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Volume element structure and roton-maxon-phonon excitations in superfluid helium beyond the Gross-Pitaevskii approximation

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We propose a theory which deals with the structure and interactions of volume elements in liquid helium II. The approach consists of two nested models linked via parametric space. The short-wavelength part describes the interior structure of the fluid element using a non-perturbative approach based on the logarithmic wave equation; it suggests the Gaussian-like behaviour of the element's interior density and interparticle interaction potential. The long-wavelength part is the quantum many-body theory of such elements which deals with their dynamics and interactions. Our approach leads to a unified description of the phonon, maxon and roton excitations, and has noteworthy agreement with experiment: with one essential parameter to fit we reproduce at high accuracy not only the roton minimum but also the neighboring local maximum as well as the sound velocity and structure factor.

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1. INTRODUCTION

The microscopical structure of quantum liquids is different from that of classical ones. The latter are discrete fluids consisting of localized atoms or molecules which thermal de Broglie length is smaller than the average atomic or molecular separation. For instance, the thermal de Broglie wavelength of a water molecule at room temperature is less than the radius of the hydrogen atom. Therefore, classical liquids can be treated as continuous media only on the length scales which are larger than atomic or molecular separation. In quantum liquids it is other way around: atoms become delocalized in space because their de Broglie lengths are always larger than the inter-atomic distance, and overlapping with each other. As a result, instead of discrete atoms, quantum liquids must be described in terms of the fluid volume elements and elementary excitations such that non-locality and continuity are preserved down to very short scales [1–3]. Historically, it was Landau [4] who developed the theory of superfluidity based on the mechanics of continuous media which qualitatively agreed with experiment [5–7].

However, some questions remain open in this regard - in particular, those about the interplay between original degrees of freedom (^{4}He atoms) and emergent collective ones - such as the fluid volume elements (also called the fluid particles in the Lagrangian description [8]). Indeed, the fluid-dynamical description of liquid helium presumes that usage of the volume element's notion must be physically justified before, since it is the volume element, not the atom itself, which is supposed to be the most elementary object of the fluid approach. Thus, the understanding and describing of this notion in the quantum fluid theory is more important than in the classical one because for classical fluids the continuity is just the long-wavelength approximation whereas for quantum ones it is an empirically established fact. Therefore, searches for the most suitable collective degrees of freedom in superfluids continue, and the issue of how the formation and stability of the volume element of liquid helium as a continuous medium can be explained from first principles remains a subject of active studies nowadays [9, 10].

The Bose-Einstein condensation (BEC) is another aspect of the superfluidity phenomenon which must be taken into account [11]. Being predicted almost a century ago, it was used by London and Tisza [12] to explain the superfluidity of liquid helium discovered by Kapitsa and Allen [13]. The existence of BEC in the superfluid phase has been confirmed by inelastic neutron scattering [14, 15], thus, the influence of BEC upon the properties of liquid helium is under current study as well [16–18].

Here we describe the structure of the volume element of superfluid helium as well as the phenomenological consequences which follow. Among the derived effects is the famous Landau roton spectrum for which full theoretical explanation is still pending despite the numerous efforts made towards its understanding [9, 10, 16–20]. As a matter of fact, even usage of the term "roton" itself is often a purely historical one since the explanation of the Landau spectrum in helium does not necessarily have to be based on rotating degrees of freedom of helium atoms [9, 18]. Indeed, here we show that it is possible to formulate a theory without their explicit involvement - yet obtain a remarkable agreement with experiment.

Throughout the paper we neglect temperature effects which is a standard assumption for fundamental models since below the lambda point the thermal effects can be considered as corrections (the superfluid helium has almost perfect heat conduction).

In section 2 we formulate a general theory of helium II, in sections 3 and 4 we derive the observable quantities and compare them with available experimental data. Conclusions are made in Sec. 5.

2. COLLECTIVE VARIABLE THEORY

We divide a theory into two nested parts. The shorter-wavelength part describes the interior structure of the volume element of helium superfluid using the nonlinear logarithmic quantum wave equation. The longer-wavelength part is the quantum many-body theory of these elements, treated as new collective degrees of freedom. While one assumes them to be effectively point-like objects in the long-wavelength approximation, their spatial extent and internal structure are taken into account by virtue of the nonlocal interaction term. At that, these two models are not independent: in the shorter-wavelength part we derive behavior and specific values of the parameters for the longer-scale one, in accordance with the "Russian-doll" picture of nesting of scales.

As a starting point, we assume that strongly-correlated helium atoms can form a bound state characterized by a single macroscopical wavefunction. The wave equation describing such object cannot be of the Gross-Pitaevskii (cubic Schrödinger) type for at least two reasons.

The first one is that the GP approach [21] is a perturbative one which takes into account only two-body interactions and neglects anomalous contributions to self-energy which is a good approximation for dilute system like cold gases [22], but unlikely to suffice for liquids. Indeed, according to aforesaid the atoms in quantum liquid are delocalized and thus nothing prevents them from getting involved into multiple-body interactions where the multiplicity can vary from two to the total number of atoms in the system. An example of multi-body (three and more) interactions being very important for forming bound states of bosons at low temperatures is the Efimov state [23, 24] which has been experimentally observed in helium [25], a recent review can be found in [26].

The second obstacle is that the ground-state wavefunction of the free-space GP BEC model does not describe a localized object. Instead, the free GP condensate tends to occupy all available volume - as such one needs to apply an external potential trap to confine the condensate and stabilize the system. It is difficult to imagine that upon the transition into a superfluid phase the free helium suddenly becomes surrounded, both in parts and as a whole, by a hypothetical external potential, not to mention that the coherent appearance, stability and synchronization of such domains across the bulk would be impeded by significant volatility of the liquid.

There exists, however, another candidate where these problems simply do not occur in first place. This is nonlinear Bose liquid defined by virtue of the logarithmic Schrödinger equation:

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \beta^{-1}\ln\left(\tilde{a}^3|\Psi|^2\right)\right]\Psi = 0, \qquad (1)$$

where $\Psi = \Psi(\vec{x}, t)$ is the wavefunction of condensate normalized to the number of particles N - such that particle density is determined as $n = |\Psi|^2$, m is the mass of the constituent particle (helium-4 atom in our case, i.e., $m = m_{\rm He} \approx 6.64\times10^{-24}$ g), and β and $\tilde{a}$ are constant parameters of interaction. The equation alone received attention a while ago as the simplest U(1)-symmetric wave equation (apart from the conventional Schrödinger one) which satisfies the dilatation covariance and separability properties [27, 28] but for the purposes of theory of quantum liquids it has been applied only recently [29]. While the second-quantized Hamiltonian which might lead to such equation is not known yet, the equation itself has some interesting properties and thus can be used as a starting point. The equation can be also derived from the field-theoretical action with the Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2}(\Psi\partial_t\Psi^* - \Psi^*\partial_t\Psi) + \frac{\hbar^2}{2m}|\vec{\nabla}\Psi|^2 + V_\beta(|\Psi|^2), \quad (2)$$

where the potential is defined as

$$V_\beta(n) = -\beta^{-1}n\left[\ln\left(n\tilde{a}^3\right) - 1\right]. \qquad (3)$$

The latter opens down and has local non-zero maxima at $n_{\rm ext} = \tilde{a}^{-3}$ for positive β, see Fig. 1. Despite the potential not being bounded from below as a function of $|\Psi|$, no density divergences arise since the wavefunction cannot take arbitrarily large values, due to the normalization constraint.

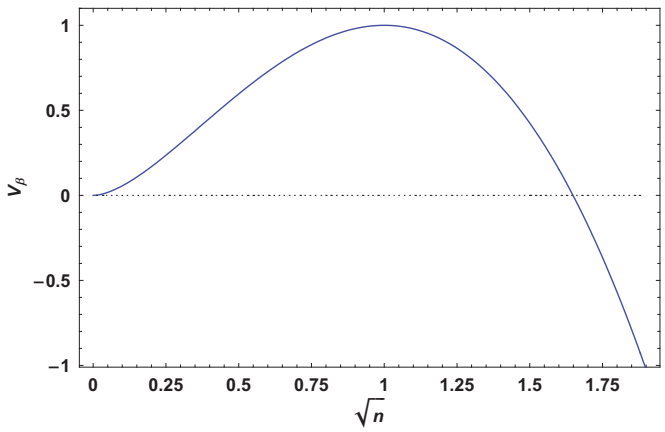

FIG. 1: Field potential for the logarithmic BEC (in units $\beta\tilde{a}^3$) versus square root of particle density (in units $\tilde{a}^{-3/2}$).

It turns out that the logarithmic Bose liquid has a number of features suitable for our objectives: it implicates not only binary but also multiple-body interactions (when more than two bodies can scatter simultaneously), and its ground state is the so-called *gausson* [28] - a spherically-symmetric object which is localized and stable even in absence of a trapping potential, with the interior density obeying the Gaussian law

$$n(\vec{x}) = n(0)\,\mathrm{e}^{-(r/a)^2}, \qquad (4)$$

where $n(0) = N/(\pi^{3/2}a^3)$ is the central particle density, $r = |\vec{x}|$, and $a = \hbar\sqrt{\beta/2m}$ is the characteristic radius.

In principle, for practical purposes the relation $a \approx \tilde{a}$ can be assumed throughout this paper, which essentially means that the parameters of the short-scale model are not entirely independent but must be bound by (at least) one physical constraint, $\beta^{-1} \approx \tilde{p}^2/2m$, where $\tilde{p} = \hbar/\tilde{a}$ is the de Broglie momentum corresponding to the length scale $\tilde{a}$.

It should be emphasized that the object described by (4) is different from the classical droplet since it does not have border in a classical sense, therefore, its stability is supported not by surface tension but by nonlinear quantum effects in the bulk [29].

The long-wavelength part can be formulated as follows. As long as we have established that the nucleation and stability of volume elements of the Gaussian type is justified on quantum-mechanical grounds, it is natural to assume that below the critical temperature the atoms tend to form the fluid elements of this kind. Therefore, new collective degrees of freedom arise, so a theory must be formulated in terms of the volume elements rather than of the helium atoms themselves [10]. Besides, as long as the bosonic liquid can no longer be assumed homogeneous one cannot use all the results of section 4 from [29]. Thus, we use the following Hamiltonian

$$\hat{H} = \int d^3x\, \hat{\psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2M} \vec{\nabla}^2 \right) \hat{\psi}(\vec{x}) + \hat{H}_{\rm int}, \qquad (5)$$

where M is the mass of the volume element and $\hat{\psi}$ is the corresponding field operator. The interaction is defined via the nonlocal term

$$\hat{H}_{\rm int} = \tfrac{1}{2} \int\int d^3x\, d^3x'\, \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}') U(|\vec{x} - \vec{x}'|)\, \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x}),$$

where $U(r)$ is the energy of interaction between volume elements. This energy can be estimated in the following way. Using (4) one can derive that the Gaussian volume element of size $r \sim a$ stores an amount of internal bulk mass-energy

$$\epsilon(r) \propto \int_0^r n(r') r'^2 dr' \propto \frac{1}{a}(r - r_0)\, \mathrm{e}^{-(r/a)^2} \left[1 + \mathcal{O}\left(r - a \right) \right],$$

where the value $r_0 = a\left[1/2 + 1/(\mathrm{e}\sqrt{\pi}\,\mathrm{erf}(1))\right] \approx 0.75\, a$ refers to the point where the dominant term of $\epsilon(r)$ changes sign. Since each element has been shown to be stable with respect to small perturbations, it tries to maintain its size and mass when interacting with immediate environment, therefore, to alter these values one has to supply the amount of energy which is proportional to $\epsilon(r)$. In absence of strong external fields this energy can come only via interaction with other volume elements hence we can conclude that $U(|\vec{x} - \vec{x}'|) \propto \epsilon(|\vec{x} - \vec{x}'|)$. Thus we can introduce the proportionality factor $U_0(\vec{x})$, assume it to be a constant in a leading-order approximation, and estimate the interparticle potential as:

$$U(r) = \frac{U_0}{a}(r - r_0)\, \mathrm{e}^{-(r/a)^2}, \qquad (6)$$

up to the terms of order $\mathcal{O}\left(r - a\right)$ which are assumed to be small. These terms can be safely omitted unless interaction deforms the elements so strongly that their interior structure cannot be neglected anymore; but then the element's spherical symmetry becomes deformed as well. The quantity $U_0 = -aU(0)/r_0 \approx -1.34\, U(0)$ becomes the free parameter of the long-wavelength part of the theory. If U_0 is positive then the critical radius r_0 determines the inter-element separation below which a pair of neighboring volume elements becomes unstable against coalescence. In principle, any possible effects of the deformations of elements can be accounted for by further upgrading the constant U_0 to a correspondingly derived function $U_0(\vec{x})$ but for the purposes of our current study the approximation $U_0(\vec{x}) \approx U_0$ will do the job, as will be shown below.

3. OBSERVABLES

In this section we derive the analytical expressions for the main observables of the theory: energy of excitations, structure factor and speed of sound.

3.1. Energy of excitations

To derive the energy spectrum of excitations one can use three alternative approaches. The first one is based on the perturbation theory [16], second is the standard Bogoliubov transform [30], and third is about analyzing small perturbations of the equation of motion for the field operator - as shown in [18] by the example of the semi-transparent sphere model. In our case one can check that all three approaches yield the same spectrum in the leading approximation. The Bogoliubov's method is the most straightforward and simple one, hence it will be used from now on. After some algebra we arrive at the following N-body Hamiltonian operator

$$\hat{H} \approx \frac{1}{2} n_0^2 \bar{U}_0 V + \sum_{\vec{p} \neq 0} E_p \hat{a}_p^\dagger \hat{a}_p, \qquad (7)$$

where $\hat{a}_p^\dagger$ and $\hat{a}_p$ are the creation and annihilation operators of the quasi-particle with momentum $\vec{p}$, $n_0 = N/V = \rho_{\rm He}/M$ is the background particle density of the liquid of N elements occupying the volume V, $\rho_{\rm He} \approx 0.145$ g/cm^3 is the liquid helium-4 density, $\bar{U}_p = \int U(x)\, \mathrm{e}^{i\vec{p}\cdot\vec{x}/\hbar} d^3x$ is the Fourier transform of the interaction potential (6), and E_p is the quasi-particle's energy counted from a ground state. The energy obeys the following dispersion relation:

$$E_p = \frac{p^2}{2M} \sqrt{1 + \frac{4 n_0 M \bar{U}_p}{p^2}}, \qquad (8)$$

with the Fourier transform being calculated as

$$\bar{U}_p = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k\, \mathrm{e}^{-(ak/2)^2} \right), \qquad (9)$$

where $f_k = r_0/a + \left[\frac{1}{2}ak - (ak)^{-1}\right]\mathrm{erfi}(ak/2)$ and $\vec{k} = \vec{p}/\hbar$ is the wave vector.

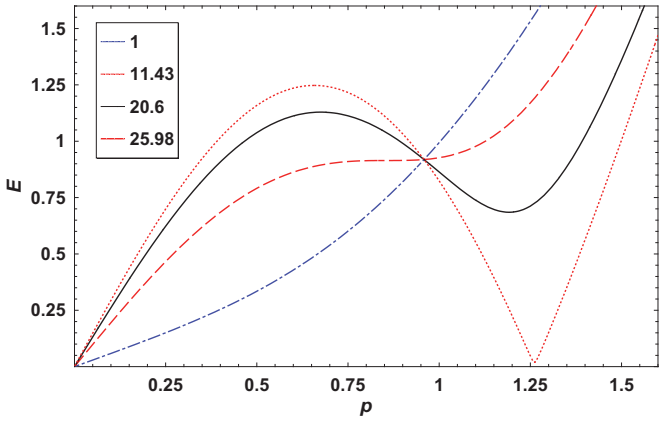

FIG. 2: Energy of quasi-particles E_p versus momentum (in units of E_a and p_a, respectively) for different values of u_0. Below the value 11.43 the roton minimum disappears, and above 25.98 energy becomes complex-valued.

Introducing the volume element's de Broglie momentum scale $p_a = 2\hbar/a$ and corresponding energy scale $E_a = p_a^2/2M$, one can re-write the dispersion relation (8) in the dimensionless form which shows that the dynamics of the quasi-particle depends on just one essential (non-scale) parameter

$$u_0 = 2\pi n_0 a^5 M U_0/\hbar^2. \tag{10}$$

Numerical analysis shows that admissible values of this parameter lie between 11.43 and 25.98 where the local (so-called roton) minimum appears, see Fig. 2. In its vicinity the dispersion relation can be written in the Landau form, $E_p \approx \Delta + \frac{(p-p_0)^2}{2\mu}$, where Δ is the roton energy gap, p_0 is the minimum value of momentum, μ is the effective roton mass. The corresponding profiles for different values of u_0 are given in Fig. 3 from where one can see that the roton mass diverges at the lower bound (the local minimum becomes a saddle point) whereas the energy becomes complex-valued above the upper bound.

We can also plot the quasi-particle's velocity $\vec{v} = v\vec{\ell}$ where $v = \partial E_p/\partial p$ and $\vec{\ell} = \vec{p}/p$. From Fig. 4 one can see that within the above-mentioned range of the parameter u_0 the liquid indeed exhibits the superfluidity feature - an interval of momentum and energy over which the effective mass p/v turns negative always exists, thus indicating that the production of dissipative excitations is suppressed. As a matter of fact, this is just a restatement of the original Landau idea of introducing the roton minimum into the excitation spectrum to explain the superfluidity - since a negative p/v range always appears if a positive-definite function has the local minimum preceded by a local maximum. One can also notice that at small momenta (phonon regime) the velocity varies very slowly as a function of frequency $\omega = E_p/\hbar$, as if there is no dispersion at all.

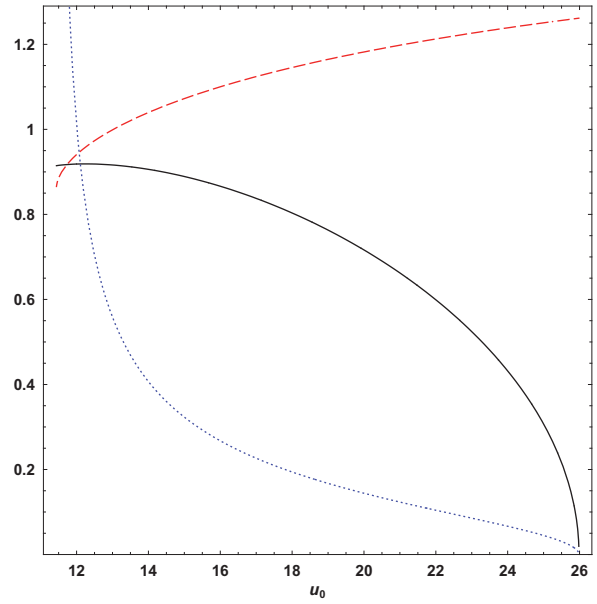

FIG. 3: Local minimum's values of momentum p_0/p_a (dashed curve), roton energy gap Δ/E_a (solid) and mass μ/M (dotted) versus the interaction coupling constant taken between $u_0 \approx 11.43$ and 25.98.

3.2. Speed of sound

By sound here we understand the conventional acoustic oscillations, the heat transfer and second sound are not considered since the temperature effects are neglected throughout the paper, as mentioned in the introduction.

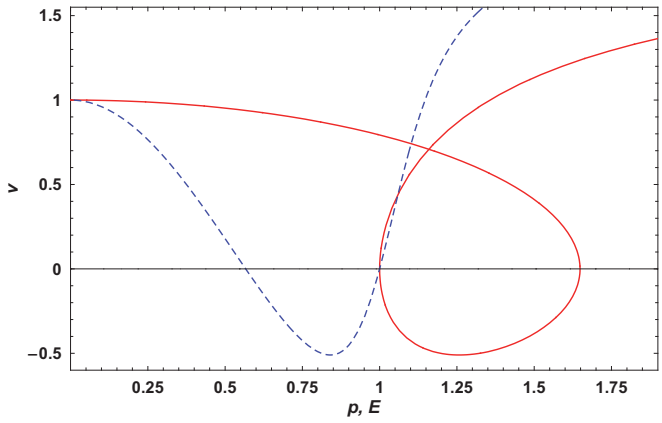

FIG. 4: Quasi-particle's velocity v (in units of $v|_{p=0}$) versus momentum (in units of p_0, dashed curve) and energy (in units of Δ, solid curve). It is plotted at $u_0 \approx 20.6$ (then $v|_{p=0} \approx 273$ m/s) but such behavior holds qualitatively for u_0 between 11.43 and 25.98.

Thus, our observable of interest is a mean value of the sound velocity $\langle v_s \rangle$ which can be estimated as follows. If size and interior structure of volume elements are ne-

glected then the speed of sound would be given just by the standard formula [22]

$$v_s^{(0)} = \sqrt{n_0 \bar{U}_0 / M} = \lim_{p \to 0} v. \tag{11}$$

However, in general one should take into account that sound propagation is affected by the volume element's "interior" which is described by the logarithmic Bose liquid in a ground state, and speed of sound inside the latter is known to be

$$c_s = 1/\sqrt{m|\beta|}, \tag{12}$$

in the leading approximation [29]. The mean speed of sound thus can be estimated as an averaged sum of the background and excitation contributions taken with the weights controlled by the interior density of a volume element (4),

$$\langle v_s \rangle = \bar{a}^{-1} \int_0^{\bar{a}} \left[\chi c_s + (1-\chi)\, v_s^{(0)} \right] dr = v_s^{(0)} \left[1 - \frac{\sqrt{\pi}}{4} \frac{a}{\bar{a}} \operatorname{erf}(\bar{a}/a) \left(1 - \frac{c_s}{v_s^{(0)}} \right) \right], \tag{13}$$

where $\chi = (1/2)\, n(\vec{x})/n(0) = \frac{1}{2} \mathrm{e}^{-(r/a)^2}$ is the weight factor, $\frac{4}{3}\pi\bar{a}^3 = 1/n_0 = M/\rho_{\mathrm{He}}$ is the effective packing volume of an element. The numeric coefficient inside the weight factor has been chosen in such a way as to satisfy the following limit case properties: if $a \to 0$ then the $v_s^{(0)}$ term dominates (zero-size limit), and if $a \to \infty$ then both terms contribute equally.

3.3. Structure factor

According to (4), our volume elements are essentially quantum states of delocalized atoms which do not have a border or interface in a classical sense. Therefore, their presence in the liquid helium can be judged only indirectly - via the structure factor S_k which determines the scattering of neutrons (or x-rays, after multiplication by the atomic structure factor) and can be measured experimentally. This situation, however, is not something extraordinary: the microscopical objects employed in previous models of liquid helium, such as vortex rings [19], hard spheres [17, 18] or stochastic clusters [9], were never claimed to be "directly seen" (whatever it could mean in our essentially quantum context) either. Instead, if a selected model's predictions, e.g. for the excitation energy and structure factor, coincide with experimentally observed values then the corresponding microscopical interpretation can be adopted - at least, until the better one arrives.

The computation of the structure factor is slightly more difficult than that of the energy of excitations: as shown in the previous section, the computation of E_p requires, basically, the knowledge of the inter-element interaction potential, such as (6), only. By contrast, to derive the structure factor one should derive the excited-state wavefunction of the liquid helium which can be done only approximately under certain simplifying assumptions. If one considers the "bare" helium atoms to be the fundamental degrees of freedom then the structure factor would be given by the formula derived in [19]. In our case it would yield

$$S_k^{(0)} = p^2/2ME_p = \left(1 + \frac{4n_0 M \bar{U}_p}{p^2} \right)^{-1/2}, \tag{14}$$

where all quantities have been defined in the previous section. However, this formula can not be blindly applied to our case since we use the collective degrees of freedom which are not "bare" helium atoms but their bound states. Besides, the logarithmic non-linearity affects not only the energy of excitations but also the correlation functions themselves. Therefore, one needs to slightly revise the Feynman's derivation by taking these effects into account - especially if one considers that Feynman and Cohen's theory gives only a qualitative agreement with experiment. Fortunately, the computation is straightforward and necessary corrections are minimal (yet important). We begin by assuming that the total energy of the system of N fluid elements comes from minimizing the integral $\mathcal{E} = \int \psi^* \mathcal{H} \psi \, d^{3N}x$, subject to the fixed normalization integral $\mathcal{J} = \int \psi^* \psi \, d^{3N}x$, where ψ is the excited-state wavefunction. Throughout this section we use $\vec{x}^N$ to denote the set of coordinates $\vec{x}_i$ of all the elements, and $\int d^{3N}x$ to represent the integral over all of them. The Hamiltonian of the system is given by a standard expression,

$$\mathcal{H} = \mathcal{H}_0 + V = -\frac{\hbar^2}{2M} \sum_i \vec{\nabla}_i^2 + V, \tag{15}$$

where V is the potential energy of the system and Latin indices run from 1 to N. Following Feynman, for the excited-state wavefunction we assume the Bijl ansatz [31]:

$$\psi = F\psi_0 = \sum_i f(\vec{x}_i)\psi_0, \quad f(\vec{x}) \equiv \mathrm{e}^{i\vec{k}\cdot\vec{x}}, \tag{16}$$

with the only difference that $\psi_0 = \psi_0(\vec{x}^N)$ is now the ground-state wavefunction of one-particle states of logarithmic quantum liquid (not plain atoms as was in the original Feynman derivation). Due to the separability property of the logarithmic Schrödinger equation, they satisfy its stationary many-body analogue:

$$\mathcal{H}_0\psi_0 = \frac{m}{M}\left[NE_0 + \beta^{-1}\ln\left(\tilde{a}^{3N}|\psi_0|^2\right)\right]\psi_0, \qquad (17)$$

where the logarithmic term has replaced the potential part, according to the two-scale structure of our theory (we remind also that throughout the paper the external potential is neglected). With the use of the separability property the N-particle equation (17) can be analytically solved for the case of a ground state, cf. section 2. One thus obtains the eigenvalue $E_0 = 3\beta^{-1}\left[1+\ln\left(\sqrt{\pi}a/\tilde{a}\right)\right] \approx 3\beta^{-1}\left(1+\frac{1}{2}\ln\pi\right)$ and $\psi_0 \propto \prod_i \Psi(\vec{x}_i) \propto \prod_i \exp\left(-|\vec{x}_i|^2/2a^2\right)$, cf. [28].

Further, with the ansatz (16) in hand the energy per particle of the system can be approximately written as

$$E_p + E_0 = \frac{\int\left[\frac{\hbar^2}{2M}\frac{1}{N}\sum_i \vec{\nabla}_i F^*\cdot\vec{\nabla}_i F + \frac{m}{M}|F|^2\left(E_0 + \beta^{-1}\ln\left(\tilde{a}^3\rho_N^{1/N}\right)\right)\right]\rho_N d^{3N}x}{\int |F|^2\rho_N d^{3N}x}, \qquad (18)$$

where

$$\rho_N = \left(\frac{1}{\pi^{3/2}a^3}\right)^N \mathrm{e}^{-\sum_i |\vec{x}_i|^2/a^2} \qquad (19)$$

is the ground-state density function. By construction, excitation energy E_p takes into account also the inter-element interaction, see Sec. 3.1. Further, using the fact that logarithm varies significantly slower than the Gaussian, we obtain

$$E_p + E_0 - \frac{m}{M}\left[E_0 + \beta^{-1}\ln\left(\tilde{a}^3\rho_N^{1/N}(0)\right)\right] = \frac{\hbar^2}{2M}\frac{\frac{1}{N}\sum_i\int\vec{\nabla}_i F^*\cdot\vec{\nabla}_i F\rho_N d^{3N}x}{\int|F|^2\rho_N d^{3N}x}, \qquad (20)$$

which after integration reduces to

$$E_p + \Sigma = \frac{\hbar^2}{2M}\frac{\int\vec{\nabla}f^*(\vec{x})\cdot\vec{\nabla}f(\vec{x})d^3x}{\int f^*(\vec{x}_1)f(\vec{x}_2)P(\vec{x}_1-\vec{x}_2)d^3x_1 d^3x_2}, \qquad (21)$$

where $P(\vec{x}_1-\vec{x}_2)$ is the probability of finding an element at $\vec{x}_2$ per unit volume if one is known to be at $\vec{x}_1$, and we denoted

$$\Sigma \approx E_0 - \frac{m}{\beta M}\left(3+\ln\pi\right) \approx 3\beta^{-1}\left[1-\frac{m}{M}+\frac{1}{2}\left(1-\frac{2}{3}\frac{m}{M}\right)\ln\pi\right].$$

The variation with respect to f^* yields the equation

$$\left(E_p+\Sigma\right)\int f(\vec{x}_2)P(\vec{x}_1-\vec{x}_2)d^3x_2 + \frac{\hbar^2}{2M}\vec{\nabla}^2 f(\vec{x}_1) = 0, \qquad (22)$$

and upon remembering (16) this gives us the final formula for the structure factor $S_k \equiv \int P(\vec{x})\mathrm{e}^{i\vec{k}\cdot\vec{x}}d^3x$ in a logarithmic theory:

$$S_k = \frac{p^2}{2M\left(E_p+\Sigma\right)} = \left[\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}}+\frac{2M\Sigma}{p^2}\right]^{-1}. \qquad (23)$$

Thus, the only difference from the "non-logarithmic" formula (14) is the appearance of an additive constant term in the denominator which is induced by the logarithmic nonlinearity.

4. THEORY VERSUS EXPERIMENT

In this section we compare the derived observables of our model with available experimental data. If we choose the value $u_0 \approx 20.6$ then the established experimental data for energy of excitations (see, for instance [5]) can be successfully fit. When fitting, we do not use the least-squares techniques but rather aim at the precise position of the roton minimum - so as to estimate the magnitude of deviations from experiment at the neighboring local

maximum $E_{\max}$ at $p = p_{\max}$ also known as the *maxon* peak. It turns out that our theory can reproduce the maximum with accuracy which is better than three percent, see the cumulative comparison between theory and experiment given in Fig.5 and Table I. The other values of primary and secondary parameters of the theory computed at $u_0 \approx 20.6$ turn out to be the following:

$$\begin{aligned} &\bar{a} \approx 2.37 \text{ Å}, \ M/m \approx 1.228, \ U_0/\Delta \approx 69, \\ &a \approx 1.25 \text{ Å}, \ p_a/p_0 \approx 0.84, \ E_a/\Delta \approx 1.46, \\ &\beta^{-1}/k_B \approx 3.84 \text{ K}, \ \beta^{-1}/E_a \approx 0.31, \end{aligned} \tag{24}$$

hence this set will be used for further comparisons with experiments in this paper. For instance, using (24) the velocity-related quantities from section 3.2 are evaluated as $c_s \approx 89$ m/s and $v_s^{(0)} \approx 273$ m/s, therefore, the mean value of the speed of sound (13) can be estimated as

$$\langle v_s \rangle \approx 231 \text{ m/s}, \tag{25}$$

which is in a good agreement with experiment as well [32].

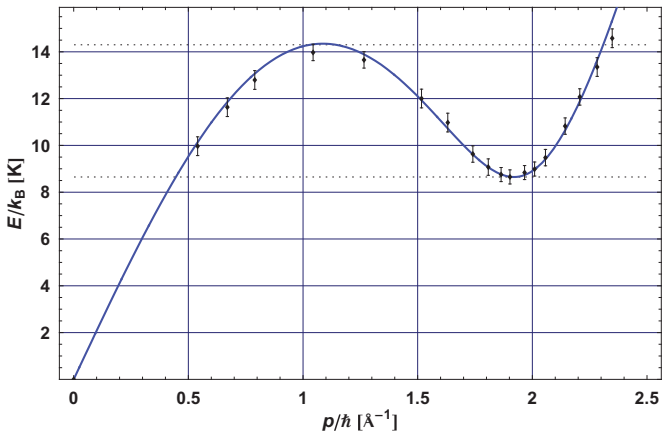

FIG. 5: Energy of quasi-particles E_p versus wave vector. Solid curve is a theoretical prediction, dots denote experimental data [5].

Finally, the structure factor (23) is also evaluated using the values (24). We compare the theoretical curve with the neutron scattering data in Fig. 6 (the neutron scattering is the most relevant method here since it is the one which is being used for measuring E_p as well). One can see that the asymptotic behaviour of the structure factor, both at small and large momenta, is compatible with known experimental data, see also the discussion in the concluding section. For instance, from the scattering data one can not surely conclude that at small momenta S_k must tend to zero as a linear function of p. Indeed, as one can see from [7] (see also the first figure from the Feynman-Cohen paper in [19]), S_k is not a straight line near origin but rather a curve which resembles more of a parabola and thus agrees with the asymptotic behaviour of (23) at $p \to 0$. Though, one can not *a priori* exclude that in real scattering experiments some side effects might appear and introduce the asymptotic corrections proportional to p and thus override the non-linear asymptotics; however, these additional effects are not a subject of our study at this stage.

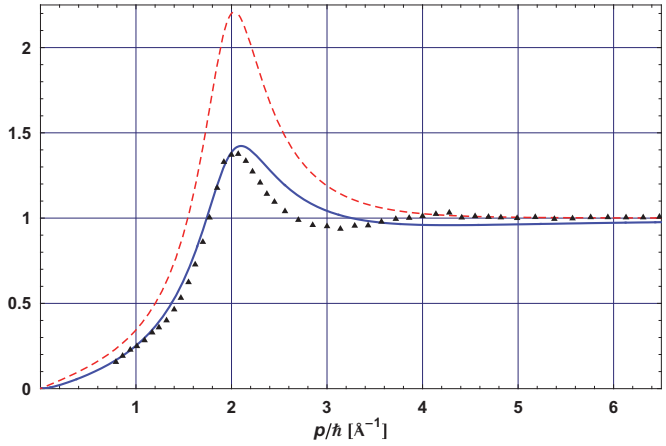

FIG. 6: Structure factor S_k versus wave vector. Solid curve is a theoretical prediction given by (23) evaluated at the values (24), triangles denote the experimental data [7]. For the sake of comparison, the result of the formula (14) is being plotted as well (dashed curve).

5. CONCLUSION

The superfluidity of liquid helium II is the complex phenomenon exhibiting a unique feature of the Landau spectrum of excitations which successfully explains the behavior of specific heat, viscosity and sound velocity. The microscopical theory of this phenomenon must address many different aspects such as the Bose-Einstein condensation, appearance of collective degrees of freedom, non-locality and continuity, formation of the volume element of the liquid, its structure and stability.

Here we proposed a concise analytical theory of structure and excitations in superfluid helium which elaborates on these aspects. It consists of two models which act on different length scales, but are connected via the parametric space: the behaviour of quantities and the values

TABLE I: Theory versus experiment

Quantity	Theory	Experiment	Difference,%
Δ/k_B, K	8.65 (fit)	8.65±0.04	0±0.4
$p_0/\hbar$, Å^{-1}	1.91 (fit)	1.91±0.01	0±0.5
μ/m	0.16	0.16±0.01	0±6
$E_{\max}/k_B$, K	14.3	13.92±0.1	2.7∓0.7
$p_{\max}/\hbar$, Å^{-1}	1.08	1.11±0.04	2.6±3.6
$\langle v_s \rangle$, m/s	231	237±2	2.6±0.08

of parameters in the long-wavelength model are derived from the short-wavelength part. The latter advocates the appearance of the collective degrees of freedom which justify the possibility of the fluid-dynamical description and can be used for characterize the volume elements of the quantum liquid. The "interior" structure of a fluid volume element is described using the non-perturbative approach based on the logarithmic wave equation. It turns out that the interior density of the element obeys the Gaussian distribution.

The long-wavelength part is the quantum many-body theory of the volume elements as effectively point-like objects - yet their spatial extent and internal structure are taken into account by virtue of the nonlocal interaction term. The corresponding interparticle interaction potential (6) was not postulated but derived from the short-wavelength part, and it appeared to be very simple as compared to the empirical and semi-empirical inter-molecular potentials widely used in perturbative approaches [20]. On the other hand, more fundamental approaches, such as those based on vortex rings [19], hard spheres [17, 18] or stochastic clusters [9], either have shown so far only the qualitative agreement with experiment, to our best knowledge, or some of observables in those models were not computed. In our theory the quantitative agreement with various experimental data has been achieved: with only one essential parameter in hand, u_0, we reproduced at high accuracy (better than three per cent) not only the roton minimum but also the neighboring local (maxon) maximum. The velocity of sound and structure factor are also computed and found to be in a very good agreement with experiment. Slight deviations of the energy spectrum and structure factor curves from experimentally observed ones at large values of momenta $p \gtrsim 2\hbar$ Å^{-1} come due to the scattering particles begin to probe the interior of fluid elements and thus affect the corresponding wavefunctions. At even higher momenta, $p \gtrsim 4\hbar$ Å^{-1}, the agreement will eventually recover since the contributions from kinetic energy come to predominate those from interaction.

To conclude, the collective variables we have been using ultimately provide a unified description of the phonon, maxon and roton excitations. In future it would be interesting to study the applications of the models with the non-polynomial ground-state wave equations and Gaussian-like interparticle potentials, like (1) and (6), in the physical situations when superfluid becomes effectively low-dimensional (cigar- or disk-shaped) and/or subjected to external fields or admixtures. A parallel direction of research would be to take into account temperature effects (see the introductory section), such as heat transfer, second sound, etc., which is necessary for a number of practical applications.

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