial t} + a'(t, x_t) \frac{\partial V_{\varphi}(t, x)}{\partial x}$$

$$+ \frac{1}{2} \operatorname{Tr} \left[b'(t, x_t) \frac{\partial^2 V_{\varphi}(t, x)}{\partial x^2} b(t, x_t) \right]. \tag{11}$$

Definition 1. The zero solution of (8) is called mean square stable if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbb{E}|x(t)|^2 < \epsilon$ for any $t \geq 0$ provided that $\|\varphi\|_1^2 < \delta$.

Definition 2. The zero solution of (8) is called asymptotically mean square stable if it is mean square stable and $\lim_{t \to \infty} \mathbb{E}|x(t)|^2 = 0$.

Definition 3. The zero solution of (8) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists $\delta > 0$ such that solution $x(t) = x(t, \varphi)$ of (8) satisfies

$$\mathbb{P}\{x(t, \varphi) > \epsilon_1\} < \epsilon_2 \tag{12}$$

for any initial function $\varphi \in H$ such that $\mathbb{P}\{|\varphi| \leq \delta\} = 1$. Here $\mathbb{P}\{\cdot\}$ is the probability of the event enclosed in braces.

Theorem 4 (see [2]). *Let there exist the functional $V(t, \varphi) \in D$ such that*

$$c_1 \mathbb{E}|x(t)|^2 \leq \mathbb{E}V(t, x_t) \leq c_2 \|x_t\|_1^2,$$

$$\mathbb{E}LV(t, x_t) \leq -c_3 \mathbb{E}|x(t)|^2, \quad c_i > 0. \tag{13}$$

Then zero solution of (8) is asymptotically mean square stable.

Theorem 5 (see [2]). *Let there exist the functional $V(t, \varphi) \in D$ such that*

$$c_1 |x(t)|^2 \leq V(t, x_t) \leq c_2 \|x_t\|_0^2, \quad LV(t, x_t) \leq 0, \ c_i > 0 \tag{14}$$

for any function $\varphi \in H$ such that $P\{|\varphi| \leq \delta\} = 1$, where $\delta > 0$ is sufficiently small. Then the zero solution of (8) is stable in probability.

3. Main Results

To prove the zero solution of system (5) is stable in probability, firstly, we will consider the linear part of system (5) and

the auxiliary system without delays. Next, we will consider the auxiliary system without delays of system (5) and we will prove that the zero solution of auxiliary system (7) is asymptotic mean square stable.

Theorem 6. *Suppose that*

$$\sigma_1^2 < 2\mu_1, \qquad \sigma_2^2 < 2\mu_2, \qquad \left(1 + \frac{q}{p^2}\right)\sigma_3^2 < 2\mu_3; \tag{15}$$

then zero solution of system (7) is asymptotic mean square stable.

Proof. Consider the following Lyapunov function:

$$V_1 = py_1^2 + y_2^2 + p^2 y_3^2 + q(y_1 + y_2 + y_3)^2, \tag{16}$$

where p, q are real positive constants to be chosen in the following. From the Itô formula (11), we have the following:

$$LV_1$$

$$= 2py_1\dot{y}_1 + 2py_2\dot{y}_2 + 2p^2 y_3\dot{y}_3 + 2q(y_1 + y_2 + y_3)$$

$$\times (\dot{y}_1 + \dot{y}_2 + \dot{y}_3) + (p + q)\sigma_1^2 y_1^2 + 2(1 + q)\sigma_2^2 y_2^2$$

$$+ 2(p^2 + q)\sigma_3^2 y_3^2$$

$$= 2(py_1 + q(y_1 + y_2 + y_3))$$

$$\times (-(\beta I^* + \mu_1) y_1 + ky_3) + 2(y_2 + q(y_1 + y_2 + y_3))$$

$$\times (\beta I^* y_1 - \beta S^* y_2) + 2(p^2 y_3 + q(y_1 + y_2 + y_3))$$

$$\times (\lambda y_2 - \mu_3 y_3 - ky_3)$$

$$+ (p + q)\sigma_1^2 y_1^2 + (1 + q)\sigma_2^2 y_2^2 + (p^2 + q)\sigma_3^2 y_3^2$$

$$+ 2q\sigma_1\sigma_2 y_1 y_2 + 2q\sigma_2\sigma_3 y_2 y_3$$

$$\leq +2q\sigma_1\sigma_3 y_1 y_3 - 2py_1^2(\beta I^* + \mu_1) - 2q(y_1 + y_2 + y_3)$$

$$\times (\mu_1 y_1 + \beta S^* y_2 - \lambda y_2 + \mu_3 y_3) + 2\beta I^* y_1 y_2 - 2\beta S^* y_2^2$$

$$- 2p^2 \mu_3 y_3^2 + (p + q)\sigma_1^2 y_1^2$$

$$+ (1 + q)\sigma_2^2 y_2^2 + p^2\lambda \left(\frac{y_2^2}{p} + py_3^2\right) + (p^2 + q)\sigma_3^2 y_3^2$$

$$+ 2q\sigma_1\sigma_2 y_1 y_2 + 2q\sigma_2\sigma_3 y_2 y_3 + 2q\sigma_1\sigma_3 y_1 y_3$$

$$= \left[-2p\beta I^* - 2p\mu_1 - 2q\mu_1 + (p + q)\sigma_1^2\right]$$

$$+ y_2^2 \left[-2q\beta S^* - 2\beta S^* + (1 + q)\sigma_2^2\right]$$

$$+ y_3^2 \left[-2q\mu_3 - 2p^2\mu_3 + p^2\lambda p - 2p^2 k + p^2\sigma_3^2\right]$$

$$+ 2y_1 y_2 \left[-q\mu_1 - q\beta S^* + q\lambda + \beta I^* + q\sigma_1\sigma_2\right]$$

$$+ 2y_2 y_3 \left[-q\mu_3 - q\beta S^* + q\lambda + q\sigma_2\sigma_3\right]$$

$$+ 2y_1 y_3 \left[-q\mu_3 - q\mu_1 + pk + q\sigma_1\sigma_3\right] + q\sigma_3^2 y_3^2$$

$$\leq y_1^2 (p+q) (\sigma_1^2 - 2\mu_1)$$
$$+ y_2^2 [(1+q)(\sigma_2^2 - 2\beta S^*) + p\lambda + 2q\lambda]$$
$$+ p^2 y_3^2 (p\lambda - 2\mu_3 + \sigma_3^2)$$
$$+ 2y_1 y_2 [\beta I^* + q(\sigma_1\sigma_2 - \mu_1 - \beta S^* + \lambda)]$$
$$+ 2y_1 y_3 [q(-\mu_3 - \mu_1 + \sigma_1\sigma_3) + pk]$$
$$+ q\sigma_3^2 y_3^2 + 2y_2 y_3 [q(-\mu_3 - \beta S^* + \lambda + \sigma_2\sigma_3)].$$

$$(17)$$

Set

$$q = \min \left\{ \frac{pk}{\mu_1 + \mu_3 - \sigma_1\sigma_3}, \frac{\beta I^*}{\mu_1 + \beta S^* - \lambda - \sigma_1\sigma_2} \right\}, \quad (18)$$

and $\sigma_3^2 < (1 + (q/p^2))\sigma_3^2 < 2\mu_3$. From the condition (15), it follows

$$|\sigma_1\sigma_2| \leq \frac{\sigma_1^2 + \sigma_2^2}{2} < \mu_1 + \beta S^* - \lambda,$$

$$|\sigma_2\sigma_3| \leq \frac{\sigma_2^2 + \sigma_3^2}{2} < \mu_3 + \beta S^* - \lambda, \quad (19)$$

$$|\sigma_1\sigma_3| \leq \frac{\sigma_1^2 + \sigma_3^2}{2} < \mu_1 + \mu_3.$$

By this way, we obtain the following:

$$LV_1 \leq -y_1^2 (p+q) (2\mu_1 - \sigma_1^2)$$
$$- y_2^2 [(1+q)(2\beta S^* - \sigma_2^2) - p\lambda - 2q\lambda] \quad (20)$$
$$- p^2 y_3^2 \left(2\mu_3 - \left(1 + \frac{q}{p^2}\right)\sigma_3^2 - p\lambda \right).$$

Choose constant p such that

$$p = \frac{1}{\lambda} \min \left\{ 2\mu_3 - \left(1 + \frac{q}{p^2}\right)\sigma_3^2, \right.$$

$$\left. (1+q)(2\beta S^* - \sigma_2^2 - 2q\lambda) \right\}. \quad (21)$$

From (15), it follows that there exists $c > 0$ such that $LV_1 \leq -c|y|^2$, where $y = (y_1, y_2, y_3)$. From Theorem 4, we have that the zero solution of system (7) is asymptotic mean square stable. This completes the proof. $\square$

In the second place, we will consider the linear part of system (5) and will prove that the zero solution of system (6) is asymptotic mean square stable.

Theorem 7. *Suppose that*

$$\sigma_1^2 < 2\mu_1, \qquad \sigma_2^2 < \frac{2q\mu_2}{1+q}, \qquad \left(1 + \frac{q}{p^2}\right)\sigma_3^2 < 2\mu_3, \quad (22)$$

where q is defined in (18); then the zero solution of the system (6) is asymptotic mean square stable.

Proof. Consider the following Lyapunov function;

$$V_2 = pz_1^2 + z_2^2 + p^2 z_3^2 + q(z_1 + z_2 + z_3)^2. \quad (23)$$

Let L be the generating operator [1] of the system (6). We compute the following:

$$LV_2$$
$$= 2 (pz_1 + q(z_1 + z_2 + z_3))$$
$$\times \left(-(\beta I^* + \mu_1) z_1 + kz_3 - \beta S^* \int_o^h f(s) z_2 (t-s) \, ds \right)$$
$$+ 2 (z_2 + q(z_1 + z_2 + z_3))$$
$$\times \left(\beta I^* z_1 - \beta S^* z_2 + \beta S^* \int_o^h f(s) z_2 (t-s) \, ds \right)$$
$$+ 2 (p^2 z_3 + q(z_1 + z_2 + z_3)) (\lambda z_2 - \mu_3 z_3 - kz_3)$$
$$+ (p+q) \sigma_1^2 z_1^2$$
$$+ (1+q) \sigma_2^2 z_2^2 + (p^2 + q) \sigma_3^2 z_3^2 + 2q\sigma_1\sigma_2 z_1 z_2$$
$$+ 2q\sigma_2\sigma_3 z_2 z_3 + 2q\sigma_1\sigma_3 z_1 z_3$$
$$\leq -z_1^2 (p+q) (2\mu_1 - \sigma_1^2)$$
$$- z_2^2 [(1+q)(2\beta S^* - \sigma_2^2) - p\lambda - 2q\lambda]$$
$$- p^2 z_3^2 (2\mu_3 - \sigma_3^2 - p\lambda) + q\sigma_3^2 z_3^2 + 2z_1 z_2$$
$$\times [\beta I^* + q(\sigma_1\sigma_2 - \mu_1 - \beta S^* + \lambda)]$$
$$+ 2z_1 z_3 [q(-\mu_3 - \mu_1 + \sigma_1\sigma_3) + pk]$$
$$+ 2z_2 z_3 [q(-\mu_3 - \beta S^* + \lambda + \sigma_2\sigma_3)]$$
$$+ 2\beta S^* (z_2 - pz_1) \int_0^h f(s) z_2 (t-s) \, ds.$$

$$(24)$$

Using (18), we obtain the following:

$$LV_2 \leq -z_1^2 q (2\mu_1 - \sigma_1^2)$$
$$- z_2^2 [(1+q)(2\beta S^* - \sigma_2^2) - p\lambda - 2q\lambda]$$
$$- p^2 z_3^2 (2\mu_3 - \sigma_3^2 - p\lambda) + q\sigma_3^2 z_3^2$$
$$+ p\beta S^* \left[z_1^2 + \int_0^h f(s) z_2^2 (t-s) \, ds \right]$$
$$+ \beta S^* \left[z_2^2 + \int_0^h f(s) z_2^2 (t-s) \, ds \right]$$

$$= -z_1^2 \left[q \left(2\mu_1 - \sigma_1^2 \right) - p\beta S^* \right]$$
$$- z_2^2 \left[(1+q) \left(2\beta S^* - \sigma_2^2 \right) - p\lambda - 2q\lambda - \beta S^* \right]$$
$$- p^2 z_3^2 \left(2\mu_3 - \sigma_3^2 - p\lambda \right) + q\sigma_3^2 z_3^2 + \beta S^* (1+p)$$
$$\times \int_0^h f(s) z_2^2 (t-s) \, ds,$$

$$(25)$$

and we define a Lyapunov function as follows:

$$V_3 = \beta S^* (1+p) \int_0^h f(s) \int_{t-s}^t z_2^2 (\tau) \, d\tau \, ds. \qquad (26)$$

With the application of the multidimensional Itô's formula [1], we obtain the following:

$$LV_3 = \beta S^* (1+p) z_2^2 (t) - \beta S^* (1+p) \int_0^h f(s) z_2^2 (t-s) \, ds. \qquad (27)$$

Finally, we define a Lyapunov function

$$V_4 = V_2 + V_3. \qquad (28)$$

By (22), (25), and (27), we have the following:

$$LV_4 \leq - z_1^2 \left[q \left(2\mu_1 - \sigma_1^2 \right) - p\beta S^* \right]$$
$$- z_2^2 \left[2q \left(\beta S^* - \lambda \right) - (1+q) \sigma_2^2 - p \left(\lambda + \beta S^* \right) \right] \quad (29)$$
$$- p^2 z_3^2 \left(2\mu_3 - \left(1 + \frac{q}{p^2} \right) \sigma_3^2 - p\lambda \right),$$

where

$$p = \min \left\{ \frac{q \left(2\mu_1 - \sigma_1^2 \right)}{\beta S^*}, \frac{2q \left(\beta S^* - \lambda \right) - (1+q) \sigma_2^2}{\lambda + \beta S^*}, \right.$$
$$\left. \frac{2\mu_3 - \left(1 + \left(q/p^2 \right) \right) \sigma_3^2}{\lambda} \right\}. \qquad (30)$$

Therefore, there exists a $c > 0$ such that $LV_4 \leq -c|z|^2$, where $z = (z_1, z_2, z_3)$. From Theorem 4, we have that the zero solution of system (6) is asymptotic mean square stable. This completes the proof. □

Theorem 8. *Suppose that the conditions of Theorem 7 hold; then the zero solution of the system (5) is stable in probability.*

Proof. Consider the following Lyapunov function:

$$V = p u_1^2 + u_2^2 + p^2 u_3^2 + q (u_1 + u_2 + u_3)^2$$
$$+ \beta S^* (1+p) \int_0^h f(s) \int_{t-s}^t u_2^2 (\tau) \, d\tau \, ds. \qquad (31)$$

Then, we obtain the following:

$$LV$$
$$= 2 \left(p u_1 + q (u_1 + u_2 + u_3) \right)$$
$$\times \left(- \left(\beta I^* + \mu_1 \right) u_1 + k u_3 - \beta S^* \int_0^h f(s) u_2 (t-s) \, ds \right.$$
$$\left. - \beta u_1 \int_0^h f(s) u_2 (t-s) \, ds \right) + 2 \left(u_2 + q (u_1 + u_2 + u_3) \right)$$
$$\times \left(\beta I^* u_1 - \beta S^* u_2 + \beta S^* \int_0^h f(s) u_2 (t-s) \, ds \right.$$
$$\left. + \beta u_1 \int_0^h f(s) u_2 (t-s) \, ds \right)$$
$$+ 2 \left(p^2 u_3 + q (u_1 + u_2 + u_3) \right)$$
$$\times \left(\lambda u_2 - \mu_3 u_3 - k u_3 \right) + (p+q) \sigma_1^2 u_1^2$$
$$+ (1+q) \sigma_2^2 u_2^2 + \left(p^2 + q \right) \sigma_3^2 u_3^2$$
$$+ 2q\sigma_1\sigma_2 u_1 u_2 + 2q\sigma_2\sigma_3 u_2 u_3 + 2q\sigma_1\sigma_3 u_1 u_3$$
$$+ (1+p) \beta S^* u_2^2 - (1+p) \beta S^* \int_0^h f(s) u_2^2 (t-s) \, ds$$
$$\leq -u_1^2 \left[q \left(2\mu_1 - \sigma_1^2 \right) - p\beta S^* \right]$$
$$- u_2^2 \left[2q \left(\beta S^* - \lambda \right) - (1+q) \sigma_2^2 - p \left(\lambda + \beta S^* \right) \right]$$
$$- p^2 u_3^2 \left(2\mu_3 - \sigma_3^2 - p\lambda \right) + 2\beta u_1 \left(u_2 - p u_1 \right)$$
$$\times \int_0^h f(s) u_2 (t-s) \, ds.$$

$$(32)$$

From Theorem 5, there exists a sufficiently small $\delta > 0$ such that $\mathbb{P}\{|u_2(s)| < \delta\} = 1$. Then

$$2\beta \left| u_1 \left(u_2 - p u_1 \right) \int_0^h f(s) u_2 (t-s) \, ds \right|$$
$$\leq \left| 2\beta u_1 u_2 \int_0^h f(s) u_2 (t-s) \, ds \right|$$
$$+ \left| 2\beta p u_1^2 \int_0^h f(s) u_2 (t-s) \, ds \right|$$
$$\leq 2\beta u_1 u_2 \delta + 2\beta p u_1^2 \delta = \beta\delta \left(2 u_1 u_2 + 2 p u_1^2 \right)$$
$$\leq \beta\delta \left(u_1^2 (1 + 2q) + u_2^2 \right). \qquad (33)$$

Therefore,

$$LV \leq -u_1^2 \left[q \left(2\mu_1 - \sigma_1^2 \right) - p\beta S^* - \beta\delta (1 + 2q) \right]$$

$$-u_2^2 \left[2q\left(\beta S^* - \lambda\right) - (1 + q)\,\sigma_2^2 - p\left(\lambda + \beta S^*\right) - \beta\delta\right]$$

$$-p^2 u_3^2 \left(2\mu_3 - \sigma_3^2 - p\lambda\right).$$

$$(34)$$

Hence, for sufficiently small $\delta > 0$, we obtain $LV \leq 0$. From Theorem 5, we have that the zero solution of system (5) is asymptotic mean square stable. This completes the proof. $\square$

4. Conclusion

In this paper, we considered the SIRS epidemic model with stochastic perturbation and distributed delays and some sufficient conditions on the stability of the zero solution are established. Further, we obtain that the white noise σ_i ($i = 1, 2, 3$) is favorable for the stability of zero solution and the distributed time delays have no impact on the stability of zero solution. The results obtained in this paper indicate that suitable stochastic perturbation can maintain the stability of zero solution.

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