Large book–cycle Ramsey numbers

Qizhong Lin* and Xing Peng†

Abstract

Let $B_n^{(k)}$ be the book graph which consists of n copies of K_{k+1} all sharing a common K_k, and let C_m be a cycle of length m. In this paper, we first determine the exact value of $r(B_n^{(2)}, C_m)$ for $\frac{8}{9}n + 112 \leq m \leq \lceil \frac{3n}{2} \rceil + 1$ and $n \geq 1000$. This answers a question of Faudree, Rousseau and Sheehan (Cycle–book Ramsey numbers, *Ars Combin.,* **31** (1991), 239–248) in a stronger form when m and n are large. Building upon this exact result, we are able to determine the asymptotic value of $r(B_n^{(k)}, C_n)$ for each $k \geq 3$. Namely, we prove that for each $k \geq 3$, $r(B_n^{(k)}, C_n) = (k + 1 + o_k(1))n$. This extends a result due to Rousseau and Sheehan (A class of Ramsey problems involving trees, *J. London Math. Soc.,* **18** (1978), 392–396).

Keywords: Ramsey number; Regularity lemma; Book; Cycle

1 Introduction

For graphs H_1 and H_2, the Ramsey number $r(H_1, H_2)$ is the minimum integer N such that every red-blue edge coloring of the complete graph K_N contains either a red H_1 or a blue H_2. Let $B_n^{(k)}$ be the book graph which consists of n copies of K_{k+1} all sharing a common K_k. When $k = 2$, we write B_n instead of $B_n^{(2)}$ for convenience. Book Ramsey numbers have attracted a lot of attention, see [12, 25, 29, 22, 23, 24] and other related references. In particular, answering a question of Erdős et al. [12], Conlon [8] established an asymptotic version of Thomason's conjecture [29] by showing

$$r(B_n^{(k)}, B_n^{(k)}) = (2^k + o_k(1))n.$$

The upper bound was improved to $2^k n + O_k\big(\frac{n}{(\log\log\log n)^{1/25}}\big)$ by using a different method, see Conlon, Fox and Wigderson [9].

Let C_m and T_m be a cycle and a tree of order m, respectively. The Ramsey numbers of book versus tree and book versus cycle also received a great deal of attention. Strengthening a classical result due to Chvátal [7], Rousseau and Sheehan [26] established that

$$r(B_n^{(k)}, T_n) = (k + 1)(n - 1) + 1. \tag{1}$$

*Center for Discrete Mathematics, Fuzhou University, Fuzhou, 350108 P. R. China. Email: linqizhong@fzu.edu.cn. Supported in part by NSFC(No. 11671088) and "New century excellent talents support plan for institutions of higher learning in Fujian province"(SJ2017-29).

†School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China. E-mail: x2peng@ahu.edu.cn. Supported by the NSFC grants (No. 11601380, 12071002) and a Start–up Fund from Anhui University.

For more book-tree Ramsey numbers, see e.g. [26, 13, 6, 16]. A natural question is whether we can prove a similar result for book-cycle Ramsey number when they have nearly equal order.

The study of book-cycle Ramsey numbers goes back to [25] by Rousseau and Sheehan. In particular, they proved $r(B_n, C_3) = 2n+3$ for $n > 1$. In [15, 17], Faudree, Rousseau and Sheehan proved some results for $r(B_n, C_4)$, and generally,

$$r(B_n, C_m) = \begin{cases} 2n+3 & \text{if } m \geq 5 \text{ is odd and } m \leq \frac{n}{4} + \frac{13}{4}, \\ 2m-1 & \text{if } m \geq 2n+2. \end{cases}$$

Improving upon the result in [17], Shi [27] obtained that $r(B_n, C_m) = 2m-1$ for $m > \frac{3n}{2} + \frac{7}{4}$. In the same paper, the author also obtained $r(B_n^{(3)}, C_m) = 3m-2$ for $m > \max\{\frac{6n+7}{4}, 70\}$. For fixed $k \geq 1$ and odd $m \geq 3$, Liu and Li [20] proved that $r(B_n^{(k)}, C_m) = 2(n+k-1)+1$ when n is large. One can easily see that $r(B_n, C_m) > 3(n-1) \geq \max\{2m-1, 2n+3\}$ for $6 \leq n \leq m \leq \frac{3n}{2} - 1$, and $r(B_n, C_m) > 3(m-1) \geq \max\{2m-1, 2n+3\}$ for $\frac{2n}{3} + 2 \leq m \leq n$. This suggests that the formula for $r(B_n, C_m)$ varies when m and n change, especially when m and n are nearly equal. As mentioned in [17], "the problem of computing $r(B_n, C_m)$ when m is odd and m and n are nearly equal provides an unanswered test of strength".

The goal of this paper is to study the Ramsey number $r(B_n^{(k)}, C_m)$ when n and m are nearly equal. First, we determine the exact value of $r(B_n, C_m)$ for $\frac{8}{9}n + 112 \leq m \leq \lceil \frac{3n}{2} \rceil + 1$ and $n \geq 1000$, which provides an answer to the question by Faudree, Rousseau and Sheehan [17] in a stronger form when m and n are large.

Theorem 1 *For $n \geq 1000$,*

$$r(B_n, C_m) = \begin{cases} 3m-2 & \text{if } \frac{8n}{9} + 112 \leq m \leq n, \\ 3n-1 & \text{if } m = n+1, \\ 3n & \text{if } n+2 \leq m \leq \frac{3n+1}{2}, \\ 2m-1 & \text{if } m = \lceil \frac{3n}{2} \rceil + 1. \end{cases}$$

Remark 1. We observe that the formula for $r(B_n, C_m)$ undergos phase transitions when $m \in \{n, n+1, n+2\}$.

So far, the value of $r(B_n, C_m)$ is known for $m \geq \frac{8n}{9}n + 112$ as well as odd m with $m \leq \frac{n}{4} + \frac{13}{4}$. It requires new ideas to determine the value of $r(B_n, C_m)$ when m and n are in other ranges.

Based on Theorem 1, we extend (1) by showing the asymptotic value of $r(B_n^{(k)}, C_n)$ for each fixed integer $k \geq 3$ as follows.

Theorem 2 *Let $k \geq 3$ be a fixed integer. We have*

$$r(B_n^{(k)}, C_n) = (k+1+o_k(1))n.$$

It is a challenge to determine the exact value of $r(B_n^{(k)}, C_m)$ when m and n are nearly equal for each $k \geq 3$.

Throughout this paper, we will use the following notation. Let G be a graph with vertex set V. For each vertex $v \in V$, we use $N_G(v)$ and $d_G(v)$ to denote the *neighborhood* and the *degree*, respectively. If $U \subseteq V$, then $G[U]$ denotes the subgraph induced by U. Moreover, let $N_G(v, U) = N_G(v) \cap U$ and $d_G(v, U) = |N_G(v, U)|$. The graph $G - v$ is the one obtained from G by deleting the vertex v and all edges incident to v.

The rest of the paper is organized as follows. In Section 2, we will collect several results which will be used to prove our results. In Section 3, we will present proofs of Theorem 1 and Theorem 2.

2 Preliminaries

In this section, we collect a number of previous results which are needed for our proofs. Crucial tools include a refined version of Szemerédi's regularity lemma and the (weakly) pancyclic properties of graphs.

2.1 Regularity method

Szemerédi's regularity lemma [28] is a powerful tool in extremal graph theory. The regularity lemma is also called the uniformity lemma, see e.g., Bollobás [3] and Gowers [18]. Applications of the regularity method are fruitful. We refer the reader to the survey of Komlós and Simonovits [19] and other related references.

A key tool in the proof of Theorem 2 is a refined version of the Szemerédi's regularity lemma by Conlon [8]. We state it and its related results as following.

Let G be a graph defined on vertex set $V = V(G)$. For $X, Y \subseteq V$, denote $e_G(X, Y)$ by the number of pairs in $X \times Y$ that are edges of G. The ratio

$$d_G(X, Y) = \frac{e_G(X, Y)}{|X||Y|}$$

is called the *edge density* of (X, Y) in G, which can be understood as the probability that a random pair (x, y) from $X \times Y$ is an edge. If $X \cap Y \neq \emptyset$, then edges in $X \cap Y$ are counted twice.

For $\epsilon > 0$, a pair (U, W) of nonempty sets $U, W \subseteq V$ is called *ϵ-regular* if

$$|d_G(X, Y) - d_G(U, W)| \leq \epsilon$$

for every $X \subseteq U, Y \subseteq W$ such that $|X| \geq \epsilon|U|$ and $|Y| \geq \epsilon|W|$. We say a subset U is ϵ-regular if the pair (U, U) is ϵ-regular.

An *equitable partition* of a graph G is a partition $V(G) = \sqcup_{i=1}^m V_i$ of the vertex set of G such that $\big||V_i| - |V_j|\big| \leq 1$ for all i and j.

We mention two properties for ϵ-regular pairs, see e.g. [19].

Fact 1 *Let (U, W) be an ϵ-regular pair with edge density d. If $Y \subseteq W$ with $|Y| \geq \epsilon|W|$, then there exists a subset $U' \subseteq U$ with $|U'| \geq (1-\epsilon)|U|$ such that each vertex in U' is adjacent to at least $(d-\epsilon)|Y|$ vertices in Y.*

Fact 2 *Let (U, W) be an ϵ-regular pair in graph G. If $X \subseteq U$, $Y \subseteq W$ with $|X| \geq \gamma|U|$ and $|Y| \geq \gamma|W|$ for some $\gamma > \epsilon$, then (X, Y) is ϵ'-regular such that $|d_G(U, W) - d_G(X, Y)| \leq \epsilon$, where $\epsilon' = \max\{\epsilon/\gamma, 2\epsilon\}$.*

We need the following refined version of the regularity lemma by Conlon [8, Lemma 3]. In the same spirit, to prove an induced removal lemma, Alon et. al [1] obtained a result in which all pairs (W_i, W_j) are ϵ-regular.

Lemma 1 *For every $0 < \epsilon < 1$ and natural number m_0, there exists a natural number M such that every graph G with at least m_0 vertices has an equitable partition $V(G) = \sqcup_{i=1}^{m} V_i$ with $m_0 \leq m \leq M$ parts and subsets $W_i \subset V_i$ such that W_i is ϵ-regular for all i and, for all but ϵm^2 pairs (i, j) with $1 \leq i \neq j \leq m, (V_i, V_j)$, (W_i, V_j) and (W_i, W_j) are ϵ-regular with $|d_G(W_i, V_j) - d_G(V_i, V_j)| \leq \epsilon$ and $|d_G(W_i, W_j) - d_G(V_i, V_j)| \leq \epsilon$.*

We will also use the following counting lemma from [8, Lemma 5].

Lemma 2 *For any $\delta > 0$ and any natural number k, there is $\eta > 0$ such that if $U_1, \ldots, U_k$, $U_{k+1}, \ldots, U_{k+\ell}$ are (not necessarily distinct) vertex sets with $(U_i, U_{i'})$ η-regular of density $d_{i,i'}$ for all $1 \leq i < i' \leq k$ and $1 \leq i \leq k < i' \leq k + \ell$ and $d_{i,i'} \geq \delta$ for all $1 \leq i < i' \leq k$, then there is a copy of K_k with vertex $u_i \in U_i$ for each $1 \leq i \leq k$ which is contained in at least*

$$\sum_{j=1}^{\ell} \left(\prod_{i=1}^{k} d_{i,k+j} - \delta \right) |U_{k+j}|$$

labeled copies of K_{k+1} with vertex u_{k+1} in $\cup_{j=1}^{\ell} U_{k+j}$.

The next lemma due to Benevides and Skokan [2] is a stronger version of the original one by Łuczak [21, Claim 3]. Both have similar proofs by using Fact 1.

Lemma 3 *For every $0 < \beta < 1$, there exists an n_0 such that for every $n > n_0$ the following holds: Let G be a bipartite graph with bipartition $V(G) = U \cup W$ such that $|U| = |W| = n$. Furthermore, let the pair (U, W) be ϵ-regular with density at least β for some ϵ satisfying $0 < \epsilon < \beta/100$. Then for each ℓ, $1 \leq \ell \leq n - 5\epsilon n/\beta$, and for each pair of vertices $u \in U$, $w \in W$ with $d_G(u) \geq 4\beta n/5$ and $d_G(w) \geq 4\beta n/5$, G contains a path of length $2\ell + 1$ connecting u and w.*

The following lemma will be used to find long odd cycles in graphs.

Lemma 4 *Suppose that (A, B) and (B, C) are ϵ-regular pairs with density at least β in a graph G, here $\epsilon < \beta/50$ and A, B and C are pairwise disjoint. If $|A| = |C| = n$ and $|E(B)| \geq |B|^2/5$, then there is an edge $xy \in E(B)$ such that $d(x, A) \geq (\beta - \epsilon)n$ and $d(y, C) \geq (\beta - \epsilon)n$.*

Proof: Let H be the $2\epsilon|B|$-core of the subgraph induced by B, i.e., H is the maximum induced subgraph of B with minimum degree at least $2\epsilon|B|$. As $|E(B)| \geq |B|^2/5$ and ϵ is small enough, we can see H is not empty and $|V(H)| \geq 2\epsilon|B|$. Since (A, B) is an ϵ-regular pair, all but at most $\epsilon|B|$ vertices in H have at least $(\beta - \epsilon)n$ neighbors in A from Fact 1. We assume x is such a vertex. The definition of H gives $N_H(x) \geq 2\epsilon|B|$. Now, (B, C) being an ϵ-regular pair yields that there is a vertex $y \in N_H(x)$ having at least $(\beta - \epsilon)n$ neighbors in C. The edge xy is a desired one and the proof is complete. $\square$

The following lemma by Łucazk [21, Claim 7] is a key ingredient in the proof of Theorem 2.

Lemma 5 *For every $0 < \delta < 10^{-15}, \alpha > 2\delta$ and $n \geq \exp(\delta^{-16}/\alpha)$ the following holds. Each graph G on n vertices which contains no odd cycles longer than αn contains subgraphs G' and G'' such that:*

(1) *$V(G') \cup V(G'') = V(G)$, $V(G') \cap V(G'') = \emptyset$ and each of the sets $V(G')$ and $V(G'')$ is either empty or contains at least $\alpha\delta n/2$ vertices;*

(2) *G' is bipartite;*

(3) *G'' contains not more than $\alpha n|V(G'')|/2$ edges;*

(4) *all but at most δn^2 edges of G belong to either G' or G''.*

2.2 Pancyclic properties of graphs

For a graph G, we use $g(G)$ and $c(G)$ to denote its *girth* and *circumference*, i.e., the length of a shortest cycle and a longest cycle of G. Similarly, the *odd girth* of G is the length of a shortest odd cycle in G. A graph is called *weakly pancyclic* if it contains cycles of every length between its girth and its circumference. A graph is *pancyclic* if it is weakly pancyclic with girth 3 and circumference $n = |V(G)|$. We say a graph is *2-connected* if it remains connected after the deletion of any vertex.

For a graph G, let $\delta(G)$ denote the minimum degree of G. The following classical result is due to Dirac [10].

Lemma 6 *Let G be a 2-connected graph of order n with minimum degree $\delta = \delta(G)$. Then $c(G) \geq \min\{2\delta, n\}$.*

Dirac's result tells us that the circumference of a 2-connected graph cannot be too small. In particular, if $\delta = \delta(G) \geq n/2$, then $c(G) = n$. This is a well-known result for a graph being hamiltonian. For the special case of $\delta \geq n/2$, the following result due to Bondy [4] tells us more about the structure of the graph.

Lemma 7 *If a graph G with n vertices satisfies $\delta(G) \geq n/2$, then G is pancyclic unless $n = 2r$ and $G = K_{r,r}$.*

The following is an elegant extension on graphs being weakly pancyclic by Brandt, Faudree and Goddard [5], which is a key ingredient in the proofs of Theorem 1 and Theorem 2 for $k = 3$.

Lemma 8 *Let G be a 2-connected nonbipartite graph of order n with $\delta(G) \geq n/4 + 250$. Then G is weakly pancyclic unless G has odd girth 7, in which case it has every cycle from 4 up to its circumference except the 5-cycle.*

We will also need the following simple fact which can be seen using the Breadth-First-Search.

Fact 3 *If a graph G with n vertices satisfies $\delta(G) \geq cn$ for some constant $c > 0$, then $g(G) \leq 4$ provided $n > c^{-2}$.*

3 Proofs of Theorems 1 and 2

In this section, we will give proofs for our main results. Throughout the proof, when considering a red-blue edge coloring of K_N, we always use R and B to denote subgraphs formed by red and blue edges, respectively. We also suppose that $n \geq 1000$ for Theorem 1 and n is sufficiently large for Theorem 2.

3.1 Proof of Theorem 1

We first give the following simple fact.

Fact 4 *Let G be a graph which consists of three connected components V_1, V_2 and V_3.*

(i) If the largest connected component has at least n vertices, then the complement $\overline{G}$ contains a B_n.

(ii) If one of sets V_1, V_2 and V_3 contains a non-edge while the other two sets have at least n vertices in total, then the complement $\overline{G}$ contains a B_n.

The proof of Theorem 1 contains three parts.

(I) $\frac{8}{9}n + 112 \le m \le n$

The lower bound $r(B_n, C_m) > 3m - 3$ holds since the graph with three disjoint copies of K_{m-1} contains no C_m and its complement contains no B_n. Thus it suffices to prove the upper bound. Let $N = 3m - 2$, and consider a red-blue edge coloring of K_N on vertex set V.

Case 1. There is a vertex $v \in V$ with $d_R(v) \ge 2n$.

We choose a subset $X \subseteq N_R(v)$ with $|X| = 2n$. If there is a vertex $x \in X$ with $d_R(x, X) \ge n$, then the red subgraph induced by x and v together with their common neighbors in X contains a B_n. Thus we assume $d_B(x, X) \ge n$ for each $x \in X$, i.e. $\delta(B[X]) \ge n$. By Lemma 7, $B[X]$ is pancyclic or $B[X] = K_{n,n}$. There will be a blue C_m if $B[X]$ is pancyclic, so we assume $B[X] = K_{n,n}$ with color classes X_1 and X_2. If there exists a vertex $y \in V \setminus (X \cup \{v\})$ such that $d_B(y, X_i) \ge 1$ for each $i \in \{1, 2\}$, then $B[X \cup \{y\}]$ contains blue cycles of length between 3 and $2n + 1$ and it definitely contains a blue C_m. Thus each vertex of $V \setminus (X \cup \{v\})$ is completely red-adjacent to X_1 or X_2. Suppose that $z \in V \setminus (X \cup \{v\})$ is red-adjacent to X_1. It follows that $R[X_1 \cup \{z, v\}]$ contains a red $K_{n+2} - e$ and definitely a red B_n. We are through in this case.

Case 2. For each vertex $v \in V$, $d_R(v) \le 2n - 1$. i.e., $\delta(B) \ge 3m - 2n - 2$.

If B is bipartite, then the larger color class of B contains at least $\frac{3m-2}{2} \ge n + 2$ vertices and induces a red clique of size at least $n + 2$. Therefore, there is a red B_n.

If B is 2-connected, then we can find a blue C_m as follows. Since

$$\delta(B) \ge 3m - 2n - 2 \ge \frac{3m - 2}{4} + 250$$

provided $m \ge (8n + 1006)/9$ which is guaranteed by the assumption that $m \ge \frac{8n}{9} + 112$, it follows from Lemma 8 that B is weakly pancyclic unless B has odd girth 7, in which case it contains every cycle of length from 4 up to its circumference $c(B)$ except the 5-cycle. Moreover, by Lemma 6, $c(B) \ge 2(3m - 2n - 2) \ge m$. Note that Fact 3 implies $g(B) \le 4$ since $\delta(B) \ge 3m - 2n - 2 > N/4$. Thus there is a blue C_m.

In the following, we assume that B is nonbipartite and not 2-connected. Suppose that $B - u$ is disconnected for some vertex $u \in V$, here it includes the case where B is disconnected. Since $\delta(B - u) \ge 3m - 2n - 3$, we have that each connected component has at least $3m - 2n - 2$ vertices. So there are at most three connected components in $B - u$. Otherwise, $4(3m - 2n - 2) > 3m - 3 = (N - 1)$, which is a contradiction.

Subcase 2.1. $B - u$ contains three connected components.

Let V_1, V_2 and V_3 be the vertex sets of these three connected components of $B - u$. We assume that V_3 is the largest one. If $|V_3| \ge m$, then we can find a blue C_m as follows. Since each connected component has size at least $3m - 2n - 2$, we have

$$|V_3| \le (3m - 3) - 2(3m - 2n - 2) = 4n - 3m + 1.$$

Since $\delta(B[V_3]) \ge \delta(B) - 1 \ge 3m - 2n - 3 > |V_3|/2$, Lemma 7 implies that $B[V_3]$ is pancyclic and contains a blue C_m. Thus we assume $|V_3| \le m - 1$. As V_3 is the largest connected component and $|V \setminus \{u\}| = 3m - 3$, we get $|V_i| = m - 1$ for each $1 \le i \le 3$.

We claim that each V_i induces a blue clique K_{m-1}. Otherwise, $R[V_1 \cup V_2 \cup V_3]$ contains a B_n by Fact 4(ii) since $2m-2 \geq n$. Because $d_B(u) \geq 3m-2n-2 \geq 4$, we get that u has at least two blue neighbors in V_i for some $1 \leq i \leq 3$, say V_1. Therefore, $B[V_1 \cup \{u\}]$ contains a C_m.

Subcase 2.2. $B-u$ contains exactly two connected components.

Let V_1 and V_2 be the vertex sets of these two connected components with $|V_1| \leq |V_2|$. Clearly, $|V_1| \leq \frac{3m-3}{2}$. If $|V_1| \geq m$, then $B[V_1]$ contains a C_m. Indeed, Lemma 7 implies that $B[V_1]$ is pancyclic as $\delta(B[V_1]) \geq 3m-2n-3 > |V_1|/2$. Hence, we assume $3m-2n-2 \leq |V_1| \leq m-1$ as each connected component has at least $3m-2n-2$ vertices. Clearly, $2m-2 \leq |V_2| \leq 2n-1$.

If $B[V_2]$ is bipartite with color classes X and Y satisfying $|X| \geq |Y|$, then we can find a red B_n as following. We notice

$$|V_1| + |X| \geq |V_1| + \frac{|V_2|}{2} > \frac{|V_1| + |V_2|}{2} \geq \frac{3m-3}{2} \geq n+2.$$

Since X induces a red clique with $|X| \geq 2$ and all edges between V_1 and X are red, $R[V_1 \cup X]$ contains a B_n.

If $B[V_2]$ is 2-connected, then we have $\delta(B[V_2]) \geq 3m-2n-3 \geq \frac{2n-1}{4} + 250 \geq \frac{|V_2|}{4} + 250$ by noting that $m \geq \frac{8n}{9} + 112$ and $n \geq 1000$. Lemma 6, Lemma 8 and Fact 3 imply that there is a blue C_m in $B[V_2]$.

Therefore, we are left to consider the case where $B[V_2]$ is nonbipartite and contains a cut vertex. Suppose that $B[V_2] - w$ is disconnected for some vertex $w \in V_2$, here it includes the case where $B[V_2]$ is not connected. As $\delta(B[V_2]) \geq 3m-2n-3$, it follows that $B[V_2] - w$ contains exactly two connected components, denoted by V_2' and V_2''.

Note that $|V_2'|, |V_2''| \leq |V_2| - (3m-2n-3) \leq 4n-3m+2$. If either $|V_2'| \geq m$ or $|V_2''| \geq m$, say $|V_2'| \geq m$, then Lemma 7 implies that $B[V_2']$ is pancyclic and contains a blue C_m since $\delta(B[V_2']) \geq 3m-2n-4 > |V_2'|/2$. Thus we assume $|V_2'| \leq m-1$ and $|V_2''| \leq m-1$. Note that $|V_1 \cup V_2' \cup V_2''| = 3m-4$ and $\max\{|V_1|, |V_2'|, |V_2''|\} \leq m-1$. We get

$$m-2 \leq |V_1|, |V_2'|, |V_2''| \leq m-1.$$

We claim that each of V_1, V_2' and V_2'' induces a blue clique. Otherwise, Fact 4(ii) implies that $R[V_1 \cup V_2' \cup V_2'']$ contains a B_n by noting $2m-3 \geq n$.

If $|V_1| = m-2$, then $|V_2'| = |V_2''| = m-1$. Since $\delta(B[V_2]) \geq 3m-2n-3 \geq 4$, we have either $d_B(w, V_2') \geq 2$ or $d_B(w, V_2'') \geq 2$ and so either $B[V_2' \cup \{w\}]$ or $B[V_2'' \cup \{w\}]$ contains a C_m.

If $|V_1| = m-1$, then we can assume $|V_2'| = m-1$ and $|V_2''| = m-2$ without loss of generality. If $d_B(u, V_1) \geq 2$, then $B[V_1 \cup \{u\}]$ contains a C_m. Hence $d_B(u, V_1) \leq 1$. Similarly, we have $d_B(u, V_2') \leq 1$. Thus $d_B(u, V_2'') \geq 3$ by noting that $\delta(B) \geq 3m-2n-2 \geq 6$. Repeating the argument above, we can show that $d_B(w, V_2'') \geq 3$. Now, $B[V_2'' \cup \{u, w\}]$ contains a C_m as desired.

We proved part I of Theorem 1. □

(II) $m = n+1$

Let G be the graph which consists of three K_n sharing a common vertex. The lower bound $r(B_n, C_{n+1}) > 3n-2$ follows from the fact that G contains no C_{n+1} and its complement contains no B_n. To show the upper bound $r(B_n, C_{n+1}) \leq 3n-1$, we consider a red-blue edge coloring of K_{3n-1} on vertex set V.

We follow the proof for Part I step by step. We will end up with the case which corresponds to Subcase 2.1 of the proof for Part I. In the following, we suppose $\delta(B) \geq n-1$. If there is a vertex $u \in V$ such that $B-u$ has three connected components, then we can easily find a red B_n by Fact 4(i) since the largest connected component must have order at least $\lceil (3n-2)/3 \rceil = n$.

Therefore, we suppose that there is a vertex u such that $B-u$ has exactly two connected components V_1 and V_2 with $|V_1| \leq |V_2|$. Furthermore, similar to Subcase 2.2, we can assume that $B[V_2]$ is nonbipartite and there is some vertex $w \in V_2$ such that $B[V_2]-w$ is disconnected. The assumption $\delta(B-u) \geq n-2$ yields that $|V_1|, |V_2| \geq n-1$. Similarly, $B[V_2]-w$ has two connected components, say V_2' and V_2'', which satisfy $|V_2'| \geq |V_2''| \geq n-2$.

If $|V_1| \geq n$, then we can definitely find a red B_n by Fact 4(i). Thus we assume $|V_1| = n-1$. As $|V_2' \cup V_2''| = 2n-2$ and $|V_2'| \geq |V_2''| \geq n-2$, we get either $|V_2'| = n$ and $|V_2''| = n-2$, or $|V_2'| = |V_2''| = n-1$. In the former case, a red B_n is ensured again by Fact 4(i). In the latter case, we have that each of V_1, V_2' and V_2'' induces a blue clique K_{n-1}. Otherwise, Fact 4(ii) gives a red B_n by noting $2n-2 \geq n$.

Since u is a cut-vertex and $w \in V_2$, we get w is completely red-adjacent to V_1. Let $x \in V_1$ be a red neighbor of w. If w has another red neighbor y in either V_2' or V_2'', say V_2', then $R[V_2'' \cup \{x, y, w\}]$ contains a B_n, where xy is the edge shared by n triangles. Hence w is completely blue-adjacent to $V_2' \cup V_2''$. If $d_B(u, V_2') \geq 2$ or $d_B(u, V_2'') \geq 2$, say V_2', then $B[V_2' \cup \{u, w\}]$ contains a C_{n+1}. Otherwise, we take two red neighbors a and b of u, where $a \in V_2'$ and $b \in V_2''$. Clearly, $R[V_1 \cup \{a, b, u\}]$ contains a B_n, where ab is the edge shared by n triangles.

The proof of Part II is complete.

(III) $n+2 \leq m \leq \frac{3n+1}{2}$

The lower bound $r(B_n, C_m) > 3n-1$ can be seen as follows. Let $K_{n-1,n-1,n-1}$ be the complete tripartite graph with color classes U_1, U_2 and U_3. Let s and t be two new vertices. If G is a graph obtained from $K_{n-1,n-1,n-1}$ by adding all edges between s and U_1, and all edges between t and U_3, then G contains no B_n and its complement contains no C_m. The lower bound follows. To show the upper bound $r(B_n, C_m) \leq 3n$, we consider a red-blue edge coloring of K_{3n} on vertex set V.

We assume $\delta(B) \geq n$, since the proof is similar to Case 1 of Part I if there is a vertex u with $d_R(u) \geq 2n$.

If the blue graph B is bipartite, then the larger color class of B induces a red clique of size at least $n+2$ and hence there is a red B_n. So we assume B is nonbipartite. If further the blue graph B is 2-connected, then the existence of a blue C_m follows from Lemma 6, Lemma 8 and Fact 3 since $\delta(B) \geq n \geq \frac{3n}{4} + 250$ for $n \geq 1000$.

Therefore, we need only to consider the case where B is nonbipartite and not 2-connected. Let $u \in V$ be a vertex such that $B-u$ is disconnected. Since $\delta(B-u) \geq n-1$, each connected component of $B-u$ has at least n vertices and hence $B-u$ has exactly two connected components.

Let V_1 and V_2 be the vertex sets of these two connected components with $|V_1| \leq |V_2|$. If $|V_1| \geq m$, then we can find a blue C_m as following. Note that $|V_1| \leq \frac{3n-1}{2}$. We have $\delta(B[V_1]) \geq n-1 > |V_1|/2$ for $n \geq 6$. Lemma 7 again implies that $B[V_1]$ is pancyclic, and so there is a blue C_m. Thus we assume $n \leq |V_1| \leq m-1$ and $3n-m \leq |V_2| \leq 2n-1$.

If $B[V_2]$ is bipartite with bipartition (X, Y), where $|X| \geq |Y|$, then we claim $R[V_1 \cup X]$

contains a B_n. To see this, we notice

$$|V_1| + |X| \geq |V_1| + \frac{|V_2|}{2} \geq \frac{|V_1| + |V| - 1}{2} \geq \frac{4n-1}{2} \geq n+2.$$

Since X induces a red clique and all edges between V_1 and X are red and $|X| \geq 2$, there is a red B_n in $R[V_1 \cup X]$ as claimed.

Moreover, if $B[V_2]$ is 2-connected, then the existence of a blue C_m in $B[V_2]$ again follows from Lemma 6, Lemma 8 and Fact 3 since $\delta(B[V_2]) \geq n - 1 \geq \frac{2n-1}{4} + 250 \geq \frac{|V_2|}{4} + 250$ for $n \geq 502$.

Therefore, we assume $B[V_2]$ is nonbiparite and not 2-connected. Let $w \in V_2$ be a vertex such that $B[V_2] - w$ is disconnected. This means that the blue subgraph induced by $V \setminus \{u, w\}$ contains three connected components exactly since $\delta(B[V \setminus \{u, w\}]) \geq n - 2 > |V \setminus \{u, w\}|/4$. Now, as the largest connected component has at least $\lceil (3n-2)/3 \rceil \geq n$ vertices, we can find a desired red B_n in $V \setminus \{u, w\}$ by Fact 4(i).

This completes the proof for part III. $\square$

(IV) $m = \lceil \frac{3n}{2} \rceil + 1$

The lower bound $r(B_n, C_m) > 2m-2$ is clear since the graph with two disjoint K_{m-1} contains no C_m and its complement contains no B_n. For the upper bounds $r(B_n, C_m) \leq 2m - 1 = 3n+1$ if n is even and $r(B_n, C_m) \leq 2m - 1 = 3n+2$ if n is odd, one can easily follow the proof of Part III step by step apart from a few modifications.

This completes the proof for part IV and hence the proof of Theorem 1. $\square$

3.2 Proof of Theorem 2

We note for $k \geq 3$, the graph with $k+1$ disjoint copies of K_{n-1} contains no C_n and its complement contains no $B_n^{(k)}$, so we have $r(B_n^{(k)}, C_n) \geq (k+1)(n-1)+1$. Therefore, it suffices to establish the upper bound. The proof is by induction on $k \geq 3$. The base case where $k = 3$ is built upon an induction idea and the case of $k = 2$.

Step 1: $B_n^{(3)}$ **versus** C_n

Let $0 < \xi < 1/10$ and $N = \lfloor (4+\xi)n \rfloor$. Consider a red-blue edge coloring of K_N on vertex set V. If there exists a vertex $v \in V$ with $d_R(v) \geq 3n-2$, then by Theorem 1, we can find either a blue C_n or a red B_n in the red neighborhood of v. We are done if there is a blue C_n, so we assume that there is a red B_n in the red neighborhood of v. Now, this red B_n together with v form a red $B_n^{(3)}$. Thus we are left to consider the case where $\delta(B) > (1+\xi)n$. Fact 3 implies $g(B) \leq 4$.

We claim B is nonbipartite. Otherwise, one of its color classes induces a red clique of size at least $N/2 \geq n+3$, which will give us a red $B_n^{(3)}$.

Moreover, we can assume that B is 2-connected. Otherwise, suppose that there exists a vertex u such that $B - u$ is disconnected. Then $B - u$ has two or three connected components as $\delta(B) > (1+\xi)n$. Let $V_1 \subseteq V$ be the smallest connected component of $B - u$. Recall the assumption $\delta(B) > (1+\xi)n$. It is clear that

$$(1+\xi)n \leq |V_1| \leq (N-1)/2 \leq (2+\xi/2)n.$$

Since $\delta(B[V_1]) \geq (1+\xi)n - 1 > |V_1|/2$, it follows from Lemma 7 that $B[V_1]$ is pancyclic, which implies that there is a blue C_n.

Now, for any $0 < \xi < 1/10$, if we take $n_0 = \lceil 1000/(3\xi) \rceil$, then $\delta(B) \geq (1+\xi)n \geq \frac{N}{4} + 250$ for all $n \geq n_0$. Therefore, we can find a blue C_n by Lemma 6, Lemma 8 and Fact 3. The proof for $k = 3$ is complete. $\square$

Step 2: $B_n^{(k)}$ versus C_n for $k \geq 4$

For $k = 3$, the result has been verified for sufficiently large n in Step 1. We now suppose that the assertion holds for some $k \geq 3$ and prove it for $k+1$.

Let $0 < \xi < 1/10$ be fixed and $N = \lfloor (k+2)(1+\xi)n \rfloor$. We consider a red-blue edge coloring of K_N on vertex set V. In the following, we will omit the ceiling and floor as it will not affect the result. We choose δ sufficiently small. To be precise,

$$0 < \delta < \min\left\{10^{-15},\ \frac{\xi^2}{600(k+2)^2}\right\}. \tag{2}$$

Lemma 2 with δ and $k+1$ gives us a constant η. Let β and ϵ be sufficiently small such that

$$\beta = \frac{\xi}{20(k+2)^2} \text{ and } 0 < \epsilon < \min\left\{\eta,\ \beta^2\right\}. \tag{3}$$

Set

$$\alpha = \frac{1}{k+2} - \beta - \sqrt{\epsilon}. \tag{4}$$

Let M be given by Lemma 1 with ϵ and large $m_0 = \lceil \frac{1}{\epsilon} \rceil$. We apply Lemma 1 to the red subgraph R and obtain an equitable partition $V(G) = \sqcup_{i=1}^m V_i$ and subsets $W_i \subset V_i$ such that W_i is ϵ-regular for all i and, for all but ϵm^2 pairs (i, j) with $1 \leq i \neq j \leq m$, (V_i, V_j) and (W_i, V_j) are ϵ-regular with $|d_R(W_i, V_j) - d_R(V_i, V_j)| \leq \epsilon$. Here we do not require pairs (W_i, W_j) for $1 \leq i, j \leq m$ to be ϵ-regular. For convenience, we will assume $|V_i| = \frac{N}{m}$ for all $1 \leq i \leq m$. If n is large enough, then $\frac{N}{m} \geq \frac{N}{M} \geq \max\{n_1, n_2\}$, where n_1 is given by Lemma 3 with $\beta - \epsilon$, and n_2 is given by Lemma 5 with δ and α. Note that a partition obtained by applying Theorem 1 to R is also such a partition for B.

Let F be the reduced graph defined on $\{v_1, v_2, \ldots, v_m\}$, in which v_i and v_j are non-adjacent in F if the pairs (V_i, V_j) and (W_i, V_j) are not all ϵ-regular with $|d_R(W_i, V_j) - d_R(V_i, V_j)| \leq \epsilon$. Then the number of edges of F is at least $(1-\epsilon)m^2$. Therefore, by deleting at most $\sqrt{\epsilon}m$ vertices, we may assume that each vertex is adjacent to at least $(1 - \sqrt{\epsilon})m$ vertices. In what follows, when referring to the reduced graph, we will assume that these vertices have been removed. For each remaining vertex v_i, we color v_i red if the density of the red subgraph induced by W_i satisfies that $d_R(W_i) \geq 1/2$, and we color v_i blue otherwise. We color an edge v_iv_j red if $d_R(V_i, V_j) \geq 1 - \beta$, or blue if $d_B(V_i, V_j) \geq \beta$. Let F_R and F_B be the subgraphs formed by red edges and blue edges of F, respectively.

For a blue vertex v_a in F, suppose $d_{F_R}(v_a) \geq m_1 := (\frac{k+1}{k+2} + \beta)m$. Recall an edge v_iv_j in F is red if and only if $d_R(V_i, V_j) \geq 1 - \beta$. By averaging, there exists a vertex $u \in V_a$ such that

$$d_R(u) \geq (1-\beta)\frac{N}{m} \cdot m_1 = (1-\beta)(k+2)(1+\xi)\left(\frac{k+1}{k+2} + \beta\right)n > (k+1)(1+\xi)n,$$

here we used the assumption (3). Thus, by the induction hypothesis, there is either a blue C_n or a red $B_n^{(k)}$ in $N_R(u)$. In the former case, we are through. In the latter case, the red $B_n^{(k)}$ together with u gives a red $B_n^{(k+1)}$. We are done with this case. Therefore, we assume that each blue vertex v_a in the reduced graph F satisfies

$$d_{F_B}(v_a) \geq m_2 := \left(\frac{1}{k+2} - \beta - \sqrt{\epsilon}\right) m. \tag{5}$$

For a red vertex v_a in F, if $d_{F_R}(v_a) \geq \frac{(1-0.5\xi)m}{k+2}$, then we can apply Lemma 2 to find a red $B_n^{(k+1)}$. Recall the red density $d_R(W_a) \geq 1/2 > \delta$, where $W_a \subseteq V_a$. We apply Lemma 2 with $U_i = W_a$ for $1 \leq i \leq k+1$ and U_{k+1+s} equal to each of the V_s for which (W_a, V_s) is ϵ-regular and the edge $v_a v_s$ is red. We conclude that there is a red K_{k+1} which is contained in at least

$$\begin{aligned}
\sum_s \left([d_R(W_a, V_s)]^{k+1} - \delta\right) |V_s| &\geq \left((1-\beta-\epsilon)^{k+1} - \delta\right) \frac{N}{m} \cdot \frac{(1-0.5\xi)}{k+2} m \\
&\geq \left((1-\beta-\epsilon)^{k+1} - \delta\right) \left(1 + \frac{\xi}{3}\right) n \\
&\geq \left(1 + \frac{\xi}{3} - 2(k+1)(\beta+\epsilon) - 2\delta\right) n
\end{aligned}$$

red K_{k+2}. We notice this quantity is at least n since β, ϵ and δ are sufficiently small in terms of $1/k$ and ξ from (2) and (3), so we are through as there is a red $B_n^{(k+1)}$. Therefore, for the rest of the proof, we assume that each red vertex v_a satisfies

$$d_{F_B}(v_a) \geq \left(\frac{k+1+0.5\xi}{k+2} - \sqrt{\epsilon}\right) m. \tag{6}$$

From (5) and (6), we have $d_{F_B}(v_a) \geq m_2 := (\frac{1}{k+2} - \beta - \sqrt{\epsilon})m$ no matter which color a vertex v_a has received. We separate the proof into two cases depending on the parity of n.

Case 1. n is even.

We apply the techniques used by Łuczak [21] and Lemma 3 to show that the blue graph B contains a C_n. By Erdős–Gallai theorem [14], F_B contains a path $v_1 v_2 \cdots v_\ell$, here $\ell \geq m_2$. For each $1 \leq i \leq \ell$, we split V_i corresponding to the vertex v_i into two subsets V_i' and V_i'', where $|V_i'|, |V_i''| \geq |V_i|/2 = \frac{N}{2m}$. We have an even "fat" cycle $V_1' V_2' \cdots V_\ell' V_{\ell-1}'' V_{\ell-2}'' \cdots V_2'' V_1'$. For convenience, we relabel it as $U_1 U_2 \cdots U_\ell U_{\ell+1} U_{\ell+2} \cdots U_{2\ell-2} U_1$. Note that Fact 2 implies that (U_i, U_{i+1}) is 3ϵ-regular with blue density at least $\beta - \epsilon$ for $1 \leq i \leq 2\ell-2$, where the sums of the indices are taken modulo $2\ell-2$. By Fact 1, there are at least $(1-6\epsilon)|U_i|$ vertices of U_i having $(\beta - 4\epsilon)\frac{N}{2m}$ neighbors in each sets of U_{i-1} and U_{i+1}. Therefore, for each $1 \leq i \leq 2\ell-2$, we can choose a vertex $u_i \in U_i$ such that $u_1 u_2 \cdots u_{2\ell-2} u_1$ form an even cycle satisfying $d_B(u_i, U_{i-1}) \geq 4\beta/5 \cdot \frac{N}{2m}$ and $d_B(u_i, U_{i+1}) \geq 4\beta/5 \cdot \frac{N}{2m}$. By Lemma 3, for each $1 \leq i \leq \ell-1$, we can replace the edge $u_{2i-1} u_{2i}$ by an odd path with endpoints u_{2i-1} and u_{2i} using vertices from U_{2i-1} and U_{2i}, here the length of the path can vary from 1 to $(1 - 5\epsilon/\beta) \cdot \frac{N}{2m}$. Therefore, we can enlarge this even cycle to all even cycles of length from $2\ell - 2$ to $(2\ell-2)(1-5\epsilon/\beta)\frac{N}{2m}$. Recall assumptions (3), $m \geq m_0 = \lceil \frac{1}{\epsilon} \rceil$ and $\ell \geq m_2 = \left(\frac{1}{k+2} - \beta - \sqrt{\epsilon}\right) m$. We have $(2\ell-2)(1-5\epsilon/\beta)\frac{N}{2m} \geq n$.

Therefore, there is a blue C_n as desired.

Case 2. n is odd.

First, we suppose that F_B contains an odd cycle $v_1v_2\dots v_{2\ell+1}v_1$ such that $2\ell+1 \geq m_2$. Similarly, by Fact 2, we can find an odd cycle $u_1u_2\dots u_{2\ell+1}u_1$ such that $u_i \in V_i$, $d_B(u_i, V_{i-1}) \geq 4\beta/5 \cdot |V_{i-1}|$, and $d_B(u_i, V_{i+1}) \geq 4\beta/5 \cdot |V_{i+1}|$ for each $1 \leq i \leq 2\ell+1$. Now, applying Lemma 3, for each $1 \leq i \leq \ell$, we replace the edge $u_{2i-1}u_{2i}$ by odd path using vertices from V_{i-1} and V_i to enlarge this odd cycle to odd cycles of length from $2\ell+1$ to $2\ell(1-5\epsilon/\beta)\frac{N}{m}$ in the blue graph B. We can definitely find an odd cycle C_n as $2\ell(1-5\epsilon/\beta)\frac{N}{m} \geq n$.

It remains to consider the case where F_B contains no odd cycles of length at least m_2. Erdős–Gallai theorem [14] already implies that F_B contains an even cycle of length at least m_2. We assume that $C := v_1v_2, \cdots v_{2s}v_1$ is such an even cycle, here $2s \geq m_2$.

Subcase 2.1. There exists a blue vertex in C, say v_{2s}.

Recall the definition of W_{2s}. We note (V_{2s-1}, W_{2s}) and (W_{2s}, V_1) are ϵ-regular pairs. Moreover, $d_B(V_{2s-1}, W_{2s}) \geq \beta - \epsilon$ and $d_B(W_{2s}, V_1) \geq \beta - \epsilon$. We apply Lemma 4 with $A = V_{2s-1}, B = W_{2s}$, $C = V_1$ and $\beta' = \beta - \epsilon$ to get a blue edge xy in W_{2s} which satisfies $d_B(x, V_{2s-1}) \geq (\beta' - \epsilon) \cdot \frac{N}{m} \geq 4\beta/5 \cdot \frac{N}{m}$ and $d_B(y, V_1) \geq (\beta' - \epsilon)\frac{N}{m}$. Similar to the case where F_B contains an odd blue cycle of length at least m_2, we can find a blue odd cycle $u_1u_2u_3\dots u_{2s+1}u_1$, where $u_i \in V_i$ for $1 \leq i \leq 2s-1$, $u_{2s} = x$, and $u_{2s+1} = y$. Furthermore, we have $d_B(u_1, V_2) \geq 4\beta/5 \cdot \frac{N}{m}$. For each $2 \leq i \leq 2s-1$, it satisfies $d_B(u_i, V_{i-1}) \geq 4\beta/5 \cdot \frac{N}{m}$ and $d_B(u_i, V_{i+1}) \geq 4\beta/5 \cdot \frac{N}{m}$. For each $1 \leq i \leq s$, we again apply Lemma 3 to replace each blue edge $u_{2i-1}u_{2i}$ by an odd blue path. When we enlarge the edge $u_{2s-1}u_{2s}$, we use vertices from $V_{2s-1} \cup (V_{2s} \setminus \{y\})$ to avoid the vertex $u_{2s+1} = y \in V_{2s}$. Thus F_B contains odd cycles of length from $2s+1$ to $2s(1-5\epsilon/\beta)\frac{N}{m}$ and there is a blue C_n as $2s(1-5\epsilon/\beta)\frac{N}{m} \geq n$.

Subcase 2.2. Each vertex of C is red.

Write $d = 2s$ for the length of the cycle C. Recalling (6), we have a lower bound on the number of edges in F_B as follows:

$$|E(F_B)| \geq \sum_{i=1}^{d} d_{F_B}(v_i) - \binom{d}{2} \geq \left(\frac{k+1+0.5\xi}{k+2} - \sqrt{\epsilon}\right) dm - \binom{d}{2}.$$

Since $d \geq m_2$ and the right hand side above is an increasing function of d when $d \geq m_2$, it follows that

$$\begin{aligned} |E(F_B)| &\geq \left(\frac{k+1+0.5\xi}{k+2} - \sqrt{\epsilon}\right)\left(\frac{1}{k+2} - \beta - \sqrt{\epsilon}\right)m^2 - \frac{1}{2}\left(\frac{1}{k+2} - \beta - \sqrt{\epsilon}\right)^2 m^2 \\ &\geq \left(\frac{2k+1}{2(k+2)^2} + \frac{0.5\xi}{(k+2)^2} - \beta - \sqrt{\epsilon} - \frac{\sqrt{\epsilon}}{k+2}\right)m^2 \\ &\geq \left(\frac{2k+1}{2(k+2)^2} + \frac{\xi}{3(k+2)^2}\right)m^2, \end{aligned} \tag{7}$$

here we note that β and $\sqrt{\epsilon}$ are sufficiently small in terms of $1/k^2$ and ξ from (3).

We next apply Lemma 5 to obtain an upper bound on $|E(F_B)|$. Actually, we will apply Lemma 5 with $G = F_B$, δ and $\alpha = \frac{1}{k+2} - \beta - \sqrt{\epsilon}$. Since F_B contains no odd cycles of length at least $m_2 = \alpha m$, there are two subgraphs $G' = F'_B$ and $G'' = F''_B$ satisfying all properties listed in the lemma. If F'_B is empty, then we get $|E(F_B)| \le \frac{1}{2}(\frac{1}{k+2} - \beta - \sqrt{\epsilon})m^2 + \delta m^2$, which clearly is a contradiction to the lower bound on $|E(F_B)|$ from (7). If F'_B is not empty, then we assume that X and Y are two color classes of F'_B.

Claim. We have $\max\{|X|, |Y|\} \le k + 1 + \frac{(1-0.5\xi)m}{k+2}$.

Proof. Suppose $|X| \ge k + 1 + \frac{(1-0.5\xi)m}{k+2}$ without loss of generality. We aim to find a book $B_n^{(k+1)}$ in the red graph. Since all but at most δm^2 edges of F_B belong to either F'_B or F''_B by (4) of Lemma 5, it follows that X contains at least $\binom{|X|}{2} - (\epsilon + \delta)m^2 \ge (1-\gamma)\binom{|X|}{2}$ red edges, where $0 \le \gamma < 3(k+2)^2(\epsilon + \delta)$. By deleting at most $\sqrt{\gamma}|X|$ vertices, we may assume that each vertex of X is red-adjacent to at least $(1 - \sqrt{\gamma})|X|$ vertices in X. We notice

$$(1 - (k+1)\sqrt{\gamma})|X| \ge m_3 := \frac{(1-0.6\xi)m}{k+2},$$

here we note $\epsilon, \delta < \frac{\xi^2}{600(k+2)^2}$ from (2) and (3). Therefore, we can choose vertices $v_1, \ldots, v_{k+1}$ from X step by step such that $v_1, \ldots, v_{k+1}$ form a red clique in the reduced graph F and $v_1, \ldots, v_{k+1}$ have at least m_3 common red neighbors, say $v_{k+2}, \ldots, v_{k+1+m_3}$. i.e., X contains a red book $B_{m_3}^{(k+1)}$. We apply Lemma 2 with $U_i = V_i$ for $1 \le i \le k+1$ and $U_{k+1+j} = V_{k+1+j}$ for $1 \le j \le m_3$, and conclude that there is a red K_{k+1} which is contained in at least

$$\begin{aligned}\sum_{j=1}^{m_3}\left(\prod_{i=1}^{k} d_R(V_i, V_{k+1+j}) - \delta\right)|V_{k+1+j}| &\ge \left((1-\beta)^{k+1} - \delta\right)\frac{N}{m}\cdot\frac{(1-0.6\xi)}{k+2}m \\ &\ge \left((1-\beta)^{k+1} - \delta\right)\left(1 + \frac{\xi}{3}\right)n\end{aligned}$$

red K_{k+2}. This quantity is at least n since β and δ are sufficiently small in terms of $1/k$ and ξ from (2) and (3). We found a red $B_n^{(k+1)}$ as desired. □

Consequently, $|E(F'_B)| \le \left(k + 1 + \frac{(1-0.5\xi)m}{k+2}\right)^2$ from the above claim. The number of edges in F_B can be bounded from above as follows:

$$\begin{aligned}|E(F_B)| &\le \left(k + 1 + \frac{(1-0.5\xi)m}{k+2}\right)^2 + \frac{1}{2}\left(\frac{1}{k+2} - \beta - \sqrt{\epsilon}\right)m^2 + \delta m^2 \\ &\le \left(\frac{k+4}{2(k+2)^2} - \frac{\xi}{(k+2)^2} + \delta\right)m^2 + 2m + (k+1)^2 \\ &\le \left(\frac{k+4}{2(k+2)^2} - \frac{\xi}{2(k+2)^2}\right)m^2,\end{aligned}$$

here we note that $m \ge m_0 \ge 1/\epsilon$, ϵ and δ are sufficiently small in terms of $1/k^2$ and ξ from (2) and (3). Recalling the inequality (7), we obtained

$$\left(\frac{2k+1}{2(k+2)^2} + \frac{\xi}{3(k+2)^2}\right)m^2 \le |E(F_B)| \le \left(\frac{k+4}{2(k+2)^2} - \frac{\xi}{2(k+2)^2}\right)m^2. \tag{8}$$

This is a contradiction provided $k \ge 3$.

This completes the proof of the induction step and hence Theorem 2. □

Remark 2: For $k = 2$, the inequality (8) indeed holds and there is no contradiction. Thus we are not able to prove $r(B_n^{(3)}, C_n) = (4 + o(1))n$ by induction and we provide a separated proof for this case.

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