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Research Article

A Numerical Algorithm for Solving a Four-Point Nonlinear Fractional Integro-Differential Equations

Er Gao,[1,2] Songhe Song,[1,2] and Xinjian Zhang[1]

[1] *Department of Mathematics and Systems Science, College of Science, National University of Defense Technology, Changsha 410073, China*
[2] *State key Laboratory of High Performance Computing, National University of Defense Technology, Changsha 410073, China*

Correspondence should be addressed to Er Gao, gao.nudter@gmail.com

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We provide a new algorithm for a four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order $q \in (1,2]$ based on reproducing kernel space method. According to our work, the analytical solution of the equations is represented in the reproducing kernel space which we construct and so the n-term approximation. At the same time, the n-term approximation is proved to converge to the analytical solution. An illustrative example is also presented, which shows that the new algorithm is efficient and accurate.

1. Introduction

In recent years, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Several authors have used fixed point theory to show the existence of solution to differential equations of fractional order, see the monographs of Bai and Liu [1], Wu and Liu [2], Hamani et al. [3] and Ahmad and Sivasundaram [4]. At the same time, there may be several methods for solving differential equations of fractional order, such as the least squares finite-element method [5], collection method [6], fractional differential transform method [7], decomposition method [8], and variational iteration method [9]. Besides these cited works, few more contributions [10, 11] have been made to the analytical and numerical study of the solutions of fractional boundary value problems.

Ahmad and Sivasundaram [4] proved the existence and uniqueness of solutions for a four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order $q \in (1,2]$ by applying some standard fixed point theorems:

$$\begin{gathered} {}^cD^q u(x) = f\big(x, u(x), (\phi u)(x), (\varphi u)(x)\big), \quad 1 < q \le 2, \\ u'(0) + au(\eta_1) = 0, \quad bu'(1) + u(\eta_2) = 0, \quad 0 < \eta_1 \le \eta_2 < 1, \end{gathered} \tag{1.1}$$

where ${}^cD$ is the Caputo's fractional derivative and $f : [0,1] \times X \to X$ is continuous.

In this paper, we consider the following nonlinear fractional integro-differential equation with four-point nonlocal boundary conditions:

$$\begin{gathered} {}^cD^q u(x) + (\phi u)(x) + (\varphi u)(x) = f(x, u(x)), \quad 1 < q \le 2, \\ u'(0) + au(\eta_1) = 0, \quad bu'(1) + u(\eta_2) = 0, \quad 0 < \eta_1 \le \eta_2 < 1, \end{gathered} \tag{1.2}$$

where ${}^cD$ is the Caputo's fractional derivative and $f : [0,1] \times X \to X$ is continuous, for $\gamma, \delta : [0,1] \times [0,1] \to [0, +\infty)$,

$$(\phi u)(x) = \int_0^x \gamma(x,t) u(t) dt, \qquad (\varphi u)(x) = \int_0^x \delta(x,t) u(t) dt, \tag{1.3}$$

and $a, b \in (0,1)$. Here, $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

Actually, we remark that the boundary conditions in (1.2) arise in the study of heat flow problems involving a bar of unit length with two controllers at $t = 0$ and $t = 1$ adding or removing heat according to the temperatures detected by two sensors at $t = \eta_1$ and $t = \eta_2$.

The rest of the paper is organized as follows. We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. Then we construct some special reproducing kernel spaces, and the new reproducing kernel method is introduced in Section 3. In Section 4 we present one examples to demonstrate the efficiency of the method.

2. Preliminaries

Let us recall some basic definition and lemmas on fractional calculus.

Definition 2.1. For a function $g : [0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q \le n,\ q > 0, \tag{2.1}$$

where Γ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \tag{2.2}$$

provided the integral exists.

Lemma 2.3 (see [4]). *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{gathered} {}^cD^q u(x) = \sigma(x), \quad 0 < x < 1, \ 1 < q \le 2, \\ u'(0) + au(\eta_1) = 0, \quad bu'(1) + u(\eta_2) = 0, \quad 0 < \eta_1 \le \eta_2 < 1, \end{gathered} \tag{2.3}$$

is given by

$$\begin{aligned} u(x) = {} & \int_0^x \frac{(x-s)^{q-1}}{\Gamma(q)} \sigma(s) ds + \frac{a(b+\eta_2 - x)}{1 + a(\eta_1 - \eta_2 - b)} \int_0^{\eta_1} \frac{(\eta_1 - s)^{q-1}}{\Gamma(q)} \sigma(s) ds \\ & + \frac{ax - (1 + a\eta_1)}{1 + a(\eta_1 - \eta_2 - b)} \left[b \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds + \int_0^{\eta_2} \frac{(\eta_2 - s)^{q-1}}{\Gamma(q)} \sigma(s) ds \right]. \end{aligned} \tag{2.4}$$

To introduce the next lemma, we need the following assumptions.

(A_1) There exist positive functions $L_1(t)$, $L_2(t)$, $L_3(t)$ such that

$$\begin{aligned} & \| f(t, u(t), (\phi u)(t), (\psi u)(t)) - f(t, v(t), (\phi v)(t), (\psi v)(t)) \| \\ & \quad \le L_1(t) \|u - v\| + L_2(t) \|\phi u - \phi v\| + L_3(t) \|\psi u - \psi v\|, \quad \forall t \in [0,1], \ u, v \in X. \end{aligned} \tag{2.5}$$

Further,

$$\begin{gathered} \gamma_0 = \sup_{t \in [0,1]} \left| \int_0^t \gamma(t,s) ds \right|, \qquad \delta_0 = \sup_{t \in [0,1]} \left| \int_0^t \delta(t,s) ds \right|, \\ I_L^q = \max\left\{ \sup_{t \in [0,1]} |I^q L_1(t)|, \sup_{t \in [0,1]} |I^q L_2(t)|, \sup_{t \in [0,1]} |I^q L_3(t)| \right\}, \\ I^{q-1} L(1) = \max\left\{ \left| I^{q-1} L_1(1) \right|, \left| I^{q-1} L_2(1) \right|, \left| I^{q-1} L_3(1) \right| \right\}, \\ I^q L(\eta_i) = \max\{ |I^q L_1(\eta_i)|, |I^q L_2(\eta_i)|, |I^q L_3(\eta_i)| \}, \quad i = 1, 2. \end{gathered} \tag{2.6}$$

(A_2) There exist a number κ such that $\Delta \leq \kappa < 1$, $t \in [0,1]$, where

$$\Delta = (1+\gamma_0+\delta_0)\left\{I_L^q + \lambda_1 I^q L(\eta_1) + \lambda_2\left(bI^{q-1}L(1) + I^q L(\eta_2)\right)\right\},$$
$$\lambda_1 = \sup_{t\in[0,1]}\left|\frac{a(b+\eta_2-t)}{1+a(\eta_1-\eta_2-b)}\right|, \qquad \lambda_2 = \sup_{t\in[0,1]}\left|\frac{at-(1+a\eta_1)}{1+a(\eta_1-\eta_2-b)}\right|. \tag{2.7}$$

Lemma 2.4 (see [4]). *Assume that $f : [0,1]\times X\times X\times X \to X$ is a jointly continuous function and satisfies assumption (A_1). Then the boundary value problem* (1.1) *has an unique solution provided $\Delta < 1$, where Δ is given in assumption (A_2).*

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. Reproducing Kernel Method

3.1. Some Reproducing Kernel Spaces

Firstly, inner space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real-valued functions, $u' \in L^2[0,1]\}$. The inner product in $W_2^1[0,1]$ is given by

$$(f,h)_{W_2^1} = f(0)h(0) + \int_0^1 f'(t)h'(t)dt, \quad f,h \in W_2^1[0,1], \tag{3.1}$$

and the norm $\|u\|_{W_2^1}$ is denoted by $\|u\|_{W_2^1} = \sqrt{(u,u)_{W_2^1}}$. From [12], $W_2^1[0,1]$ is a reproducing kernel Hilbert space and the reproducing kernel is

$$K_1(t,s) = 1 + \min\{t,s\}. \tag{3.2}$$

In order to solve (1.2) using RKM, we construct a reproducing kernel space $H_2^3[0,1]$ in which every function satisfies the boundary conditions of (1.2). Inner space $H_2^3[0,1]$ is defined as $H_2^3[0,1] = \{u(x) | u, u', u''$ are absolutely continuous real valued functions, $u''' \in L^2[0,1]$, and $u'(0) + au(c) = 0$, $bu'(1) + u(d) = 0\}$, and the inner product is defined as follows:

$$(f,h)_{H_2^3} = f(0)h(0) + \int_0^1 f'''(t)h'''(t)dt, \quad f,h \in H_2^3[0,1]. \tag{3.3}$$

Theorem 3.1. *$H_2^3[0,1]$ is a Hilbert reproducing kernel space.*

Proof. Suppose $\{v_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $H_2^3[0,1]$, that means

$$\|v_{n+p} - v_n\|^2 = (v_{n+p}(0) - v_n(0))^2 + \int_0^1 \left[v_{n+p}^{(3)}(x) - v_n^{(3)}(x)\right]^2 dx \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.4}$$

Therefore, we have $v_{n+p}(0) - v_n(0) \rightarrow 0$ and $\int_0^1 [v_{n+p}^{(3)}(x) - v_n^{(3)}(x)]^2 dx \rightarrow 0$, which shows that $\{v_n(0)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and $\{v_n^{(3)}(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in space $L^2[0,1]$. So, we have

$$\lim_{n\to\infty} v_n(0) \longrightarrow \lambda, \quad \int_0^1 \left[v_n^{(3)}(x) - h(x)\right]^2 dx \longrightarrow 0, \quad n \longrightarrow \infty, \tag{3.5}$$

where λ is a real constant and $h(x) \in L^2[0,1]$.

Let

$$g(x) = \lambda + \frac{1}{2}\int_0^x (x-t)^2 h(t)dt + a_1 x + a_2 x^2, \tag{3.6}$$

where a_1, a_2 are determined by $g'(0) + ag(c) = 0$, and $bg'(1) + g(d) = 0$.

From $h(x) \in L^2[0,1]$, $g''(x) = \int_0^x h(t)dt + 2a_2$ is absolutely continuous in $[0,1]$ and $g'''(x) = h(x) \in L^2[0,1]$ is almost true everywhere in $[0,1]$. Consequently, $g(x) \in H_2^3[0,1]$. Moreover,

$$\|v_n - g(x)\|^2 = (v_n(0) - \lambda)^2 + \int_0^1 \left[v_n^{(3)}(x) - h(x)\right]^2 dx \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.7}$$

That means that, $H_2^3[0,1]$ is complete.

Similar to [13], we can prove that the point-evaluation functional $x^*(x^*(x) = u(x), x \in [0,1])$ of $H_2^3[0,1]$ is bounded. So $H_2^3[0,1]$ is a Hilbert reproducing kernel space. □

From [12, 14], we have the following.

Theorem 3.2. *The reproducing kernel of* $H_2^3[0,1]$ *is*

$$\begin{aligned} R(t,s) &= \frac{1}{120}\frac{R_1(t,s)}{\Delta^2} + \frac{R_2(t,s) + R_3(t,s) + R_2(s,t) + R_3(s,t)}{\Delta} \\ &\quad + \begin{cases} \dfrac{1}{120}s^3\left(s^2 - 5st + 10t^2\right), & t \geq s, \\ \dfrac{1}{120}t^3\left(10s^2 - 5st + t^2\right), & t < s, \end{cases} \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \Delta &= b(-2 + a(-2+c)c) - d(d + ac(-c+d)), \\ R_1&(t,s) \\ &= -s\Big(\left(5b(-4+d)d^3 - 6d^5\right)(s + ac(-c+s)) \\ &\quad + ac^3\left(c^2 - 5cd + 10d^2\right)(b(-2+s) + d(-d+s)) \\ &\quad - b\Big(\left(-40b + 5(-4+d)d^3\right)(ac(c-s) - s) \\ &\quad + 5a(-4+c)c^3(b(-2+s) + d(-d+s))\Big)\Big)t(t + ac(-c+t)) \end{aligned}$$

$$
\begin{aligned}
&+ac^3s\Big(-5b(-4+c)(ac(c-s)-s)+\left(c^2-5cd+10d^2\right)(ac(c-s)-s)\\
&\qquad +6ac^2(b(-2+s)+d(-d+s))\Big)t(b(-2+t)+d(-d+t))\\
&+120\Big((d-s)(-d+a(c-d)(c-s)-s)+b\Big(-2+a\Big(-2c+c^2-(-2+s)s\Big)\Big)\Big)\\
&\times\Big((d-t)(-d+a(c-d)(c-t)-t)+b\Big(-2+a\Big(-2c+c^2-(-2+t)t\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
R_2(t,s)&=t(t+ac(-c+t))\left(-\frac{1}{24}b(-4+s)s^3+\begin{cases}\dfrac{1}{120}s^3\left(10d^2-5ds+s^2\right), & d\geq s\\ \dfrac{1}{120}d^3\left(d^2-5ds+10s^2\right), & d<s\end{cases}\right),\\
R_3(t,s)&=-at(b(-2+t)+d(-d+t))=\begin{cases}\dfrac{1}{120}s^3\left(10c^2-5cs+s^2\right), & c\geq s,\\ \dfrac{1}{120}c^3\left(c^2-5cs+10s^2\right), & c<s.\end{cases}
\end{aligned}
\tag{3.9}
$$

Actually, it is easy to prove that for every $x \in [0,1]$ and $u(y) \in H_2^3[0,1]$, $R(x,y) \in H_2^3[0,1]$ and $(u(y), R(x,y)) = u(x)$ holds, that is, $R(x,y)$ is the reproducing kernel of $H_2^3[0,1]$.

3.2. The Reproducing Kernel Method

In recent years, there has been a growing interest in using a reproducing kernel to solve the operator equation. In this section, the representation of analytical solution of (1.2) is given in the reproducing kernel space $H_2^3[0,1]$.

Note $Lu = {}^cD^q u(x) + (\phi u)(x) + (\varphi u)(x) + \beta(x)u(x)$ and $F(x,u(x)) = f(x,u(x)) + \beta(x)u(x)$. We can convert (1.2) into an equivalent equation $Lu(x) = F(x,u(x))$. It is clear that $L : H_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator.

Put $\varphi_i(x) = K_1(x_i,x)$, $\Psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. Then

$$
\begin{aligned}
\Psi_i(x) &= \left(L^*\varphi_i(y), R(x,y)\right)\\
&= \left(\varphi_i(y), L_y R(x,y)\right)\\
&= \overline{\left(L_y R(x,y), \varphi_i(x)\right)} = L_y R(x,y)|_{y=x_i}.
\end{aligned}
\tag{3.10}
$$

Similar to [15], we can prove the following.

Lemma 3.3. *Under the previous assumptions, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\Psi_i(x)\}_{i=1}^{\infty}$ is the complete basis of $H_2^3[0,1]$.*

The orthogonal system $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$ of $H_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$, and

$$\overline{\Psi}_i(x) = \sum_{j=1}^{i} \beta_{ij} \Psi_j(x). \tag{3.11}$$

We also can prove the following theorem.

Theorem 3.4. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of (1.2) is unique, the solution can be expressed in the form*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k)) \overline{\Psi}_i(x). \tag{3.12}$$

The approximate solution of the (1.2) is

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k)) \overline{\Psi}_i(x). \tag{3.13}$$

If (1.2) is linear, that is $F(x, u(x)) = F(x)$, then the approximate solution of (1.2) can be obtained directly from (3.13). Else, the approximate process could be modified into the following form:

$$\begin{gathered} u_0(x) = 0, \\ u_{n+1}(x) = \sum_{i=1}^{n+1} B_i \overline{\Psi}_i(x), \end{gathered} \tag{3.14}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u_n(x_k))$.

4. Convergent Theorem of the Numerical Method

In this section, we will give the following convergent theorem of our algorithm.

Lemma 4.1. *There exists a constant M, satisfied $|u(x)| \leq M\|u\|_{H_2^3}$, for all $u(x) \in H_2^3[0,1]$.*

Proof. For all the $x \in [0,1]$ and $u \in H_2^3[0,1]$, there are

$$|u(x)| = |(u(\cdot), K_3(\cdot, x))| \leq \|K_3(\cdot, x)\|_{H_2^3} \cdot \|u\|_{H_2^3} \tag{4.1}$$

$K_3(\cdot, x) \in H_2^3[0,1]$, and note that

$$M = \max_{x \in [0,1]} \|K_3(\cdot, x)\|_{H_2^3}. \tag{4.2}$$

That is, $|u(x)| \leq M\|u\|_{H_2^3}$. □

By Lemma 4.1, it is easy to obtain the following lemma.

Lemma 4.2. *If* $u_n \xrightarrow{\|\cdot\|} \overline{u}$ $(n \to \infty)$, $\|u_n\|$ *is bounded,* $x_n \to y$ $(n \to \infty)$ *and* $F(x, u(x))$ *is continuous, then* $F(x_n, u_{n-1}(x_n)) \to F(y, \overline{u}(y))$.

Theorem 4.3. *Suppose that* $\|u_n\|$ *is bounded in* (3.13) *and* (1.2) *has a unique solution. If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0, 1]$, *then the n-term approximate solution* $u_n(x)$ *derived from the above method converges to the analytical solution* $u(x)$ *of* (1.2).

Proof. First, we will prove the convergence of $u_n(x)$.

From (3.14), we infer that

$$u_{n+1}(x) = u_n(x) + B_{n+1}\overline{\Psi_{n+1}}(x). \tag{4.3}$$

The orthonormality of $\{\overline{\Psi}_i\}_{i=1}^{\infty}$ yields that

$$\|u_{n+1}\|^2 = \|u_n\|^2 + (B_{n+1})^2 = \cdots = \sum_{i=1}^{n+1}(B_i)^2. \tag{4.4}$$

That means $\|u_{n+1}\| \geq \|u_n\|$. Due to the condition that $\|u_n\|$ is bounded, $\|u_n\|$ is convergent and there exists a constant ℓ such that

$$\sum_{i=1}^{\infty}(B_i)^2 = \ell. \tag{4.5}$$

If $m > n$, then

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|^2. \tag{4.6}$$

In view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned}\|u_m - u_n\|^2 &= \|u_m - u_{m-1}\|^2 + \|u_{m-1} - u_{m-2}\|^2 + \cdots + \|u_{n+1} - u_n\|^2 \\ &= \sum_{i=n+1}^{m}(B_i)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{4.7}$$

The completeness of $H_2^3[0, 1]$ shows that $u_n \to \overline{u}$ as $n \to \infty$ in the sense of $\|\cdot\|_{H_2^3}$.

Secondly, we will prove that $\overline{u}$ is the solution of (1.2).

Taking limits in (3.12), we get

$$\overline{u}(x) = \sum_{i=1}^{\infty}B_i\overline{\Psi}_i(x). \tag{4.8}$$

So

$$\begin{aligned}L\overline{u}(x) &= \sum_{i=1}^{\infty}B_iL\overline{\Psi}_i(x), \\ (L\overline{u})(x_n) &= \sum_{i=1}^{\infty}B_i\left(L\overline{\Psi}_i, \varphi_n\right) = \sum_{i=1}^{\infty}B_i\left(\overline{\Psi}_i, L^*\varphi_n\right) = \sum_{i=1}^{\infty}B_i\left(\overline{\Psi}_i, \Psi_n\right).\end{aligned} \tag{4.9}$$

Therefore,

$$\sum_{i=1}^{n}\beta_{nj}(L\overline{u})(x_n) = \sum_{i=1}^{\infty}B_i\left(\overline{\Psi_i}, \sum_{j=1}^{n}\beta_{nj}\Psi_j\right) = \sum_{i=1}^{\infty}B_i\left(\overline{\Psi_i}, \overline{\Psi_n}\right) = B_n. \tag{4.10}$$

If $n = 1$, then

$$L\overline{u}(x_1) = F(x_1, u_0(x_1)). \tag{4.11}$$

If $n = 2$, then

$$\beta_{21}L\overline{u}(x_1) + \beta_{22}L\overline{u}(x_2) = \beta_{21}F(x_1, u_0(x_1)) + \beta_{22}F(x_2, u_1(x_2)). \tag{4.12}$$

It is clear that

$$(L\overline{u})(x_2) = F(x_2, u_1(x_2)). \tag{4.13}$$

Moreover, it is easy to see by induction that

$$(L\overline{u})(x_j) = F\left(x_j, u_{j-1}(x_j)\right), \quad j = 1, 2, \ldots. \tag{4.14}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, for all $Y \in [0,1]$, there exists a subsequence $\{x_{nj}\}_{j=1}^{\infty}$ such that

$$x_{nj} \longrightarrow Y \quad \text{as } j \longrightarrow \infty. \tag{4.15}$$

It is easy to see that $(L\overline{u})(x_{nj}) = F(x_{nj}, u_{nj-1}(x_{nj}))$. Let $j \to \infty$; by the continuity of $F(x, u(x))$ and Lemma 4.2, we have

$$(L\overline{u})(Y) = F(Y, \overline{u}(Y)). \tag{4.16}$$

At the same time, $\overline{u} \in H_2^3[0,1]$; clearly, u satisfies the boundary conditions of (1.2).

That is, $\overline{u}$ is the solution of (1.2).

The proof is complete. □

In fact, $u_n(x)$ is just the orthogonal projection of exact solution $\overline{u}(x)$ onto the space $\overline{\mathrm{Span}\{\overline{\Psi_i}\}_{i=1}^{n}}$.

5. Numerical Example

To give a clear overview of the methodology as a numerical tool, we consider one example in this section. We apply the reproducing kernel method and results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other. Also, the numerical results obtained are compared with the corresponding experimental results obtained by the methods presented in [8, 9].

Table 1: Absolute errors for Example 5.1.

x	True solution	DM [8]	VIM [9]	RKM (u_5^{11})	RKM (u_5^{101})
0	0.05566	5.61689$E-5$	1.48366$E-4$	2.16993$E-5$	2.11552$E-7$
0.1	0.19798	5.47421$E-5$	1.46246$E-4$	1.92463$E-5$	1.90591$E-7$
0.2	0.39473	5.43973$E-5$	1.49244$E-4$	1.70817$E-5$	1.59788$E-7$
0.3	0.60560	5.47579$E-5$	1.55807$E-4$	1.49046$E-5$	1.36933$E-7$
0.4	0.77891	5.56624$E-5$	1.65449$E-4$	1.26031$E-5$	1.14128$E-7$
0.5	0.83516	5.68976$E-5$	1.78225$E-4$	1.02477$E-5$	9.25561$E-8$
0.6	0.70036	5.80389$E-5$	1.94554$E-4$	7.88466$E-6$	6.84821$E-8$
0.7	0.50144	5.82441$E-5$	2.1518$E-4$	5.65515$E-6$	4.82468$E-8$
0.8	0.29175	5.5968$E-5$	2.41183$E-4$	3.54481$E-6$	2.9536$E-8$
0.9	0.11797	4.85764$E-5$	2.74062$E-4$	1.50572$E-6$	9.8474$E-9$
1.0	0.01485	3.18397$E-5$	3.15881$E-4$	9.79078$E-7$	1.3896$E-8$

Example 5.1. Consider the following boundary value problem:

$$
\begin{gathered}
{}^cD^{3/2}u(t) = \frac{1}{5}\int_0^t \frac{e^{-(s-t)} + e^{-(s-t)/2}}{5}u(s)ds + u^2(t) - 2t^2u(t) + \frac{20t}{17}u(t) + \frac{454}{153}u(t) + f(t), \quad t \in [0,1], \\
u'(0) + \frac{1}{2}u\left(\frac{1}{3}\right) = 0, \qquad \frac{1}{4}u'(1) + u\left(\frac{2}{3}\right) = 0,
\end{gathered}
\tag{5.1}
$$

where $f(t) = 1674244/585225 - 2354e^{t/2}/3825 - 169e^t/3825 + 4\sqrt{t}/\sqrt{\pi} - 3316t/2601 - 162647t^2/65025 + 20t^3/17 + t^4$. According to Lemma 2.4, the boundary value problem (5.1) has a unique solution on $[0,1]$. $u(t) = t^2 + 10t/17 - 227/153$ is the solution of (5.1), so it is the one and the only one solution. Using our method, taking $x_i = (i-1)/(N-1)$, $i = 1, 2, \ldots, N$, $N = 11, 101$, the numerical results are given in Table 1.

6. Conclusion

In this paper, RKM is presented to solve four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order $q \in (1,2]$. The results of numerical examples demonstrate that the present method is more accurate than the existing methods.

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