Gauge conservation laws in a general setting. Superpotential

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The fact that the conserved current of a gauge symmetry is reduced to a superpotential is proved in a very general setting.

I. INTRODUCTION

The fact that the conserved current of a gauge symmetry is reduced to a superpotential has been stated in different particular variants, e.g., gauge theory of principal connections and gauge gravitation theory.¹⁻⁴ We aim to prove this assertion in a very general setting.

Generic higher-order Lagrangian theory of even and odd fields on an *n*-dimensional smooth manifold X and its variational generalized supersymmetries (henceforth symmetries) are considered.⁵⁻⁸ These symmetries form a real vector space \mathcal{G}_L . In a general setting, a gauge symmetry of a Lagrangian L is defined as a \mathcal{G}_L -valued linear differential operator on some Grassmann-graded projective $C^{\infty}(X)$ -module of finite rank.^{7,8} Note that any Lagrangian possesses gauge symmetries which therefore must be separated into the trivial and non-trivial ones. However, there is a problem of defining non-trivial gauge symmetries.⁷

In contrast with gauge symmetries, non-trivial Noether identities of Lagrangian field theory are well described in homology terms.⁶⁻⁸ Therefore, we define non-trivial gauge symmetries as those associated to complete non-trivial Noether identities in accordance with the second Noether theorem (Theorem 5).

Given a non-trivial gauge symmetry of a Lagrangian L, the corresponding current \mathcal{J} (12) is conserved by virtue of the first Noether theorem (Theorem 3). We prove that this current takes the superpotential form

$$\mathcal{J}^{\mu} = W^{\mu} + d_{\nu}U^{\nu\mu}$$

where the term W^{μ} vanishes on the kernel of the Euler–Lagrange operator δL (3) of L and $U^{\nu\mu} = -U^{\mu\nu}$ is a superpotential (Theorem 6).

II. LAGRANGIAN THEORY OF EVEN AND ODD FIELDS

Lagrangian theory of even and odd fields is adequately formulated in terms of the Grassmann-graded variational bicomplex on fiber bundles and graded manifolds.^{5,8,9} In a

very general setting, let us consider a composite bundle $F \to Y \to X$ where $F \to Y$ is a vector bundle provided with bundle coordinates $(x^{\lambda}, y^{i}, q^{a})$. The jet manifolds $J^{r}F$ of $F \to X$ also are vector bundles $J^{r}F \to J^{r}Y$ coordinated by $(x^{\lambda}, y^{i}_{\Lambda}, q^{a}_{\Lambda}), 0 \leq |\Lambda| \leq r$, where $\Lambda = (\lambda_{1}...\lambda_{k}), |\Lambda| = k$, denotes a symmetric multi-index. Let $(J^{r}Y, \mathcal{A}_{r})$ be a graded manifold whose body is $J^{r}Y$ and whose structure ring \mathcal{A}_{r} of graded functions consists of sections of the exterior bundle

$$\wedge (J^r F)^* = \mathbb{R} \oplus (J^r F)^* \oplus \bigwedge^2 (J^r F)^* \oplus \cdots,$$

where $(J^r F)^*$ is the dual of $J^r F \to J^r Y$. The local odd basis for this ring is $\{c_{\Lambda}^a\}, 0 \leq |\Lambda| \leq r$. Let $\mathcal{S}_r^*[F;Y]$ be the differential graded algebra (henceforth DGA) of graded differential forms on the graded manifold $(J^r Y, \mathcal{A}_r)$. The inverse system of jet manifolds $J^{r-1}Y \leftarrow J^r Y$ yields the direct system of DGAs

$$\mathcal{S}^*[F;Y] \longrightarrow \mathcal{S}^*_1[F;Y] \longrightarrow \cdots \mathcal{S}^*_r[F;Y] \longrightarrow \cdots$$

Its direct limit $\mathcal{S}^*_{\infty}[F;Y]$ is the DGA of all graded differential forms on graded manifolds (J^rY, \mathcal{A}_r) . It is a $C^{\infty}(Y)$ -algebra locally generated by elements $(c^a_{\Lambda}, dx^{\lambda}, dy^i_{\Lambda}, dc^a_{\Lambda}), 0 \leq |\Lambda|$. Let us recall the formulas

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'| + [\phi][\phi']} \phi' \wedge \phi, \qquad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi,$$

where $[\phi]$ denotes the Grassmann parity. The collective symbol (s^A) further stands for the tuple (y^i, c^a) , called the local basis for the DGA $\mathcal{S}^*_{\infty}[F; Y]$. Let us denote $[A] = [s^A] = [s^A_{\Lambda}]$.

The DGA $\mathcal{S}^*_{\infty}[F;Y]$ is split into the Grassmann-graded variational bicomplex of $\mathcal{S}^0_{\infty}[F;Y]$ modules $\mathcal{S}^{k,r}_{\infty}[F;Y]$ of *r*-horizontal and *k*-contact graded forms locally generated by the oneforms dx^{λ} and $\theta^A_{\Lambda} = ds^A_{\Lambda} - s^A_{\lambda+\Lambda} dx^{\lambda}$. This bicomplex contains the variational subcomplex

$$0 \to \mathbb{R} \longrightarrow \mathcal{S}^{0}_{\infty}[F;Y] \xrightarrow{d_{H}} \mathcal{S}^{0,1}_{\infty}[F;Y] \cdots \xrightarrow{d_{H}} \mathcal{S}^{0,n}_{\infty}[F;Y] \xrightarrow{\delta} \mathcal{S}^{1,n}_{\infty}[F;Y],$$
(1)

whose coboundary operator

$$d_H(\phi) = dx^{\lambda} \wedge d_{\lambda}\phi = dx^{\lambda} \wedge (\partial_{\lambda} + \sum_{0 \le |\Lambda|} s^A_{\lambda\Lambda} \partial^{\Lambda}_A)\phi,$$

$$d_H \circ h_0 = h_0 \circ d, \qquad h_0(\theta^A_{\Lambda}) = 0, \qquad h_0(dx^{\lambda}) = dx^{\lambda},$$

is the total differential, and whose elements

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_{\infty}[F;Y], \qquad \omega = dx^1 \wedge \dots \wedge dx^n, \tag{2}$$

$$\delta L = \theta^A \wedge \mathcal{E}_A \omega = \sum_{0 \le |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda (\partial_A^\Lambda \mathcal{L}) \omega, \qquad d_\Lambda = d_{\lambda_1} \cdots d_{\lambda_k}, \tag{3}$$

are graded Lagrangians and their Euler–Lagrange operators. Further, a pair $(\mathcal{S}^*_{\infty}[F;Y],L)$ denotes Lagrangian field theory.

Cohomology of the Grassmann-graded variational bicomplex has been obtained.^{8,10} Let us mention the following relevant results.

Theorem 1: Cohomology of the variational complex (1) equals the de Rham cohomology of a fiber bundle Y.

In particular, any odd element of this complex possesses trivial cohomology.

Theorem 2: Given a graded Lagrangian L, there is the decomposition

$$dL = \delta L - d_H \Xi_L, \qquad \Xi \in \mathcal{S}_{\infty}^{n-1}[F;Y], \tag{4}$$

$$\Xi_L = L + \sum_{s=0} \theta^A_{\nu_s \dots \nu_1} \wedge F^{\lambda \nu_s \dots \nu_1}_A \omega_\lambda, \qquad \omega_\lambda = \partial_\lambda \rfloor \omega, \tag{5}$$

$$F_A^{\nu_k\dots\nu_1} = \partial_A^{\nu_k\dots\nu_1} \mathcal{L} - d_\lambda F_A^{\lambda\nu_k\dots\nu_1} + \psi_A^{\nu_k\dots\nu_1}, \qquad k = 1, 2, \dots,$$

where local graded functions ψ obey the relations

$$\psi_A^{\nu} = 0, \qquad \psi_A^{(\nu_k \nu_{k-1})\dots\nu_1} = 0.$$

The form Ξ_L (5) provides a global Lepage equivalent of a graded Lagrangian L. In particular, one can locally choose Ξ_L (5) where all functions ψ vanish.

The corollaries of Theorem 2 are the first variational formula (9) and the first Noether theorem (Theorem 3).

III. THE FIRST NOETHER THEOREM

In order to treat symmetries of Lagrangian field theory $(\mathcal{S}^*_{\infty}[F;Y],L)$ in a very general setting, we consider graded derivations of the \mathbb{R} -ring $\mathcal{S}^0_{\infty}[F;Y]$.^{5,8} They take the form

$$\vartheta = \vartheta^{\lambda} \partial_{\lambda} + \sum_{0 \le |\Lambda|} \vartheta^{A}_{\Lambda} \partial^{\Lambda}_{A}, \qquad \partial^{\Lambda}_{A} (s^{B}_{\Sigma}) = \partial^{\Lambda}_{A} \rfloor ds^{B}_{\Sigma} = \delta^{B}_{A} \delta^{\Lambda}_{\Sigma}.$$
(6)

Any graded derivation ϑ (6) yields the Lie derivative

$$\begin{split} \mathbf{L}_{\vartheta}\phi &= \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), \qquad \phi \in \mathcal{S}_{\infty}^{*}[F;Y], \\ \mathbf{L}_{\vartheta}(\phi \wedge \sigma) &= \mathbf{L}_{\vartheta}(\phi) \wedge \sigma + (-1)^{[\vartheta][\phi]}\phi \wedge \mathbf{L}_{\vartheta}(\sigma), \end{split}$$

of the DGA $\mathcal{S}^*_{\infty}[F;Y]$.

A graded derivation ϑ (6) is called contact if the Lie derivative \mathbf{L}_{ϑ} preserves the ideal of contact graded forms of the DGA $\mathcal{S}^*_{\infty}[F;Y]$. Any contact graded derivation admits the decomposition

$$\vartheta = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + \left[\upsilon^A \partial_A + \sum_{|\Lambda|>0} d_\Lambda (\upsilon^A - s^A_\mu \upsilon^\mu) \partial^\Lambda_A\right] \tag{7}$$

into the horizontal and vertical parts v_H and v_V . A glance at the expression (7) shows that a contact graded derivation ϑ is an infinite order jet prolongation of its restriction

$$v = v^{\lambda} \partial_{\lambda} + v^{A} \partial_{A} \tag{8}$$

to the graded commutative ring $S^0[F;Y]$. One calls v (8) the generalized graded vector field. It is a graded vector field if its components v^{λ} , v^A are independent of jets s^A_{Λ} . Note that generalized symmetries of Lagrangian systems have been intensively studied.^{5,11-13}

Given a contact graded derivation (7), a corollary of the decomposition (4) is the above mentioned first variational formula

$$\mathbf{L}_{\vartheta}L = v_V \rfloor \delta L + d_H (h_0(\vartheta \rfloor \Xi_L)) + d_V (v_H \rfloor \omega) \mathcal{L}, \tag{9}$$

where Ξ_L is the Lepage equivalent (5) of L.

Given a Lagrangian L (2), a contact graded derivation ϑ (7) is said to be its variational symmetry (strictly speaking a variational generalized supersymmetry) if the Lie derivative $\mathbf{L}_{\vartheta}L$ is d_H -exact, i.e.

$$\mathbf{L}_{\vartheta}L = d_H \sigma. \tag{10}$$

A variational symmetry ϑ of a Lagrangian L is called its exact symmetry if the Lie derivative $\mathbf{L}_{\vartheta}L$ vanishes.

An immediate corollary of the first variational formula (9) is the following first Noether theorem.

Theorem 3: If a contact graded derivation ϑ (7) is a variational symmetry (10) of a Lagrangian L, the first variational formula (9) restricted to Ker δL leads to the weak conservation law

$$d_H(\sigma - h_0(\vartheta \rfloor \Xi_L)) \approx 0 \tag{11}$$

of the current

$$\mathcal{J}_{\vartheta} = \mathcal{J}_{\vartheta}^{\mu} \omega_{\mu} = \sigma - h_0(\vartheta \rfloor \Xi_L).$$
(12)

Obviously, the conserved current (12) is defined up to a d_H -closed horizontal (n-1)-form

$$U = \frac{1}{2} U^{\nu\mu} \omega_{\nu\mu}, \qquad \omega_{\nu\mu} = \partial_{\nu} \rfloor \omega_{\mu}, \qquad (13)$$

called the superpotential.

Lemma 4: A glance at the expression (9) shows the following.⁵

(i) A contact graded derivation ϑ is a variational symmetry only if the generalized vector field v (8) is projected onto X, i.e., $v^{\lambda}\partial_{\lambda}$ is a vector field on X.

(ii) A contact graded derivation ϑ is a variational symmetry iff its vertical part v_V is well.

(iii) Any projectable contact graded derivation is a variational symmetry of a variationally trivial Lagrangian.

(iv) A contact graded derivation ϑ is a variational symmetry iff the graded density $v_V | \delta L$ is d_H -exact.

Variational symmetries of a Lagrangian L constitute a real vector space \mathcal{G}_L . By virtue of item (iii) of Lemma 4, the Lie superbracket

$$\mathbf{L}_{[\vartheta, \vartheta']} = [\mathbf{L}_{\vartheta}, \mathbf{L}_{\vartheta'}]$$

of variational symmetries is a variational symmetry. Consequently, the vector space \mathcal{G}_L of variational symmetries is a real Lie superalgebra.

By virtue of item (ii) of Lemma 4, we further restrict our consideration to vertical contact graded derivations

$$\vartheta = \upsilon^A \partial_A + \sum_{0 < |\Lambda|} d_\Lambda \upsilon^A \partial_A^\Lambda.$$
(14)

A graded derivation ϑ (14) is called nilpotent if $\mathbf{L}_{\vartheta}(\mathbf{L}_{\vartheta}\phi) = 0$ for any horizontal form $\phi \in \mathcal{S}_{\infty}^{0,*}[F;Y]$. One can show that ϑ (14) is nilpotent only if it is odd and iff $\vartheta(v) = 0.5^{\circ}$

For the sake of brevity, the common symbol v further stands for a generalized graded vector field $v = v^A \partial_A$, the vertical contact graded derivation ϑ (14) determined by v, and the Lie derivative \mathbf{L}_{ϑ} . We agree to call v the graded derivation of the DGA $\mathcal{S}^*_{\infty}[F;Y]$. The right graded derivations $\overleftarrow{v} = \overleftarrow{\partial}_A v^A$ of $\mathcal{S}^*_{\infty}[F;Y]$ also are considered.

IV. GAUGE SYMMETRIES

Without a loss of generality, let a Lagrangian L be even. To describe Noether identities of Lagrangian field theory $(\mathcal{S}^*_{\infty}[F;Y], L)$, let us introduce the following notation. Given a vector bundle $E \to X$, we call

$$\overline{E} = E^* \otimes \bigwedge^n T^* X$$

the density-dual of E. The density dual of a graded vector bundle $E = E^0 \oplus E^1$ is $\overline{E} = \overline{E}^1 \oplus \overline{E}^0$. Given a graded vector bundle $E = E^0 \oplus E^1$ over Y, we consider the composite

bundle $E \to E^0 \to X$ and denote $\mathcal{P}^*_{\infty}[E;Y] = \mathcal{S}^*_{\infty}[E;E^0]$. Let VF be the vertical tangent bundle of $F \to X$, the density-dual of the vector bundle $VF \to F$ is

$$\overline{VF} = V^*F \mathop{\otimes}_F \mathop{\wedge}^n T^*X.$$

Let us enlarge $\mathcal{S}^*_{\infty}[F;Y]$ to the DGA $\mathcal{P}^*_{\infty}[\overline{VF};Y]$ possessing the local basis (s^A, \overline{s}_A) , $[\overline{s}_A] = ([A] + 1) \mod 2$. Its elements \overline{s}_A are called antifields. The DGA $\mathcal{P}^*_{\infty}[\overline{VF};Y]$ is endowed with the odd right graded derivation $\overline{\delta} = \overleftarrow{\partial}^{-A} \mathcal{E}_A$, where \mathcal{E}_A are the variational derivatives (3). This graded derivation is obviously nilpotent. Then we have the chain complex

$$0 \leftarrow \operatorname{Im} \overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_{\infty} [\overline{VF}; Y]_1 \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_{\infty} [\overline{VF}; Y]_2$$
(15)

of graded densities of antifield number ≤ 2 . Its one-cycles

$$\overline{\delta}\Phi = 0, \qquad \Phi = \sum_{0 \le |\Lambda|} \Phi^{A,\Lambda} \overline{s}_{\Lambda A} d^n x \in \mathcal{P}^{0,n}_{\infty}[\overline{VF};Y]_1, \tag{16}$$

define Noether identities of Lagrangian field theory $(\mathcal{S}^*_{\infty}[F;Y], L)$. In particular, one-chains $\Phi \in \mathcal{P}^{0,n}_{\infty}[\overline{VF};Y]_1$ are necessarily Noether identities if they are boundaries. Therefore, these Noether identities are called trivial. Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the first homology $H_1(\overline{\delta})$ of the complex (15).^{6,8}

Let us assume that the homology $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a projective $C^{\infty}(X)$ -module $\mathcal{C} \subset H_1(\overline{\delta})$ of finite rank possessing the local basis $\{\Delta_r\}$ such that any element $\Phi \in H_1(\overline{\delta})$ factorizes as

$$\Phi = \sum_{0 \le |\Xi|} G^{r,\Xi} d_{\Xi} \Delta_r d^n x, \qquad \Delta_r = \sum_{0 \le |\Lambda|} \Delta_r^{A,\Lambda} \overline{s}_{\Lambda A}, \qquad G^{r,\Xi}, \Delta_r^{A,\Lambda} \in \mathcal{S}^0_{\infty}[F;Y], \tag{17}$$

through elements of C. Thus, all non-trivial Noether identities (16) result from the Noether identities

$$\overline{\delta}\Delta_r = \sum_{0 \le |\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{18}$$

called the complete Noether identities. By virtue of the generalized Serre–Swan theorem,⁸ the module \mathcal{C} is isomorphic to the $C^{\infty}(X)$ -module of sections of the density-dual \overline{E} of some graded vector bundle $E \to X$.

We define a non-trivial gauge symmetry of Lagrangian field theory $(\mathcal{S}^*_{\infty}[F;Y],L)$ as that associated to the Noether identities (18) by means of the inverse second Noether theorem.^{6–8}

Let us enlarge the DGA $\mathcal{P}^*_{\infty}[\overline{VF};Y]$ to the DGA $\mathcal{P}^*_{\infty}[\overline{VF} \bigoplus_Y E;Y]$ possessing the local basis $(s^A, \overline{s}_A, c^r)$. Its elements c^r of Grassmann parity $[c_r] = [\Delta_r]$ are called the ghosts. The

graded derivation $\overline{\delta}$ is naturally prolonged to the DGA $\mathcal{P}^*_{\infty}[\overline{VF} \bigoplus_Y E; Y]$. Let us extend an original Lagrangian L to the even Lagrangian

$$L_e = L + c^r \Delta_r \omega \in \mathcal{P}^{0,n}_{\infty}[\overline{VF} \bigoplus_Y E; Y].$$
(19)

It is readily observed that, by virtue of the Noether identities (18), the graded derivation $\overline{\delta}$ is an exact symmetry of L_e (19). It follows from item (iv) of Lemma 4 that

$$\frac{\delta(c^r \Delta_r)}{\delta \overline{s}_A} \mathcal{E}_A \omega = \sum_{0 \le |\Lambda|} (-1)^{|\Lambda|} d_\Lambda(c^r \Delta_r^{A,\Lambda}) \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma.$$
(20)

Then by the same reason, the odd graded derivation

$$u = u^{A} \frac{\partial}{\partial s^{A}}, \qquad u^{A} = \sum_{0 \le |\Lambda|} c^{r}_{\Lambda} \eta(\Delta^{A}_{r})^{\Lambda}, \qquad (21)$$

of $\mathcal{P}^*_{\infty}[\overline{VF}; Y]$ is a variational symmetry of an original Lagrangian L.

A glance at the expression (21) shows that the variational symmetry u is a linear differential operator on the $C^{\infty}(X)$ -module \mathcal{C} of ghosts with values in the real space \mathcal{G}_L of variational symmetries. It is called the gauge symmetry of a Lagrangian L which is associated to the complete non-trivial Noether identities (18).

This association is unique due to the following. The variational derivative of the equality (20) with respect to ghosts c^r leads to the equalities

$$\delta_r(u^A \mathcal{E}_A \omega) = \sum_{0 \le |\Lambda|} (-1)^{|\Lambda|} d_\Lambda(\eta(\Delta_r^A)^\Lambda \mathcal{E}_A) = \sum_{0 \le |\Lambda|} \eta(\eta(\Delta_r^A))^\Lambda d_\Lambda \mathcal{E}_A = \sum_{0 \le |\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0$$

which reproduce the complete non-trivial Noether identities (18).

Moreover, the gauge symmetry u (21) is complete in the following sense. Let

$$\sum_{0 \le |\Xi|} C^R G_R^{r,\Xi} d_{\Xi} \Delta_r \omega$$

be some projective $C^{\infty}(X)$ -module of finite rank of non-trivial Noether identities parameterized by the corresponding ghosts C^{R} . A direct computation shows that the graded derivation

$$d_{\Lambda} (\sum_{0 \le |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R) u_r^{A,\Lambda} \frac{\partial}{\partial s^A}$$

is a variational symmetry of a Lagrangian L and, consequently, its gauge symmetry parameterized by ghosts $C^{R,7,8}$ It factorizes through the gauge symmetry (21) by putting ghosts

$$c^r = \sum_{0 \le |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R.$$

Thus, we come to the following second Noether theorem.

Theorem 5: The odd graded derivation u (21) is a complete non-trivial gauge symmetry of a Lagrangian L associated to the complete non-trivial Noether identities (18).

V. GAUGE CONSERVATION LAWS

Being a variational symmetry, the gauge symmetry u (21) defines the weak conservation law (11). The peculiarity of this conservation law is that the conserved current \mathcal{J}_u (12) is reduced to a superpotential as follows.

Theorem 6: If u (21) is a gauge symmetry of a Lagrangian L, the corresponding conserved current \mathcal{J}_u (12) takes the form

$$\mathcal{J}_{u} = W + d_{H}U = (W^{\mu} + d_{\nu}U^{\nu\mu})\omega_{\mu}, \qquad (22)$$

where the form W is $\overline{\delta}$ -exact (i.e. it vanishes on-shell) and U is a superpotential (13).

Proof: Let the gauge symmetry u (21) be at most of jet order N in ghosts. Then the conserved current \mathcal{J}_u (22) is decomposed into the sum

$$\mathcal{J}_{u}^{\mu} = J_{r}^{\mu\mu_{1}...\mu_{M}} c_{\mu_{1}...\mu_{M}}^{r} + \sum_{1 < k < M} J_{r}^{\mu\mu_{k}...\mu_{M}} c_{\mu_{k}...\mu_{M}}^{r} + J_{r}^{\mu} c^{r} + J^{\mu}, \qquad N \le M,$$
(23)

and the first variational formula (9) takes the form

$$0 = \left[\sum_{k=1}^{N} u_{Vr}^{i} \mu_{k} \dots \mu_{N} c_{\mu_{k} \dots \mu_{N}}^{r} + u_{Vr}^{i} c^{r}\right] \mathcal{E}_{i} - d_{\mu} \left(\sum_{k=1}^{M} J_{r}^{\mu\mu_{k} \dots \mu_{M}} c_{\mu_{k} \dots \mu_{M}}^{r} + J_{r}^{\mu} c^{r} + J^{\mu}\right).$$

This equality provides the following set of equalities for each $c^r_{\mu\mu_1...\mu_M}$, $c^r_{\mu_k...\mu_M}$ (k = 1, ..., M - N - 1), $c^r_{\mu_k...\mu_N}$ (k = 1, ..., N - 1), c^r_{μ} and c^r :

$$0 = J_r^{(\mu\mu_1)\dots\mu_M},$$
 (24)

$$0 = J_r^{(\mu_k \mu_{k+1})\dots\mu_M} + d_\nu J_r^{\nu\mu_k\dots\mu_M}, \qquad 1 \le k < M - N,$$
(25)

$$0 = u_{Vr}^{i} {}^{\mu_k \dots \mu_N} \mathcal{E}_i - J_r^{(\mu_k \mu_{k+1}) \dots \mu_N} - d_\nu J_r^{\nu \mu_k \dots \mu_N}, \qquad 1 \le k < N,$$
(26)

$$0 = u_{Vr}^{i}{}^{\mu}\mathcal{E}_{i} - J_{r}^{\mu} - d_{\nu}J_{r}^{\nu\mu}, \tag{27}$$

where $(\mu\nu)$ means symmetrization of indices in accordance with the splitting

$$J_r^{\mu_k\mu_{k+1}...\mu_N} = J_r^{(\mu_k\mu_{k+1})...\mu_N} + J_r^{[\mu_k\mu_{k+1}]...\mu_N}$$

We also have the equalities

$$0 = u_{Vr}^i \mathcal{E}_i - d_\mu J_r^\mu, \tag{28}$$

$$0 = d_{\mu}J^{\mu}.\tag{29}$$

With the equalities (24) - (27), the decomposition (23) takes the form

$$\begin{split} \mathcal{J}_{u}^{\mu} &= J_{r}^{[\mu\mu_{1}]\dots\mu_{M}} c_{\mu_{1}\dots\mu_{M}}^{r} + \\ &\sum_{1 < k \leq M-N} [(J_{r}^{[\mu\mu_{k}]\dots\mu_{M}} - d_{\nu}J_{r}^{\nu\mu\mu_{k}\dots\mu_{M}})c_{\mu_{k}\dots\mu_{M}}^{r}] + \\ &\sum_{1 < k < N} [(u_{V_{r}}^{i}\mu^{\mu\mu_{k}\dots\mu_{N}}\mathcal{E}_{i} - d_{\nu}J_{r}^{\nu\mu\mu_{k}\dots\mu_{N}} + J_{r}^{[\mu\mu_{k}]\dots\mu_{N}})c_{\mu_{k}\dots\mu_{N}}^{r}] + \\ &(u_{V_{r}}^{i}\mu^{\mu\nu_{N}}\mathcal{E}_{i} - d_{\nu}J_{r}^{\nu\mu\mu_{N}} + J_{r}^{[\mu\mu_{N}]})c_{\mu_{N}}^{r} + (u_{V_{r}}^{i}\mathcal{E}_{i} - d_{\nu}J_{r}^{\nu\mu})c^{r} + J^{\mu}. \end{split}$$

A direct computation

$$\begin{split} \mathcal{J}_{u}^{\mu} &= d_{\nu} (J_{r}^{[\mu\nu]\mu_{2}...\mu_{M}} c_{\mu_{2}...\mu_{M}}^{r}) - d_{\nu} J_{r}^{[\mu\nu]\mu_{2}...\mu_{M}} c_{\mu_{2}...\mu_{M}}^{r} + \\ &\sum_{1 < k \leq M-N} [d_{\nu} (J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{M}} c_{\mu_{k+1}...\mu_{M}}^{r}) - \\ d_{\nu} J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{M}} c_{\mu_{k+1}...\mu_{M}}^{r} - d_{\nu} J_{r}^{\nu\mu\mu_{k}...\mu_{M}} c_{\mu_{k}...\mu_{M}}^{r}] + \\ &\sum_{1 < k < N} [(u_{Vr}^{i}\mu^{\mu_{k}...\mu_{N}} \mathcal{E}_{i} - d_{\nu} J_{r}^{\nu\mu\mu_{k}...\mu_{N}}) c_{\mu_{k}...\mu_{N}}^{r} + \\ d_{\nu} (J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{N}} c_{\mu_{k+1}...\mu_{N}}^{r}) - d_{\nu} J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{N}} c_{\mu_{k+1}...\mu_{N}}^{r}] + \\ [(u_{Vr}^{i}\mu^{\mu_{K}} \mathcal{E}_{i} - d_{\nu} J_{r}^{\nu\mu\mu_{N}}) c_{\mu_{N}}^{r} + d_{\nu} (J_{r}^{[\mu\nu]} c^{r}) - d_{\nu} J_{r}^{[\mu\nu]} c^{r}] + \\ (u_{Vr}^{i}\mu^{\mu_{k}} \mathcal{E}_{i} - d_{\nu} J_{r}^{\nu\mu_{N}}) c^{r} + J^{\mu} \\ = d_{\nu} (J_{r}^{[\mu\nu]\mu_{2}...\mu_{M}} c_{\mu_{k+1}...\mu_{M}}^{r}) - d_{\nu} J_{r}^{(\nu\mu)\mu_{k}...\mu_{M}} c_{\mu_{k}...\mu_{M}}^{r}] + \\ \sum_{1 < k \leq M-N} [d_{\nu} (J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{N}} c_{\mu_{k+1}...\mu_{M}}^{r}) - d_{\nu} J_{r}^{(\nu\mu)\mu_{k}...\mu_{M}} c_{\mu_{k}...\mu_{M}}^{r}] + \\ \sum_{1 < k \leq M-N} [(u_{Vr}^{i}\mu^{\mu_{k}...\mu_{N}} \mathcal{E}_{i} - d_{\nu} J_{r}^{(\nu\mu)\mu_{k}...\mu_{N}}) c_{\mu_{k}...\mu_{N}}^{r} + \\ d_{\nu} (J_{r}^{[\mu\nu]\mu_{k+1}...\mu_{N}} c_{\mu_{k+1}...\mu_{N}}^{r})] + \\ [(u_{Vr}^{i}\mu^{\mu_{K}} \mathcal{E}_{i} - d_{\nu} J_{r}^{(\nu\mu)\mu_{N}}) c_{\mu_{N}}^{r} + d_{\nu} (J_{r}^{[\mu\nu]} c^{r})] + (u_{Vr}^{i}\mu^{\mu_{k}} \mathcal{E}_{i} - d_{\nu} J_{r}^{(\nu\mu)\mu_{N}}) c_{r}^{r} + J^{\mu} \\ \end{bmatrix}$$

leads to the expression

$$\mathcal{J}_{u}^{\mu} = \left(\sum_{1 < k \le N} u_{V_{r}}^{i \ \mu\mu_{k}...\mu_{N}} c_{\mu_{k}...\mu_{N}}^{r} + u_{V_{r}}^{i \ \mu} c^{r}\right) \mathcal{E}_{i} - \left(\sum_{1 < k \le M} d_{\nu} J^{(\nu\mu)\mu_{k}...\mu_{M}} c_{\mu_{k}...\mu_{M}}^{r} + d_{\nu} J_{r}^{(\nu\mu)} c^{r}\right) -$$
(30)

$$d_{\nu} \left(\sum_{1 < k \le M} J^{[\nu\mu]\mu_k...\mu_M} c^r_{\mu_k...\mu_M} + J^{[\nu\mu]}_r c^r \right) + J^{\mu}.$$

The first summand of this expression vanishes on-shell. Its second one contains the terms $d_{\nu}J^{(\nu\mu_k)\mu_{k+1}...\mu_M}$, k = 1, ..., M. By virtue of the equalities (25) – (26), every $d_{\nu}J^{(\nu\mu_k)\mu_{k+1}...\mu_M}$ is expressed into the terms vanishing on-shell and the term $d_{\nu}J^{(\nu\mu_{k-1})\mu_k...\mu_M}$. Iterating the procedure and bearing in mind the equality (24), one can easily show that the second summand of the expression (30) also vanishes on-shell. Finally, the condition (29) means that the odd (n-1)-form $J^{\mu}\omega_{\mu}$ is d_H -closed and, consequently, it is d_H -exact in accordance with Theorem 1. Thus, the current \mathcal{J}_u takes the form (22).

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