

# On the spectral radius of uniform weighted hypergraph

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Let  $\mathbb{Q}_{k,n}$  be the set of the connected  $k$ -uniform weighted hypergraphs with  $n$  vertices, where  $k, n \geq 3$ . For a hypergraph  $G \in \mathbb{Q}_{k,n}$ , let  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  be its adjacency tensor, Laplacian tensor and signless Laplacian tensor, respectively. The spectral radii of  $\mathcal{A}(G)$  and  $\mathcal{Q}(G)$  are investigated. Some basic properties of the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the  $H^{++}$ -eigenvalue of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  are presented. Several lower and upper bounds of the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the  $H^{++}$ -eigenvalue for  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  are established. The largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$  and the smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  are characterized. A relationship among the  $H$ -eigenvalues of  $\mathcal{L}(G)$ ,  $\mathcal{Q}(G)$  and  $\mathcal{A}(G)$  is also given.

Keywords:  $k$ -uniform weighted hypergraph; Adjacency tensor; Laplacian tensor; Signless Laplacian tensor; Spectrum

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## 1. INTRODUCTION

Weighted hypergraphs are a natural extension of hypergraphs. They are of interest in real life and have many applications in graph theory. For example, the circuit is mathematically modeled by a weighted hypergraph and weighted hypergraphs are closely related to the specific application of circuit division [1].

A weighted hypergraph is obtained from a hypergraph  $G^\Delta = (V(G^\Delta), E(G^\Delta))$  by assigning a weight (namely, nonzero real number) to each edge of  $G^\Delta$ . We denote such a weighted hypergraph by  $G = (V(G), E(G), W(G))$ , where  $V(G) = V(G^\Delta) = \{v_1, \dots, v_n\}$ ,  $E(G) = E(G^\Delta) = \{e_1, e_2, \dots, e_m\}$ , and  $W(G) = \{w_G(e) \in \mathbb{R} : e \in E(G)\}$  are the vertex set, the edge set, and the weight set of  $G$ , respectively. Here  $w_G(e)$  is the weight on the edge  $e$  of  $G$  and  $\mathbb{R}$  is the set of real numbers. A weighted hypergraph is simple if it has no loops or multiple edges. In this paper, we consider the weighted hypergraph which is simple and connected and satisfies that the weight of each edge is a positive real number.

A simple weighted hypergraph  $G$  is  $k$ -uniform if each edge of  $G$  has  $k$  vertices, where  $k \geq 2$ . If  $k = 2$ , then  $G$  is a simple weighted graph. A hypergraph  $G$  is called linear if any two edges in  $G$  intersect on at most one common vertex. A hypergraph  $G$  is connected if there exists a path between every pair of vertices in  $V(G)$ . Here a path of length  $p$  ( $p \geq 1$ ) between  $v_1$  and  $v_{p+1}$  is denoted by  $P = (v_1, e_1, v_2, \dots, v_p, e_p, v_{p+1})$ , where all  $v_i$  and all  $e_i$  are distinct, and  $v_i, v_{i+1} \in e_i$  for  $1 \leq i \leq p$ .

Let  $G$  be a weighted hypergraph and  $u, v \in V(G)$ . A vertex  $v$  is said to be incident with an edge  $e \in E(G)$  if  $v \in e$ . If  $\{u, v\} \subseteq e \in E(G)$ , then we say that  $u$  and  $v$  are adjacent. Let  $E_G(v)$  be the set of all the edges incident with  $v$  of  $G$ , i.e.,  $E_G(v) = \{e \in E(G) : v \in e\}$ . The degree of  $v$  is denoted by  $d_G(v)$ . Namely  $d_G(v) = |E_G(v)|$ . If each vertex of  $G$  has degree  $r$  ( $r \geq 1$ ), then we say that  $G$  is  $r$ -regular. We use  $N_G(u)$  to denote the set of vertices which are adjacent with  $u$ , where  $u \in V(G)$ . For simplicity, let  $\Delta = \max_{v \in V(G)} |E_G(v)|$  and  $W_0 = \max_{e \in E(G)} w_G(e)$ . Hereinafter, if each edge of  $G$  has the same weight, then we denote the weight by  $W_0$ . For  $v_i \in V(G)$ , let  $w_{v_i} = \sum_{e \in E_G(v_i)} w_G(e)$  and we call  $w_{v_i}$  the weight of vertex  $v_i$  of  $G$ , where  $i = 1, \dots, n$ . Let  $\alpha = \max\{w_{v_i} : i \in [n]\}$  and  $\delta = \min\{w_{v_i} : i \in [n]\}$ , where  $[n] = \{1, 2, \dots, n\}$ .

A real tensor (or hypermatrix)  $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$  of order  $k$  and dimension  $n$  is a multi-dimensional array with entries  $a_{i_1 i_2 \dots i_k}$ , where  $a_{i_1 i_2 \dots i_k} \in \mathbb{R}$  with  $i_1, i_2, \dots, i_k \in [n]$ .

The concept of tensor eigenvalues and the spectra of tensors were introduced by Qi [2] and Lim [3] in 2005 independently. Let  $\mathbb{C}$  be the set of complex numbers and  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$  an  $n$ -dimensional complex column vector. Let  $\mathbf{x}^{[k]} = (x_1^k, x_2^k, \dots, x_n^k)^T$ , where  $k$  is a positive integer. Then  $\mathcal{A}\mathbf{x}$  is a vector in  $\mathbb{C}^n$  whose  $i$ -th component is given by

$$(\mathcal{A}\mathbf{x})_i = \sum_{i_2, \dots, i_k=1}^n a_{ii_2 \dots i_k} x_{i_2} \cdots x_{i_k}, \text{ for each } i \in [n]. \quad (1)$$

Furthermore, we have

$$\mathbf{x}^T(\mathcal{A}\mathbf{x}) = \sum_{i_1, i_2, \dots, i_k=1}^n a_{i_1 i_2 \dots i_k} x_{i_1} \cdots x_{i_k}. \quad (2)$$

Let  $\mathbb{T}_{k,n}$  be the set of tensors of order  $k$  and dimension  $n$ , where  $k, n \geq 3$ .

**Definition 1.1** Let  $\mathcal{A} \in \mathbb{T}_{k,n}$ , where  $k, n \geq 3$ . If for any vector  $\mathbf{x} \in \mathbb{R}^n$ , we have

$$\mathbf{x}^T(\mathcal{A}\mathbf{x}) = \sum_{i_1, i_2, \dots, i_k=1}^n a_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \geq 0,$$

then  $\mathcal{A}$  is called a positive semi-definite tensor. If for any vector  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{x} \neq \mathbf{0}$ , we have  $\mathbf{x}^T(\mathcal{A}\mathbf{x}) > 0$ , then  $\mathcal{A}$  is said to be a positive definite tensor.

For  $\lambda \in \mathbb{C}$  and  $\mathbf{x} \in \mathbb{C}^n$ , if they satisfy  $\mathcal{A}\mathbf{x} = \lambda \mathbf{x}^{[k-1]}$ , namely,  $(\mathcal{A}\mathbf{x})_i = \lambda x_i^{k-1}$  for any  $i \in [n]$ , then  $\lambda$  is called an eigenvalue of  $\mathcal{A}$  and  $\mathbf{x}$  an eigenvector of  $\mathcal{A}$  corresponding to  $\lambda$ . The largest modulus of the eigenvalues of  $\mathcal{A}$  is called the spectral radius of  $\mathcal{A}$ . If  $\mathbf{x}$  is a real eigenvector of  $\mathcal{A}$ , then  $\lambda$  is also real and is referred to as an  $H$ -eigenvalue and  $\mathbf{x}$  an  $H$ -eigenvector. Let  $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, i \in [n]\}$  and  $\mathbb{R}_{++}^n = \{\mathbf{x} \in \mathbb{R}^n : x_i > 0, i \in [n]\}$ . If  $\mathbf{x} \in \mathbb{R}_+^n$ , then  $\lambda$  is an  $H^+$ -eigenvalue of  $\mathcal{A}$ . If  $\mathbf{x} \in \mathbb{R}_{++}^n$ , then  $\lambda$  is an  $H^{++}$ -eigenvalue of  $\mathcal{A}$ .

The adjacency tensor of a  $k$ -uniform weighted hypergraph  $G$  is defined as follows.

**Definition 1.2** Let  $G$  be a  $k$ -uniform weighted hypergraph with  $n$  vertices. The adjacency tensor of  $G$  is the  $k$ -ordered and  $n$ -dimensional adjacency tensor  $\mathcal{A}(G) = (a_{i_1 i_2 \dots i_k})$  whose  $(i_1 i_2 \cdots i_k)$ -entry is

$$a_{i_1 i_2 \dots i_k} = \begin{cases} \frac{w_G(e)}{(k-1)!}, & \text{if } e = \{i_1, i_2, \dots, i_k\} \in E(G), \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where each  $i_j$  runs from 1 to  $n$  for  $j \in [k]$ .

In Definition 1.2, if  $w_G(e) = 1$  for each edge of  $G$ , then  $\mathcal{A}(G)$  is just the tensor defined by Cooper and Dutle [4] in 2012 for a  $k$ -uniform hypergraph with  $n$  vertices. For a real tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$ , if  $a_{i_1 i_2 \dots i_k}$  is invariant under any permutation of the indices  $i_1, i_2, \dots, i_k$ , then  $\mathcal{A}$  is said to be symmetric. A tensor is called nonnegative if all its entries are nonnegative. Let  $G$  be a  $k$ -uniform weighted hypergraph. Obviously, the adjacency tensor  $\mathcal{A}(G)$  of  $G$  is always nonnegative and symmetric. The spectral radius of  $\mathcal{A}(G)$ , denoted by  $\rho(G)$ , is called the spectral radius of  $G$ .

Inspired by the definitions of the Laplacian tensor and the signless Laplacian tensor of a  $k$ -uniform hypergraph which were introduced by Qi [5], in this paper, we introduce the definitions of the Laplacian tensor and the signless Laplacian tensor for a  $k$ -uniform weighted hypergraph. Let  $\mathbb{Q}_{k,n}$  be the set of the connected  $k$ -uniform weighted hypergraphs with  $n$  vertices, where  $k, n \geq 3$ . Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . We use  $\mathcal{D}(G) = (d_{i_1 i_2 \dots i_k})$  to denote a diagonal tensor of order  $k$  and dimension  $n$ , where  $k, n \geq 3$ ,  $d_{i \dots i} = w_{v_i}$  for  $i \in [n]$  and  $d_{i_1, \dots, i_k} = 0$  otherwise. Let  $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$  and  $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ . We call  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  the Laplacian tensor and the signless Laplacian tensor of  $G$ , respectively.

The research on the spectral radius of the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor for hypergraphs has attracted a lot of interests. For the three tensors of hypergraphs, many interesting results about the characterization of the hypergraph with extremal spectral radius are derived, and some properties and bounds for the extremal spectral radii have been obtained. Interested readers can find Refs. [5–24].

Xie and Chang [25] obtained some bounds on the smallest and the largest  $Z$ -eigenvalues of the adjacency tensor for uniform hypergraphs. Xie and Chang [18, 26] introduced the signless Laplacian tensor for even uniform hypergraphs, and derived several properties of the smallest and the largest  $H$ -eigenvalues and  $Z$ -eigenvalues of the signless Laplacian tensor for an even uniform hypergraph. Qi [5] defined the Laplacian and the signless Laplacian tensors of a uniform hypergraph for the study on their  $H^+$ -eigenvalues and  $H^{++}$ -eigenvalues, and established some bounds for the largest signless Laplacian  $H^+$ -eigenvalue. Hu et al. [19] obtained a tight lower bound for the largest Laplacian  $H$ -eigenvalue of a  $k$ -uniform hypergraph and derived the tight lower and upper bounds for the largest signless Laplacian  $H$ -eigenvalue of a connected hypergraph. Yue et

al. [21] obtained the upper bounds of the largest Laplacian  $H$ -eigenvalue for a  $k$ -uniform loose path with a length not less than 3. All the results are related with the unweighted hypergraph.

Inspired by the above results, in this paper, we investigate the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the  $H^{++}$ -eigenvalue of adjacency tensor, Laplacian tensor and signless Laplacian tensor for the  $k$ -uniform weighted hypergraph  $G$ . This article is organized as follows. In Section 2, some notations and necessary lemmas which are useful for subsequent proofs are introduced and some basic properties of the eigenvalues of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$ , and  $\mathcal{Q}(G)$  are presented. In Section 3, we study the lower and upper bounds of the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the  $H^{++}$ -eigenvalue for  $\mathcal{L}(G)$ . The largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$  is characterized. A relationship among the  $H$ -eigenvalues of  $\mathcal{L}(G)$ ,  $\mathcal{Q}(G)$  and  $\mathcal{A}(G)$  is also given. In Section 4, we consider the lower and upper bounds of the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the spectral radius of  $\mathcal{Q}(G)$ . The smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  is characterized. A property of the  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  is derived. Finally, the lower and upper bounds of the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the spectral radius of  $\mathcal{A}(G)$  are derived in Section 5. A property of the  $H^+$ -eigenvalue for  $\mathcal{A}(G)$  is also deduced.

## 2. PRELIMINARY

In this section, we first define some notations and introduce necessary lemmas. Then we derive some fundamental properties about the eigenvalues of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$ , and  $\mathcal{Q}(G)$  for a  $k$ -uniform weighted hypergraph  $G$ .

Let  $\mathbf{x} = (x_1, \dots, x_n)^T$  be an  $n$ -dimensional eigenvector of  $\mathcal{A}(G)$  ( $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$ ) and  $x_i$  the component of  $\mathbf{x}$  which corresponds to vertex  $v_i$  ( $i = 1, \dots, n$ ) of  $G$ , where  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ . Let  $U$  be a subset of  $[n]$ . Let

$$x^U = \prod_{i \in U} x_i.$$

By (1) and (3), for  $i \in [n]$ , we get

$$(\mathcal{A}(G)\mathbf{x})_i = \sum_{e \in E_G(v_i)} w_G(e) x^{e \setminus \{v_i\}}. \quad (4)$$

Furthermore, we have

$$(\mathcal{Q}(G)\mathbf{x})_i = w_{v_i}x_i^{k-1} + \sum_{e \in E_G(v_i)} w_G(e)x^{e \setminus \{v_i\}}, \quad (5)$$

$$(\mathcal{L}(G)\mathbf{x})_i = w_{v_i}x_i^{k-1} - \sum_{e \in E_G(v_i)} w_G(e)x^{e \setminus \{v_i\}}. \quad (6)$$

Friedland et al. [27] defined the nonnegative weakly irreducible tensor and Yang et al. [28] restated it as follows.

**Definition 2.1** [28] *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_k})$  be a nonnegative tensor of order  $k$  and dimension  $n$ . If for any nonempty proper index subset  $I \subset [n]$ , there is at least an entry  $a_{i_1 i_2 \dots i_k} > 0$ , where  $i_1 \in I$  and at least an  $i_j \in [n] \setminus I$  for  $j = 2, 3, \dots, k$ , then  $\mathcal{A}$  is called a nonnegative weakly irreducible tensor.*

**Lemma 2.1** [27, 29] *Let  $\mathcal{A}$  be a nonnegative tensor of order  $k$  and dimension  $n$ , where  $k \geq 2$ . Then we have the following statements.*

- (i).  $\rho(\mathcal{A})$  is an eigenvalue of  $\mathcal{A}$  with a nonnegative eigenvector  $\mathbf{x} \in \mathbb{R}_+^n$  corresponding to it.
- (ii). If  $\mathcal{A}$  is weakly irreducible, then  $\rho(\mathcal{A})$  is the only eigenvalue of  $\mathcal{A}$  with a positive eigenvector  $\mathbf{x} \in \mathbb{R}_{++}^n$ , up to a positive scaling coefficient.

Let  $\mathbb{S}_{k,n}$  be the set of real symmetric tensors of order  $k$  and dimension  $n$ , where  $k, n \geq 3$ .

**Lemma 2.2** [2] *We have the following conclusions on the eigenvalues of  $\mathcal{A} \in \mathbb{S}_{k,n}$ , where  $k, n \geq 3$ .*

- (i). A number  $\lambda \in \mathbb{C}$  is an eigenvalue of  $\mathcal{A}$  if and only if it is a root of the characteristic polynomial  $\phi(\lambda) = \det(\mathcal{A} - \lambda\mathcal{I})$ , where  $\mathcal{I}$  is the unit tensor.
- (ii). The number of eigenvalues of  $\mathcal{A}$  is  $n(k-1)^{n-1}$ . Their product is equal to  $\det(\mathcal{A})$ .
- (iii). The sum of all the eigenvalues of  $\mathcal{A}$  is  $(k-1)^{n-1} \text{tr}(\mathcal{A})$ .

**Lemma 2.3** [2] *Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_k}) \in \mathbb{S}_{k,n}$ , where  $k, n \geq 3$ . The following conclusions hold for  $\mathcal{A}$ .*

- (i). Assume that  $k$  is even.  $\mathcal{A}$  always has  $H$ -eigenvalues.  $\mathcal{A}$  is positive definite (positive semi-definite) if and only if all of its  $H$ -eigenvalues are positive (nonnegative).

(ii). The eigenvalues of  $\mathcal{A}$  lie in the union of  $n$  disks in  $\mathbb{C}$ . These  $n$  disks have the diagonal elements  $a_{i,\dots,i}$  of the supersymmetric tensor as their centers, and the sums of the absolute values of the off-diagonal elements  $\sum_{i_2,\dots,i_k=1;\{i_2,\dots,i_k\}\neq\{i,\dots,i\}}^n |a_{ii_2\dots i_k}|$  as their radii, where  $i \in [n]$ .

Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_k}) \in \mathbb{R}^{n \times \dots \times n}$  be a nonnegative tensor of order  $k$  and dimension  $n$ . Based on  $\mathcal{A}$ , we define a directed graph  $\Gamma_{\mathcal{A}}$  as follows. The vertex set of  $\Gamma_{\mathcal{A}}$  is  $V(\Gamma_{\mathcal{A}}) = \{1, \dots, n\}$  and the arc set of  $\Gamma_{\mathcal{A}}$  is

$$E(\Gamma_{\mathcal{A}}) = \{(i, j) : a_{ii_2 \dots i_k} > 0, j \in \{i_2 \dots i_k\}\}. \quad (7)$$

A graph is strongly connected if it contains a directed path from  $i$  to  $j$  and a directed path from  $j$  to  $i$  for every pair of vertices  $i$  and  $j$ . A tensor  $\mathcal{A}$  is called weakly irreducible if  $\Gamma_{\mathcal{A}}$  is strongly connected [27, 30, 31].

According to the definitions of the weakly irreducible tensor and the adjacency tensor of weighted hypergraph, we have Theorem 2.1 as follows.

**Theorem 2.1** *Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Any two of the three conclusions are equivalent.*

- (i).  $G$  is connected.
- (ii).  $\mathcal{A}(G)$  is weakly irreducible.
- (iii).  $\mathcal{Q}(G)$  is weakly irreducible.

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Let the directed graph associated with  $G$  be  $\Gamma_{\mathcal{A}(G)} = (V(\Gamma_{\mathcal{A}(G)}), E(\Gamma_{\mathcal{A}(G)}))$ , where  $V(\Gamma_{\mathcal{A}(G)}) = \{1, 2, \dots, n\}$  and  $E(\Gamma_{\mathcal{A}(G)}) = \{(i, j) : e = \{v_i, v_j, v_{j_3}, \dots, v_{j_k}\} \in E(G), j_3, \dots, j_k \in [n] \setminus \{i, j\}\}$  (by (3) and (7)).

Let  $i$  and  $j$  be any two different vertices in  $V(\Gamma_{\mathcal{A}(G)})$ . By the definition of  $E(\Gamma_{\mathcal{A}(G)})$ , for  $i, j \in [n]$  and  $i \neq j$ , we obtain

$$(i, j), (j, i) \in E(\Gamma_{\mathcal{A}(G)}) \Leftrightarrow v_i, v_j \in e \in E(G). \quad (8)$$

Since  $G$  is connected, by (8), we can get that  $\Gamma_{\mathcal{A}(G)}$  is strongly connected. Namely,  $\mathcal{A}(G)$  is weakly irreducible.

If  $\Gamma_{\mathcal{A}(G)}$  is strongly connected, then for any two different vertices  $i$  and  $j$  in  $V(\Gamma_{\mathcal{A}(G)})$ , there exist  $j_1, \dots, j_t \in V(\Gamma_{\mathcal{A}(G)})$  such that  $(i, j_1), (j_1, j_2), \dots, (j_t, j), (j, j_t), \dots, (j_2, j_1), (j_1,$

$i) \in E(\Gamma_{\mathcal{A}}(G))$ , where  $t \geq 0$ . It follows from (8) that there exist  $e_1, e_2, \dots, e_{t+1} \in E(G)$  such that  $v_i, v_{j_1} \in e_1, v_{j_1}, v_{j_2} \in e_2, \dots, v_{j_t}, v_j \in e_{t+1}$ , where  $t \geq 0$ . Namely, for  $v_i, v_j \in V(G)$ , there exists a path in  $G$  connecting  $v_i$  and  $v_j$ . Thus, we get that  $G$  is connected.

Therefore, we have  $(i) \Leftrightarrow (ii)$ . Similarly, we can get  $(i) \Leftrightarrow (iii)$ . Thus, we have Theorem 2.1.  $\square$

**Theorem 2.2** *Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ .  $\rho(\mathcal{A}(G))$  ( $\rho(\mathcal{Q}(G))$ ) is the only eigenvalue of  $\mathcal{A}(G)$  ( $\mathcal{Q}(G)$ ) with a unique positive eigenvector  $\mathbf{x} \in \mathbb{R}_{++}^n$ , up to a positive scaling coefficient.*

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Since  $G$  is connected, by Theorem 2.1,  $\mathcal{A}(G)$  and  $\mathcal{Q}(G)$  are weakly irreducible. Furthermore, by Lemma 2.1(ii), we get Theorem 2.2.  $\square$

By Lemmas 2.2 and 2.3, we obtain some basic properties of the eigenvalues of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$ , and  $\mathcal{Q}(G)$ , where  $G \in \mathbb{Q}_{k,n}$ , which are shown in Theorem 2.3.

**Theorem 2.3** *Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . We have the five conclusions as follows.*

(i). *A number  $\lambda \in \mathbb{C}$  is an eigenvalue of  $\mathcal{A}(G)$  ( $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$ ) if and only if it is a root of the characteristic polynomial  $\phi(\mathcal{A}(G))$  ( $\phi(\mathcal{L}(G))$  and  $\phi(\mathcal{Q}(G))$ ).*

(ii). *The number of the eigenvalues of  $\mathcal{A}(G)$  ( $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$ ) is  $n(k-1)^{n-1}$ . Their product is equal to  $\det(\mathcal{A}(G))$  ( $\det(\mathcal{L}(G))$  and  $\det(\mathcal{Q}(G))$ ).*

(iii). *The sum of all the eigenvalues of  $\mathcal{A}(G)$  is zero and the sum of all the eigenvalues of  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  is  $(k-1)^{n-1} \sum_{i=1}^n w_{v_i} = k(k-1)^{n-1} \sum_{e \in E(G)} w_G(e)$ .*

(iv). *All the eigenvalues of  $\mathcal{A}(G)$  lie in the disks  $\{\lambda : |\lambda| \leq W_0 \Delta\}$  and all the eigenvalues of  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  lie in the disks  $\{\lambda : |\lambda - W_0 \Delta| \leq W_0 \Delta\}$ .*

(v). *When  $k$  is even,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  are positive semi-definite tensors.*

**Proof.** (i). The proof of Theorem 2.3(i)–(iii).

By Lemma 2.2(i)–(iii) and the definition of the tensor of the weighted hypergraph, we can directly get Theorem 2.3(i)–(iii), respectively.

(ii). The proof of Theorem 2.3(iv).

By (3), for  $i \in [n]$ , we have

$$\sum_{i_2, \dots, i_k=1; \{i_2, \dots, i_k\} \neq \{i, \dots, i\}}^n a_{ii_2 \dots i_k} = \sum_{e \in E_G(v_i)} w_G(e) = w_{v_i}.$$



Let  $\lambda$  be an arbitrary eigenvalue of  $\mathcal{A}(G)$ . Let  $\bigcirc_i = \{\lambda : |\lambda| \leq w_{v_i}\}$  be a disk, where  $i = 1, \dots, n$ . By Lemma 2.3(ii), we obtain  $\lambda \in \bigcup_{i=1}^n \bigcirc_i$ . Let  $e$  be an arbitrary edge in  $E(G)$ . Since  $w_G(e) \leq W_0$  and  $|E_G(v_i)| \leq \Delta$  for  $i = 1, \dots, n$ , we get

$$|\lambda| \leq w_{v_i} = \sum_{e \in E_G(v_i)} w_G(e) \leq W_0 \sum_{e \in E_G(v_i)} 1 = W_0 |E_G(v_i)| \leq W_0 \Delta. \quad (9)$$

Let  $\mu$  be an arbitrary eigenvalue of  $\mathcal{L}(G)$  ( $\mathcal{Q}(G)$ ). Let  $\odot_i = \{\mu : |\mu - w_{v_i}| \leq w_{v_i}\}$  be a disk, where  $i = 1, \dots, n$ . By Lemma 2.3(ii), we get  $\mu \in \bigcup_{i=1}^n \odot_i$ . Since  $w_{v_i} = \sum_{e \in E_G(v_i)} w_G(e) \leq W_0 \Delta$ , we have  $\mu \in \bigcup_{i=1}^n \odot_i \subseteq \{\mu : |\mu - W_0 \Delta| \leq W_0 \Delta\}$ .  
(iii). The proof of Theorem 2.3(v).

When  $k$  is even, by Theorem 2.3(iv) and Lemma 2.3(i), we obtain that  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  are positive semi-definite tensors.  $\square$

### 3. THE EIGENVALUES OF $\mathcal{L}(G)$

In this section, we study the eigenvalues of  $\mathcal{L}(G)$ , where  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ . We obtain the upper and lower bounds of the  $H$ -eigenvalue and the  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ , which are shown in Theorems 3.1 and 3.2, respectively. The largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$  is given in Theorem 3.4. Two results of the  $H^+$ -eigenvalue and the  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$  are derived in Theorems 3.3 and 3.5, respectively. A relationship among the  $H$ -eigenvalues of  $\mathcal{L}(G)$ ,  $\mathcal{Q}(G)$  and  $\mathcal{A}(G)$  is shown in Theorem 3.6.

**Theorem 3.1** (*The bounds for the  $H$ -eigenvalue of  $\mathcal{L}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\mathcal{L}(G)$  has an  $H$ -eigenvalue  $\lambda$  and  $0 \leq \lambda \leq 2W_0\Delta$ .

**Proof.** By (6) and  $(\mathcal{L}(G)\mathbf{x})_i = \lambda x_i^{k-1}$  ( $i = 1, \dots, n$ ), we get

$$(\mathcal{L}(G)\mathbf{1})_i = w_{v_i} - \sum_{e \in E_G(v_i)} w_G(e) = 0 = 0 \cdot [\mathbf{1}]_i. \quad (10)$$

By (10), zero is an  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$  and  $\mathbf{1}$  is the eigenvector of  $\mathcal{L}(G)$  corresponding to zero. Thus,  $\mathcal{L}(G)$  has  $H$ -eigenvalues. Let  $\lambda$  be an  $H$ -eigenvalue of  $\mathcal{L}(G)$ . By Theorem 2.3(iv),  $0 \leq \lambda \leq 2W_0\Delta$ . Therefore, we obtain Theorem 3.1.  $\square$

**Theorem 3.2** (*The bounds for the  $H^+$ -eigenvalue of  $\mathcal{L}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\mathcal{L}(G)$  has an  $H^+$ -eigenvalue  $\lambda$  and  $0 \leq \lambda \leq \alpha$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Since  $\mathcal{L}(G)$  has an  $H^{++}$ -eigenvalue zero (by (10)),  $\mathcal{L}(G)$  has  $H^+$ -eigenvalues. Let  $\lambda$  be an  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ . By Theorem 3.1,  $\lambda \geq 0$ . Let  $\mathbf{x}$  be an  $H^+$ -eigenvector of  $\mathcal{L}(G)$  corresponding to  $\lambda$ . Then  $\mathbf{x} \in \mathbb{R}_+^n$ . Thus, the largest component of  $\mathbf{x}$  is positive. Without loss of generality, we assume the largest component of  $\mathbf{x}$  is 1. Let  $u \in V(G)$  and  $x_u = 1$ . Therefore, by (6) and  $(\mathcal{L}(G)\mathbf{x})_u = \lambda x_u^{k-1}$ , we have

$$w_u - \lambda = \sum_{e \in E_G(u)} w_G(e) x^{e \setminus \{u\}}.$$

Since  $\mathbf{x} \in \mathbb{R}_+^n$  and  $w_G(e) > 0$  for any edge  $e$  in  $E(G)$ , we have  $w_u - \lambda \geq 0$ . Namely,  $\lambda \leq w_u$ . Since  $w_u \leq \alpha$ , we get  $0 \leq \lambda \leq \alpha$ .  $\square$

Let  $\mathbf{e}^{(i)}$  be an  $n$ -dimensional vector satisfying  $e_j^{(i)} = 1$  if  $j = i$  and  $e_j^{(i)} = 0$  if  $j \neq i$ , where  $i, j = 1, 2, \dots, n$ .

**Theorem 3.3** ( *$H^+$ -eigenvalue of  $\mathcal{L}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then for any  $i \in [n]$ ,  $w_{v_i}$  is an  $H^+$ -eigenvalue of  $\mathcal{L}(G)$  and  $\mathbf{e}^{(i)}$  is an  $H^+$ -eigenvector of  $\mathcal{L}(G)$  corresponding to  $w_{v_i}$ .

**Proof.** Let  $i, j \in [n]$ . If  $j = i$ , since  $e_j^{(i)} = 1$ , we get

$$\begin{aligned} (\mathcal{L}(G)\mathbf{e}^{(i)})_j &= (\mathcal{D}(G)\mathbf{e}^{(i)})_j - (\mathcal{A}(G)\mathbf{e}^{(i)})_j \\ &= \sum_{j_2, \dots, j_k=1}^n d_{jj_2 \dots j_k} e_{j_2}^{(i)} \cdots e_{j_k}^{(i)} - \sum_{j_2, \dots, j_k=1}^n a_{jj_2 \dots j_k} e_{j_2}^{(i)} \cdots e_{j_k}^{(i)} \\ &= d_{jj \dots j} e_j^{(i)} \cdots e_j^{(i)} \\ &= d_{ii \dots i} = w_{v_i} = w_{v_i} e_j^{(i)}. \end{aligned}$$

If  $j \neq i$ , since  $e_j^{(i)} = 0$ , we obtain

$$\begin{aligned} (\mathcal{L}(G)\mathbf{e}^{(i)})_j &= (\mathcal{D}(G)\mathbf{e}^{(i)})_j - (\mathcal{A}(G)\mathbf{e}^{(i)})_j \\ &= \sum_{j_2, \dots, j_k=1}^n d_{jj_2 \dots j_k} e_{j_2}^{(i)} \cdots e_{j_k}^{(i)} - \sum_{j_2, \dots, j_k=1}^n a_{jj_2 \dots j_k} e_{j_2}^{(i)} \cdots e_{j_k}^{(i)} \\ &= d_{jj \dots j} e_j^{(i)} \cdots e_j^{(i)} \\ &= 0 = w_{v_i} e_j^{(i)}. \end{aligned}$$

Therefore, we get Theorem 3.3.  $\square$

**Theorem 3.4** (*The largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\alpha$  is the largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ .

**Proof.** By Theorem 3.2,  $\mathcal{L}(G)$  has  $H^+$ -eigenvalues. Let  $\lambda$  be an  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ . It follows from Theorem 3.2 that  $\lambda \leq \alpha$ . Let  $\lambda_0$  be the largest  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ . Then, we have  $\lambda_0 \leq \alpha$ . By Theorem 3.3,  $\alpha$  is an  $H^+$ -eigenvalue of  $\mathcal{L}(G)$ . Therefore,  $\alpha \leq \lambda_0$ . Thus, we obtain  $\lambda_0 = \alpha$ .  $\square$

**Theorem 3.5** ( *$H^{++}$ -eigenvalue of  $\mathcal{L}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then zero is the unique  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$ .

**Proof.** By (10), zero is an  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$ . Thus,  $\mathcal{L}(G)$  has  $H^{++}$ -eigenvalues. Let  $\lambda$  be an  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$ . Next, we prove  $\lambda = 0$ .

Let  $\mathbf{x}$  be an  $H^{++}$ -eigenvector of  $\mathcal{L}(G)$  corresponding to  $\lambda$ . Then  $\mathbf{x} \in \mathbb{R}_{++}^n$ . Without loss of generality, we assume that the smallest component of  $\mathbf{x}$  is 1. Let  $v \in V(G)$  and  $x_v = 1$ . Therefore, by (6) and  $(\mathcal{L}(G)\mathbf{x})_v = \lambda x_v^{k-1}$ , we have

$$\lambda = w_v - \sum_{e \in E_G(v)} w_G(e) x_v^{e \setminus \{v\}} \leq w_v - \sum_{e \in E_G(v)} w_G(e) = 0. \quad (11)$$

It is noted that (11) holds since  $x_{v'} \geq 1$  for any  $v' \in V(G)$  and  $w_G(e) > 0$  for any edge  $e \in E(G)$ . Therefore, we obtain  $\lambda \leq 0$ . Furthermore, by Theorem 3.1,  $\lambda \geq 0$ . Therefore, we get  $\lambda = 0$ . Namely, zero is the unique  $H^{++}$ -eigenvalue of  $\mathcal{L}(G)$ .  $\square$

Let  $G^\Delta$  be a  $k$ -uniform hypergraph. It is interesting that Qi [5] used the methods of optimization theory to obtain a result about the largest  $H^+$ -eigenvalue of  $\mathcal{L}(G^\Delta)$  (namely, Theorem 5.1 in [5]) which is similar to Theorem 3.4, and obtained some results about the  $H^+$ -eigenvalue and the unique  $H^{++}$ -eigenvalue of  $\mathcal{L}(G^\Delta)$  (namely, Theorem 3.2 in [5]) which are similar to Theorems 3.3 and 3.5.

For  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ , we obtain a relationship of the  $H$ -eigenvalues of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$ , which is shown in Theorem 3.6. We can prove Theorem 3.6 by using the relationship among the tensors of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$ . However, to enrich the diversity of proof, we prove it by using different methods.

**Theorem 3.6** (*The relationship among the  $H$ -eigenvalues of  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Furthermore, we suppose that  $G$  is  $r$ -regular ( $r \geq 1$ ) and each edge of  $G$  has the same weight  $W_0$ . If  $\lambda$  is an  $H$ -eigenvalue of  $\mathcal{L}(G)$ , then

(i).  $2W_0r - \lambda$  is an  $H$ -eigenvalue of  $\mathcal{Q}(G)$ .

(ii).  $W_0r - \lambda$  is an  $H$ -eigenvalue of  $\mathcal{A}(G)$ .

**Proof.** Let  $G$  be as described in Theorem 3.6. By Theorem 3.1,  $\mathcal{L}(G)$  has  $H$ -eigenvalues. Let  $\lambda$  be an  $H$ -eigenvalue of  $\mathcal{L}(G)$  and  $\mathbf{x}$  be an  $H$ -eigenvector of  $\mathcal{L}(G)$  corresponding to  $\lambda$ . Thus, we have  $\mathbf{x} \in \mathbb{R}^n$ . For any  $i \in [n]$ , by (6) and  $(\mathcal{L}(G)\mathbf{x})_i = \lambda x_i^{k-1}$ , we get

$$\begin{aligned} \lambda x_i^{k-1} &= w_{v_i} x_i^{k-1} - \sum_{e \in E_G(v_i)} w_G(e) x^{e \setminus \{v_i\}} \\ &= W_0 \cdot r \cdot x_i^{k-1} - W_0 \sum_{e \in E_G(v_i)} x^{e \setminus \{v_i\}}. \end{aligned} \quad (12)$$

It is noted that (12) holds since  $w_{v_i} = \sum_{e \in E_G(v_i)} w_G(e)$ ,  $G$  is  $r$ -regular, and each edge of  $G$  has weight  $W_0$ .

For any  $i \in [n]$ , we have

$$\begin{aligned} (2W_0 r - \lambda) x_i^{k-1} &= 2W_0 \cdot r \cdot x_i^{k-1} - \lambda x_i^{k-1} \\ &= W_0 \cdot r \cdot x_i^{k-1} + W_0 \sum_{e \in E_G(v_i)} x^{e \setminus \{v_i\}} \end{aligned} \quad (13)$$

$$\begin{aligned} &= w_{v_i} x_i^{k-1} + \sum_{e \in E_G(v_i)} w_G(e) x^{e \setminus \{v_i\}} \\ &= (\mathcal{Q}(G)\mathbf{x})_i. \end{aligned} \quad (14)$$

It is noted that (13) follows from (12), and (14) holds since  $w_{v_i} = \sum_{e \in E_G(v_i)} w_G(e)$ ,  $G$  is  $r$ -regular and each edge of  $G$  has the same weight  $W_0$ .

Therefore, we obtain that  $2W_0 r - \lambda$  is an  $H$ -eigenvalue of  $\mathcal{Q}(G)$ . Namely, we get Theorem 3.6(i). By the methods similar to those for theorem 3.6(i), we get Theorem 3.6(ii).  $\square$

#### 4. THE EIGENVALUES OF $\mathcal{Q}(G)$

In this section, we investigate the eigenvalues of  $\mathcal{Q}(G)$ , where  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ . The upper and lower bounds for the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the spectral radius of  $\mathcal{Q}(G)$  are shown in Theorems 4.1–4.3, respectively. The weighted hypergraph with the largest  $H$ -eigenvalue of  $\mathcal{Q}(G)$  is also characterized in Theorem 4.1. A property of the  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  is given in Theorem 4.4. The smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  is obtained in Theorem 4.5.

**Theorem 4.1** (*The bounds for the  $H$ -eigenvalue of  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then (i)  $\mathcal{Q}(G)$  has an  $H$ -eigenvalue  $\lambda$  and  $0 \leq \lambda \leq 2W_0\Delta$ ; (ii)  $2W_0\Delta$  is an

$H$ -eigenvalue of  $\mathcal{Q}(G)$  if and only if  $G$  is  $\Delta$ -regular and each edge of  $G$  has the same weight  $W_0$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Since  $G$  is connected, by Theorem 2.2,  $\rho(\mathcal{Q}(G))$  is an  $H^{++}$ -eigenvalue of  $\mathcal{Q}(G)$ . Thus,  $\mathcal{Q}(G)$  has  $H$ -eigenvalues. Let  $\lambda$  be an  $H$ -eigenvalue of  $\mathcal{Q}(G)$ . By Theorem 2.3(iv),  $0 \leq \lambda \leq 2W_0\Delta$ . Thus, we get Theorem 4.4(i).

Next, we prove Theorem 4.1(ii).

If  $G$  is  $\Delta$ -regular and each edge of  $G$  has the same weight  $W_0$ , by (5), for  $i \in [n]$ , we get

$$(\mathcal{Q}(G)\mathbf{1})_i = w_{v_i} + \sum_{e \in E_G(v_i)} w_G(e) = 2 \sum_{e \in E_G(v_i)} w_G(e) = 2W_0\Delta = 2W_0\Delta \cdot [\mathbf{1}]_i.$$

Thus,  $2W_0\Delta$  is an  $H$ -eigenvalue of  $\mathcal{Q}(G)$  and  $\mathbf{1}$  is the eigenvector of  $\mathcal{Q}(G)$  corresponding to  $2W_0\Delta$ .

We assume that  $2W_0\Delta$  is an  $H$ -eigenvalue of  $\mathcal{Q}(G)$ . Next, we prove that  $G$  is  $\Delta$ -regular and each edge of  $G$  has the same weight  $W_0$ . Let  $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$  be the eigenvector of  $\mathcal{Q}(G)$  corresponding to  $2W_0\Delta$  with  $\sum_{i=1}^n x_i^k = 1$ . Without loss of generality, let  $|x_j| = \max_{1 \leq i \leq n} |x_i|$ , where  $j \in [n]$ . Obviously,  $|x_j| > 0$ . By (5) and  $(\mathcal{Q}(G)\mathbf{x})_j = 2W_0\Delta x_j^{k-1}$ , we have

$$w_{v_j}x_j^{k-1} + \sum_{e=\{v_j, v_{j_2}, \dots, v_{j_k}\} \in E(G)} w_G(e)x_{j_2} \cdots x_{j_k} = 2W_0\Delta x_j^{k-1}. \quad (15)$$

Since  $w_{v_j} = \sum_{e \in E_G(v_j)} w_G(e)$ , we have  $w_{v_j} < 2W_0\Delta$ . Thus, we get

$$2W_0\Delta - w_{v_j} = \left| \sum_{e=\{v_j, v_{j_2}, \dots, v_{j_k}\} \in E(G)} w_G(e) \frac{x_{j_2}}{x_j} \cdots \frac{x_{j_k}}{x_j} \right| \quad (16)$$

$$\leq \sum_{e=\{v_j, v_{j_2}, \dots, v_{j_k}\} \in E(G)} w_G(e) \left| \frac{x_{j_2}}{x_j} \right| \cdots \left| \frac{x_{j_k}}{x_j} \right| \quad (17)$$

$$\leq \sum_{e \in E_G(v_j)} w_G(e) = w_{v_j}. \quad (18)$$

It is noted that (16) is obtained from (15) by first subtracting  $w_{v_j}x_j^{k-1}$  from both sides of (15), then dividing  $x_j^{k-1}$  at the same time, and finally taking the modulus on both sides to get (16). (17) follows from the property of absolute value inequality and (18) follows from  $|x_j| = \max_{1 \leq i \leq n} |x_i|$ . Thus, we get  $W_0\Delta \leq w_{v_j} = \sum_{e \in E_G(v_j)} w_G(e)$ .

For a fixed  $j \in [n]$  with  $|x_j| = \max_{1 \leq i \leq n} |x_i|$ , if  $|E_G(v_j)| < \Delta$  holds or there exists one  $e \in E_G(v_j)$  such that  $w_G(e) < W_0$ , then  $w_{v_j} < W_0\Delta$ . Obviously, this contradicts  $W_0\Delta \leq w_{v_j}$ . Therefore, we have  $|E_G(v_j)| = \Delta$  for a fixed  $j \in [n]$  with  $|x_j| = \max_{1 \leq i \leq n} |x_i|$  and  $w_G(e) = W_0$  for any  $e \in E_G(v_j)$ . Thus, the two equalities in (17) and (18) hold simultaneously. Namely,  $|x_v| = |x_j| = \max_{1 \leq i \leq n} |x_i|$  for any  $v \in N_G(v_j)$ , and  $x^{e_1 \setminus \{v_j\}}$  and  $x^{e_2 \setminus \{v_j\}}$  have the same symbol, where  $e_1, e_2 \in E_G(v_j)$ . Since  $G$  is connected, there exists a path between every pair of vertices in  $V(G)$ . By repeatedly using the same analysis as above, we get that  $w_G(e) = W_0$  for any  $e \in E(G)$  and  $|E_G(v_i)| = \Delta$  for  $i = 1, \dots, n$ . Therefore, if  $G$  is connected and  $2W_0\Delta$  is an  $H$ -eigenvalue of  $\mathcal{Q}(G)$ , then  $G$  is  $\Delta$ -regular and each edge of  $G$  has the same weight  $W_0$ . Namely, Theorem 4.1(ii) holds.  $\square$

**Theorem 4.2** (*The bounds for the  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\mathcal{Q}(G)$  has an  $H^+$ -eigenvalue  $\lambda$  and  $\delta \leq \lambda \leq 2\alpha$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . By Theorem 2.2,  $\rho(\mathcal{Q}(G))$  is an  $H^{++}$ -eigenvalue of  $\mathcal{Q}(G)$ . Thus,  $\mathcal{Q}(G)$  has  $H^+$ -eigenvalues. Let  $\lambda$  be an  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$ . Let  $\mathbf{x}$  be an  $H^+$ -eigenvector of  $\mathcal{Q}(G)$  corresponding to  $\lambda$ . Then,  $\mathbf{x} \in \mathbb{R}_+^n$ . Thus, the largest component of  $\mathbf{x}$  is positive. Without loss of generality, we assume that the largest component of  $\mathbf{x}$  is 1. Let  $u \in V(G)$  and  $x_u = 1$ . By (5) and  $(\mathcal{Q}(G)\mathbf{x})_u = \lambda x_u^{k-1}$ , we get

$$w_u + \sum_{e \in E_G(u)} w_G(e) x^{e \setminus \{u\}} = \lambda. \quad (19)$$

Since  $\mathbf{x} \in \mathbb{R}_+^n$ ,  $w_G(e) > 0$  for any  $e \in E(G)$ , and  $0 \leq x_v \leq 1$  for any  $v \in V(G)$ , we obtain

$$0 \leq \lambda - w_u = \sum_{e \in E_G(u)} w_G(e) x^{e \setminus \{u\}} \leq \sum_{e \in E_G(u)} w_G(e) = w_u. \quad (20)$$

Thus, we get  $w_u \leq \lambda \leq 2w_u$ . Since  $\delta \leq w_u \leq \alpha$ , we obtain  $\delta \leq \lambda \leq 2\alpha$ .  $\square$

**Theorem 4.3** (*The bounds for the spectral radius of  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $2\delta \leq \rho(\mathcal{Q}(G)) \leq 2\alpha$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Since  $G$  is connected, by Theorem 2.2,  $\rho(\mathcal{Q}(G))$  is the only  $H^{++}$ -eigenvalue of  $\mathcal{Q}(G)$ . Let  $\mathbf{x}$  be the  $H^{++}$ -eigenvector of  $\mathcal{Q}(G)$  corresponding to  $\rho(\mathcal{Q}(G))$ . We have  $\mathbf{x} \in \mathbb{R}_{++}^n$ . Let  $v \in V(G)$  and  $x_v$  be the smallest component of  $\mathbf{x}$ . By (5) and  $(\mathcal{Q}(G)\mathbf{x})_v = \rho(\mathcal{Q}(G))x_v^{k-1}$ , we get

$$w_v x_v^{k-1} + \sum_{e \in E_G(v)} w_G(e) x^{e \setminus \{v\}} = \rho(\mathcal{Q}(G))x_v^{k-1}.$$

Since  $\mathbf{x} \in \mathbb{R}_{++}^n$ , we have  $x_v > 0$ . Thus, we obtain

$$\rho(\mathcal{Q}(G)) = w_v + \sum_{e \in E_G(v)} w_G(e) \frac{x^{e \setminus \{v\}}}{x_v^{k-1}}.$$

Since  $0 < x_v \leq x_{v'}$  for any  $v' \in V(G)$  and  $w_G(e) > 0$  for any  $e \in E(G)$ , we obtain  $\rho(\mathcal{Q}(G)) \geq w_v + \sum_{e \in E_G(v)} w_G(e) \geq 2\delta$ . Furthermore, by Theorem 4.2, we get  $\rho(\mathcal{Q}(G)) \leq 2\alpha$ . Thus, we obtain Theorem 4.3.  $\square$

By using the methods similar to those for Theorem 3.3, we get Theorem 4.4.

**Theorem 4.4** ( *$H^+$ -eigenvalue of  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then for any  $i \in [n]$ ,  $w_{v_i}$  is an  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$  and  $\mathbf{e}^{(i)}$  is an  $H^+$ -eigenvector of  $\mathcal{Q}(G)$  corresponding to  $w_{v_i}$ .

In Theorem 4.5, we obtain the smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$ . The proof of Theorem 4.5 is omitted since we can apply Theorems 4.2 and 4.4 and use the same methods similar to those for Theorem 3.4 to get it.

**Theorem 4.5** (*The smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\delta$  is the smallest  $H^+$ -eigenvalue of  $\mathcal{Q}(G)$ .

It is interesting that Qi [5] used the methods of optimization theory to obtain a result about the smallest  $H^+$ -eigenvalue of the signless Laplacian tensor of a  $k$ -uniform hypergraph (shown in Theorem 7.1 in [5]) which is similar to Theorem 4.5.

## 5. THE EIGENVALUES OF $\mathcal{A}(G)$

In this section, we study the eigenvalues of  $\mathcal{A}(G)$ , where  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ . The upper and lower bounds for the  $H$ -eigenvalue, the  $H^+$ -eigenvalue and the spectral radius of  $\mathcal{A}(G)$  are derived in Theorems 5.1–5.3, respectively. The weighted hypergraph with the largest  $H$ -eigenvalue of  $\mathcal{A}(G)$  is also presented in Theorem 5.1. We find that zero is an  $H^+$ -eigenvalue of  $\mathcal{A}(G)$  and  $\mathbf{e}^{(i)}$  is an  $H^+$ -eigenvector of  $\mathcal{A}(G)$  corresponding to zero, where  $i \in [n]$ , which is shown in Theorem 5.4.

**Theorem 5.1** (*The bound for the  $H$ -eigenvalue of  $\mathcal{A}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then (i)  $\mathcal{A}(G)$  has an  $H$ -eigenvalue  $\lambda$  and  $|\lambda| \leq W_0\Delta$ ; (ii)  $W_0\Delta$  is an  $H$ -eigenvalue of  $\mathcal{A}(G)$  if and only if  $G$  is  $\Delta$ -regular and each edge of  $G$  has the same weight  $W_0$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Since  $G$  is connected, by Theorem 2.2,  $\mathcal{A}(G)$  has an  $H^{++}$ -eigenvalue  $\rho(G)$ . Thus,  $\mathcal{A}(G)$  has an  $H$ -eigenvalue  $\lambda$ . By Theorem 2.3(iv),  $|\lambda| \leq W_0\Delta$ . Therefore, we get Theorem 5.1(i). By the methods similar to those for Theorem 4.1, we obtain Theorem 5.1(ii).  $\square$

Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Let  $X$  be a non-empty subset of  $V(G)$ . We use  $E_t(X)$  to denote the set of edges of  $G$  which share  $t$  common vertices with  $X$ , where  $t \geq 1$ . Namely,  $E_t(X) = \{e : e \in E(G) \text{ and } |e \cap X| = t\}$ . Furthermore, let  $E_t^v(X) = \{e : e \in E(G), v \in e \text{ and } |e \cap X| = t\}$ . We define  $e_G(u, v)$  as the number of the edges of  $G$  which contain  $u$  and  $v$ , where  $u, v \in V(G)$ .

**Theorem 5.2** (*The bounds for the  $H^+$ -eigenvalue of  $\mathcal{A}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\mathcal{A}(G)$  has an  $H^+$ -eigenvalue  $\lambda$  and

$$0 \leq \lambda \leq \sqrt{\frac{W_0^2}{k-1} \sum_{t=1}^k \sum_{e \in E_t(N_G(u))} \sum_{v \in (e \cap N_G(u))} e_G(u, v)},$$

where  $u$  is the vertex of  $G$  which has the largest component of the principal eigenvector corresponding to  $\lambda$ .

**Proof.** Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . By Theorem 2.2,  $\rho(G)$  is an  $H^{++}$ -eigenvalue of  $\mathcal{A}(G)$ . Thus,  $\mathcal{A}(G)$  has  $H^+$ -eigenvalues. Let  $\lambda$  be an  $H^+$ -eigenvalue of  $\mathcal{A}(G)$  and  $\mathbf{x}$  be an  $H^+$ -eigenvector of  $\mathcal{A}(G)$  corresponding to  $\lambda$ . Thus,  $\mathbf{x} \in \mathbb{R}_+^n$ . For all  $i \in [n]$ , by (4) and  $(\mathcal{A}(G)\mathbf{x})_i = \lambda x_i^{k-1}$ , we get

$$\lambda x_i^{k-1} = \sum_{e=\{v_i, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) x_{i_2} \cdots x_{i_k}. \quad (21)$$

Since  $\mathbf{x} \in \mathbb{R}_+^n$ , the largest component of  $\mathbf{x}$  is positive. Without loss of generality, we assume the largest component of  $\mathbf{x}$  is 1. Let  $u \in V(G)$  and  $x_u = 1$ . In (21), let  $i = u$ . Since  $\mathbf{x} \in \mathbb{R}_+^n$  and  $w_G(e) > 0$  for any  $e \in E(G)$ , by (21), we obtain

$$0 \leq \lambda = \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) x_{i_2} \cdots x_{i_k} \quad (22)$$

$$\leq \frac{1}{k-1} \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) (x_{i_2}^{k-1} + \cdots + x_{i_k}^{k-1}), \quad (23)$$

where (23) follows from the AM-GM equality.



Multiplying both sides of (23) by  $\lambda$  and bearing (21) in mind, we have

$$\lambda^2 \leq \frac{1}{k-1} \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) \times \left[ \sum_{f_2 \in E_G(v_{i_2})} w_G(f_2) x^{f_2 \setminus \{v_{i_2}\}} + \dots + \sum_{f_k \in E_G(v_{i_k})} w_G(f_k) x^{f_k \setminus \{v_{i_k}\}} \right]. \quad (24)$$

Since  $0 \leq x_v \leq 1$  for any  $v \in V(G)$  and  $w_G(e) \geq 0$  for any  $e \in E(G)$ , by (24), we get

$$\lambda^2 \leq \frac{1}{k-1} \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) \left[ \sum_{f_2 \in E_G(v_{i_2})} w_G(f_2) + \dots + \sum_{f_k \in E_G(v_{i_k})} w_G(f_k) \right].$$

For any  $f_s \in E_G(v_{i_s})$ , since  $v_{i_s} \in N_G(u)$ , we have  $1 \leq |f_s \cap N_G(u)| \leq k$ , where  $s = 2, \dots, k$ .

Thus,  $E_G(v_{i_s}) = \bigcup_{t=1}^k E_t^{v_{i_s}}(N_G(u))$ . Therefore, we obtain

$$\lambda^2 \leq \frac{1}{k-1} \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} w_G(e) \times \left[ \sum_{t=1}^k \left( \sum_{f_2 \in E_t^{v_{i_2}}(N_G(u))} w_G(f_2) + \dots + \sum_{f_k \in E_t^{v_{i_k}}(N_G(u))} w_G(f_k) \right) \right]. \quad (25)$$

Since  $0 < w_G(e) \leq W_0$  for any  $e \in E(G)$ , we have

$$\lambda^2 \leq \frac{W_0^2}{k-1} \sum_{e=\{u, v_{i_2}, \dots, v_{i_k}\} \in E(G)} \left[ \sum_{t=1}^k \left( \sum_{f_2 \in E_t^{v_{i_2}}(N_G(u))} 1 + \dots + \sum_{f_k \in E_t^{v_{i_k}}(N_G(u))} 1 \right) \right].$$

Obviously, the upper bound of  $\lambda^2$  is related with these edges which contain at least one vertex in  $N_G(u)$ . Let  $f = \{v_{j_1}, v_{j_2}, \dots, v_{j_k}\} \in E(G)$  with  $f \cap N_G(u) \neq \emptyset$ . Without loss of generality, we suppose  $f \cap N_G(u) = \{v_{j_1}, v_{j_2}, \dots, v_{j_t}\}$ , where  $1 \leq t \leq k$ . Then  $f$  appears  $e_G(u, v_{j_1}) + e_G(u, v_{j_2}) + \dots + e_G(u, v_{j_t})$  times. Therefore, we obtain

$$\begin{aligned} \lambda^2 &\leq \frac{W_0^2}{k-1} \left[ \sum_{e \in E_1(N_G(u))} \sum_{v \in (e \cap N_G(u))} e_G(u, v) + \dots + \sum_{e \in E_k(N_G(u))} \sum_{v \in (e \cap N_G(u))} e_G(u, v) \right] \\ &= \frac{W_0^2}{k-1} \sum_{t=1}^k \sum_{e \in E_t(N_G(u))} \sum_{v \in (e \cap N_G(u))} e_G(u, v). \end{aligned} \quad (26)$$

Thus, we get Theorem 5.2.  $\square$

It is noted that Theorem 5.2 is a generalization of Lemma 1 in [32] obtained by Hou et al. In Theorem 5.2, if  $W_0 = 1$ , then Theorem 5.2 is Lemma 1 in [32].

We get the upper and lower bounds for the spectral radius of  $\mathcal{A}(G)$  in Theorem 5.3, where  $G \in \mathbb{Q}_{k,n}$  with  $k, n \geq 3$ . Since the proofs for Theorem 5.3 are similar to those for Theorem 4.3, we omit it here.

**Theorem 5.3** (*The bounds for the spectral radius of  $\mathcal{A}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then  $\delta \leq \rho(G) \leq \alpha$ .

**Theorem 5.4** ( *$H^+$ -eigenvalue of  $\mathcal{A}(G)$* ) Let  $G \in \mathbb{Q}_{k,n}$ , where  $k, n \geq 3$ . Then zero is an  $H^+$ -eigenvalue of  $\mathcal{A}(G)$  and  $\mathbf{e}^{(i)}$  is an  $H^+$ -eigenvector of  $\mathcal{A}(G)$  corresponding to zero, where  $i \in [n]$ .

**Proof.** For any  $i, j \in [n]$ , we get

$$\begin{aligned} (\mathcal{A}(G)\mathbf{e}^{(i)})_j &= \sum_{j_2, \dots, j_k=1}^n a_{jj_2 \dots j_k} e_{j_2}^{(i)} \cdots e_{j_k}^{(i)} \\ &= a_{ii \dots i} e_i^{(i)} \cdots e_i^{(i)} = 0 = 0 \cdot e_j^{(i)}. \end{aligned} \quad (27)$$

It is noted that (27) follows from (3). Thus, zero is an  $H^+$ -eigenvalue of  $\mathcal{A}(G)$  and  $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n)}$  are the  $H^+$ -eigenvectors of  $\mathcal{A}(G)$  corresponding to zero.  $\square$

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- [1] M. P. Leng, L. Y. Sun, K. Q. Guo, J. N. Bian, P. Zhu, Comparative experiment of weighted hypergraph partitioning algorithms based on circuit partitioning, *Computer Engineering and Applications* **48** (2012) 74–79.
  - [2] L. Q. Qi, Eigenvalues of a real supersymmetric tensor, *Journal of Symbolic Computation* **40** (2005) 1302–1324.
  - [3] L. H. Lim, Singular values and eigenvalues of tensors: a variational approach, in: *Proceedings of the 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2005, pp. 129–132.

- [4] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, *Linear Algebra and its Applications* **436** (2012) 3268–3292.
- [5] L. Q. Qi,  $H^+$ -eigenvalues of Laplacian and signless Laplacian tensor, *Communications in Mathematical Sciences* **12** (2014) 1045–1064.
- [6] X. Y. Yuan, J. Y. Shao, H. Y. Shan, Ordering of some uniform supertrees with larger spectral radii, *Linear Algebra and its Applications* **495** (2016) 206–222.
- [7] W. H. Wang, L. Yuan, Uniform supertrees with extremal spectral radii, *Frontiers of Mathematics in China* **15** (2020) 1211–1229.
- [8] B. Ergemlidze, E. Győri, A. Methuku, Asymptotics for Turán numbers of cycles in 3-uniform linear hypergraphs, *Journal of Combinatorial Theory, Series A* **163** (2019) 163–181.
- [9] P. Xiao, L. G. Wang, Y. Lu, The maximum spectral radii of uniform supertrees with given degree sequences, *Linear Algebra and its Applications* **523** (2017) 33–45.
- [10] L. Zhang, A. Chang, Spectral radius of  $r$ -uniform supertrees with perfect matchings, *Frontiers Mathematics in China* **13** (2018) 1489–1499.
- [11] H. Y. Guo, B. Zhou, On the spectral radius of uniform hypertrees, *Linear Algebra and its Applications* **558** (2018) 236–249.
- [12] W. H. Wang, The minimum spectral radius of the  $r$ -uniform supertree having two vertices of maximum degree, *Linear and Multilinear Algebra* (2020) doi.org/10.1080/03081087.2020.1819188.
- [13] P. Xiao, L. G. Wang, The maximum spectral radius of uniform hypergraphs with given number of pendant edges, *Linear and Multilinear Algebra* **67** (2019) 1392–1403.
- [14] Y. Z. Fan, Y. Y. Tan, X. X. Peng, A. H. Liu, Maximizing spectral radii of uniform hypergraphs with few edges, *Discussiones Mathematicae Graph Theory* **36** (2016) 845–856.
- [15] L. Y. Kang, L. L. Liu, L. Q. Qi, X. Y. Yuan, Spectral radii of two kinds of uniform hypergraphs, *Applied Mathematics and Computation* **338** (2018) 661–668.
- [16] C. Ouyang, L. Q. Qi, X. Y. Yuan, The first few unicyclic and bicyclic hypergraphs with largest spectral radii, *Linear Algebra and its Applications* **527** (2017) 141–162.
- [17] G. Y. Li, L. Q. Qi, G. H. Yu, The  $Z$ -eigenvalues of a symmetric tensor and its application to

- spectral hypergraph theory, *Numerical Linear Algebra with Applications* **20** (2013) 1001–1029.
- [18] J. S. Xie, A. Chang, On the  $Z$ -eigenvalues of the signless Laplacian tensor for an even uniform hypergraph, *Numerical Linear Algebra with Applications* **20** (2013) 1030–1045.
- [19] S. L. Hu, L. Q. Qi, J. S. Xie, The largest Laplacian and signless Laplacian  $H$ -eigenvalues of a uniform hypergraph, *Linear Algebra and its Applications* **469** (2015) 1–27.
- [20] S. L. Hu, L. Q. Qi, J. Y. Shao, Cored hypergraphs, power hypergraphs and their Laplacian  $H$ -eigenvalues, *Linear Algebra and its Applications* **439** (2013) 2980–2998.
- [21] J. J. Yue, L. P. Zhang, M. Lu, Largest adjacency, signless Laplacian, and Laplacian  $H$ -eigenvalues of loose paths, *Frontiers of Mathematics in China* **11** (2016) 623–645.
- [22] J. J. Yue, L. P. Zhang, Computing all Laplacian  $H$ -eigenvalues for a uniform loose path of length three, *Computational and Applied Mathematics* **39** (2020) 121.
- [23] J. J. Yue, L. P. Zhang, Finding all  $H$ -eigenvalues of signless Laplacian tensor for a uniform loose path of length three, *Asia-Pacific Journal of Operational Research* **37** (2020) 2040007.
- [24] X. Y. Yuan, L. Q. Qi, J. Y. Shao, The proof of a conjecture on largest Laplacian and signless Laplacian  $H$ -eigenvalues of uniform hypergraphs, *Linear Algebra and its Applications* **490** (2016) 18–30.
- [25] J. S. Xie, A. Chang, On the  $Z$ -eigenvalues of the adjacency tensors for uniform hypergraphs, *Linear Algebra and its Applications* **439** (2013) 2195–2204.
- [26] J. S. Xie, A. Chang,  $H$ -eigenvalues of signless Laplacian tensor for an even uniform hypergraph, *Frontiers of Mathematics in China* **8** (2013) 107–127.
- [27] S. Friedland, S. Gaubert, L. Han, Perron–Frobenius theorem for nonnegative multilinear forms and extensions, *Linear Algebra and its Applications* **438** (2013) 738–749.
- [28] Y. N. Yang, Q. Z. Yang, On some properties of nonnegative weakly irreducible tensors, arXiv:1111.0713 (2011).
- [29] Y. N. Yang, Q. Z. Yang, Further results for Perron–Frobenius theorem for nonnegative tensors, *SIAM Journal on Matrix Analysis and Applications* **31** (2010) 2517–2530.
- [30] K. J. Pearson, T. Zhang, On spectral hypergraph theory of the adjacency tensor, *Graphs and Combinatorics* **30** (2014) 1233–1248.
- [31] C. J. Bu, Y. P. Wei, L. Z. Sun, J. Zhou, Brualdi-type eigenvalue inclusion sets of tensors,

*Linear Algebra and its Applications* **480** (2015) 168–175.

- [32] Y. Hou, A. Chang, J. Cooper, Spectral extremal results for hypergraphs, *The Electronic Journal of Combinatorics* **28** (2021) #P3.46.