# Gauge conservation laws in a general setting. Superpotential 

G. SARDANASHVILY<br>Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

The fact that the conserved current of a gauge symmetry is reduced to a superpotential is proved in a very general setting.

## I. INTRODUCTION

The fact that the conserved current of a gauge symmetry is reduced to a superpotential has been stated in different particular variants, e.g., gauge theory of principal connections and gauge gravitation theory. ${ }^{1-4}$ We aim to prove this assertion in a very general setting.

Generic higher-order Lagrangian theory of even and odd fields on an $n$-dimensional smooth manifold $X$ and its variational generalized supersymmetries (henceforth symmetries) are considered. ${ }^{5-8}$ These symmetries form a real vector space $\mathcal{G}_{L}$. In a general setting, a gauge symmetry of a Lagrangian $L$ is defined as a $\mathcal{G}_{L}$-valued linear differential operator on some Grassmann-graded projective $C^{\infty}(X)$-module of finite rank. ${ }^{7,8}$ Note that any Lagrangian possesses gauge symmetries which therefore must be separated into the trivial and non-trivial ones. However, there is a problem of defining non-trivial gauge symmetries. ${ }^{7}$

In contrast with gauge symmetries, non-trivial Noether identities of Lagrangian field theory are well described in homology terms. ${ }^{6-8}$ Therefore, we define non-trivial gauge symmetries as those associated to complete non-trivial Noether identities in accordance with the second Noether theorem (Theorem 5).

Given a non-trivial gauge symmetry of a Lagrangian $L$, the corresponding current $\mathcal{J}$ (12) is conserved by virtue of the first Noether theorem (Theorem 3). We prove that this current takes the superpotential form

$$
\mathcal{J}^{\mu}=W^{\mu}+d_{\nu} U^{\nu \mu}
$$

where the term $W^{\mu}$ vanishes on the kernel of the Euler-Lagrange operator $\delta L(3)$ of $L$ and $U^{\nu \mu}=-U^{\mu \nu}$ is a superpotential (Theorem 6).

## II. LAGRANGIAN THEORY OF EVEN AND ODD FIELDS

Lagrangian theory of even and odd fields is adequately formulated in terms of the Grassmann-graded variational bicomplex on fiber bundles and graded manifolds. ${ }^{5,8,9}$ In a
very general setting, let us consider a composite bundle $F \rightarrow Y \rightarrow X$ where $F \rightarrow Y$ is a vector bundle provided with bundle coordinates $\left(x^{\lambda}, y^{i}, q^{a}\right)$. The jet manifolds $J^{r} F$ of $F \rightarrow X$ also are vector bundles $J^{r} F \rightarrow J^{r} Y$ coordinated by $\left(x^{\lambda}, y_{\Lambda}^{i}, q_{\Lambda}^{a}\right), 0 \leq|\Lambda| \leq r$, where $\Lambda=\left(\lambda_{1} \ldots \lambda_{k}\right),|\Lambda|=k$, denotes a symmetric multi-index. Let $\left(J^{r} Y, \mathcal{A}_{r}\right)$ be a graded manifold whose body is $J^{r} Y$ and whose structure ring $\mathcal{A}_{r}$ of graded functions consists of sections of the exterior bundle

$$
\wedge\left(J^{r} F\right)^{*}=\mathbb{R} \oplus\left(J^{r} F\right)^{*} \oplus{ }^{2}\left(J^{r} F\right)^{*} \oplus \cdots,
$$

where $\left(J^{r} F\right)^{*}$ is the dual of $J^{r} F \rightarrow J^{r} Y$. The local odd basis for this ring is $\left\{c_{\Lambda}^{a}\right\}, 0 \leq|\Lambda| \leq$ $r$. Let $\mathcal{S}_{r}^{*}[F ; Y]$ be the differential graded algebra (henceforth DGA) of graded differential forms on the graded manifold $\left(J^{r} Y, \mathcal{A}_{r}\right)$. The inverse system of jet manifolds $J^{r-1} Y \leftarrow J^{r} Y$ yields the direct system of DGAs

$$
\mathcal{S}^{*}[F ; Y] \longrightarrow \mathcal{S}_{1}^{*}[F ; Y] \longrightarrow \cdots \mathcal{S}_{r}^{*}[F ; Y] \longrightarrow \cdots
$$

Its direct limit $\mathcal{S}_{\infty}^{*}[F ; Y]$ is the DGA of all graded differential forms on graded manifolds $\left(J^{r} Y, \mathcal{A}_{r}\right)$. It is a $C^{\infty}(Y)$-algebra locally generated by elements $\left(c_{\Lambda}^{a}, d x^{\lambda}, d y_{\Lambda}^{i}, d c_{\Lambda}^{a}\right), 0 \leq|\Lambda|$. Let us recall the formulas

$$
\phi \wedge \phi^{\prime}=(-1)^{|\phi|\left|\phi^{\prime}\right|+[\phi]\left[\phi^{\prime}\right]} \phi^{\prime} \wedge \phi, \quad d\left(\phi \wedge \phi^{\prime}\right)=d \phi \wedge \phi^{\prime}+(-1)^{|\phi|} \phi \wedge d \phi
$$

where $[\phi]$ denotes the Grassmann parity. The collective symbol $\left(s^{A}\right)$ further stands for the tuple $\left(y^{i}, c^{a}\right)$, called the local basis for the DGA $\mathcal{S}_{\infty}^{*}[F ; Y]$. Let us denote $[A]=\left[s^{A}\right]=\left[s_{\Lambda}^{A}\right]$.

The DGA $\mathcal{S}_{\infty}^{*}[F ; Y]$ is split into the Grassmann-graded variational bicomplex of $\mathcal{S}_{\infty}^{0}[F ; Y]$ modules $\mathcal{S}_{\infty}^{k, r}[F ; Y]$ of $r$-horizontal and $k$-contact graded forms locally generated by the oneforms $d x^{\lambda}$ and $\theta_{\Lambda}^{A}=d s_{\Lambda}^{A}-s_{\lambda+\Lambda}^{A} d x^{\lambda}$. This bicomplex contains the variational subcomplex

$$
\begin{equation*}
0 \rightarrow \mathbb{R} \longrightarrow \mathcal{S}_{\infty}^{0}[F ; Y] \xrightarrow{d_{H}} \mathcal{S}_{\infty}^{0,1}[F ; Y] \cdots \xrightarrow{d_{H}} \mathcal{S}_{\infty}^{0, n}[F ; Y] \xrightarrow{\delta} \mathcal{S}_{\infty}^{1, n}[F ; Y], \tag{1}
\end{equation*}
$$

whose coboundary operator

$$
\begin{aligned}
& d_{H}(\phi)=d x^{\lambda} \wedge d_{\lambda} \phi=d x^{\lambda} \wedge\left(\partial_{\lambda}+\sum_{0 \leq|\Lambda|} s_{\lambda \Lambda}^{A} \partial_{A}^{\Lambda}\right) \phi, \\
& d_{H} \circ h_{0}=h_{0} \circ d, \quad h_{0}\left(\theta_{\Lambda}^{A}\right)=0, \quad h_{0}\left(d x^{\lambda}\right)=d x^{\lambda}
\end{aligned}
$$

is the total differential, and whose elements

$$
\begin{align*}
& L=\mathcal{L} \omega \in \mathcal{S}_{\infty}^{0, n}[F ; Y], \quad \omega=d x^{1} \wedge \cdots \wedge d x^{n},  \tag{2}\\
& \delta L=\theta^{A} \wedge \mathcal{E}_{A} \omega=\sum_{0 \leq|\Lambda|}(-1)^{|\Lambda|} \theta^{A} \wedge d_{\Lambda}\left(\partial_{A}^{\Lambda} \mathcal{L}\right) \omega, \quad d_{\Lambda}=d_{\lambda_{1}} \cdots d_{\lambda_{k}} \tag{3}
\end{align*}
$$

are graded Lagrangians and their Euler-Lagrange operators. Further, a pair $\left(\mathcal{S}_{\infty}^{*}[F ; Y], L\right)$ denotes Lagrangian field theory.

Cohomology of the Grassmann-graded variational bicomplex has been obtained. ${ }^{8,10}$ Let us mention the following relevant results.

Theorem 1: Cohomology of the variational complex (1) equals the de Rham cohomology of a fiber bundle $Y$.

In particular, any odd element of this complex possesses trivial cohomology.
Theorem 2: Given a graded Lagrangian $L$, there is the decomposition

$$
\begin{align*}
& d L=\delta L-d_{H} \Xi_{L}, \quad \Xi \in \mathcal{S}_{\infty}^{n-1}[F ; Y],  \tag{4}\\
& \left.\Xi_{L}=L+\sum_{s=0} \theta_{\nu_{s} \ldots \nu_{1}}^{A} \wedge F_{A}^{\lambda \nu_{s} \ldots \nu_{1}} \omega_{\lambda}, \quad \omega_{\lambda}=\partial_{\lambda}\right\rfloor \omega,  \tag{5}\\
& F_{A}^{\nu_{k} \ldots \nu_{1}}=\partial_{A}^{\nu_{k} \ldots \nu_{1}} \mathcal{L}-d_{\lambda} F_{A}^{\lambda \nu_{k} \ldots \nu_{1}}+\psi_{A}^{\nu_{k} \ldots \nu_{1}}, \quad k=1,2, \ldots,
\end{align*}
$$

where local graded functions $\psi$ obey the relations

$$
\psi_{A}^{\nu}=0, \quad \psi_{A}^{\left(\nu_{k} \nu_{k-1}\right) \ldots \nu_{1}}=0
$$

The form $\Xi_{L}(5)$ provides a global Lepage equivalent of a graded Lagrangian $L$. In particular, one can locally choose $\Xi_{L}(5)$ where all functions $\psi$ vanish.

The corollaries of Theorem 2 are the first variational formula (9) and the first Noether theorem (Theorem 3).

## III. THE FIRST NOETHER THEOREM

In order to treat symmetries of Lagrangian field theory $\left(\mathcal{S}_{\infty}^{*}[F ; Y], L\right)$ in a very general setting, we consider graded derivations of the $\mathbb{R}$-ring $\mathcal{S}_{\infty}^{0}[F ; Y]{ }^{5,8}$ They take the form

$$
\begin{equation*}
\left.\vartheta=\vartheta^{\lambda} \partial_{\lambda}+\sum_{0 \leq|\Lambda|} \vartheta_{\Lambda}^{A} \partial_{A}^{\Lambda}, \quad \partial_{A}^{\Lambda}\left(s_{\Sigma}^{B}\right)=\partial_{A}^{\Lambda}\right\rfloor d s_{\Sigma}^{B}=\delta_{A}^{B} \delta_{\Sigma}^{\Lambda} \tag{6}
\end{equation*}
$$

Any graded derivation $\vartheta(6)$ yields the Lie derivative

$$
\begin{aligned}
& \left.\left.\mathbf{L}_{\vartheta} \phi=\vartheta\right] d \phi+d(\vartheta] \phi\right), \quad \phi \in \mathcal{S}_{\infty}^{*}[F ; Y] \\
& \mathbf{L}_{\vartheta}(\phi \wedge \sigma)=\mathbf{L}_{\vartheta}(\phi) \wedge \sigma+(-1)^{[\vartheta][\phi]} \phi \wedge \mathbf{L}_{\vartheta}(\sigma),
\end{aligned}
$$

of the DGA $\mathcal{S}_{\infty}^{*}[F ; Y]$.

A graded derivation $\vartheta(6)$ is called contact if the Lie derivative $\mathbf{L}_{\vartheta}$ preserves the ideal of contact graded forms of the DGA $\mathcal{S}_{\infty}^{*}[F ; Y]$. Any contact graded derivation admits the decomposition

$$
\begin{equation*}
\vartheta=v_{H}+v_{V}=v^{\lambda} d_{\lambda}+\left[v^{A} \partial_{A}+\sum_{|\Lambda|>0} d_{\Lambda}\left(v^{A}-s_{\mu}^{A} v^{\mu}\right) \partial_{A}^{\Lambda}\right] \tag{7}
\end{equation*}
$$

into the horizontal and vertical parts $v_{H}$ and $v_{V}$. A glance at the expression (7) shows that a contact graded derivation $\vartheta$ is an infinite order jet prolongation of its restriction

$$
\begin{equation*}
v=v^{\lambda} \partial_{\lambda}+v^{A} \partial_{A} \tag{8}
\end{equation*}
$$

to the graded commutative ring $S^{0}[F ; Y]$. One calls $v(8)$ the generalized graded vector field. It is a graded vector field if its components $v^{\lambda}, v^{A}$ are independent of jets $s_{\Lambda}^{A}$. Note that generalized symmetries of Lagrangian systems have been intensively studied. ${ }^{5,11-13}$

Given a contact graded derivation (7), a corollary of the decomposition (4) is the above mentioned first variational formula

$$
\begin{equation*}
\left.\left.\left.\mathbf{L}_{\vartheta} L=v_{V}\right\rfloor \delta L+d_{H}\left(h_{0}(\vartheta\rfloor \Xi_{L}\right)\right)+d_{V}\left(v_{H}\right\rfloor \omega\right) \mathcal{L} \tag{9}
\end{equation*}
$$

where $\Xi_{L}$ is the Lepage equivalent (5) of $L$.
Given a Lagrangian $L(2)$, a contact graded derivation $\vartheta(7)$ is said to be its variational symmetry (strictly speaking a variational generalized supersymmetry) if the Lie derivative $\mathbf{L}_{\vartheta} L$ is $d_{H}$-exact, i.e.

$$
\begin{equation*}
\mathbf{L}_{\vartheta} L=d_{H} \sigma \tag{10}
\end{equation*}
$$

A variational symmetry $\vartheta$ of a Lagrangian $L$ is called its exact symmetry if the Lie derivative $\mathbf{L}_{\vartheta} L$ vanishes.

An immediate corollary of the first variational formula (9) is the following first Noether theorem.

Theorem 3: If a contact graded derivation $\vartheta(7)$ is a variational symmetry (10) of a Lagrangian $L$, the first variational formula (9) restricted to Ker $\delta L$ leads to the weak conservation law

$$
\begin{equation*}
\left.d_{H}\left(\sigma-h_{0}(\vartheta\rfloor \Xi_{L}\right)\right) \approx 0 \tag{11}
\end{equation*}
$$

of the current

$$
\begin{equation*}
\left.\mathcal{J}_{\vartheta}=\mathcal{J}_{\vartheta}^{\mu} \omega_{\mu}=\sigma-h_{0}(\vartheta\rfloor \Xi_{L}\right) \tag{12}
\end{equation*}
$$

Obviously, the conserved current (12) is defined up to a $d_{H}$-closed horizontal ( $n-1$ )-form

$$
\begin{equation*}
\left.U=\frac{1}{2} U^{\nu \mu} \omega_{\nu \mu}, \quad \omega_{\nu \mu}=\partial_{\nu}\right\rfloor \omega_{\mu} \tag{13}
\end{equation*}
$$

called the superpotential.
Lemma 4: A glance at the expression (9) shows the following. ${ }^{5}$
(i) A contact graded derivation $\vartheta$ is a variational symmetry only if the generalized vector field $v(8)$ is projected onto $X$, i.e., $v^{\lambda} \partial_{\lambda}$ is a vector field on $X$.
(ii) A contact graded derivation $\vartheta$ is a variational symmetry iff its vertical part $v_{V}$ is well.
(iii) Any projectable contact graded derivation is a variational symmetry of a variationally trivial Lagrangian.
(iv) A contact graded derivation $\vartheta$ is a variational symmetry iff the graded density $\left.v_{V}\right\rfloor \delta L$ is $d_{H}$-exact.

Variational symmetries of a Lagrangian $L$ constitute a real vector space $\mathcal{G}_{L}$. By virtue of item (iii) of Lemma 4, the Lie superbracket

$$
\mathbf{L}_{\left[\vartheta, \vartheta^{\prime}\right]}=\left[\mathbf{L}_{\vartheta}, \mathbf{L}_{\vartheta^{\prime}}\right]
$$

of variational symmetries is a variational symmetry. Consequently, the vector space $\mathcal{G}_{L}$ of variational symmetries is a real Lie superalgebra.

By virtue of item (ii) of Lemma 4, we further restrict our consideration to vertical contact graded derivations

$$
\begin{equation*}
\vartheta=v^{A} \partial_{A}+\sum_{0<|\Lambda|} d_{\Lambda} v^{A} \partial_{A}^{\Lambda} . \tag{14}
\end{equation*}
$$

A graded derivation $\vartheta(14)$ is called nilpotent if $\mathbf{L}_{\vartheta}\left(\mathbf{L}_{\vartheta} \phi\right)=0$ for any horizontal form $\phi \in \mathcal{S}_{\infty}^{0, *}[F ; Y]$. One can show that $\vartheta(14)$ is nilpotent only if it is odd and iff $\vartheta(v)=0 .{ }^{5}$

For the sake of brevity, the common symbol $v$ further stands for a generalized graded vector field $v=v^{A} \partial_{A}$, the vertical contact graded derivation $\vartheta$ (14) determined by $v$, and the Lie derivative $\mathbf{L}_{\vartheta}$. We agree to call $v$ the graded derivation of the DGA $\mathcal{S}_{\infty}^{*}[F ; Y]$. The right graded derivations $\overleftarrow{v}=\overleftarrow{\partial}_{A} v^{A}$ of $\mathcal{S}_{\infty}^{*}[F ; Y]$ also are considered.

## IV. GAUGE SYMMETRIES

Without a loss of generality, let a Lagrangian $L$ be even. To describe Noether identities of Lagrangian field theory $\left(\mathcal{S}_{\infty}^{*}[F ; Y], L\right)$, let us introduce the following notation. Given a vector bundle $E \rightarrow X$, we call

$$
\bar{E}=E^{*} \otimes \wedge^{n} T^{*} X
$$

the density-dual of $E$. The density dual of a graded vector bundle $E=E^{0} \oplus E^{1}$ is $\bar{E}=$ $\bar{E}^{1} \oplus \bar{E}^{0}$. Given a graded vector bundle $E=E^{0} \oplus E^{1}$ over $Y$, we consider the composite
bundle $E \rightarrow E^{0} \rightarrow X$ and denote $\mathcal{P}_{\infty}^{*}[E ; Y]=\mathcal{S}_{\infty}^{*}\left[E ; E^{0}\right]$. Let $V F$ be the vertical tangent bundle of $F \rightarrow X$, the density-dual of the vector bundle $V F \rightarrow F$ is

$$
\overline{V F}=V^{*} F \underset{F}{\otimes} \wedge T^{*} X
$$

Let us enlarge $\mathcal{S}_{\infty}^{*}[F ; Y]$ to the DGA $\mathcal{P}_{\infty}^{*}[\overline{V F} ; Y]$ possessing the local basis $\left(s^{A}, \bar{s}_{A}\right)$, $\left[\bar{s}_{A}\right]=([A]+1) \bmod 2$. Its elements $\bar{s}_{A}$ are called antifields. The DGA $\mathcal{P}_{\infty}^{*}[\overline{V F} ; Y]$ is endowed with the odd right graded derivation $\bar{\delta}=\overleftarrow{\partial}^{A} \mathcal{E}_{A}$, where $\mathcal{E}_{A}$ are the variational derivatives (3). This graded derivation is obviously nilpotent. Then we have the chain complex

$$
\begin{equation*}
0 \leftarrow \operatorname{Im} \bar{\delta} \stackrel{\bar{\delta}}{\longleftarrow} \mathcal{P}_{\infty}^{0, n}[\overline{V F} ; Y]_{1} \stackrel{\bar{\delta}}{\longleftarrow} \mathcal{P}_{\infty}^{0, n}[\overline{V F} ; Y]_{2} \tag{15}
\end{equation*}
$$

of graded densities of antifield number $\leq 2$. Its one-cycles

$$
\begin{equation*}
\bar{\delta} \Phi=0, \quad \Phi=\sum_{0 \leq|\Lambda|} \Phi^{A, \Lambda} \bar{s}_{\Lambda A} d^{n} x \in \mathcal{P}_{\infty}^{0, n}[\overline{V F} ; Y]_{1} \tag{16}
\end{equation*}
$$

define Noether identities of Lagrangian field theory $\left(\mathcal{S}_{\infty}^{*}[F ; Y], L\right)$. In particular, one-chains $\Phi \in \mathcal{P}_{\infty}^{0, n}[\overline{V F} ; Y]_{1}$ are necessarily Noether identities if they are boundaries. Therefore, these Noether identities are called trivial. Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the first homology $H_{1}(\bar{\delta})$ of the complex (15). ${ }^{6,8}$

Let us assume that the homology $H_{1}(\bar{\delta})$ is finitely generated. Namely, there exists a projective $C^{\infty}(X)$-module $\mathcal{C} \subset H_{1}(\bar{\delta})$ of finite rank possessing the local basis $\left\{\Delta_{r}\right\}$ such that any element $\Phi \in H_{1}(\bar{\delta})$ factorizes as

$$
\begin{equation*}
\Phi=\sum_{0 \leq|\Xi|} G^{r, \Xi} d_{\Xi} \Delta_{r} d^{n} x, \quad \Delta_{r}=\sum_{0 \leq|\Lambda|} \Delta_{r}^{A, \Lambda} \bar{s}_{\Lambda A}, \quad G^{r, \Xi}, \Delta_{r}^{A, \Lambda} \in \mathcal{S}_{\infty}^{0}[F ; Y] \tag{17}
\end{equation*}
$$

through elements of $\mathcal{C}$. Thus, all non-trivial Noether identities (16) result from the Noether identities

$$
\begin{equation*}
\bar{\delta} \Delta_{r}=\sum_{0 \leq|\Lambda|} \Delta_{r}^{A, \Lambda} d_{\Lambda} \mathcal{E}_{A}=0 \tag{18}
\end{equation*}
$$

called the complete Noether identities. By virtue of the generalized Serre-Swan theorem, ${ }^{8}$ the module $\mathcal{C}$ is isomorphic to the $C^{\infty}(X)$-module of sections of the density-dual $\bar{E}$ of some graded vector bundle $E \rightarrow X$.

We define a non-trivial gauge symmetry of Lagrangian field theory $\left(\mathcal{S}_{\infty}^{*}[F ; Y], L\right)$ as that associated to the Noether identities (18) by means of the inverse second Noether theorem. ${ }^{6-8}$

Let us enlarge the DGA $\mathcal{P}_{\infty}^{*}[\overline{V F} ; Y]$ to the DGA $\mathcal{P}_{\infty}^{*}[\overline{V F} \underset{Y}{\oplus} E ; Y]$ possessing the local basis $\left(s^{A}, \bar{s}_{A}, c^{r}\right)$. Its elements $c^{r}$ of Grassmann parity $\left[c_{r}\right]=\left[\Delta_{r}\right]$ are called the ghosts. The
graded derivation $\bar{\delta}$ is naturally prolonged to the $\mathrm{DGA} \mathcal{P}_{\infty}^{*}[\overline{V F} \underset{Y}{\oplus} E ; Y]$. Let us extend an original Lagrangian $L$ to the even Lagrangian

$$
\begin{equation*}
L_{e}=L+c^{r} \Delta_{r} \omega \in \mathcal{P}_{\infty}^{0, n}[\overline{V F} \underset{Y}{\oplus} E ; Y] . \tag{19}
\end{equation*}
$$

It is readily observed that, by virtue of the Noether identities (18), the graded derivation $\bar{\delta}$ is an exact symmetry of $L_{e}$ (19). It follows from item (iv) of Lemma 4 that

$$
\begin{equation*}
\frac{\overleftarrow{\delta}\left(c^{r} \Delta_{r}\right)}{\delta \bar{s}_{A}} \mathcal{E}_{A} \omega=\sum_{0 \leq|\Lambda|}(-1)^{|\Lambda|} d_{\Lambda}\left(c^{r} \Delta_{r}^{A, \Lambda}\right) \mathcal{E}_{A} \omega=u^{A} \mathcal{E}_{A} \omega=d_{H} \sigma \tag{20}
\end{equation*}
$$

Then by the same reason, the odd graded derivation

$$
\begin{equation*}
u=u^{A} \frac{\partial}{\partial s^{A}}, \quad u^{A}=\sum_{0 \leq|\Lambda|} c_{\Lambda}^{r} \eta\left(\Delta_{r}^{A}\right)^{\Lambda} \tag{21}
\end{equation*}
$$

of $\mathcal{P}_{\infty}^{*}[\overline{V F} ; Y]$ is a variational symmetry of an original Lagrangian $L$.
A glance at the expression (21) shows that the variational symmetry $u$ is a linear differential operator on the $C^{\infty}(X)$-module $\mathcal{C}$ of ghosts with values in the real space $\mathcal{G}_{L}$ of variational symmetries. It is called the gauge symmetry of a Lagrangian $L$ which is associated to the complete non-trivial Noether identities (18).

This association is unique due to the following. The variational derivative of the equality (20) with respect to ghosts $c^{r}$ leads to the equalities

$$
\delta_{r}\left(u^{A} \mathcal{E}_{A} \omega\right)=\sum_{0 \leq|\Lambda|}(-1)^{|\Lambda|} d_{\Lambda}\left(\eta\left(\Delta_{r}^{A}\right)^{\Lambda} \mathcal{E}_{A}\right)=\sum_{0 \leq|\Lambda|} \eta\left(\eta\left(\Delta_{r}^{A}\right)\right)^{\Lambda} d_{\Lambda} \mathcal{E}_{A}=\sum_{0 \leq|\Lambda|} \Delta_{r}^{A, \Lambda} d_{\Lambda} \mathcal{E}_{A}=0
$$

which reproduce the complete non-trivial Noether identities (18).
Moreover, the gauge symmetry $u(21)$ is complete in the following sense. Let

$$
\sum_{0 \leq|\Xi|} C^{R} G_{R}^{r, \Xi} d_{\Xi} \Delta_{r} \omega
$$

be some projective $C^{\infty}(X)$-module of finite rank of non-trivial Noether identities parameterized by the corresponding ghosts $C^{R}$. A direct computation shows that the graded derivation

$$
d_{\Lambda}\left(\sum_{0 \leq|\Xi|} \eta\left(G_{R}^{r}\right)^{\Xi} C_{\Xi}^{R}\right) u_{r}^{A, \Lambda} \frac{\partial}{\partial s^{A}}
$$

is a variational symmetry of a Lagrangian $L$ and, consequently, its gauge symmetry parameterized by ghosts $C^{R} .{ }^{7,8}$ It factorizes through the gauge symmetry (21) by putting ghosts

$$
c^{r}=\sum_{0 \leq|\Xi|} \eta\left(G_{R}^{r}\right)^{\Xi} C_{\Xi}^{R} .
$$

Thus, we come to the following second Noether theorem.
Theorem 5: The odd graded derivation $u(21)$ is a complete non-trivial gauge symmetry of a Lagrangian $L$ associated to the complete non-trivial Noether identities (18).

## V. GAUGE CONSERVATION LAWS

Being a variational symmetry, the gauge symmetry $u(21)$ defines the weak conservation law (11). The peculiarity of this conservation law is that the conserved current $\mathcal{J}_{u}$ (12) is reduced to a superpotential as follows.

Theorem 6: If $u(21)$ is a gauge symmetry of a Lagrangian $L$, the corresponding conserved current $\mathcal{J}_{u}$ (12) takes the form

$$
\begin{equation*}
\mathcal{J}_{u}=W+d_{H} U=\left(W^{\mu}+d_{\nu} U^{\nu \mu}\right) \omega_{\mu} \tag{22}
\end{equation*}
$$

where the form $W$ is $\bar{\delta}$-exact (i.e. it vanishes on-shell) and $U$ is a superpotential (13).
Proof: Let the gauge symmetry $u(21)$ be at most of jet order $N$ in ghosts. Then the conserved current $\mathcal{J}_{u}$ (22) is decomposed into the sum

$$
\begin{gather*}
\mathcal{J}_{u}^{\mu}=J_{r}^{\mu \mu_{1} \ldots \mu_{M}} c_{\mu_{1} \ldots \mu_{M}}^{r}+\sum_{1<k<M} J_{r}^{\mu \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}+  \tag{23}\\
J_{r}^{\mu \mu_{M}} c_{\mu_{M}}^{r}+J_{r}^{\mu} c^{r}+J^{\mu}, \quad N \leq M,
\end{gather*}
$$

and the first variational formula (9) takes the form

$$
\begin{aligned}
& 0= {\left[\sum_{k=1}^{N} u_{V r}^{i} \mu_{k} \ldots \mu_{N}\right.} \\
&\left.c_{\mu_{k} \ldots \mu_{N}}^{r}+u_{V r}^{i} c^{r}\right] \mathcal{E}_{i}- \\
& d_{\mu}\left(\sum_{k=1}^{M} J_{r}^{\mu \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}+J_{r}^{\mu} c^{r}+J^{\mu}\right) .
\end{aligned}
$$

This equality provides the following set of equalities for each $c_{\mu \mu_{1} \ldots \mu_{M}}^{r}, c_{\mu_{k} \ldots \mu_{M}}^{r}(k=1, \ldots, M-$ $N-1), c_{\mu_{k} \ldots \mu_{N}}^{r}(k=1, \ldots, N-1), c_{\mu}^{r}$ and $c^{r}$ :

$$
\begin{align*}
& 0=J_{r}^{\left(\mu \mu_{1}\right) \ldots \mu_{M}}  \tag{24}\\
& 0=J_{r}^{\left(\mu_{k} \mu_{k+1}\right) \ldots \mu_{M}}+d_{\nu} J_{r}^{\nu \mu_{k} \ldots \mu_{M}}, \quad 1 \leq k<M-N  \tag{25}\\
& 0=u_{V r}^{i} \mu_{k} \ldots \mu_{N} \mathcal{E}_{i}-J_{r}^{\left(\mu_{k} \mu_{k+1}\right) \ldots \mu_{N}}-d_{\nu} J_{r}^{\nu \mu_{k} \ldots \mu_{N}}, \quad 1 \leq k<N,  \tag{26}\\
& 0=u_{V r}^{i}{ }_{r}^{\mu} \mathcal{E}_{i}-J_{r}^{\mu}-d_{\nu} J_{r}^{\nu \mu} \tag{27}
\end{align*}
$$

where $(\mu \nu)$ means symmetrization of indices in accordance with the splitting

$$
J_{r}^{\mu_{k} \mu_{k+1} \ldots \mu_{N}}=J_{r}^{\left(\mu_{k} \mu_{k+1}\right) \ldots \mu_{N}}+J_{r}^{\left[\mu_{k} \mu_{k+1}\right] \ldots \mu_{N}} .
$$

We also have the equalities

$$
\begin{align*}
& 0=u_{V r}^{i} \mathcal{E}_{i}-d_{\mu} J_{r}^{\mu},  \tag{28}\\
& 0=d_{\mu} J^{\mu} . \tag{29}
\end{align*}
$$

With the equalities (24) - (27), the decomposition (23) takes the form

$$
\begin{aligned}
\mathcal{J}_{u}^{\mu}= & J_{r}^{\left[\mu \mu_{1}\right] \ldots \mu_{M}} c_{\mu_{1} \ldots \mu_{M}}^{r}+ \\
& \sum_{1<k \leq M-N}\left[\left(J_{r}^{\left[\mu \mu_{k}\right] \ldots \mu_{M}}-d_{\nu} J_{r}^{\nu \mu \mu_{k} \ldots \mu_{M}}\right) c_{\mu_{k} \ldots \mu_{M}}^{r}\right]+ \\
& \sum_{1<k<N}\left[\left(u_{V r}^{i}{ }_{V}^{\mu \mu_{k} \ldots \mu_{N}} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu \mu_{k} \ldots \mu_{N}}+J_{r}^{\left[\mu \mu_{k}\right] \ldots \mu_{N}}\right) c_{\mu_{k} \ldots \mu_{N}}^{r}\right]+ \\
& \left(u_{V r}^{i \mu \mu_{N}} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu \mu_{N}}+J_{r}^{\left[\mu \mu_{N}\right]}\right) c_{\mu_{N}}^{r}+\left(u_{V}^{i}{ }_{r}^{\mu} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu}\right) c^{r}+J^{\mu} .
\end{aligned}
$$

A direct computation

$$
\begin{aligned}
& \mathcal{J}_{u}^{\mu}= d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{2} \ldots \mu_{M}} c_{\mu_{2} \ldots \mu_{M}}^{r}\right)-d_{\nu} J_{r}^{[\mu \nu] \mu_{2} \ldots \mu_{M}} c_{\mu_{2} \ldots \mu_{M}}^{r}+ \\
& \sum_{1<k \leq M-N}\left[d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{M}} c_{\mu_{k+1} \ldots \mu_{M}}^{r}\right)-\right. \\
&\left.d_{\nu} J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{M}} c_{\mu_{k+1} \ldots \mu_{M}}^{r}-d_{\nu} J_{r}^{\nu \mu \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}\right]+ \\
& \sum_{1<k<N}\left[\left(u_{V r}^{i}{ }_{r}^{\mu \mu_{k} \ldots \mu_{N}} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu \mu_{k} \ldots \mu_{N}}\right) c_{\mu_{k} \ldots \mu_{N}}^{r}+\right. \\
&\left.d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{N}} c_{\mu_{k+1} \ldots \mu_{N}}^{r}\right)-d_{\nu} J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{N}} c_{\mu_{k+1} \ldots \mu_{N}}^{r}\right]+ \\
& {\left[\left(u_{V r}^{i \mu \mu_{N}} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu \mu_{N}}\right) c_{\mu_{N}}^{r}+d_{\nu}\left(J_{r}^{[\mu \nu]} c^{r}\right)-d_{\nu} J_{r}^{[\mu \nu]} c^{r}\right]+} \\
&\left(u_{V r}^{i \mu} \mathcal{E}_{i}-d_{\nu} J_{r}^{\nu \mu}\right) c^{r}+J^{\mu} \\
&= d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{2} \ldots \mu_{M}} c_{\mu_{2} \ldots \mu_{M}}^{r}\right)+ \\
& \sum_{1<k \leq M-N}\left[d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{M}} c_{\mu_{k+1} \ldots \mu_{M}}^{r}\right)-d_{\nu} J_{r}^{(\nu \mu) \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}\right]+ \\
& \sum_{1<k<N}\left[\left(u_{V r}^{i} \mu \mu_{k} \ldots \mu_{N}\right.\right. \\
&\left.\mathcal{E}_{i}-d_{\nu} J_{r}^{(\nu \mu) \mu_{k} \ldots \mu_{N}}\right) c_{\mu_{k} \ldots \mu_{N}}^{r}+ \\
&\left.d_{\nu}\left(J_{r}^{[\mu \nu] \mu_{k+1} \ldots \mu_{N}} c_{\mu_{k+1} \ldots \mu_{N}}^{r}\right)\right]+ \\
& {\left[\left(u_{V r}^{i \mu \mu_{N}} \mathcal{E}_{i}-d_{\nu} J_{r}^{(\nu \mu) \mu_{N}}\right) c_{\mu_{N}}^{r}+d_{\nu}\left(J_{r}^{[\mu \nu]} c^{r}\right)\right]+\left(u_{V r}^{i \mu} \mathcal{E}_{i}-d_{\nu} J_{r}^{(\nu \mu)}\right) c^{r}+J^{\mu} }
\end{aligned}
$$

leads to the expression

$$
\begin{align*}
\mathcal{J}_{u}^{\mu}= & \left(\sum_{1<k \leq N} u_{V r}^{i}{ }_{r}^{\mu \mu_{k} \ldots \mu_{N}} c_{\mu_{k} \ldots \mu_{N}}^{r}+u_{V r}^{i}{ }_{r}^{\mu} c^{r}\right) \mathcal{E}_{i}-  \tag{30}\\
& \left(\sum_{1<k \leq M} d_{\nu} J^{(\nu \mu) \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}+d_{\nu} J_{r}^{(\nu \mu)} c^{r}\right)-
\end{align*}
$$

$$
d_{\nu}\left(\sum_{1<k \leq M} J^{[\nu \mu] \mu_{k} \ldots \mu_{M}} c_{\mu_{k} \ldots \mu_{M}}^{r}+J_{r}^{[\nu \mu]} c^{r}\right)+J^{\mu}
$$

The first summand of this expression vanishes on-shell. Its second one contains the terms $d_{\nu} J^{\left(\nu \mu_{k}\right) \mu_{k+1} \ldots \mu_{M}}, k=1, \ldots, M$. By virtue of the equalities (25) - (26), every $d_{\nu} J^{\left(\nu \mu_{k}\right) \mu_{k+1} \ldots \mu_{M}}$ is expressed into the terms vanishing on-shell and the term $d_{\nu} J^{\left(\nu \mu_{k-1}\right) \mu_{k} \ldots \mu_{M}}$. Iterating the procedure and bearing in mind the equality (24), one can easily show that the second summand of the expression (30) also vanishes on-shell. Finally, the condition (29) means that the odd $(n-1)$-form $J^{\mu} \omega_{\mu}$ is $d_{H}$-closed and, consequently, it is $d_{H}$-exact in accordance with Theorem 1. Thus, the current $\mathcal{J}_{u}$ takes the form (22).
${ }^{1}$ B.Julia and S. Silva, Currents and superpotentials in classical gauge inveriant theories. Local results with applications to perfect dluids and General Relativity, Class. Quant. Grav. 15 (1998) 2173.
${ }^{2}$ M.Gotay and J.Marsden, Stress-energy-momentum tensors and the BelinfanteRosenfeld formula, Contemp. Math. 132 (1992) 367.
${ }^{3}$ L.Fatibene, M.Ferraris and M.Francaviglia, (1994). Nöther formalism for conserved quantities in classical gauge field theories. J. Math. Phys. 35 (1994) 1644.
${ }^{4}$ G.Giachetta, L.Mangiarotti and G.Sardanashvily, New Lagrangianm and Hamiltonian Methods in Field Theory (World Scientific, Singapore, 1997).
${ }^{5}$ G.Giachetta, L.Mangiarotti, and G.Sardanashvily, Lagrangian supersymmetries depending on derivatives. Global analysis and cohomology. Commun. Math. Phys. $\mathbf{2 5 9}$ (2005) 103; E-print arXiv: hep-th/0407185.
${ }^{6}$ D.Bashkirov, G.Giachetta, L.Mangiarotti and G.Sardanashvily, The KT-BRST complex of a degenerate Lagrangian system, Lett. Math. Phys. 83 (2008) 237; E-print arXiv: math-ph/0702097.
${ }^{7}$ G.Giachetta, L.Mangiarotti and G.Sardanashvily, On the notion of gauge symmetries of generic Lagrangian field theory, J. Math. Phys. 50 (2009) 012903; E-print arXiv: 0807.3003.
${ }^{8}$ G.Giachetta, L.Mangiarotti and G.Sardanashvily, Advanced Classical Field Theory (World Scientific, Singapore, 2009).
${ }^{9}$ G.Barnich, F.Brandt and M.Henneaux, Local BRST cohomology in gauge theories, Phys. Rep. 338 (2000) 439.
${ }^{10}$ G.Sardanashvily, Graded infinite order jet manifolds, Int. J. Geom. Methods Mod. Phys. 4 (2007) 1335; E-print arXiv: 0708.2434.
${ }^{11}$ P. Olver, Applications of Lie Groups to Differential Equations (Springer, Berlin, 1986).
${ }^{12}$ L.Fatibene, M.Ferraris, M.Francaviglia and R.McLenaghan, Generalized symmetries in mechanics and field theories, J. Math.. Phys. 43 (2002) 3147.
${ }^{13}$ R.Bryant, P.Griffiths and D.Grossman, Exterior Differential Systems and EulerLagrange Partial Differential Equations (Univ. of Chicago Press, Chicago, IL, 2003).

