## Explicit formula of deformation quantization with separation of variables for complex two-dimensional locally symmetric Kähler manifold

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#### Abstract

We give a complex two-dimensional noncommutative locally symmetric Kähler manifold via a deformation quantization with separation of variables. We present an explicit formula of its star product by solving the system of recurrence relations given by Hara-Sako. In the two-dimensional case, this system of recurrence relations gives two types of equations corresponding to the two coordinates. From the two types of recurrence relations, symmetrized and antisymmetrized recurrence relations are obtained. The symmetrized one gives the solution of the recurrence relation. From the antisymmetrized one, the identities satisfied by the solution are obtained. The star products for  $\mathbb{C}^2$  and  $\mathbb{C}P^2$  are constructed by the method obtained in this study, and we verify that these star products satisfy the identities.

### 1 Introduction

Deformation quantization is one of the quantization method based on a deformation for a Poisson algebra and is known as a method of constructing noncommutative differentiable manifolds. There are two types of this, "formal deformation quantization" proposed by Bayen et al. [4], and "strict deformation quantization", based on  $C^*$ -algebra proposed by Rieffel [50]. In this paper, we study formal deformation quantization, and in the following, "deformation quantization" is used in the sense of "formal deformation quantization".

**Definition 1.1.** Let M be a Poisson manifold,  $C^{\infty}(M)$  be a set of  $C^{\infty}$  functions on M,  $\{\cdot, \cdot\} : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$  be a Poisson bracket, and  $C^{\infty}(M) \llbracket \hbar \rrbracket := \{f | f = \sum_{k} f_{k} \hbar^{k}, f_{k} \in C^{\infty}(M)\}$  be the ring of formal power series over  $C^{\infty}(M)$ , where  $\hbar$  is a formal parameter. Let a product \* on  $C^{\infty}(M) \llbracket \hbar \rrbracket$ , called the star product, be a product denoted by

$$f * g = \sum_{k=0}^{\infty} C_k (f, g) \hbar^k$$

satisfying the following conditions:

1. \* is associative, i.e. for any  $f, g, h \in C^{\infty}(M)$   $\llbracket \hbar \rrbracket$ , f \* (g \* h) = (f \* g) \* h.

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2. Each  $C_k(\cdot, \cdot) : C^{\infty}(M) \llbracket \hbar \rrbracket \times C^{\infty}(M) \llbracket \hbar \rrbracket \to C^{\infty}(M) \llbracket \hbar \rrbracket$  is a bi-differential operator, i.e. for any  $f, g \in C^{\infty}(M) \llbracket \hbar \rrbracket$ ,  $C_k(f, g)$  can be written as

$$C_k(f,g) = \sum_{I,J} a_{I,J} \partial^I f \partial^J g,$$

where I, J are multi-indices.

3. For any  $f, g \in C^{\infty}(M)$ ,

$$C_0(f,g) = fg,$$
  
 $C_1(f,g) - C_1(g,f) = \{f,g\}.$ 

4. For any  $f \in C^{\infty}(M) \llbracket \hbar \rrbracket$ , f \* 1 = 1 \* f = f.

The pair  $(C^{\infty}(M) \llbracket \hbar \rrbracket, *)$  is called a deformation quantization for M.

For a more detailed review of deformation quantization, see e.g. [21]. The construction method of deformation quantization for symplectic manifolds has been known by de Wilde-Lecomte [16], Omori-Maeda-Yoshioka [44] and Fedosov [19]. After these works, a method for Poisson manifolds was proposed by Kontsevich [35]. For any Kähler manifold, Karabegov studied a construction method of deformation quantization with separation of variables [30, 31].

**Definition 1.2.** Let M be a Kähler manifold. A star product \* on M is the separation of variables if the following two conditions are satisfied for any open set U of M and  $f \in C^{\infty}(U)$ :

- 1. For a holomorphic function a on U, a \* f = af.
- 2. For an anti-holomorphic function b on U, f \* b = fb.

In this paper, we propose an explicit formula that gives a deformation quantization with separation of variables for a complex two-dimensional locally symmetric Kähler manifold. This main result that is given in Theorem 3.3 in Section 3 gives the explicit star product which is expanded in differential operators whose coefficients consist of covariantly constant. Each coefficient is explicitly determined by some matrix multiplications, and it contains the Riemann curvature tensor. This theorem is shown by solving the recurrence relations given by Hara-Sako [22, 23]. To explain our main result, we must introduce several definitions. So we shall not state our main theorem concretely, here.

This paper is organized mainly into four Sections and three Appendices. In Section 2, we review the previous works by Karabegov [30, 31] and Hara-Sako [22, 23], as the background concerning a deformation quantization with separation of variables for Kähler manifolds. In Section 3, we show our main results which are the explicit formula to give the star product and the identities. In Section 4, we construct concrete examples for  $\mathbb{C}^2$  and  $\mathbb{C}P^2$  and they reproduce the previous results. In Section 5, we state future works related to our main results from both mathematical and physical perspectives. In each Appendices A–C, we describe the properties and detailed calculations used in this paper. In Appendix A, we summarize some properties of Kähler manifolds used in this paper. In Appendix B, we calculate in detail the identity (31) in Subsection 3.2 for the 2nd order. In Appendix C, the Hermiteness of the coefficients of a star product is shown.

### 2 Review of Noncommutative Kähler manifolds

The Quantization of Kähler manifolds was studied by Berezin [5, 6], Moreno [40, 41], Cahen-Gutt-Rawnsley [11, 12, 13, 14], Karabegov [30, 31], Omori-Maeda-Miyazaki-Yoshioka [46], Schlichenmaier [60, 61, 62, 63, 64, 65, 66], Karabegov-Schlichenmaier [32], Sako-Suzuki-Umetsu [54, 55] and Hara-Sako [22, 23]. In particular, Karabegov's method was proposed as a way to give noncommutative Kähler manifolds via a deformation quantization with separation of variables. After that, Sako-Suzuki-Umetsu and Hara-Sako methods were proposed for a locally symmetric case, inspired by this method. In addition, Sako-Suzuki-Umetsu was mentioned the Fock representations of noncommutative  $\mathbb{C}P^N$  and  $\mathbb{C}H^N$ . Moreover, these previous results were generalized for any noncommutative Kähler manifolds by Sako-Umetsu [57, 58, 59]. In Section 2.1, we review the methods by Karabegov as the background of this paper, and in Section 2.2, we review the method by Hara-Sako since our result is obtained from the recurrence relations in this method.

### 2.1 Noncommutative Kähler manifolds

Berezin proposed a general definition of quantization and constructed the quantization of Kähler manifolds in the case of phase space via symbol algebras [5, 6]. The coherent states of Kähler manifolds arising from the geometric quantization of Kostant [36] and Souriau [67] have also been studied by Rawnsley [49]. It is known that this coherent state is related to Berezin quantization. See [48] for more detail. After that, the deformation quantization of Kähler manifolds have been provided by Moreno [40, 41] and Omori-Maeda-Miyazaki-Yoshioka [46]. The relations between deformation quantization and Berezin quantization have been studied by Cahen-Gutt-Rawnsley [11, 12, 13, 14]. It has also studied the quantization of Kähler manifolds via Toeplitz quantization by Bordemann et al. [8]. Furthermore, Karabegov and Schlichenmaier have provided Berezin-Toeplitz quantization in the case of compact Kähler ones [60, 61, 62, 63, 64, 66]. These previous works related to Berezin-Toeplitz quantizations for Kähler manifolds were reviewed by Schlichenmaier [65]. From the other angle of the quantization, the construction method of noncommutative Kähler manifolds was studied via the deformation quantization with separation of variables by Karabegov [30, 31]. Moreover, for any noncommutative Kähler manifolds obtained by Karabegov's construction, Sako-Umetsu constructed Fock representations of them [57, 58]. In this subsection, we review Karabegov's method and Fock representations of noncommutative Kähler manifolds by Sako-Umetsu.

Let M be an N-dimensional Kähler manifold and  $U \subset M$  be a holomorphic coordinate neighborhood of M. We choose a local holomorphic coordinates by  $(z^1, \dots, z^N)$ . For a Kähler manifolds, a Kähler 2-form  $\omega$  and a Kähler metric g can be locally expressed by using a Kähler potential  $\Phi$  as follows:

$$\omega = ig_{k\bar{l}}dz^k \wedge d\bar{z}^l, \qquad g_{k\bar{l}} = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}$$

Note that we use the Einstein summation convention on the above. We also denote the inverse matrix of  $(g_{\overline{k}l})$  by  $(g^{\overline{k}l})$ . Here we introduce the differential operators  $D^k, D^{\overline{k}}$  defined by

$$D^k := g^{k\overline{l}}\partial_{\overline{l}}, \quad D^{\overline{k}} := g^{\overline{k}l}\partial_l.$$

We define the set of differential operators

$$\mathcal{S} := \left\{ A \middle| A = \sum_{\alpha} a_{\alpha} D^{\alpha}, \, a_{\alpha} \in C^{\infty} \left( U \right) \right\},\,$$

where  $\alpha = (\alpha_1, \ldots, \alpha_N)$  is a multi-index, i.e.  $D^{\alpha} := (D^{\bar{1}})^{\alpha_1} \cdots (D^{\bar{N}})^{\alpha_N}$ . We can construct the left star-multiplication operator  $L_f$  for  $f \in C^{\infty}(U)$  such that  $L_f g := f * g$ :

**Theorem 2.1** (Karabegov[30]). Let M be an N-dimensional Kähler manifold, U be a holomorphic coordinate neighborhood on M, and  $\omega$  be a Kähler form on M. Then, there is the left star-multiplication operator

$$L_{f} = \sum_{n=0}^{\infty} \hbar^{n} A_{n}, \qquad f \in C^{\infty}(U),$$

where  $A_n := a_{n,\alpha}(f) D^{\alpha} \in S$  are differential operators whose coefficients  $a_{n,\alpha}(f) \in C^{\infty}(U)$  depend on f.  $L_f$  is determined by the following conditions:

1.  $\left[L_f, R_{\partial_{\overline{l}} \Phi}\right] = 0$ , where  $R_{\partial_{\overline{l}} \Phi} = \partial_{\overline{l}} \Phi + \hbar \partial_{\overline{l}}$ ,

2. 
$$L_f 1 = f * 1 = f$$
,

3. For any  $g, h \in C^{\infty}(U)$ , the left star-multiplication operator is associative, i.e.

$$L_f(L_g h) = f * (g * h) = (f * g) * h = L_{L_f g} h$$

By using the definition of the separation of variables and the commutation relations of the star-multiplication operators, we obtain the following commutation relations concerning  $z^i$ ,  $\overline{z}^i$ ,  $\partial_i \Phi$  and  $\partial_{\overline{i}} \Phi$ .

$$\left[\frac{1}{\hbar}\partial_i\Phi, z^j\right]_* = \delta_{ij}, \qquad \left[z^i, z^j\right]_* = 0, \qquad \left[\partial_i\Phi, \partial_j\Phi\right]_* = 0 \tag{1}$$

$$\left[\overline{z}^{i}, \frac{1}{\hbar}\partial_{\overline{j}}\Phi\right]_{*} = \delta_{ij}, \qquad \left[\overline{z}^{i}, \overline{z}^{j}\right]_{*} = 0, \qquad \left[\partial_{\overline{i}}\Phi, \partial_{\overline{j}}\Phi\right]_{*} = 0.$$
<sup>(2)</sup>

Note that the commutator  $[\cdot, \cdot]_*$  is defined by  $[A, B]_* := A * B - B * A$ . Here, we introduce the creation and annihilation operators as follows:

$$a_i^{\dagger} = z^i, \qquad \underline{a}_i = \frac{1}{\hbar} \partial_i \Phi, \qquad a_i = \overline{z}^i, \qquad \underline{a}_i^{\dagger} = \frac{1}{\hbar} \partial_{\overline{i}} \Phi \qquad (i = 1, \cdots, N).$$
 (3)

Then, the commutation relations (1) and (2) can be rewritten as

$$\left[\underline{a}_{i}, a_{j}^{\dagger}\right]_{*} = \delta_{ij}, \qquad \left[a_{i}^{\dagger}, a_{j}^{\dagger}\right]_{*} = 0, \qquad \left[\underline{a}_{i}, \underline{a}_{j}\right]_{*} = 0 \tag{4}$$

$$\left[a_i, \underline{a}_j^{\dagger}\right]_* = \delta_{ij}, \qquad \left[a_i, a_j\right]_* = 0, \qquad \left[\underline{a}_i^{\dagger}, \underline{a}_j^{\dagger}\right]_* = 0.$$
<sup>(5)</sup>

Since  $[a_i, a_i^{\dagger}]_* \neq 0$ ,  $[\underline{a}_i, \underline{a}_j^{\dagger}]_* \neq 0$  in general, these relations are slightly different from the ordinary canonical commutation relation. From the above operators, the (twisted) Fock space is defined by a vector space spanned by the basis

$$|\vec{n}\rangle = |n_1, \cdots, n_N\rangle = \frac{1}{\sqrt{\vec{n!}}} \left(a_1^{\dagger}\right)_*^{n_1} * \cdots * \left(a_N^{\dagger}\right)_*^{n_N} * |\vec{0}\rangle, \tag{6}$$

where a vacuum  $|\vec{0}\rangle = |0, \cdots, 0\rangle$  is the vector such that

$$\underline{a}_i * |\vec{0}\rangle = 0, \qquad (i = 1, \cdots, N), \tag{7}$$

and  $\vec{n}! = n_1! \cdots n_N!$ . Note that  $(A)^n_*$  is the product of multiplying n by the star product \*, i.e.  $(A)^n_* := \underbrace{A*\cdots*A}_{n \text{ times by }*}$ . Similarly, the dual basis for  $|\vec{n}\rangle$  is defined by

$$\underline{\langle \vec{m}|} = \underline{\langle m_1, \cdots, m_N|} = \langle \vec{0}| * (\underline{a}_1)^{m_1}_* * \cdots * (\underline{a}_N)^{m_N}_* \frac{1}{\sqrt{\vec{m}!}},\tag{8}$$

where  $\langle \vec{0} |$  is the (dual) vector for a vacuum  $|\vec{0}\rangle$  such that

$$\langle \vec{0} | * a_i^{\dagger} = 0, \qquad (i = 1, \cdots, N),$$
(9)

and  $\vec{m}! = m_1! \cdots m_N!$ . Note that  $\langle \vec{m} |$  does not imply Hermitian conjugate of  $|\vec{m}\rangle$ , i.e.  $\langle \vec{m} | \neq |\vec{m}\rangle^{\dagger}$ .

**Definition 2.2.** Let M be a Kähler manifold and U be a holomorphic coordinate neighborhood on M. Then, the (local) twisted Fock algebra (representation)  $F_U$  is defined by

$$F_U := \left\{ \sum_{\vec{n},\vec{m}} A_{\vec{n}\vec{m}} | \vec{n} \rangle \underline{\langle \vec{m} |} \, \middle| \, A_{\vec{n}\vec{m}} \in \mathbb{C} \right\}.$$
(10)

 $F_U$  is defined as the algebra which is given by the creation and annihilation operators in (3) and starmultiplication between each element of  $F_U$ . Moreover, we can concretely express the coefficient functions  $A_{\vec{n}\vec{m}}$  which are the elements of  $F_U$ . We expand a function  $\exp \Phi(z, \bar{z})/\hbar$  as a power series,

$$e^{\Phi(z,\bar{z})/\hbar} = \sum_{\vec{m},\vec{n}} H_{\vec{m},\vec{n}}(z)^{\vec{m}}(\bar{z})^{\vec{n}}$$
(11)

where  $(z)^{\vec{n}} = (z^1)^{n_1} \cdots (z^N)^{n_N}$  and  $(\bar{z})^{\vec{n}} = (\bar{z}^1)^{n_1} \cdots (\bar{z}^N)^{n_N}$ . The creation and annihilation operators  $a_i^{\dagger}, \underline{a}_i$  act on the bases as follows,

$$a_i^{\dagger} * |\vec{m}\rangle \underline{\langle \vec{n}|} = \sqrt{m_i + 1} |\vec{m} + \vec{e}_i\rangle \underline{\langle \vec{n}|}, \qquad \underline{a}_i * |\vec{m}\rangle \underline{\langle \vec{n}|} = \sqrt{m_i} |\vec{m} - \vec{e}_i\rangle \underline{\langle \vec{n}|}, \tag{12}$$

$$|\vec{m}\rangle\underline{\langle\vec{n}|} * a_i^{\dagger} = \sqrt{n_i}|\vec{m}\rangle\underline{\langle\vec{n}-\vec{e_i}|}, \qquad |\vec{m}\rangle\underline{\langle\vec{n}|} * \underline{a_i} = \sqrt{n_i+1}|\vec{m}\rangle\underline{\langle\vec{n}+\vec{e_i}|}, \tag{13}$$

where  $\vec{e}_i$  is a unit vector,  $(\vec{e}_i)_j = \delta_{ij}$ . The action of  $a_i$  and  $\underline{a}_i^{\dagger}$  is derived by the Hermitian conjugation of the above equations. The results of the twisted Fock representation of the noncommutative Kähler manifolds are summarized as the dictionary in the Table 1:

 Table 1: Functions - Fock operators Dictionary

Functions	Fock operators
$e^{-\Phi/\hbar}$	$ \vec{0}\rangle\langle\vec{0} $
$z_i$	$a_i^\dagger$
$\frac{1}{\hbar}\partial_i \Phi$	$\underline{a}_i$
$ar{z}^i$	$a_i = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} H_{\vec{m},\vec{k}} H_{\vec{k}+\vec{e}_i,\vec{n}}^{-1}  \vec{m}\rangle \underline{\langle \vec{n} }$
$rac{1}{\hbar}\partial_{\overline{i}}\Phi$	$\underline{a}_{i}^{\dagger} = \sum \sqrt{\frac{\vec{m}!}{\vec{n}!}} (k_{i}+1) H_{\vec{m},\vec{k}+\vec{e}_{i}} H_{\vec{k},\vec{n}}^{-1}  \vec{m}\rangle \underline{\langle \vec{n} }$

For physics, it is difficult to interpret formal power series. So this Fock representation is useful to construct physics theories like field theories on noncommutative Kähler manifolds. We will discuss this point in Section 5.

# 2.2 Deformation quantization with separation of variables for locally symmetric Kähler manifold

Let M be a complex N-dimensional locally symmetric Kähler manifold, U be a holomorphic coordinate neighborhood of M. For any  $f, g \in C^{\infty}(U)$ , we assume the following form for a star product with separation of variables on M by Sako-Suzuki-Umetsu [54, 55] and Hara-Sako [22, 23] :

$$f * g = L_f g := \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} \left( D^{\overrightarrow{\alpha_n}} f \right) \left( D^{\overrightarrow{\beta_n^*}} g \right).$$
(14)

Here  $L_f$  is a left star-multiplication operator with respect to f, and  $D^{\overrightarrow{\alpha_n}}, D^{\overrightarrow{\beta_n^*}}$  are differential operators defined by

$$\begin{split} D^{k} &:= g^{k\overline{l}}\partial_{\overline{l}}, \ D^{\overline{k}} := g^{\overline{k}l}\partial_{l}, \\ D^{\overrightarrow{\alpha_{n}}} &:= D^{\alpha_{1}^{n}} \cdots D^{\alpha_{N}^{n}}, \quad D^{\alpha_{k}^{n}} := \left(D^{k}\right)^{\alpha_{k}^{n}}, \\ D^{\overrightarrow{\beta_{n}^{*}}} &:= \overline{D^{\overrightarrow{\beta_{n}^{*}}}} = \overline{D^{\beta_{1}^{n}}} \cdots \overline{D^{\beta_{N}^{n}}}, \quad \overline{D^{\beta_{k}^{n}}} := \left(D^{\overline{k}}\right)^{\beta_{k}^{n}}, \\ \overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}} \in \left\{ \left(\gamma_{1}^{n}, \cdots, \gamma_{N}^{n}\right) \in \mathbb{Z}^{N} \middle| \sum_{k=1}^{N} \gamma_{k}^{n} = n \right\}, \end{split}$$

respectively, where  $\left\{ (\gamma_1^n, \cdots, \gamma_N^n) \in \mathbb{Z}^N \middle| \sum_{k=1}^N \gamma_k^n = n \right\}$  is a *N*-dimensional module such that a sum of all components is a non-negative integer *n*. If there exists at least one negative  $\alpha_k^n \notin \mathbb{Z}_{\geq 0}$  for  $k \in \{1, \cdots, N\}$ , then we define  $D^{\overrightarrow{\alpha_n}} := 0$ . We define  $D^{\overrightarrow{\beta_n^*}} := 0$  in the same way when  $\overrightarrow{\beta_n^*}$  has negative components. Note that (14) is not a power series in the formal parameter  $\hbar$ , but a power series in the differential operators  $D^k$  and  $D^{\overrightarrow{k}}$ . Since *M* is locally symmetric, the coefficients of the star product  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  can be assumed to be covariantly constants. If  $\overrightarrow{\alpha_n} \notin \mathbb{Z}_{\geq 0}^n$  or  $\overrightarrow{\beta_n} \notin \mathbb{Z}_{\geq 0}^n$ , then we define  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}} := 0$  as well as  $D^{\overrightarrow{\alpha_n}}$  and  $D^{\overrightarrow{\beta_n}}$ . In the following discussion, we shall omit \* when  $\overrightarrow{\beta_n^*}$  is expressed in explicit form. For example, if  $\overrightarrow{\beta_n^*} = (1, 2, 3)^*$ , then we denote  $(1, 2, 3)^* = (1, 2, 3)$ . It is known that the coefficient  $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$  for the zeroth and first orders in f \* g are given below.

**Proposition 2.3** (Hara-Sako[22, 23]). For a star product with separation of variables \* on U,

$$T^{0}_{\overrightarrow{\alpha_{0}},\overrightarrow{\beta_{0}^{*}}} = 1, \ T^{1}_{\overrightarrow{e_{i}},\overrightarrow{e_{j}}} = \hbar g_{i\overline{j}},$$

where  $\overrightarrow{e_i} = (\delta_{1i}, \cdots, \delta_{Ni})$ .

In other words, Proposition 2.3 states that for any complex N-dimensional locally symmetric Kähler manifold, the zeroth and first orders are completely determined.

**Theorem 2.4** (Hara-Sako[22, 23]). For  $f, g \in C^{\infty}(M)$ , there exists a star product with separation of variables \* such that

$$L_{f}g := f * g = \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}^{*}}} T_{\overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}^{*}}}^{n} \left( D^{\overrightarrow{\alpha_{n}}} f \right) \left( D^{\overrightarrow{\beta_{n}^{*}}} g \right), \ (f, g \in C^{\infty} \left( M \right)) \in C^{\infty} \left( M \right)$$

where the coefficient  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  satisfies the following recurrence relation :

$$\begin{split} &\sum_{d=1}^{N} \hbar g_{\overline{i}d} T^{n-1}_{\overrightarrow{\alpha_{n}} - \overrightarrow{e_{d}}, \overrightarrow{\beta_{n}^{2}} - \overrightarrow{e_{i}}} \\ &= \beta_{i}^{n} T^{n}_{\overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}^{2}}} + \sum_{k=1}^{N} \sum_{\rho=1}^{N} \frac{\hbar \left(\beta_{k}^{n} - \delta_{k\rho} - \delta_{ik} + 1\right) \left(\beta_{k}^{n} - \delta_{k\rho} - \delta_{ik} + 2\right)}{2} R_{\overline{\rho}}^{\overline{k} \overline{k}} \overline{i} T^{n}_{\overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}^{2}} - \overrightarrow{e_{\rho}^{2}} + \overrightarrow{e_{k}^{2}} - \overrightarrow{e_{i}^{2}}} \\ &+ \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\rho=1}^{N} \hbar \left(\beta_{k}^{n} - \delta_{k\rho} - \delta_{ik} + 1\right) \left(\beta_{k+l}^{n} - \delta_{k+l,\rho} - \delta_{i,k+l} + 1\right) R_{\overline{\rho}}^{\overline{k+l} \overline{k}} \overline{i} T^{n}_{\overrightarrow{\alpha_{n}}, \overrightarrow{\beta_{n}^{2}} - \overrightarrow{e_{\rho}^{2}} + \overrightarrow{e_{k}^{2}} - \overrightarrow{e_{i}^{2}}} \end{split}$$

These recurrence relations in Theorem 2.4 are equivalent to the equations (6.9) in [54]. Hence, from Theorem 2.4, a star product with separation of variables on any complex N-dimensional locally symmetric Kähler manifold is obtained. However, finding a general term  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  that satisfies this system of recurrence relations is not easy, except in the one-dimensional case.

**Proposition 2.5** (Hara-Sako[22, 23]). Let M be a one-dimensional locally symmetric Kähler manifold, and U be an open set of M. For  $f, g \in C^{\infty}(U)$ , the star product f \* g is given by

$$f * g = \sum_{n=0}^{\infty} \left[ (g_{1\overline{1}})^n \left\{ \prod_{k=1}^n \frac{4\hbar}{4k + \hbar k \left(k - 1\right) R} \right\} \left\{ \left( g^{1\overline{1}} \frac{\partial}{\partial \overline{z}} \right)^n f \right\} \left\{ \left( g^{1\overline{1}} \frac{\partial}{\partial z} \right)^n g \right\} \right],$$

where  $R = 2R_{\overline{1}} \overline{11}_{\overline{1}}$  is the scalar curvature on M.

Note that Proposition 2.5 is corrected some errata for the one-dimensional formula in [22, 23]. This proposition can be shown by direct calculations since the recurrence relation is a simple expression in the one-dimensional case.

On the other hand,  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n}}$  were only obtained for n = 0, 1 and 2 for the two-dimensional case. The following proposition is shown by directly solving the recurrence relation in Theorem 2.4 for n = 2.

**Proposition 2.6** (Hara-Sako[22, 23]). Let M be a complex two-dimensional locally symmetric Kähler manifold. Then the coefficients  $T^2_{\overrightarrow{\alpha_2},\overrightarrow{\beta_2^*}}$  are given by

$$(T_{ij}^2) = \hbar^2 A_2 X_2^{-1},$$

where

$$T_{ij}^2 := T_{(3-i,i-1),(3-j,j-1)}^2,$$

$$\begin{split} A_2 &:= \begin{pmatrix} \left(g_{\bar{1}1}\right)^2 & g_{\bar{1}1}g_{\bar{2}1} & \left(g_{\bar{2}1}\right)^2 \\ 2g_{\bar{1}1}g_{\bar{1}2} & g_{\bar{1}2}g_{\bar{2}1} + g_{\bar{1}1}g_{2\bar{2}} & 2g_{\bar{2}1}g_{\bar{2}2} \\ \left(g_{\bar{1}2}\right)^2 & g_{\bar{2}1}g_{\bar{2}2} & \left(g_{\bar{2}2}\right)^2 \end{pmatrix}, \\ X_2 &:= \begin{pmatrix} 2 + \hbar R_{\bar{1}}^{\ 1\bar{1}} & \hbar R_{\bar{2}}^{\ 1\bar{1}} & \hbar R_{\bar{2}}^{\ 1\bar{1}} \\ \hbar R_{\bar{1}}^{\ 2\bar{1}} & 1 + \hbar R_{\bar{2}}^{\ 2\bar{1}} & \hbar R_{\bar{2}}^{\ 2\bar{1}} \\ \hbar R_{\bar{1}}^{\ 2\bar{2}} & \hbar R_{\bar{2}}^{\ 2\bar{2}} & 2 + \hbar R_{\bar{2}}^{\ 2\bar{2}} \\ \end{pmatrix}. \end{split}$$

We have reviewed the previous works. In general, solving recurrence relations is not easy. In particular, when we attempt to obtain the general term using a matrix representation, we need a square matrix of order n+1. In addition, the matrix size increases with increasing order. Therefore, it had been considered difficult to obtain the general term. In this paper, we find that the expression of the general term does not become unlimitedly complex, and succeed in getting the general term by using this fact. We shall see that in the following sections. In Section 3, we are going to describe the coefficients  $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$  for any  $n \in \mathbb{Z}_{\geq 0}$ .

### 3 Star product with separation of variables for a complex two-dimensional locally symmetric Kähler manifold

In this section, we construct the formula that explicitly determines a star product with separation of variables for a complex two-dimensional locally symmetric Kähler manifold. In other words, we construct the solution of the recurrence relations in Theorem 2.4 for the two-dimensional case in this section.

#### 3.1 Complex two-dimensional formula

The explicit formula for the coefficient  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  for any order  $n \in \mathbb{Z}_{\geq 0}$  is obtained by not dealing with the recurrence relations independently but attributing them to one recurrence relation.

**Theorem 3.1.** Let M be a complex two-dimensional locally symmetric Kähler manifold, U be an open set of M, and \* be a star product with separation of variables on U such that

$$\begin{split} f * g &= \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} T_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}^n \left( D^{\overrightarrow{\alpha_n}} f \right) \left( D^{\overrightarrow{\beta_n^*}} g \right) \\ &= \sum_{n=0}^{\infty} \sum_{i,j=1}^{n+1} T_{ij}^n \left\{ \left( D^1 \right)^{n-i+1} \left( D^2 \right)^{i-1} f \right\} \left\{ \left( D^{\overline{1}} \right)^{n-j+1} \left( D^{\overline{2}} \right)^{j-1} g \right\} \end{split}$$

for  $f, g \in C^{\infty}(U)$ . Then,

$$T_n X_n = \hbar A'_n,\tag{15}$$

where

$$T_n := (T_{ij}^n) \in M_{n+1} (\mathbb{C} \llbracket \hbar \rrbracket),$$
  
$$T_{ij}^n := T_{(n-i+1,i-1),(n-j+1,j-1)}^n,$$

$$A'_{n} = \left(A'_{ij}^{n}\right) = \left(\sum_{k,l=1}^{2} g_{\overline{k}l} T^{n-1}_{i-\delta_{2l},j-\delta_{2k}}\right) \in M_{n+1}\left(\mathbb{C}\left[\!\left[\hbar\right]\!\right]\right),$$

and  $X_n \in M_{n+1}(\mathbb{C} \llbracket \hbar \rrbracket)$  is a pentadiagonal matrix such that these components are given as follows :

$$\begin{split} X_{j-2,j}^n &= \binom{n-j+3}{2} \hbar R_{\bar{2}}^{\bar{1}\bar{1}}{}_{\bar{2}}, \\ X_{j-1,j}^n &= 2\binom{n-j+2}{2} \hbar R_{\bar{2}}^{\bar{1}\bar{1}}{}_{\bar{1}} + (n-j+2) \left(j-2\right) \hbar R_{\bar{2}}^{\bar{2}\bar{1}}{}_{\bar{2}}, \\ X_{j,j}^n &= n + \binom{n-j+1}{2} \hbar R_{\bar{1}}^{\bar{1}\bar{1}}{}_{\bar{1}} + \binom{j-1}{2} \hbar R_{\bar{2}}^{\bar{2}\bar{2}}{}_{\bar{2}} + 2 \left(n-j+1\right) \left(j-1\right) \hbar R_{\bar{2}}^{\bar{2}\bar{1}}{}_{\bar{1}}, \\ X_{j+1,j}^n &= 2\binom{j}{2} \hbar R_{\bar{2}}^{\bar{2}\bar{2}}{}_{\bar{1}} + j \left(n-j\right) \hbar R_{\bar{1}}^{\bar{2}\bar{1}}{}_{\bar{1}}, \\ X_{j+2,j}^n &= \binom{j+1}{2} \hbar R_{\bar{1}}^{\bar{2}\bar{2}}{}_{\bar{1}}, \\ X_{j,k}^n &= 0 \ \left(|j-k| > 2\right), \end{split}$$

where  $\binom{m}{n}$  is a binomial coefficient.

**Proof.** Note that,  $\overrightarrow{\alpha_n} = (\alpha_1^n, \alpha_2^n)$  and  $\overrightarrow{\beta_n} = (\beta_1^n, \beta_2^n)$  satisfy  $\alpha_1^n + \alpha_2^n = \beta_1^n + \beta_2^n = n$ . All possible combinations of  $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$  are  $(n+1)^2$  ways :

$$\begin{array}{ccccc} T_{(n,0),(n,0)}^n & \cdots & T_{(n,0),(0,n)}^n, \\ \vdots & \ddots & \vdots \\ T_{(0,n),(n,0)}^n & \cdots & T_{(0,n),(0,n)}^n. \end{array}$$

The recurrence relations for these  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  satisfies

$$\sum_{d=1}^{2} \hbar g_{\overline{i}d} T_{\overline{\alpha_{n}},\overline{e_{d}},\overline{\beta_{n}}^{*}}^{n-1} = \overrightarrow{e_{i}}$$

$$= \beta_{i}^{n} T_{\overline{\alpha_{n}},\overline{\beta_{n}}}^{n} + \sum_{k=1}^{2} \sum_{c=1}^{2} \frac{\hbar \left(\beta_{k}^{n} - \delta_{kc} - \delta_{ik} + 1\right) \left(\beta_{k}^{n} - \delta_{kc} - \delta_{ik} + 2\right)}{2} R_{\overline{c}} \overline{k_{\overline{k}}} T_{\overline{\alpha_{n}},\overline{\beta_{n}}^{*}}^{n} = \overrightarrow{e_{c}} + 2\overrightarrow{e_{k}} = \overrightarrow{e_{i}}$$

$$+ \sum_{c=1}^{2} \hbar \left(\beta_{1}^{n} - \delta_{1c} - \delta_{i1} + 1\right) \left(\beta_{2}^{n} - \delta_{2c} - \delta_{i2} + 1\right) R_{\overline{c}} \overline{t_{i}} T_{\overline{\alpha_{n}},\overline{\beta_{n}}^{*}}^{n} = \overrightarrow{e_{c}} + \overrightarrow{e_{i}} = \overrightarrow{e_{i}}$$

$$(16)$$

from Theorem 2.4, and two types of them for i = 1, 2 exist. Both sides of these recurrence relations are linear combinations of some  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$ . So we now make one new recurrence relation below by summing over the index i on both sides of them.

$$\sum_{i,d=1}^{2} \hbar g_{\overline{i}d} T^{n-1}_{\overrightarrow{\alpha_n} - \overrightarrow{e_d}, \overrightarrow{\beta_n^*} - \overrightarrow{e_i}}$$

$$= \sum_{i=1}^{2} \beta_i^n T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} + \sum_{i,k,c=1}^{2} \frac{\hbar \left(\beta_k^n - \delta_{kc} - \delta_{ik} + 1\right) \left(\beta_k^n - \delta_{kc} - \delta_{ik} + 2\right)}{2} R_{\overline{c}}^{\overline{k}\overline{k}} \overline{i} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*} - \overrightarrow{e_c} + 2\overrightarrow{e_k^*} - \overrightarrow{e_i}}$$

$$+ \sum_{i,c=1}^{2} \hbar \left(\beta_1^n - \delta_{1c} - \delta_{i1} + 1\right) \left(\beta_2^n - \delta_{2c} - \delta_{i2} + 1\right) R_{\overline{c}}^{\overline{2}\overline{1}} \overline{i} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*} - \overrightarrow{e_c} + \overrightarrow{e_i}} \cdot$$
(17)

Using the fact  $\alpha_1^n + \alpha_2^n = n$  and  $\beta_1^n + \beta_2^n = n$ , we can redefine the coefficients  $T_{\overrightarrow{\alpha_n,\beta_n^*}}$  as

$$T_{i,j}^n := T_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}^n, \ i := \alpha_2^n + 1, \ j := \beta_2^n + 1.$$

Then the recurrence relation (17) is rewritten as

$$\hbar \sum_{k,l=1}^{2} g_{\overline{k}l} T_{i-\delta_{2l},j-\delta_{2k}}^{n-1} = X_{j-2}^{n} T_{i,j-2}^{n} + X_{j-1}^{n} T_{i,j-1}^{n} + X_{j}^{n} T_{i,j}^{n} + X_{j+1}^{n} T_{i,j+1}^{n} + X_{j+1}^{n} T_{i,j+2}^{n},$$
(18)

where each  $X_j^n$  is given as

$$\begin{split} X_{j-2}^n &= \binom{n-j+3}{2} \hbar R_{\bar{2}}^{-11}{\bar{2}}, \\ X_{j-1}^n &= 2\binom{n-j+2}{2} \hbar R_{\bar{2}}^{-\bar{1}\bar{1}}{\bar{1}} + (n-j+2) \left(j-2\right) \hbar R_{\bar{2}}^{-\bar{2}\bar{1}}{\bar{2}}, \\ X_j^n &= n + \binom{n-j+1}{2} \hbar R_{\bar{1}}^{-\bar{1}\bar{1}}{\bar{1}} + \binom{j-1}{2} \hbar R_{\bar{2}}^{-\bar{2}\bar{2}}{\bar{2}} + 2 \left(n-j+1\right) \left(j-1\right) \hbar R_{\bar{2}}^{-\bar{2}\bar{1}}{\bar{1}}, \\ X_{j+1}^n &= 2\binom{j}{2} \hbar R_{\bar{2}}^{-\bar{2}\bar{2}}{\bar{1}} + j \left(n-j\right) \hbar R_{\bar{1}}^{-\bar{2}\bar{1}}{\bar{1}}, \\ X_{j+2}^n &= \binom{j+1}{2} \hbar R_{\bar{1}}^{-\bar{2}\bar{2}}{\bar{1}}, \end{split}$$

respectively. We put that  $X_j^n := 0$  when j < 0 or j > n + 1. Here, introducing each  $X_{k,j}^n$  for  $k = 1, \dots, j, \dots, n+1$  by

$$\begin{pmatrix} 0 & \cdots & 0 & X_{j-2,j}^n & X_{j-1,j}^n & X_{j,j}^n & X_{j+1,j}^n & X_{j+2,j}^n & 0 & \cdots & 0 \end{pmatrix}^T := \begin{pmatrix} 0 & \cdots & 0 & X_{j-2}^n & X_{j-1}^n & X_j^n & X_{j+1}^n & X_{j+2}^n & 0 & \cdots & 0 \end{pmatrix}^T,$$

then the right-hand side of the recurrence relation (18) can be written as

Furthermore, to summarize the recurrence relation (18) by a matrix representation, we introduce  $A_{ij}^{'n}$  for the left-hand side and  $\mathbf{X}_j$  for the right-hand side as

$$A_{ij}^{\prime n} := \sum_{k,l=1}^{2} g_{\overline{k}l} T_{i-\delta_{2l},j-\delta_{2k}}^{n-1}$$
  
$$\mathbf{X}_{j} := \begin{pmatrix} 0 & \cdots & 0 & X_{j-2,j}^{n} & X_{j-1,j}^{n} & X_{j,j}^{n} & X_{j+1,j}^{n} & X_{j+2,j}^{n} & 0 & \cdots & 0 \end{pmatrix}^{T},$$

respectively. Thus, by summarizing the recurrence relation (18) for each i and using a matrix representation, we have

 $\hbar A'_n = T_n X_n$ 

for the coefficient  $T_{i,j}^n = T_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}^n$ , where  $T_n := \left(T_{ij}^n\right)$ ,  $A'_n = \left(A'_{ij}\right) \in M_{n+1} (\mathbb{C} \llbracket \hbar \rrbracket)$ , and

 $X_n = (\mathbf{X}_1, \cdots, \mathbf{X}_{n+1})$ 

This proof was completed.  $\blacksquare$ 

Here, we note the fact with respect to  $X_k$  that each  $X_k^{-1}$ , the inverse matrix of each  $X_k$ , is determined by a formal power series with respect to a matrix  $H_k \in M_{k+1}(\mathbb{C})$ . Here  $H_k$  is a pentadiagonal matrix such that each component depends on Riemann curvature tensors on M. We decompose  $X_k$  into the two matrices:

$$X_k = k \mathrm{Id}_{k+1} + \hbar H_k,$$

where  $H_k$  is given as follows :

$$\begin{split} H_{j-2,j}^{k} &= \binom{k-j+3}{2} R_{\bar{2}}^{\,\bar{1}\bar{1}}{}_{\bar{2}}, \\ H_{j-1,j}^{k} &= 2\binom{k-j+2}{2} R_{\bar{2}}^{\,\bar{1}\bar{1}}{}_{\bar{1}}^{\,+} \left(k-j+2\right) \left(j-2\right) R_{\bar{2}}^{\,\bar{2}\bar{1}}{}_{\bar{2}}, \\ H_{j,j}^{k} &= \binom{k-j+1}{2} R_{\bar{1}}^{\,\bar{1}\bar{1}}{}_{\bar{1}}^{\,+} \binom{j-1}{2} R_{\bar{2}}^{\,\bar{2}\bar{2}}{}_{\bar{2}}^{\,+} + 2 \left(k-j+1\right) \left(j-1\right) R_{\bar{2}}^{\,\bar{2}\bar{1}}{}_{\bar{1}}, \\ H_{j+1,j}^{k} &= 2\binom{j}{2} R_{\bar{2}}^{\,\bar{2}\bar{2}}{}_{\bar{1}}^{\,+} + j \left(k-j\right) R_{\bar{1}}^{\,\bar{2}\bar{1}}{}_{\bar{1}}, \\ H_{j+2,j}^{k} &= \binom{j+1}{2} R_{\bar{1}}^{\,\bar{2}\bar{2}}{}_{\bar{1}}, \\ H_{j,l}^{k} &= 0 \ \left(|j-l| > 2\right). \end{split}$$

Then,  $X_k^{-1}$  is given by

$$X_k^{-1} = \sum_{p=0}^{\infty} \frac{(-\hbar)^p}{k^{p+1}} (H_k)^p.$$
(19)

Note that, since a power of a pentadiagonal matrix is not a pentadiagonal matrix in general,  $X_k$  is pentadiagonal but  $X_k^{-1}$  is not always pentadiagonal.

For each  $n \in \mathbb{Z}_{\geq 0}$ ,  $T_n \in M_{n+1} (\mathbb{C} \llbracket \hbar \rrbracket)$ , (15) is also a square matrix of order n + 1. These matrices are not a unified expression, as the size of matrices depends on n, then it is inconvenient to solve the general term. This problem can be solved by embedding "finite-dimensional" matrices into "infinite-dimensional" matrices, and such a procedure provides a unified expression for  $n \ge 2$ . For matrices  $A \in M_n (\mathbb{C} \llbracket \hbar \rrbracket)$  the embedding of A is carried out so that the component whose row or column is greater than n is 0. That is, the embedding is

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto A(\infty) := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & 0 \\ a_{n1} & \cdots & a_{nn} \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{pmatrix} = \begin{pmatrix} A & 0 \\ \hline 0 & 0 \\ \hline 0 \\ \hline \end{bmatrix} .$$
(20)

In the following calculations, we use  $\mathrm{Id}_{n}$ ,  $\mathrm{Id}(\infty)$ ,  $F_{d,n}$ ,  $F_{d}(\infty)$   $F_{r,n}$  and  $F_{r}(\infty)$  given by

$$\operatorname{Id}_{n} = \begin{pmatrix} 1 & 0 \\ & \ddots & \\ 0 & 1 \end{pmatrix}, \qquad \qquad \operatorname{Id}(\infty) = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & \ddots \end{pmatrix}, \qquad (21)$$

$$F_{d,n} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad F_d(\infty) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix},$$
(22)

$$F_{r,n} = F_{d,n}^T, \qquad \qquad F_r(\infty) = F_d(\infty)^T.$$
(23)

**Remark 3.2.** Note that multiplying  $F_{d,n}$  from the left corresponds to "shifting the components of the matrix downward by one position, with zeros appearing in the top row". And multiplying  $F_{r,n}$  from the right corresponds to "shifting the components of the matrix rightward by one position, with zeros appearing in the first column".  $F_d(\infty)$  and  $F_r(\infty)$  correspond to their infinite-dimensional versions.

From Proposition 2.3, the embeddings of  $T_0(\infty)$  and  $T_1(\infty)$  are

$$T_{0} = (1) \mapsto T_{0}(\infty) := \begin{pmatrix} T_{0} & 0 \\ \hline 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 \\ \vdots & \ddots \end{pmatrix},$$
$$T_{1} = \hbar \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} \mapsto T_{1}(\infty) := \begin{pmatrix} T_{1} & 0 \\ \hline 0 & 0 \end{pmatrix} = \hbar \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} & 0 & \cdots \\ g_{2\bar{1}} & g_{2\bar{2}} & 0 \\ 0 & 0 & 0 \\ \vdots & \ddots \end{pmatrix},$$

respectively. Similarly, (15) in Theorem 3.1 is embedded as

$$T_n(\infty) = \hbar A'_n(\infty) X_n^{-1}(\infty), \qquad (24)$$

where  $A_{n}^{\prime}\left(\infty\right)$  and  $X_{n}^{-1}\left(\infty\right)$  are

$$A'_{n}(\infty) := \left(\begin{array}{c|c} A'_{n} & 0\\ \hline 0 & 0 \end{array}\right), \qquad X_{n}^{-1}(\infty) := \left(\begin{array}{c|c} X_{n}^{-1} & 0\\ \hline 0 & 0 \end{array}\right).$$

Note that  $X_n^{-1}(\infty)$  is not the inverse matrix of  $X_n(\infty)$ . As we saw in Theorem 3.1,  $A'_n \in M_{n+1}(\mathbb{C}[\![\hbar]\!])$  is expressed as

$$A'_{n} = \left(\sum_{k,l=1}^{2} g_{\overline{k}l} T^{n-1}_{i-\delta_{2l},j-\delta_{2k}}\right) = \left(g_{\overline{1}1} T^{n-1}_{i,j} + g_{\overline{1}2} T^{n-1}_{i-1,j} + g_{\overline{2}1} T^{n-1}_{i,j-1} + g_{\overline{2}2} T^{n-1}_{i-1,j-1}\right)$$

Recall that  $T_{i,j}^{n-1} = 0$  if i < 0, i > n-1, j < 0 or j > n-1 by definition. By a matrix representation

$$\begin{pmatrix} g_{\overline{1}1}T_{i,j}^{n-1} + g_{\overline{1}2}T_{i-1,j}^{n-1} + g_{\overline{2}1}T_{i,j-1}^{n-1} + g_{\overline{2}2}T_{i-1,j-1}^{n-1} \end{pmatrix}$$
  
=  $g_{\overline{1}1} \begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & \mathbf{0} \end{pmatrix} + g_{\overline{1}2} \begin{pmatrix} \mathbf{0}_{n-1}^T & \mathbf{0} \\ T_{n-1} & \mathbf{0}_{n-1} \end{pmatrix} + g_{\overline{2}1} \begin{pmatrix} \mathbf{0}_{n-1} & T_{n-1} \\ \mathbf{0} & \mathbf{0}_{n-1}^T \end{pmatrix} + g_{\overline{2}2} \begin{pmatrix} \mathbf{0} & \mathbf{0}_{n-1}^T \\ \mathbf{0}_{n-1} & T_{n-1} \end{pmatrix}$ 

with  $T_{n-1}$  which is the square matrix of order n, where  $\mathbf{0}_{n-1}$  is an n-1 dimensional zero vector. Since these matrices can be expressed as

$$\begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} & 0 \end{pmatrix} = \mathrm{Id}_{n+1} \begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} & 0 \end{pmatrix} \mathrm{Id}_{n+1}, \\ \begin{pmatrix} \mathbf{0}_{n-1}^T & \mathbf{0} \\ \overline{T_{n-1}} & \mathbf{0}_{n-1} \end{pmatrix} = F_{d,n+1} \begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} & 0 \end{pmatrix} \mathrm{Id}_{n+1}, \\ \begin{pmatrix} \mathbf{0}_{n-1} & T_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} \end{pmatrix} = \mathrm{Id}_{n+1} \begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} & 0 \end{pmatrix} F_{r,n+1}, \\ \begin{pmatrix} \mathbf{0} & \mathbf{0}_{n-1}^T \\ \overline{\mathbf{0}_{n-1}} & T_{n-1} \end{pmatrix} = F_{d,n+1} \begin{pmatrix} T_{n-1} & \mathbf{0}_{n-1} \\ \overline{\mathbf{0}_{n-1}^T} & 0 \end{pmatrix} F_{r,n+1},$$

using  $Id_{n+1}$ ,  $F_{d,n+1}$  and  $F_{r,n+1}$ , respectively, then we have

$$g_{\overline{1}1}\left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^T & 0 \end{array}\right) + g_{\overline{1}2}\left(\begin{array}{c|c} \mathbf{0}_{n-1}^T & 0 \\ \hline T_{n-1} & \mathbf{0}_{n-1} \end{array}\right) + g_{\overline{2}1}\left(\begin{array}{c|c} \mathbf{0}_{n-1} & T_{n-1} \\ \hline \mathbf{0} & \mathbf{0}_{n-1}^T \end{array}\right) + g_{\overline{2}2}\left(\begin{array}{c|c} \mathbf{0} & \mathbf{0}_{n-1}^T \\ \hline \mathbf{0}_{n-1} & T_{n-1} \end{array}\right) \\ = g_{\overline{1}1} \mathrm{Id}_{n+1}\left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^T & 0 \end{array}\right) \mathrm{Id}_{n+1} + g_{\overline{1}2} F_{d,n+1}\left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^T & 0 \end{array}\right) \mathrm{Id}_{n+1} \\ + g_{\overline{2}1} \mathrm{Id}_{n+1}\left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^T & 0 \end{array}\right) F_{r,n+1} + g_{\overline{2}2} F_{d,n+1}\left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^T & 0 \end{array}\right) F_{r,n+1}. \end{array}$$

By introducing some functions  $\Theta^p_{\bar{\mu}}, \Theta^p_{\nu} \in C^{\infty}(U)$   $(p \in \{1, \dots, 4\})$  such that  $g_{\bar{\mu}\nu} = \Theta^p_{\bar{\mu}}\Theta^p_{\nu}, A'_n$  can be expressed as

$$A'_{n} = \sum_{p=1}^{4} \left(\Theta_{1}^{p} \mathrm{Id}_{n+1} + \Theta_{2}^{p} F_{d,n+1}\right) \left(\frac{T_{n-1} \mid \mathbf{0}_{n-1}}{\mathbf{0}_{n-1}^{T} \mid \mathbf{0}}\right) \left(\Theta_{1}^{p} \mathrm{Id}_{n+1} + \Theta_{2}^{p} F_{r,n+1}\right).$$

These  $\Theta^p_{\bar{\mu}}$  and  $\Theta^p_{\nu}$  play a similar role of "a vierbein", but not a vierbein because they are not necessarily to be orthonormal. Furthermore, by introducing the new matrices  $F^p_n$ ,  $B^p_n \in M_{n+1}(\mathbb{C} \llbracket \hbar \rrbracket)$  as

$$F_n^p := \Theta_1^p \mathrm{Id}_{n+1} + \Theta_2^p F_{d,n+1}, \qquad B_n^p := \Theta_{\bar{1}}^p \mathrm{Id}_{n+1} + \Theta_{\bar{2}}^p F_{r,n+1}$$

respectively,  $A'_n$  is expressed as

$$A'_{n} = \sum_{p=1}^{4} F_{n}^{p} \left( \begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_{n-1}^{T} & \mathbf{0} \end{array} \right) B_{n}^{p},$$

so  $A_{n}^{\prime}\left(\infty\right)$ , that is, the embedding matrix for  $A_{n}^{\prime}$ , is expressed as

$$A'_{n}(\infty) = \sum_{p=1}^{4} F_{n}^{p}(\infty) T_{n-1}(\infty) B_{n}^{p}(\infty).$$

Here,  $F_n^p$  and  $B_n^p$  were replaced by  $F_n^p(\infty)$  and  $B_n^p(\infty)$ , where

$$F_n^p(\infty) := \Theta_1^p \mathrm{Id}(\infty) + \Theta_2^p F_d(\infty), \qquad B_n^p(\infty) := \Theta_1^p \mathrm{Id}(\infty) + \Theta_2^p F_r(\infty).$$
(25)

Therefore,  $T_n(\infty)$  is obtained as

$$T_{n}(\infty) = \hbar \sum_{p=1}^{4} F_{n}^{p}(\infty) T_{n-1}(\infty) B_{n}^{p}(\infty) X_{n}^{-1}(\infty).$$

By using recursively the above procedure, we obtain the following main theorem.

**Theorem 3.3** (Main result). Let M be a complex two-dimensional locally symmetric Kähler manifold, U be an open set of M, and \* be a star product with separation of variables on M. For  $f, g \in C^{\infty}(U)$ , f \* g is given by

$$f * g = \sum_{n=0}^{\infty} \sum_{i,j=1}^{n+1} T_{ij}^n \left\{ \left( D^1 \right)^{n-i+1} \left( D^2 \right)^{i-1} f \right\} \left\{ \left( D^{\bar{1}} \right)^{n-j+1} \left( D^{\bar{2}} \right)^{j-1} g \right\}$$

Here each of the coefficient  $T_{ij}^n$  is

$$T_n(\infty)_{ij} = \begin{cases} T_{ij}^n & (1 \le i, j \le n+1) \\ 0 & (\text{otherwise}) \end{cases}$$

,

and  $T_n(\infty)$  is determined by

$$T_{n}(\infty) = \hbar^{n} \sum_{p_{1},\cdots,p_{n}=1}^{4} F_{n}^{p_{n}}(\infty) \cdots F_{1}^{p_{1}}(\infty) T_{0}(\infty) \left( B_{1}^{p_{1}}(\infty) X_{1}^{-1}(\infty) \right) \cdots \left( B_{n}^{p_{n}}(\infty) X_{n}^{-1}(\infty) \right), \quad (26)$$

where each  $F_{k}^{p_{k}}(\infty)$ ,  $B_{k}^{p_{k}}(\infty)$  and  $X_{k}^{-1}(\infty)$  are given as above.

From Theorem 3.3, we obtain a deformation quantization with separation of variables for complex twodimensional locally symmetric Kähler manifold is realized by this star product.

#### 3.2Another formula

The formula (26) was obtained by summation with respect to the index i of the complex coordinate in the recurrence relation (16). In Subsection 3.1, we made one recurrence relation by adding two recurrence relations and determined the solution using only that recurrence relation. On the other hand, we have not yet considered another recurrence relation. In this subsection, we consider another formula obtained by "subtracting" for i = 1, 2 rather than "adding" two recurrence relations for i = 1, 2 as we did in Subsection 3.1.

**Theorem 3.4.** Let M be a locally symmetric Kähler manifold, U be an open set of M, and  $f,g \in$  $C^{\infty}\left(U
ight)$ , \* be a star product with separation of variables on U such that

$$f * g = \sum_{n=0}^{\infty} \sum_{i,j=1}^{n+1} T_{ij}^n \left\{ \left( D^1 \right)^{n-i+1} \left( D^2 \right)^{i-1} f \right\} \left\{ \left( D^{\bar{1}} \right)^{n-j+1} \left( D^{\bar{2}} \right)^{j-1} g \right\}.$$

Then

$$T_n Y_n = \hbar C'_n \tag{27}$$

or equivalently

where

$$Y_n^{\dagger} T_n = \hbar C_n^{\prime \dagger}, \tag{28}$$

$$C'_{n} = \left(C'_{ij}\right) = \left(\sum_{k,l=1}^{2} (-1)^{\delta_{k2}} g_{\overline{k}l} T^{n-1}_{i-\delta_{2l},j-\delta_{2k}}\right) \in M_{n+1} \left(\mathbb{C} \left[\!\left[\hbar\right]\!\right]\right)$$

 $Y^{\dagger}T_{r} = \hbar C^{\prime \dagger}$ 

and  $Y_n \in M_{n+1}(\mathbb{C} \llbracket \hbar \rrbracket)$  is a pentadiagonal matrix such that its components are given as follows :

$$\begin{split} Y_{j-2,j}^n &= -\binom{n-j+3}{2} \hbar R_{\bar{2}}^{-11} {}_{\bar{2}}, \\ Y_{j-1,j}^n &= -\left(n-j+2\right) \left(j-2\right) \hbar R_{\bar{2}}^{-\bar{2}\bar{1}} {}_{\bar{2}}, \\ Y_{j,j}^n &= n-2j+2 + \binom{n-j+1}{2} \hbar R_{\bar{1}}^{-1} {}_{\bar{1}}^{-1} - \binom{j-1}{2} \hbar R_{\bar{2}}^{-\bar{2}\bar{2}} {}_{\bar{2}}, \\ Y_{j+1,j}^n &= j \left(n-j\right) \hbar R_{\bar{1}}^{-\bar{2}\bar{1}} {}_{\bar{1}}, \\ Y_{j+2,j}^n &= \binom{j+1}{2} \hbar R_{\bar{1}}^{-\bar{2}\bar{2}} {}_{\bar{1}}, \\ Y_{j,k}^n &= 0 \ \left(|j-k| > 2\right). \end{split}$$

**Proof.** This is shown in the same way as in Theorem 3.3.

. .

Note that equation (27) is also expressed as an embedding version

$$T_n(\infty) Y_n(\infty) = \hbar C'_n(\infty)$$
<sup>(29)</sup>

or equivalently

$$Y_n^{\dagger}(\infty) T_n(\infty) = \hbar C_n^{\prime \dagger}(\infty) , \qquad (30)$$

where the embedding  $Y_n(\infty)$  for  $Y_n$  is as in the way of (20) in Subsection 3.1. Here we use  $T_n = T_n^{\dagger}$ . Its derivation is in Appendix C.

It is known from Karabegov's result that there is always a star product with separation of variables on a Kähler manifold [30, 31]. A star product with separation of variables on a locally symmetric Kähler manifold is determined by (26) in Theorem 3.3. Therefore,  $T_n(\infty)$  given by (26) should satisfy (29). To ensure that the result obtained by Theorem 3.3 does not contradict Theorem 3.4, we consider the following equation:

$$Y_n^{\dagger} T_n X_n = \hbar C_n^{\prime \dagger} X_n. \tag{31}$$

Here, (31) is obtained by multiplying (28) by  $X_n$  from the right. This is merely a change to a form that allows direct substitution of the result of Theorem 3.3. The reason for using (28) rather than (27) is that  $X_n^{-1}$  appears if we try to check (27) directly, and it is difficult to calculate because it is an infinite power series matrix. In this discussion, we shall simply show only the cases n = 1 and 2.

Case: 
$$n = 1$$

Since Proposition 2.3 and

$$X_{1} = \begin{pmatrix} X_{11}^{1} & X_{12}^{1} \\ X_{21}^{1} & X_{22}^{1} \end{pmatrix} = \mathrm{Id}_{2},$$
$$Y_{1} = \begin{pmatrix} Y_{11}^{1} & Y_{12}^{1} \\ Y_{21}^{1} & Y_{22}^{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

it follows that

$$Y_1^{\dagger} T_1 X_1 = \hbar \begin{pmatrix} g_{1\overline{1}} & g_{2\overline{1}} \\ -g_{1\overline{2}} & -g_{2\overline{2}} \end{pmatrix},$$
$$C_1^{\dagger} X_1 = \begin{pmatrix} g_{\overline{1}1} & -g_{\overline{2}1} \\ g_{\overline{1}2} & -g_{\overline{2}2} \end{pmatrix}^{\dagger} = \begin{pmatrix} g_{1\overline{1}} & g_{2\overline{1}} \\ -g_{1\overline{2}} & -g_{2\overline{2}} \end{pmatrix}.$$

Therefore  $Y_1^{\dagger}T_1X_1 = \hbar C_1'X_1$ , equivalently  $Y_1^{\dagger}(\infty) T_1(\infty) X_1(\infty) = \hbar C_1'(\infty) X_1(\infty)$ .

Case: 
$$n = 2$$

Calculations for both sides of (31) yields

$$Y_{2}^{\dagger}T_{2}X_{2} = \hbar^{2} \begin{pmatrix} 2 + \hbar R_{1}^{11} & \hbar R_{1}^{21} & \hbar R_{1}^{22} \\ 0 & 0 & 0 \\ -\hbar R_{2}^{11} & -\hbar R_{2}^{21} & -2 - \hbar R_{2}^{22} \end{pmatrix} \begin{pmatrix} (g_{\overline{1}1})^{2} & 2g_{\overline{1}1}g_{\overline{2}1} & (g_{\overline{2}1})^{2} \\ 2g_{\overline{1}1}g_{\overline{1}2} & 2(g_{\overline{1}1}g_{\overline{2}2} + g_{\overline{1}2}g_{\overline{2}1}) & 2g_{\overline{2}1}g_{\overline{2}2} \\ (g_{\overline{1}2})^{2} & 2g_{\overline{1}2}g_{\overline{2}2} & (g_{\overline{2}2})^{2} \end{pmatrix}$$
(32)

and

$$\hbar C_2' X_2 = \hbar^2 \begin{pmatrix} (g_{\overline{1}1})^2 & 2g_{\overline{1}1}g_{\overline{2}1} & (g_{\overline{2}1})^2 \\ 0 & 0 & 0 \\ -(g_{\overline{1}2})^2 & -2g_{\overline{1}2}g_{\overline{2}2} & -(g_{\overline{2}2})^2 \end{pmatrix} \begin{pmatrix} 2 + \hbar R_{\bar{1}}^{\ 1\overline{1}} & 2\hbar R_{\bar{2}}^{\ 1\overline{1}} & \hbar R_{\bar{2}}^{\ 1\overline{1}} \\ \hbar R_{\bar{1}}^{\ 2\overline{1}} & 2 + 2\hbar R_{\bar{2}}^{\ 2\overline{1}} & \hbar R_{\bar{2}}^{\ 2\overline{1}} \\ \hbar R_{\bar{1}}^{\ 2\overline{2}} & 2 + \hbar R_{\bar{2}}^{\ 2\overline{2}} & 2 + \hbar R_{\bar{2}}^{\ 2\overline{2}} \\ \end{pmatrix}$$
(33)

respectively. Here we use

$$T_2 X_2 = \hbar A'_2 = \begin{pmatrix} (g_{\overline{1}1})^2 & 2g_{\overline{1}1}g_{\overline{2}1} & (g_{\overline{2}1})^2 \\ 2g_{\overline{1}1}g_{\overline{1}2} & 2(g_{\overline{1}1}g_{\overline{2}2} + g_{\overline{1}2}g_{\overline{2}1}) & 2g_{\overline{2}1}g_{\overline{2}2} \\ (g_{\overline{1}2})^2 & 2g_{\overline{1}2}g_{\overline{2}2} & (g_{\overline{2}2})^2 \end{pmatrix}.$$
(34)

Furthermore, each component of both sides of (31) coincides with each other from the calculations (57)–(74) in Appendix B. Hence, it is shown that (31) for n = 2 holds.

### 4 Examples

In this section, we describe the deformation quantization with separation of variables for  $\mathbb{C}^2$  and  $\mathbb{C}P^2$  as concrete examples realized from Theorem 3.3. The deformation quantizations were already given by the other methods as we will see in the next subsection. To compare our results with them, we can check if Theorem 3.3 works well. Moreover, we show that the concrete examples satisfy the identities in Theorem 3.4.

### 4.1 Previous results for $\mathbb{C}^2$ and $\mathbb{C}P^2$

It is known that noncommutative  $\mathbb{R}^{2N}$  is the most trivial noncommutative manifold. We can say that the quantization map from  $\mathbb{R}^{2N}$  to noncommutative  $\mathbb{R}^{2N}$  is introduced by Dirac [17] because the canonical quantization is the quantization of phase space  $\mathbb{R}^{2N}$ . In the strict deformation quantization case, the noncommutative plane has also been proposed by Rieffel from the perspective of  $C^*$ -algebra [51]. On the other hand, the deformation quantization of  $\mathbb{R}^{2N}$  is also the most trivial example in the formal deformation quantization case. In particular, the Moyal product [42] and the Voros product [68] are well-known examples of star products on  $\mathbb{R}^{2N}$ . Here, a noncommutative  $\mathbb{R}^{2N}$  can be regarded as a noncommutative  $\mathbb{C}^N$ . For noncommutative  $\mathbb{C}^N$ , the concrete construction is provided by using the deformation quantization with separation of variables by Karabegov as an example [30]. As another concrete example, we shall consider here a deformation quantization with separation of variables for  $\mathbb{C}P^2$ . Previous works related to noncommutative  $\mathbb{C}P^N$  are known from deformation quantizations and fuzzy geometry. Noncommutative  $\mathbb{C}P^N$  via a deformation quantizations had constructed by Omori-Maeda-Yoshioka [45], Bordemann et al. [7], Sako-Suzuki-Umetsu [54, 55], and Hara-Sako [22, 23]. In addition, the (twisted) Fock representation of noncommutative  $\mathbb{C}P^N$  was given by Sako-Suzuki-Umetsu [54, 55] and Sako-Umetsu [57, 58, 59]. It is known that this representation is essentially equivalent to ordinary Fock representation. On the other hand, in fuzzy physics, noncommutative  $\mathbb{C}P^N$ , called fuzzy  $\mathbb{C}P^N$ , had constructed via a matrix algebra. The construction methods of fuzzy  $S^2$ , i.e. fuzzy  $\mathbb{C}P^1$ , had proposed by Hoppe [29] and Madore [37]. Hayasaka-Nakayama-Takaya constructed a star product on  $S^2$  from their fuzzy  $S^2$  [24]. Furthermore, more general works were proposed by Grosse-Strohmaier [20], Alexanian et al. [1] for fuzzy  $\mathbb{C}P^2$ , and by Balachandran et al. [3], Carow-Watamura-Steinacker-Watamura [15] for  $\mathbb{C}P^N$ . See [38] for other references. In particular, the Fock representation on fuzzy  $\mathbb{C}P^N$  was also discussed by Alexanian-Pinzul-Stern [2] and Carow-Watamura-Steinacker-Watamura [15]. In this subsection, we introduce the results of noncommutative  $\mathbb{C}^2$  by Karabegov and Voros, and noncommutative  $\mathbb{C}P^2$  by Sako-Suzuki-Umetsu to compare them with results given by Theorem 3.3 in the following sections.

**Theorem 4.1** (Karabegov[31]). For any  $f, g \in C^{\infty}(\mathbb{C}^2)$ , the star product with separation of variables on  $\mathbb{C}^2$  is locally given by

$$f * g = f \exp \nu \left( \overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}} + \overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}} \right) g.$$
(35)

It is known as the star product which is called "Voros product" [68] and isomorphic to Moyal product [42]. A simple proof for the isomorphism of them is given by, for example, Section 2.2 in [2]. The star

product (35) is sometimes called the "Wick product" [7, 9, 10]. For  $\mathbb{C}P^N$ , the noncommutative one had been constructed by several previous works. In Section 4.3, we compare the star product on  $\mathbb{C}P^2$  obtained by Theorem 3.3 with the previous work by Sako-Suzuki-Umetsu. So we denote the result for  $\mathbb{C}P^2$  by them.

**Theorem 4.2** (Sako-Suzuki-Umetsu[54, 55]). For any  $f, g \in C^{\infty}(\mathbb{C}P^2)$ , the star product with separation of variables on  $\mathbb{C}P^2$  is locally given by

$$f * g = \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - n + \frac{1}{\hbar}\right) g_{\overline{\mu_1}\nu_1} \cdots g_{\overline{\mu_n}\nu_n}}{n!\Gamma\left(1 + \frac{1}{\hbar}\right)} \left(D^{\nu_1} \cdots D^{\nu_n}f\right) \left(D^{\overline{\mu_1}} \cdots D^{\overline{\mu_n}}g\right).$$
(36)

Here  $\mu_k, \nu_k = 1, 2$ . Note that we use the Einstein summation convention in Theorem 4.2. In [54, 55], it is confirmed that the star product on  $\mathbb{C}P^N$  coincides with one obtained by Bordemann et al. [7]. See [54, 55] for more detail.

### 4.2 Example of $\mathbb{C}^2$

We shall construct a deformation quantization with separation of variables for  $\mathbb{C}^2$ , the simplest concrete example via Theorem 3.3, and show that it also reproduces the star product with separation of variables by Karabegov [31]. Since  $\mathbb{C}^2$  is flat, i.e.  $R_{ABC}^{\ \ D} = 0$ , each  $X_k(\infty)$  is a constant multiple of the identity matrix, and  $T_n(\infty)$  can be obtained easily.

**Proposition 4.3.** The formula for determining a star product with separation of variables on  $\mathbb{C}^2$  is given by

$$T_{n}(\infty) = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \begin{pmatrix} \binom{n}{0} & 0 \\ & \ddots & & \\ & & \binom{n}{n} & \\ 0 & & 0 \\ & & & \ddots \end{pmatrix}.$$
 (37)

**Proof.** We now take the canonical coordinates  $z^1 = x^1 + iy^1$ ,  $z^2 = x^2 + iy^2$  as the local coordinates. Since  $(g_{AB})$ , the component matrix of the Kähler metric on  $\mathbb{C}^2$ , is

$$(g_{AB}) = \begin{pmatrix} g_{11} & g_{12} & g_{1\overline{1}} & g_{1\overline{2}} \\ g_{21} & g_{22} & g_{2\overline{1}} & g_{2\overline{2}} \\ g_{\overline{1}1} & g_{\overline{1}2} & g_{\overline{1}\overline{1}} & g_{\overline{1}\overline{2}} \\ g_{\overline{2}1} & g_{\overline{2}2} & g_{\overline{2}\overline{1}} & g_{\overline{2}\overline{2}} \end{pmatrix} := \begin{pmatrix} g_{z^{1}z^{1}} & g_{z^{1}z^{2}} & g_{z^{1}\overline{z}^{1}} & g_{z^{1}\overline{z}^{2}} \\ g_{z^{2}z^{1}} & g_{z^{2}z^{2}} & g_{\overline{z}^{2}\overline{z}^{1}} & g_{\overline{z}^{2}\overline{z}^{2}} \\ g_{\overline{z}^{1}z^{1}} & g_{\overline{z}^{1}z^{2}} & g_{\overline{z}^{1}\overline{z}^{1}} & g_{\overline{z}^{1}\overline{z}^{2}} \\ g_{\overline{z}^{2}z^{1}} & g_{\overline{z}^{2}z^{2}} & g_{\overline{z}^{2}\overline{z}^{1}} & g_{\overline{z}^{2}\overline{z}^{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

so the inverse matrix  $(g^{AB})$  is

$$(g^{AB}) = \begin{pmatrix} g^{11} & g^{12} & g^{1\overline{1}} & g^{1\overline{2}} \\ g^{21} & g^{22} & g^{2\overline{1}} & g^{2\overline{2}} \\ g^{\overline{1}1} & g^{\overline{1}2} & g^{\overline{1}\overline{1}} & g^{\overline{1}\overline{2}} \\ g^{\overline{2}1} & g^{\overline{2}2} & g^{\overline{2}\overline{1}} & g^{\overline{2}\overline{2}} \end{pmatrix} := \begin{pmatrix} g^{z^{1}z^{1}} & g^{z^{1}z^{2}} & g^{z^{1}\overline{z}^{1}} & g^{z^{1}\overline{z}^{2}} \\ g^{z^{2}z^{1}} & g^{z^{2}z^{2}} & g^{\overline{z}^{1}\overline{z}^{1}} & g^{\overline{z}^{2}\overline{z}^{2}} \\ g^{\overline{z}^{2}z^{1}} & g^{\overline{z}^{2}z^{2}} & g^{\overline{z}^{2}\overline{z}^{1}} & g^{\overline{z}^{2}\overline{z}^{2}} \\ g^{\overline{z}^{2}z^{1}} & g^{\overline{z}^{2}z^{2}} & g^{\overline{z}^{2}\overline{z}^{1}} & g^{\overline{z}^{2}\overline{z}^{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

$$F_{k}^{i_{k}} = \frac{1}{\sqrt{2}} \left( \delta_{1}^{i_{k}} \mathrm{Id}_{k+1} + \delta_{2}^{i_{k}} F_{d,k+1} \right), \ B_{k}^{i_{k}} = \frac{1}{\sqrt{2}} \left( \delta_{1}^{i_{k}} \mathrm{Id}_{k+1} + \delta_{2}^{i_{k}} F_{d,k+1} \right).$$

By  $X_k^{-1} = \frac{1}{k} \operatorname{Id}_k$ , we obtain

$$T_{n}(\infty) = \hbar^{n} \sum_{i_{1},\cdots,i_{n}=1}^{4} F_{n}^{i_{n}}(\infty) \cdots F_{1}^{i_{1}}(\infty) T_{0}(\infty) \left( B_{1}^{i_{1}}(\infty) X_{1}^{-1}(\infty) \right) \cdots \left( B_{n}^{i_{n}}(\infty) X_{n}^{-1}(\infty) \right) .$$
  
$$= \frac{\hbar^{n}}{n!} \left( \frac{1}{\sqrt{2}} \right)^{2n} \sum_{i_{1},\cdots,i_{n}=1}^{4} \left( \delta^{i_{n}}_{1} \mathrm{Id}(\infty) + \delta^{i_{n}}_{2} F_{d}(\infty) \right) \cdots \left( \delta^{i_{1}}_{1} \mathrm{Id}(\infty) + \delta^{i_{1}}_{2} F_{d}(\infty) \right) T_{0}(\infty) .$$
  
$$\cdot \left( \delta^{i_{1}}_{1} \mathrm{Id}(\infty) + \delta^{i_{1}}_{2} F_{r}(\infty) \right) \cdots \left( \delta^{i_{n}}_{1} \mathrm{Id}(\infty) + \delta^{i_{n}}_{2} F_{r}(\infty) \right) .$$

By using

$$\left(\begin{array}{c|c} A_{k+1} & 0\\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} X_k^{-1} & 0\\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} A_{k+1} & 0\\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} X_k^{-1} & 0\\ \hline 0 & B \end{array}\right)$$

for any  $A_{k+1} \in M_{k+1} (\mathbb{C} \llbracket \hbar \rrbracket)$  and  $B \in M_{\infty} (\mathbb{C} \llbracket \hbar \rrbracket)$ . The above equation can be written as

$$T_{n}(\infty) = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \delta_{\mu_{1}\nu_{1}}\cdots\delta_{\mu_{n}\nu_{n}} (F_{d}(\infty))^{\gamma(\nu^{(n)})} T_{0}(\infty) (F_{r}(\infty))^{\gamma(\mu^{(n)})} \\ = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \sum_{\mu_{1},\cdots,\mu_{n}=1}^{2} (F_{d}(\infty))^{\gamma(\mu^{(n)})} T_{0}(\infty) (F_{r}(\infty))^{\gamma(\mu^{(n)})} ,$$

where  $\gamma(\cdot): \{1,2\}^n \to \mathbb{Z}_{\geq 0}$  is defined by

$$\gamma\left(\mu^{(n)}\right) := \sum_{k=1}^{n} \left[\frac{\mu_k}{2}\right] = \sum_{k=1}^{n} \delta_{\mu_k 2}$$

for multi-indices  $\mu^{(n)} = (\mu_1, \dots, \mu_n)$ , and  $[\cdot] : \mathbb{R} \to \mathbb{Z}$  is Gauss symbol. We define  $\gamma(\nu^{(n)})$  in the same way. In other words,  $\gamma(\cdot) : \{1,2\}^n \to \mathbb{Z}_{\geq 0}$  gives the number of each  $\mu_k$  (or  $\nu_k$ ) that is 2. We can rewrite

$$\left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \sum_{\mu_{1},\cdots,\mu_{n}=1}^{2} \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \left(F_{d}\left(\infty\right)\right)^{j} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{j} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{j$$

since there are  $\binom{n}{j}$  combinations of each  $\mu_k$  such that  $\gamma(\mu^{(n)}) = j$ . After summing over  $j \in \{1, \dots, n\}$ , this is concretely expressed as

$$\left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (F_{d}(\infty))^{j} T_{0}(\infty) (F_{r}(\infty))^{j} = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \begin{pmatrix} \binom{n}{0} & 0 \\ & \ddots & \\ & & \binom{n}{n} \\ 0 & & 0 \\ & & \ddots \end{pmatrix}$$

by matrix representations. This proof was completed.  $\blacksquare$ 

By Theorem 4.3, the coefficients  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  (or  $T^n_{ij}$  ) is determined as

$$T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}} = T^n_{ij} = \begin{cases} \frac{\binom{n}{i-1}h^n}{2^n \cdot n!} & (i=j)\\ 0 & (i\neq j) \end{cases},\tag{38}$$

where  $\alpha_2^n = i - 1$  and  $\beta_2^n = j - 1$ . Hence, the star product with separation of variables on  $\mathbb{C}^2$  is obtained as follows :

$$f * g = \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} \left( D^{\overrightarrow{\alpha_n}} f \right) \left( D^{\overrightarrow{\beta_n^*}} g \right)$$
$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} T^n_{ii} \left\{ \left( D^1 \right)^{n-(i-1)} \left( D^2 \right)^{i-1} f \right\} \left\{ \left( D^{\overline{1}} \right)^{n-(i-1)} \left( D^{\overline{2}} \right)^{i-1} g \right\}.$$
(39)

Substituting (38) into the right-hand side of (39),

$$\sum_{n=0}^{\infty} \sum_{i=1}^{n+1} T_{ii}^{n} \left\{ \left( D^{1} \right)^{n-(i-1)} \left( D^{2} \right)^{i-1} f \right\} \left\{ \left( D^{\overline{1}} \right)^{n-(i-1)} \left( D^{\overline{2}} \right)^{i-1} g \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} \left( \frac{\hbar}{2} \right)^{n} \frac{1}{n!} {\binom{n}{i-1}} \left\{ \left( g^{1\overline{1}} \partial_{\overline{1}} \right)^{n-(i-1)} \left( g^{2\overline{2}} \partial_{\overline{2}} \right)^{i-1} f \right\} \left\{ \left( g^{\overline{1}1} \partial_{1} \right)^{n-(i-1)} \left( g^{\overline{2}2} \partial_{2} \right)^{i-1} g \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(2\hbar)^{n}}{n!} \sum_{j=0}^{n} {\binom{n}{j}} \left( \partial_{\overline{z}^{1}}^{n-j} \partial_{\overline{z}^{2}}^{j} f \right) \left( \partial_{z^{1}}^{n-j} \partial_{z^{2}}^{j} g \right), \tag{40}$$

where we rewrite i - 1 as j in the last line in (40). Here, using the left (right) differential operator

$$\overrightarrow{\partial_{z^A}} := \partial_{z^A} f, \ \overrightarrow{\partial_{z^A}} g := \partial_{z^A} g \ (A \in \{1, 2, \overline{1}, \overline{2}\}),$$

we obtain the star product on  $\mathbb{C}^2$  as

$$f * g = \sum_{n=0}^{\infty} \frac{(2\hbar)^n}{n!} \sum_{j=0}^n \binom{n}{j} \left(\partial_{\overline{z}^1}^{n-j} \partial_{\overline{z}^2}^j f\right) \left(\partial_{z^1}^{n-j} \partial_{z^2}^j g\right)$$

$$=\sum_{n=0}^{\infty} \frac{(2\hbar)^n}{n!} f\left\{\sum_{j=0}^n \binom{n}{j} \left(\overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}}\right)^{n-j} \left(\overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}}\right)^j\right\} g.$$
(41)

Let us compare the previous result (35) to verify that Theorem 3.3 works well. By using the binomial theorem for the left and right differential operators in (41), we have

$$\sum_{n=0}^{\infty} \frac{(2\hbar)^n}{n!} f\left\{\sum_{j=0}^n \binom{n}{j} \left(\overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}}\right)^{n-j} \left(\overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}}\right)^j\right\} g = \sum_{n=0}^{\infty} \frac{(2\hbar)^n}{n!} f\left(\overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}} + \overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}}\right)^n g$$
$$= f \exp 2\hbar \left(\overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}} + \overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}}\right) g. \tag{42}$$

Here, if we put the formal parameter  $\nu$  as  $\nu := 2\hbar$ , we obtain  $f * g = f \exp \nu \left( \overleftarrow{\partial_{\overline{z}^1}} \overrightarrow{\partial_{z^1}} + \overleftarrow{\partial_{\overline{z}^2}} \overrightarrow{\partial_{z^2}} \right) g$ , which is the star product with separation of variables by Karabegov (35).

We stated in Subsection 3.1 that  $T_n(\infty)$  given in (26) solved via Theorem 3.3 satisfies the formula (29) in Theorem 3.4. So we shall directly verify that  $T_n(\infty)$  obtained via Theorem 3.3 actually satisfies the formula (29) in Theorem 3.4 in the case of  $\mathbb{C}^2$ . By Theorem 3.4,  $Y_n$  is obtained as

$$Y_n = \begin{pmatrix} n & & & 0 \\ & n-2 & & & \\ & & \ddots & & \\ & & & -(n-2) & \\ 0 & & & & -n \end{pmatrix}$$

The embedding  $Y_n(\infty)$  for  $Y_n$  is as in the way of (20). Thus,

$$T_{n}(\infty) Y_{n}(\infty) = \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \begin{pmatrix} n\binom{n}{0} & 0 & \\ & (n-2)\binom{n}{1} & & \\ & \ddots & & \\ & & -(n-2)\binom{n}{n-1} & \\ 0 & & & -n\binom{n}{n} & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}.$$

On the other hand,  $\hbar C_{n}^{\prime}(\infty)$  is calculated as

$$\hbar C'_{n}(\infty) = \hbar \left(\frac{1}{2}T_{n-1}(\infty) - \frac{1}{2}F_{d}(\infty)T_{n-1}(\infty)F_{r}(\infty)\right).$$

We substitute (37) for n - 1, (22), and (23) into the the right-hand side of the above. Then,

$$\left(\frac{\hbar}{2}\right)^{n} \frac{1}{(n-1)!} \left\{ \begin{pmatrix} \binom{n-1}{0} & 0 \\ & \ddots & & \\ & \binom{n-1}{n-1} & \\ 0 & 0 \\ & & 0 \end{pmatrix} - \begin{pmatrix} 0 & \binom{n-1}{0} & & \\ & \ddots & & \\ 0 & & \binom{n-1}{n-1} & \\ 0 & & \binom{n-1}{n-1} & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix} \right\}$$
$$= \left(\frac{\hbar}{2}\right)^{n} \frac{1}{n!} \begin{pmatrix} \binom{n\binom{n}{0}}{0} & & & \\ & (n-2)\binom{n}{1} & & & \\ 0 & & & \ddots & \\ & & \ddots & \\ 0 & & & & -(n-2)\binom{n}{n-1} & \\ & & & & \\ 0 & & & & \ddots \end{pmatrix}$$

Hence, for any  $n \in \mathbb{Z}_{\geq 0}$ , the matrix  $T_n(\infty)$  for  $\mathbb{C}^2$  satisfies (30).

### 4.3 Example of $\mathbb{C}P^2$

In this subsection, we confirm that the deformation quantization with separation of variables for  $\mathbb{C}P^2$  obtained via our main result (Theorem 3.3) satisfies the identities in Theorem 3.4 or equivalently (29) in Subsection 3.2.

**Proposition 4.4.** The formula for determining a star product with separation of variables on  $\mathbb{C}P^2$  is given by

$$T_{n}(\infty) = \frac{\Gamma\left(1-n+\frac{1}{\hbar}\right)}{n!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \left(F_{d}(\infty)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}(\infty) \left(F_{r}(\infty)\right)^{\gamma\left(\mu^{(n)}\right)}, \quad (43)$$

where each  $g_{\overline{\mu_k}\nu_k}$  is Fubini-Study metric and  $\gamma(\cdot): \{1,2\}^n \to \mathbb{Z}_{\geq 0}$  is defined by

$$\gamma\left(\mu^{(n)}\right) := \sum_{k=1}^{2} \left[\frac{\mu_{k}}{2}\right] = \sum_{k=1}^{2} \delta_{\mu_{k}2}, \qquad \gamma\left(\nu^{(n)}\right) := \sum_{k=1}^{2} \left[\frac{\nu_{k}}{2}\right] = \sum_{k=1}^{2} \delta_{\nu_{k}2}$$
(44)

for multi-indices  $\mu^{(n)} = (\mu_1, \cdots, \mu_n), \nu^{(n)} = (\nu_1, \cdots, \nu_n).$ 

We can prove this proposition by direct calculations of (26) in Theorem 3.3. See [47] for detailed proof of it. By Proposition 4.4, the star product with separation of variables on  $\mathbb{C}P^2$  is obtained by

$$f * g = \sum_{n=0}^{\infty} \frac{\Gamma\left(1 - n + \frac{1}{\hbar}\right) g_{\overline{\mu_1}\nu_1} \cdots g_{\overline{\mu_n}\nu_n}}{n!\Gamma\left(1 + \frac{1}{\hbar}\right)} \left(D^{\nu_1} \cdots D^{\nu_n}f\right) \left(D^{\overline{\mu_1}} \cdots D^{\overline{\mu_n}}g\right).$$
(45)

Detailed calculations for deriving equation (45) from (43) are also given in [47]. This star product coincides with (36) by Sako-Suzuki-Umetsu. This result implies that Theorem 3.3 works well.

As in the case of  $\mathbb{C}^2$ , we shall see that (43) satisfies (29) in Theorem 3.4. By Theorem 3.4, the components of  $Y_n$  for  $\mathbb{C}P^2$  are calculated as

$$Y_{i,j}^{n} = \begin{cases} \hbar \left\{ n - 2 \left( i - 1 \right) \right\} \left( 1 - n + \frac{1}{\hbar} \right) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

 $Y_n(\infty)$  is written as

$$Y_{n}(\infty) = \hbar \left( 1 - n + \frac{1}{\hbar} \right) \begin{pmatrix} n & 0 \\ n - 2 & & \\ & \ddots & & \\ & & -(n - 2) & \\ 0 & & -n & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}.$$
(46)

To calculate  $T_n(\infty) Y_n(\infty)$  we use the following proposition.

Proposition 4.5.

$$(F_{d}(\infty))^{k} T_{0}(\infty) (F_{r}(\infty))^{l} \begin{pmatrix} n & 0 \\ n-2 & & \\ & \ddots & \\ & -(n-2) \\ 0 & & -n \\ & & 0 \\ & & & 0 \end{pmatrix}$$
$$= (n-2l) (F_{d}(\infty))^{k} T_{0}(\infty) (F_{r}(\infty))^{l}, \qquad (k,l=0,\cdots,n).$$

**<u>Proof.</u>** Since the properties of  $F_d(\infty)$  and  $F_r(\infty)$  in Remark 3.2,

$$\left(F_d\left(\infty\right)\right)^k T_0\left(\infty\right) \left(F_r\left(\infty\right)\right)^l = \left(\begin{array}{c|c} (\delta_{i,k}\delta_{j,l}) & 0\\ \hline 0 & 0 \end{array}\right),\tag{47}$$

where  $(\delta_{i,k}\delta_{j,l}) \in M_{n+1}(\mathbb{C}[\![\hbar]\!])$  is the square matrix of order n+1 such that the (k,l) component is 1 and the others are 0. And note that the indices k and l are fixed. Then we get the equation.

By Proposition 4.5,  $T_n(\infty) Y_n(\infty)$  is calculated as

$$T_n(\infty) Y_n(\infty)$$

$$= \frac{\hbar \left(1 - n + \frac{1}{\hbar}\right) \Gamma \left(1 - n + \frac{1}{\hbar}\right)}{n! \Gamma \left(1 + \frac{1}{\hbar}\right)} \sum_{\mu_1, \nu_1, \cdots, \mu_n, \nu_n = 1}^2 g_{\overline{\mu_1}\nu_1} \cdots g_{\overline{\mu_n}\nu_n}$$

$$\cdot (F_{d}(\infty))^{\gamma(\nu^{(n)})} T_{0}(\infty) (F_{r}(\infty))^{\gamma(\mu^{(n)})} \begin{pmatrix} n & 0 \\ n-2 & & \\ & \ddots & \\ & -(n-2) \\ 0 & & n \\ & & 0 \\ & & & 0 \\ & & & \ddots \end{pmatrix}$$

$$= \frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\substack{\mu_{1},\cdots,\mu_{n}=1\\\nu_{1},\cdots,\nu_{n}=1}}^{2} g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \left(1-\frac{2}{n}\gamma\left(\mu^{(n)}\right)\right) (F_{d}(\infty))^{\gamma(\nu^{(n)})} T_{0}(\infty) (F_{r}(\infty))^{\gamma(\mu^{(n)})}.$$

$$(48)$$

Next, we calculate the right-hand side of (29). By Theorem 3.4,

$$\hbar C_{n}'(\infty) = \hbar \left( g_{\bar{1}1} T_{n-1}(\infty) + g_{\bar{1}2} F_{d}(\infty) T_{n-1}(\infty) - g_{\bar{2}1} T_{n-1}(\infty) F_{r}(\infty) - g_{\bar{2}2} F_{d}(\infty) T_{n-1}(\infty) F_{r}(\infty) \right),$$

$$= \frac{\hbar \Gamma \left( 1 - (n-1) + \frac{1}{\hbar} \right)}{(n-1)! \Gamma \left( 1 + \frac{1}{\hbar} \right)} \sum_{p_{n}=1}^{4} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n-1},\nu_{n-1}=1}^{2} g_{\overline{\mu_{1}}\nu_{1}} \cdots g_{\overline{\mu_{n-1}}\nu_{n-1}} \left( \Theta_{1}^{p_{n}} \mathrm{Id}(\infty) + \Theta_{2}^{p_{n}} F_{d}(\infty) \right) \\
\cdot \left( F_{d}(\infty) \right)^{\gamma \left( \nu^{(n-1)} \right)} T_{0}(\infty) \left( F_{r}(\infty) \right)^{\gamma \left( \mu^{(n-1)} \right)} \left( \Theta_{\overline{1}}^{p_{n}} \mathrm{Id}(\infty) - \Theta_{\overline{2}}^{p_{n}} F_{r}(\infty) \right).$$
(49)

Here  $\Theta_{\bar{\mu}}^{i_k}, \Theta_{\nu}^{i_k} \in C^{\infty}(U)$  are the  $C^{\infty}$  functions such that  $g_{\bar{\mu}\nu} = \sum_{i_k=1}^4 \Theta_{\bar{\mu}}^{i_k} \Theta_{\nu}^{i_k}$ . We can choose  $\Theta_{\bar{\mu}}^{i_k}$  and  $\Theta_{\nu}^{i_k}$  concretely [47], but we do not have to. This is because we can carry out the following calculations without using concrete expressions. The right-hand side of (49) can be rewritten as

$$\sum_{p_n=1}^{4} \left(\Theta_1^{p_n} \mathrm{Id}\left(\infty\right) + \Theta_2^{p_n} F_d\left(\infty\right)\right) \left(F_d\left(\infty\right)\right)^{\gamma\left(\nu^{(n-1)}\right)} T_0\left(\infty\right) \left(F_r\left(\infty\right)\right)^{\gamma\left(\mu^{(n-1)}\right)} \left(\Theta_{\bar{1}}^{p_n} \mathrm{Id}\left(\infty\right) - \Theta_{\bar{2}}^{p_n} F_r\left(\infty\right)\right)$$
$$= \sum_{\mu_n,\nu_n=1}^{2} \left(1 - 2\left[\frac{\mu_n}{2}\right]\right) g_{\overline{\mu_n}\nu_n} \left(F_d\left(\infty\right)\right)^{\left(\sum_{k=1}^{n-1}\left[\frac{\nu_k}{2}\right]\right) + \left[\frac{\nu_n}{2}\right]} T_0\left(\infty\right) \left(F_r\left(\infty\right)\right)^{\left(\sum_{k=1}^{n-1}\left[\frac{\mu_k}{2}\right]\right) + \left[\frac{\mu_n}{2}\right]}, \tag{50}$$

where we use  $\gamma\left(\mu^{(n-1)}\right) = \sum_{k=1}^{n-1} \left[\frac{\mu_k}{2}\right]$ ,  $\gamma\left(\nu^{(n-1)}\right) = \sum_{k=1}^{n-1} \left[\frac{\nu_k}{2}\right]$  and also introduce a new indices  $\mu_n$  and  $\nu_n$ .  $\left(1 - 2\left[\frac{\mu_n}{2}\right]\right)$  in the right-hand side of (50) is sign  $\pm 1$  using the Gauss symbol and -1 when  $\mu_n = 2$ . Then we obtain

$$\hbar C'_{n}(\infty) = \frac{\hbar \Gamma \left(1 - (n-1) + \frac{1}{\hbar}\right)}{(n-1)!\Gamma \left(1 + \frac{1}{\hbar}\right)} \left(1 - 2\left[\frac{\mu_{n}}{2}\right]\right) \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \\ \cdot (F_{d}(\infty))^{\sum_{k=1}^{n-1} \left(\left[\frac{\nu_{k}}{2}\right]\right) + \left[\frac{\nu_{n}}{2}\right]} T_{0}(\infty) (F_{r}(\infty))^{\sum_{k=1}^{n-1} \left(\left[\frac{\mu_{k}}{2}\right]\right) + \left[\frac{\mu_{n}}{2}\right]}.$$

Using

$$\gamma\left(\mu^{(n)}\right) = \left(\sum_{k=1}^{n-1} \left[\frac{\mu_k}{2}\right]\right) + \left[\frac{\mu_n}{2}\right], \qquad \gamma\left(\nu^{(n)}\right) = \left(\sum_{k=1}^{n-1} \left[\frac{\nu_k}{2}\right]\right) + \left[\frac{\nu_n}{2}\right],$$

we have

$$\frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-2\left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n-1}}\nu_{n-1}}g_{\overline{\mu_{n}}\nu_{n}} \\
\cdot \left(F_{d}\left(\infty\right)\right)^{\left(\sum_{k=1}^{n-1}\left[\frac{\nu_{k}}{2}\right]\right)+\left[\frac{\nu_{n}}{2}\right]} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\left(\sum_{k=1}^{n-1}\left[\frac{\mu_{k}}{2}\right]\right)+\left[\frac{\mu_{n}}{2}\right]} \\
= \frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-2\left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \\
= \frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-\frac{2}{n}\cdot n\left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \\
\cdot \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)}.$$
(51)

By replacing the indices, the above is expressed as

$$\sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1 - \frac{2}{n} \cdot n \left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}} \cdots g_{\overline{\mu_{n}}\nu_{n}} \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \right)$$

$$= \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1 - \frac{2}{n} \left[\frac{\mu_{n}}{2}\right] - \cdots - \frac{2}{n} \left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}} \cdots g_{\overline{\mu_{n}}\nu_{n}} \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \right)$$

$$= \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1 - \frac{2}{n} \left[\frac{\mu_{1}}{2}\right] - \cdots - \frac{2}{n} \left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}} \cdots g_{\overline{\mu_{n}}\nu_{n}} \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \right).$$
(52)

Substituting (52) into (51), we have

$$\frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-\frac{2}{n}\cdot n\left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \\
\cdot \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \\
= \frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-\frac{2}{n}\left[\frac{\mu_{1}}{2}\right]-\cdots-\frac{2}{n}\left[\frac{\mu_{n}}{2}\right]\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \\
\cdot \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)} \\
= \frac{\hbar\Gamma\left(1-(n-1)+\frac{1}{\hbar}\right)}{(n-1)!\Gamma\left(1+\frac{1}{\hbar}\right)} \sum_{\mu_{1},\nu_{1},\cdots,\mu_{n},\nu_{n}=1}^{2} \left(1-\frac{2}{n}\gamma\left(\mu^{(n)}\right)\right) g_{\overline{\mu_{1}}\nu_{1}}\cdots g_{\overline{\mu_{n}}\nu_{n}} \\
\cdot \left(F_{d}\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_{0}\left(\infty\right) \left(F_{r}\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)}.$$

Hence,

$$\hbar C_n'(\infty) = \frac{\hbar \Gamma \left(1 - (n-1) + \frac{1}{\hbar}\right)}{(n-1)! \Gamma \left(1 + \frac{1}{\hbar}\right)} \sum_{\mu_1, \nu_1, \cdots, \mu_n, \nu_n = 1}^2 \left(1 - \frac{2}{n} \gamma \left(\mu^{(n)}\right)\right) g_{\overline{\mu_1}\nu_1} \cdots g_{\overline{\mu_n}\nu_n}$$

$$\cdot \left(F_d\left(\infty\right)\right)^{\gamma\left(\nu^{(n)}\right)} T_0\left(\infty\right) \left(F_r\left(\infty\right)\right)^{\gamma\left(\mu^{(n)}\right)}.$$
(53)

From (48) and (53) it is shown that (43) satisfies (29) in Subsection 3.2 for the  $\mathbb{C}P^2$  case.

### 5 Discussions

In this paper, we obtained the explicit star products with separation of variables for two-dimensional locally symmetric Kähler manifolds by solving the recurrence relations in [22, 23]. By using Theorem 3.3, we can construct a deformation quantization with separation of variables for any complex two-dimensional locally symmetric Kähler manifold. Furthermore, we gave the noncommutative  $\mathbb{C}^2$  and  $\mathbb{C}P^2$  as concrete examples. We verified these noncommutative  $\mathbb{C}^2$  and  $\mathbb{C}P^2$  coincide with the previous results by Karabegov [30] and Sako-Suzuki-Umetsu [54, 55], respectively.

In this section, we discuss what can be done in the future using the theorems obtained in this paper. We give an example of a two-dimensional symmetric Kähler manifold to which Theorem 3.3 is applicable and for which deformation quantization has not yet been obtained. The concrete example of two dimensional locally symmetric Kähler manifold is a quadric surface  $Q^2(\mathbb{C}) = SO(4)/(SO(2) \times U(1))$ . However, the deformation quantization for  $Q^2(\mathbb{C})$  have not yet been constructed. It is known that the Kähler potential of  $Q^2(\mathbb{C})$  is given by

$$\Phi_{Q^2(\mathbb{C})} = \log\left(1 + \sum_{k=1}^2 |z^k|^2 + \frac{1}{4} \sum_{k,l=1}^2 (\overline{z}^k)^2 (z^l)^2\right).$$
(54)

See [18, 26, 43] for more detail. Since Kähler metric and curvature are expressed by using a Kähler potential, it is possible to explicitly construct the noncommutative  $Q^2(\mathbb{C})$  by using Theorem 3.3, explicitly. Supersymmetric nonlinear sigma models whose target spaces are Kähler manifolds are studied by Higashijima-Nitta [26, 27, 28], Higashijima-Kimura-Nitta [25] and Kondo-Takahashi [34]. Some models of them are defined on  $\mathbb{C}P^N$  and  $Q^2(\mathbb{C})$  as the target spaces. It should be possible to extend the nonlinear sigma models to ones on noncommutative  $\mathbb{C}P^N$  and  $Q^2(\mathbb{C})$ , naively.

We can expect to develop the physics on noncommutative locally symmetric Kähler manifolds from our result. For example, the constructions of some field theories and gauge theories on noncommutative ones are expected. Someone might think that in contrast to strict deformation quantization, formal deformation quantization is difficult to interpret some physics. However, it is possible to construct Fock representations from the formal deformation quantization in our cases. In fact, (twisted) Fock representations can be constructed by using a deformation quantization with separation of variables as discussed in Section 2. See [54, 55, 57, 58, 59] for more detailed discussions. Recalling that field theories can be described by using Fock representations, we can expect to propose the field theories on noncommutative Kähler manifold. In addition, it is expected to clarify the relationship between fuzzy manifolds and deformation quantization, since Fock representations can be interpreted by using a matrix representation. Furthermore, we can concretely construct gauge theories by using (twisted) Fock representations. They have been already studied by Maeda-Sako-Suzuki-Umetsu [39] and Sako-Suzuki-Umetsu [56] on noncommutative homogeneous Kähler manifolds. See [53] for the review of these facts. For example, if the noncommutative  $Q^2$  ( $\mathbb{C}$ ) are given by Theorem 3.3, we can propose gauge theories on noncommutative  $Q^2$  ( $\mathbb{C}$ ). As described above, it is expected that various physical theories on complex two-dimensional noncommutative locally symmetric Kähler manifold can be obtained by using the Theorem 3.3. They are left for future work.

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### A Some properties from Kähler geometry

We review some properties for Kähler manifolds that we use in this paper. See Kobayashi-Nomizu for more details [33]. Let M be a complex N-dimensional Kähler manifold, U be a holomorphic coordinate neighborhood of M, and  $\nabla$  be the Levi-Civita connection on M. For  $\partial_B, \partial_C \in \Gamma(TM \mid_U)$ , the Christoffel symbol  $\Gamma_{BC}^A$  is defined by  $\nabla_{\partial_B}\partial_C = \Gamma_{BC}^A\partial_A$ , where  $A, B, C \in \{1, \dots, N, \overline{1}, \dots, \overline{N}\}$ ,  $\partial_A := \frac{\partial}{\partial z^A}$ , and  $z^{\overline{i}} := \overline{z}^i$  for  $i \in \{1, \dots, N\}$ . In particular,  $\Gamma_{BC}^A$  is given by

$$\Gamma^{A}_{BC} = \frac{1}{2} g^{AD} \left( \partial_{B} g_{DC} + \partial_{C} g_{DB} - \partial_{D} g_{BC} \right)$$

by using the components of the Kähler metric g. Since M is Kähler, the non-trivial Christoffel symbols are  $\Gamma_{ij}^k$  and  $\Gamma_{ij}^{\overline{k}}$ , where  $i, j, k \in \{1, \dots, N\}$ . Furthermore, the covariant derivative for (k, l)-tensor field  $Y_{B_1 \dots B_l}^{A_1 \dots A_k} \partial_{A_1} \otimes \dots \otimes \partial_{A_k} \otimes dz^{B_1} \otimes \dots \otimes dz^{B_l} \in \Gamma((TM)^{\otimes k} \otimes (T^*M)^{\otimes l})$  is given by using the Christoffel symbol as follows :

$$\nabla_{\partial_C} \left( Y_{B_1 \cdots B_l}^{A_1 \cdots A_k} \partial_{A_1} \otimes \cdots \otimes \partial_{A_k} \otimes dz^{B_1} \otimes \cdots \otimes dz^{B_l} \right) \\ = \left( \partial_C Y_{\nu_1 \cdots \nu_l \overline{\nu_1} \cdots \overline{\nu_n}}^{\mu_1 \cdots \overline{\mu_m}} + \sum_{q=1}^k \Gamma_{CD}^{A_q} Y_{B_1 \cdots B_l}^{A_1 \cdots A_{q-1} D A_{q+1} \cdots A_k} - \sum_{q=1}^l \Gamma_{CB_q}^D Y_{B_1 \cdots B_{q-1} D B_{q+1} \cdots B_l}^{A_1 \cdots A_k} \right) \\ \times \partial_{A_1} \otimes \cdots \otimes \partial_{A_k} \otimes dz^{B_1} \otimes \cdots \otimes dz^{B_l},$$

where  $A_1, \dots, A_k, B_1, \dots, B_l, C, D \in \{1, \dots, N, \overline{1}, \dots, \overline{N}\}$ . Next, we define the Riemann curvature tensor  $R^{\nabla} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$  on M for the vector fields  $X, Y \in \Gamma(TM)$  and its component  $R^{\nabla}(\partial_A, \partial_B) \partial_C$  by

$$R^{\nabla}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$
  
$$R^{\nabla}(\partial_A, \partial_B) \partial_C := R_{ABC}{}^D \partial_D,$$

respectively. Note that the notation  $R_{ABC}^{D}$  used in this paper can be expressed by the relation

$$R_{ABC}{}^D = \mathfrak{R}^D_{\ CAB} \tag{55}$$

using the notation

$$R^{\nabla}\left(\partial_{A},\partial_{B}\right)\partial_{C} := \mathfrak{R}^{D}_{CAB}\partial_{D}$$

by Kobayashi-Nomizu [33]. In this paper, we fix the position of the indices of the components of the Riemann curvature tensor in the above equation. The components  $R_{ABC}^{\ \ D}$  are also given by

$$R_{ABC}{}^D = \partial_A \Gamma^D_{BC} - \partial_B \Gamma^D_{AC} + \Gamma^E_{BC} \Gamma^D_{AE} - \Gamma^E_{AC} \Gamma^D_{BE}$$

by using the Christoffel symbols, and their non-trivial ones are

$$R_{i\overline{j}k}{}^l = -\partial_{\overline{j}}\Gamma_{ik}^l,$$
$$R_{i\overline{j}k}{}^{\overline{l}} = \partial_i\Gamma_{\overline{j}k}^{\overline{l}}.$$

For  $R_{ABCD} = g_{DE} R_{ABC}^{E}$ , it is also confirmed that the non-trivial components are  $R_{i\bar{j}k\bar{l}}$  and  $R_{i\bar{j}k\bar{l}}$ , which are given by

$$\begin{split} R_{i\overline{j}k\overline{l}} &= -g_{\overline{l}p}\partial_{\overline{j}}\Gamma^p_{ik} = -\partial_i\partial_{\overline{j}}\partial_k\partial_{\overline{l}}\Phi + g^{p\overline{q}}\left(\partial_i\partial_{\overline{q}}\partial_k\Phi\right)\left(\partial_p\partial_{\overline{j}}\partial_{\overline{l}}\Phi\right),\\ R_{i\overline{j}\overline{k}l} &= g_{l\overline{p}}\partial_i\Gamma^{\overline{p}}_{\overline{j}\overline{k}} = \partial_{\overline{j}}\partial_i\partial_{\overline{k}}\partial_l\Phi - g^{\overline{p}q}\left(\partial_{\overline{j}}\partial_q\partial_{\overline{k}}\Phi\right)\left(\partial_{\overline{p}}\partial_i\partial_l\Phi\right). \end{split}$$

respectively, where  $\Phi$  is the Kähler potential, i.e.  $\Phi$  is a function on M such that  $g_{i\overline{j}} = \partial_i \partial_{\overline{j}} \Phi$ . Here we consider  $R_{\overline{i}} {}^{\overline{k}\overline{l}}_{\overline{c}} = g^{\overline{k}p} g^{\overline{l}q} R_{\overline{i}pq\overline{c}}$  which often appears in this paper. We refer to  $R_{\overline{i}} {}^{\overline{k}\overline{l}}_{\overline{c}}$  simply as "curvature". This curvature  $R_{\overline{i}} {}^{\overline{k}\overline{l}}_{\overline{c}}$  has the following symmetries concerning the indices  $\overline{i}, \overline{c}, \overline{k}$ , and  $\overline{l}$ :

$$R_{\overline{i}}{}^{\overline{k}\overline{l}}{}_{\overline{c}} = R_{\overline{c}}{}^{\overline{k}\overline{l}}{}_{\overline{i}} = R_{\overline{i}}{}^{\overline{l}\overline{k}}{}_{\overline{c}} = R_{\overline{c}}{}^{\overline{l}\overline{k}}{}_{\overline{i}}.$$
(56)

This property plays an important role in this paper.

### **B** Calculations for both sides of (31) in Subsection 3.2

We show that (31) for n = 2 holds in Subsection 3.2. In this appendix, we denote the detailed calculations for each component of both sides of (31) in Subsection 3.2. They can be enumerated as follows.

The left-hand side

$$(1,1) = \hbar^{2} \left\{ \left( 2 + \hbar R_{1}^{11} \right) (g_{\overline{1}1})^{2} + 2\hbar R_{1}^{21} g_{\overline{1}1} g_{\overline{1}2} + \hbar R_{1}^{22} (g_{\overline{1}2})^{2} \right\}$$
  
=  $\hbar^{2} \left\{ 2 (g_{\overline{1}1})^{2} + \hbar R_{1\overline{1}\overline{1}1} \right\},$  (57)

$$(1,2) = 2\hbar^{2} \left\{ \left( 2 + \hbar R_{1}^{11} \right) g_{\overline{1}1} g_{\overline{2}1} + \hbar R_{1}^{21} \left( g_{\overline{1}1} g_{\overline{2}2} + g_{\overline{1}2} g_{\overline{2}1} \right) + \hbar R_{1}^{22} g_{\overline{2}2} \right\} \\ = 2\hbar^{2} \left\{ g_{\overline{1}1} g_{\overline{2}1} + \hbar R_{1} \overline{2}\overline{1}_{1} \right\},$$

$$(58)$$

$$(1,3) = \hbar^{2} \left\{ \left( 2 + \hbar R_{1}^{11} \right) (g_{\overline{2}1})^{2} + 2\hbar R_{1}^{21} g_{\overline{2}1} g_{\overline{2}2} + \hbar R_{1}^{22} (g_{\overline{2}2})^{2} \right\} = \hbar^{2} \left\{ 2 (g_{\overline{2}1})^{2} + \hbar R_{1\overline{2}\overline{2}1} \right\},$$
(59)

$$(2,1) = 0,$$
 (60)

$$(2,2) = 0, (61)$$

$$(2,3) = 0,$$

$$(3,1) = -\hbar^{2} \left\{ \hbar R_{2}^{11} (g_{\overline{1}1})^{2} + 2\hbar R_{2}^{21} g_{\overline{1}1} g_{\overline{1}2} + \left(2 + \hbar R_{2}^{22} \right) (g_{\overline{1}2})^{2} \right\}$$

$$= -\hbar^{2} \left\{ 2 (g_{\overline{1}2})^{2} + \hbar R_{2\overline{1}\overline{1}2} \right\},$$

$$(63)$$

$$(3,2) = -2\hbar^{2} \left\{ \hbar R_{2}^{11} g_{\overline{2}1} + \hbar R_{2}^{21} (g_{\overline{1}1} g_{\overline{2}2} + g_{\overline{1}2} g_{\overline{2}1}) + (2 + \hbar R_{2}^{22}) g_{\overline{1}2} g_{\overline{2}2} \right\} = -2\hbar^{2} \left\{ 2g_{\overline{1}2} g_{\overline{2}2} + \hbar R_{2\overline{2}\overline{1}2} \right\},$$

$$(64)$$

$$(3,3) = -\hbar^{2} \left\{ \hbar R_{2}^{11} (g_{\overline{2}1})^{2} + 2\hbar R_{2}^{21} g_{\overline{2}1} g_{\overline{2}2} + \left(2 + \hbar R_{2}^{22} \right) (g_{\overline{2}2})^{2} \right\}$$
$$= -\hbar^{2} \left\{ 2 (g_{\overline{2}2})^{2} + \hbar R_{2\overline{2}\overline{2}2} \right\}.$$
(65)

The right-hand side

$$(1,1) = \hbar^{2} \left\{ \left( 2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}}_{\bar{1}} \right) (g_{\bar{1}1})^{2} + 2\hbar R_{\bar{1}}^{\bar{2}\bar{1}}_{\bar{1}} g_{\bar{1}1} g_{\bar{2}1} + \hbar R_{\bar{1}}^{\bar{2}\bar{2}}_{\bar{1}} (g_{\bar{2}1})^{2} \right\} = \hbar^{2} \left\{ 2 (g_{\bar{1}1})^{2} + \hbar R_{1\bar{1}\bar{1}1} \right\},$$
(66)

$$(1,2) = 2\hbar^{2} \left\{ \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} (g_{\bar{1}1})^{2} + \left( 2 + 2\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} \right) g_{\bar{1}1} g_{\bar{2}1} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} (g_{\bar{2}1})^{2} \right\} = 2\hbar^{2} \left\{ g_{\bar{1}1} g_{\bar{2}1} + \hbar R_{1\bar{2}\bar{1}1} \right\},$$

$$(67)$$

$$(1,3) = \hbar^{2} \left\{ \hbar R_{\bar{2}}^{\bar{1}\bar{1}} (g_{\bar{1}1})^{2} + 2\hbar R_{\bar{2}}^{\bar{2}\bar{1}} g_{\bar{1}1} g_{\bar{2}1} + \left(2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}}\right) (g_{\bar{2}1})^{2} \right\} = \hbar^{2} \left\{ 2 (g_{\bar{2}1}) + \hbar R_{1\bar{2}\bar{2}1} \right\},$$

$$(68)$$

$$(2,1) = 0, (69) (70)$$

$$(2,2) = 0, (70) (2,3) = 0, (71)$$

$$(3,1) = -\hbar^{2} \left\{ \left( 2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}} \right) (g_{\bar{1}2})^{2} + 2\hbar R_{\bar{1}}^{\bar{2}\bar{1}}{}_{\bar{1}}g_{\bar{1}2}g_{\bar{2}2} + \hbar R_{\bar{1}}^{\bar{2}\bar{2}}{}_{\bar{1}} (g_{\bar{2}2})^{2} \right\} = -\hbar^{2} \left\{ 2 (g_{\bar{1}2})^{2} + \hbar R_{2\bar{1}\bar{1}2} \right\},$$

$$(72)$$

$$(3,2) = -2\hbar^{2} \left\{ \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}_{\bar{1}} (g_{\bar{1}2})^{2} + \left( 2 + 2\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}_{\bar{1}} \right) g_{\bar{1}2} g_{\bar{2}2} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}_{\bar{1}} (g_{\bar{2}2})^{2} \right\} = -2\hbar^{2} \left\{ 2g_{\bar{1}2} g_{\bar{2}2} + \hbar R_{2\bar{2}\bar{1}2} \right\},$$

$$(73)$$

$$(3,3) = -\hbar^{2} \left\{ \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}} (g_{\bar{1}2})^{2} + 2\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}} g_{\bar{1}2} g_{\bar{2}2} + \left(2 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\right) (g_{\bar{2}2})^{2} \right\} = -\hbar^{2} \left\{ 2 (g_{\bar{2}2})^{2} + \hbar R_{2\bar{2}\bar{2}2} \right\}.$$

$$(74)$$

Note that we denote each (i, j) component of both sides simply as (i, j). Hence, the calculations (57)–(74) show that (31) in Subsection 3.2 for n = 2 holds.

### C Hermiteness of $T_n$

In Subsection 3.2, we use the fact that  $T_n$  is Hermitian conjugate, i.e.  $T_n = T_n^{\dagger}$ . So we shall derive this property of  $T_n$  in this appendix.

**Lemma C.1** (Sako-Umetsu[57]). The coefficients  $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$  satisfy

$$T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}} = \overline{T^n_{\overrightarrow{\beta_n},\overrightarrow{\alpha_n^*}}}$$

or equivalently

$$T_n = T_n^{\dagger},\tag{75}$$

**Proof.** From Proposition 3.1. in [57],

$$\overline{f * g} = \overline{g} * \overline{f}.$$
(76)

Substituting (14) into the left-hand side of (76), we have

$$\overline{f * g} = \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n, \beta_n^*}} \overline{T_{\overrightarrow{\alpha_n, \beta_n^*}}^n} \left\{ \left( D^{\overline{1}} \right)^{\alpha_1^n} \cdots \left( D^{\overline{N}} \right)^{\alpha_N^n} \overline{f} \right\} \left\{ \left( D^{1} \right)^{\beta_1^n} \cdots \left( D^{N} \right)^{\beta_N^n} \overline{g} \right\}$$
$$= \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n, \beta_n^*}} \overline{T_{\overrightarrow{\beta_n, \alpha_n^*}}^n} \left( D^{\overrightarrow{\alpha_n}} \overline{g} \right) \left( D^{\overrightarrow{\beta_n^*}} \overline{f} \right).$$

Here we rewrote the dummy indices in the last equation. On the other hand, since

$$\overline{g} * \overline{f} = \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} \left( D^{\overrightarrow{\alpha_n}} \overline{g} \right) \left( D^{\overrightarrow{\beta_n^*}} \overline{f} \right)$$

by the assumption for the star product with separation of variables, we obtain

$$T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}} = \overline{T^n_{\overrightarrow{\beta_n},\overrightarrow{\alpha_n^*}}}$$

by comparing the coefficients on both sides of (76). The above equation can be expressed as an equivalent equation  $T_n = T_n^{\dagger}$ .

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