EQUIVARIANT COHOMOLOGY OF MOMENT-ANGLE COMPLEXES WITH RESPECT TO COORDINATE SUBTORI

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Abstract. We compute the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ of moment-angle complexes $\mathcal{Z}_\mathcal{K}$ with respect to the action of coordinate subtori $T_I \subset T^m$. We give a criterion for the equivariant formality of $\mathcal{Z}_\mathcal{K}$ and obtain specifications for the cases of flag complexes and graphs.

1. Introduction

Let $\mathcal{K}$ be a simplicial complex on an m-element set V, and let $\mathcal{Z}_\mathcal{K}$ be the corresponding moment-angle complex $\mathcal{Z}_\mathcal{K}$. We study the equivariant cohomology of $\mathcal{Z}_\mathcal{K}$ with respect to the action of coordinate subtori $T_I \subset T^m$, where $I = \{i_1, \ldots, i_k\} \subset V$.

We construct two commutative integral dga models for $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$. The first is given by

$$\big(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d\big), \quad du_i = v_i,\ dv_i = 0,$$

where $\Lambda[u_i\colon i \notin I]$ is the exterior algebra on degree-one generators u_i, $i \notin I$, and $\mathbb{Z}[\mathcal{K}]$ is the face ring of $\mathcal{Z}_\mathcal{K}$. The second dga model $R_I(\mathcal{K})$ is given by the quotient of the first one by the ideal generated by u_iv_i and v_i^2 with $i \notin I$. As a result we obtain

Theorem 3.3. *There are isomorphisms of rings*

$$\begin{aligned} H^*_{T_I}(\mathcal{Z}_\mathcal{K}) &\cong H\big(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d\big) \cong H^*\big(R_I(\mathcal{K})\big) \\ &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\big(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}]\big), \end{aligned}$$

where $\mathbb{Z}[v_i\colon i \in I]$ is the $\mathbb{Z}[v_1, \ldots, v_m]$-module via the homomorphism sending v_i to 0 for $i \notin I$.

When $I = V$, the dga model above reduces to the face ring $\mathbb{Z}[\mathcal{K}]$ with zero differential, and we recover the integral formality result of [11].

When $I = \varnothing$, Theorem 3.3 gives the description of the ordinary integral cohomology of $\mathcal{Z}_\mathcal{K}$ of [2] and [5].

The additive (or $\mathbb{Z}[v_1, \ldots, v_m]$-module) isomorphism

$$H^*_{T_I}(\mathcal{Z}_\mathcal{K}) \cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\big(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}]\big) \cong \operatorname{Tor}_{\mathbb{Z}[v_i\colon i \notin I]}\big(\mathbb{Z}, \mathbb{Z}[\mathcal{K}]\big)$$

follows from the result of [8].

Next, we study the equivariant formality of $\mathcal{Z}_\mathcal{K}$, that is, whether $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ is a free module over the polynomial ring $H^*_{T_I}(pt) = H^*(BT_I) = \mathbb{Z}[v_i\colon i \in I]$. We prove

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Theorem 4.8. *Let $\mathcal{K}$ be a simplicial complex on a finite set V. The following conditions are equivalent:*

(a) *For any $I \in \mathcal{K}$, the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$ is a free module over $H^*(BT_I)$.*

(b) *There is a partition $V = V_1 \sqcup \cdots \sqcup V_p \sqcup U$ such that*

$$\mathcal{K} = \partial\Delta(V_1) * \cdots * \partial\Delta(V_p) * \Delta(U),$$

where $\Delta(U)$ denotes a full simplex on U, and $\partial\Delta(V_i)$ denotes the boundary of a simplex on V_i.

(c) *The rational face ring $\mathbb{Q}[\mathcal{K}]$ is a complete intersection ring (the quotient of the polynomial ring by an ideal generated by a regular sequence).*

In the case of flag complexes we have the following specification:

Theorem 4.9. *Let $\mathcal{K}$ be a flag complex on V. Then the following conditions are equivalent:*

(a) *$H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is a free module over $\mathbb{Z}[v_i]$ for all i.*

(b) *$\mathcal{K} = \partial\Delta(V_1) * \cdots * \partial\Delta(V_p) * \Delta(U)$ where $|V_k| = 2$ for $k = 1, \ldots, p$.*

A similar criterion holds in the case when $\mathcal{K}$ is a simple graph:

Theorem 4.11. *Let $\mathcal{K}$ be a one-dimensional complex (a simple graph). Then the following conditions are equivalent:*

(a) *$H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is a free module over $\mathbb{Z}[v_i]$ for any i.*

(b) *$\mathcal{K}$ is the one of the following: $\partial\Delta^2$, $\partial\Delta^1 * \partial\Delta^1$, $\partial\Delta^1$, Δ^1, $\partial\Delta^1 * \Delta^0$, Δ^0.*

Along the way we establish some additional properties of the equivariant cohomology of $\mathcal{Z}_\mathcal{K}$ and give illustrative examples.

2. Preliminaries

Let $\mathcal{K}$ be a simplicial complex on a finite m-element set V, which we often identify with the index set $[m] = \{1, 2, \ldots, m\}$. We refer to a subset $I = \{i_1, \ldots, i_k\} \subset V$ that is contained in $\mathcal{K}$ as a *simplex*. We assume that $\varnothing \in \mathcal{K}$ and allow *ghost vertices*, that is, one-element subsets $\{i\} \in V$ such that $\{i\} \notin \mathcal{K}$.

Let $(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$ be a sequence of m pairs of pointed CW-complexes, $A_i \subset X_i$. For each subset $I \subset V$, define

$$(\boldsymbol{X}, \boldsymbol{A})^I := \Bigl\{(x_1, \ldots, x_m) \in \prod_{j=1}^m X_j \colon\; x_j \in A_j \quad \text{for } j \notin I\Bigr\}.$$

The *polyhedral product* of $(\boldsymbol{X}, \boldsymbol{A})$ corresponding to $\mathcal{K}$ is

$$(\boldsymbol{X}, \boldsymbol{A})^\mathcal{K} := \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} X_i \times \prod_{i \notin I} A_i\Bigr).$$

Using the categorical language, denote by $\mathrm{CAT}(\mathcal{K})$ the face category of $\mathcal{K}$, with objects $I \in \mathcal{K}$ and morphisms $I \subset J$. Define the $\mathrm{CAT}(\mathcal{K})$-diagram

$$\mathcal{D}_\mathcal{K}(\boldsymbol{X}, \boldsymbol{A}) \colon \mathrm{CAT}(\mathcal{K}) \longrightarrow \mathrm{TOP},$$
$$I \longmapsto (\boldsymbol{X}, \boldsymbol{A})^I,$$

which maps the morphism $I \subset J$ of $\mathrm{CAT}(\mathcal{K})$ to the inclusion of spaces $(\boldsymbol{X}, \boldsymbol{A})^I \subset (\boldsymbol{X}, \boldsymbol{A})^J$. Then we have

$$(\boldsymbol{X}, \boldsymbol{A})^\mathcal{K} = \operatorname{colim} \mathcal{D}_\mathcal{K}(\boldsymbol{X}, \boldsymbol{A}) = \operatorname*{colim}_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I.$$

In the case when all the pairs (X_i, A_i) are the same, i. e. $X_i = X$ and $A_i = A$ for $i = 1, \ldots, m$, we use the notation $(X, A)^\mathcal{K}$ for $(\boldsymbol{X}, \boldsymbol{A})^\mathcal{K}$.

The *moment-angle complex* $\mathcal{Z}_\mathcal{K}$ is the polyhedral product $(D^2, S^1)^\mathcal{K}$. We refer to [3, Chapter 4] for more details and examples.

The *face ring* of $\mathcal{K}$ is the quotient ring

$$\mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_\mathcal{K},$$

where $\mathcal{I}_\mathcal{K}$ is the ideal generated by the square-free monomials $v_I = \prod_{i \in I} v_i$ for which $I \subset V$ is not a simplex of $\mathcal{K}$.

3. Equivariant cohomology

For an action of a topological group G on a space X, the *Borel construction* is

$$EG \times_G X := EG \times X/(e \cdot g^{-1}, g \cdot x) \sim (e, x),$$

where EG is the universal right G-space, $e \in EG$, $g \in G$, $x \in X$. There is the *Borel fibration* $EG \times_G X \to BG$ over the classifying space $BG = EG/G$ with fibre X. The *equivariant cohomology* of X is

$$H^*_G(X) := H^*(EG \times_G X).$$

The torus $T^m = (S^1)^m$ acts on $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$ coordinatewise. The universal bundle $ES^1 \to BS^1$ is the infinite-dimensional Hopf bundle $S^\infty \to \mathbb{C}P^\infty$.

We consider the equivariant cohomology of $\mathcal{Z}_\mathcal{K}$ with respect to the action of coordinate subtori

$$T_I = \{(t_1, \dots, t_m) \in T^m \colon t_j = 1 \text{ for } j \notin I\},$$

where $I = \{i_1, \dots, i_k\} \subset V$.

Proposition 3.1. *There is a homotopy equivalence*

$$ET_I \times_{T_I} \mathcal{Z}_\mathcal{K} \xrightarrow{\simeq} (\boldsymbol{Y}, \boldsymbol{B})^\mathcal{K},$$

where

$$Y_i = \begin{cases} \mathbb{C}P^\infty, & i \in I, \\ D^2, & i \notin I, \end{cases} \qquad B_i = \begin{cases} pt, & i \in I, \\ S^1, & i \notin I. \end{cases}$$

Proof. We have

$$ET_I \times_{T_I} \mathcal{Z}_\mathcal{K} = ET_I \times_{T_I} (D^2, S^1)^\mathcal{K} = (\boldsymbol{X}, \boldsymbol{A})^\mathcal{K},$$

where

$$X_i = \begin{cases} S^\infty \times_{S^1} D^2, & i \in I, \\ D^2, & i \notin I, \end{cases} \qquad A_i = \begin{cases} S^\infty \times_{S^1} S^1, & i \in I, \\ S^1, & i \notin I. \end{cases}$$

The result follows from the homotopy equivalence of pairs

$$(S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1) \xrightarrow{\simeq} (\mathbb{C}P^\infty, pt),$$

as in [3, Theorem 4.3.2] where the case $I = [m]$ is treated. □

Next we introduce two commutative dga models for the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_\mathcal{K})$. First, consider the dga

$$\big(\Lambda[u_i \colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d\big), \quad du_i = v_i, \ dv_i = 0,$$

where $\Lambda[u_i \colon i \notin I]$ is the exterior algebra on generators indexed by $V - I$. The grading is given by $\deg u_i = 1$, $\deg v_i = 2$.

Second, consider the quotient dga

$$R_I(\mathcal{K}) := \Lambda[u_i \colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}] \big/ (u_i v_i = v_i^2 = 0, \ i \notin I),$$

noting that the ideal generated by $u_i v_i$ and v_i^2 with $i \notin I$ is d-invariant.

We denote by $C_*(X)$ and $C^*(X)$ the *normalised* singular chain dg coalgebra and singular cochain dg algebra of a space X, respectively. (A singular cochain is normalised if it vanishes on degenerate singular simplices [9, §VIII.6]; passing to normalised cochains does not change the quasi-isomorphism type of $C^*(X)$.)

Theorem 3.2. *The singular cochain algebra $C^*(ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}})$ is quasi-isomorphic to $(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d)$ and $R_I(\mathcal{K})$. The quasi-isomorphisms are natural with respect to inclusion of subcomplexes.*

Proof. We combine the arguments of [11], [3, §4.5, §8.1] and [6].

The acyclicity of the ideal generated by v_i^2 and u_iv_i for $i \notin I$ in $\Lambda[u_i\colon i \notin I]\otimes\mathbb{Z}[\mathcal{K}]$ is established in the same way as [3, Lemma 3.2.6], where the case $I = \varnothing$ is treated. This gives a quasi-isomorphism $\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}] \xrightarrow{\simeq} R_I(\mathcal{K})$. For the remaining quasi-isomorphism $R_I(\mathcal{K}) \simeq C^*(ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}})$, we use the homotopy equivalent polyhedral product model $(\boldsymbol{Y}, \boldsymbol{B})^{\mathcal{K}}$ of Proposition 3.1.

Throughout the proof, we use the following zig-zag of quasi-isomorphisms of dgas [10, §7.2]:

$$C^*(X) \otimes C^*(Y) \xrightarrow{\simeq} \mathrm{Hom}(C_*(X) \otimes C_*(Y), \mathbb{Z}) \xleftarrow{\simeq} C^*(X \times Y), \tag{3.1}$$

where the arrow on the right is the dual of the Eilenberg–Zilber map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$, which is a quasi-isomorphism of dg coalgebras [4, (17.6)].

First consider the case $\mathcal{K} = \varnothing$ with m ghost vertices. Then $\mathcal{Z}_{\mathcal{K}} = T^m$ and

$$ET_I \times_{T_I} \mathcal{Z}_{\mathcal{K}} \simeq T^m/T_I = (\boldsymbol{Y}, \boldsymbol{B})^{\mathcal{K}} = \prod_{i \notin I} S^1,$$

whereas $R_I(\mathcal{K}) = \Lambda[u_i\colon i \notin I]$. There is a quasi-isomorphism $\Lambda[u] = H^*(S^1) \to C^*(S^1)$ mapping u to its representing singular 1-cocycle (here it is important that we work with normalised cochains). Applying (3.1) we obtain the required quasi-isomorphism $\Lambda[u_i\colon i \notin I] \simeq C^*(\prod_{i\notin I} S^1)$.

Now consider the case $m = 1$ and $\mathcal{K} = \Delta^0$, a 0-simplex. If $I = \varnothing$, then $(\boldsymbol{Y}, \boldsymbol{B})^{\mathcal{K}} = D^2$ and $R_I(\mathcal{K}) = \Lambda[u] \otimes \mathbb{Z}[v]/(uv = v^2 = 0)$. Let $\varphi\colon [01] \to D^2$ be the standard parametrisation of the boundary circle S^1, viewed as a singular 1-simplex. Let $\psi\colon [012] \to D^2$ be a singular 2-simplex such that $\psi|_{[12]} = \varphi$ and $\psi|_{[02]}, \psi|_{[01]}$ are constant maps to the basepoint $1 \in S^1$. Then $\partial\varphi = 0$ and $\partial\psi = \varphi$, as we work with the normalised chains. Now if $\alpha \in C^1(D^2)$ is the cochain dual to φ and $\beta \in C^2(D^2)$ is dual to ψ, then $\Lambda[u] \otimes \mathbb{Z}[v]/(uv = v^2 = 0) \to C^*(D^2)$ mapping u to α and v to β is a quasi-isomorphism. If $I = \{1\}$, then $(\boldsymbol{Y}, \boldsymbol{B})^{\mathcal{K}} = \mathbb{C}P^\infty$ and $R_I(\mathcal{K}) = \mathbb{Z}[v]$. There is a quasi-isomorphism $\mathbb{Z}[v] = H^*(\mathbb{C}P^\infty) \to C^*(\mathbb{C}P^\infty)$ mapping v to its representing singular 2-cocycle.

Next consider the case $\mathcal{K} = \Delta^{m-1} = \Delta[m]$, the full simplex on $[m]$. Applying (3.1) and the Künneth theorem, we obtain a zig-zag of quasi-isomorphisms

$$\begin{aligned} R_I(\Delta[m]) &= \Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[v_1, \ldots, v_m]/(u_iv_i, v_i^2\colon i \notin I) \\ &= \bigotimes_{i\in I} \mathbb{Z}[v_i] \otimes \bigotimes_{i\notin I} \bigl(\Lambda[u_i] \otimes \mathbb{Z}[v_i]/(u_iv_i, v_i^2)\bigr) \xrightarrow{\simeq} \bigotimes_{i\in I} C^*(\mathbb{C}P^\infty) \otimes \bigotimes_{i\notin I} C^*(D^2) \\ &\xrightarrow{\simeq} \cdots \xleftarrow{\simeq} C^*\Bigl(\prod_{i\in I} \mathbb{C}P^\infty \times \prod_{i\notin I} D^2\Bigr) = C^*\bigl((\boldsymbol{Y}, \boldsymbol{B})^{\Delta[m]}\bigr), \end{aligned} \tag{3.2}$$

which completes the proof for the case $\mathcal{K} = \Delta[m]$.

The general case is proved by induction on the number of simplices in $\mathcal{K}$ using the naturality with respect of inclusion of subcomplexes and the Mayer–Vietoris sequence, as in [6, Theorem 1]. Namely, we add simplices one by one to the empty simplicial complex on $[m]$ and use the zig-zag of dga maps between the two short exact sequences for any two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ on $[m]$:

$$0 \to R_I(\mathcal{K}_1 \cup \mathcal{K}_2) \to R_I(\mathcal{K}_1) \oplus R_I(\mathcal{K}_2) \to R_I(\mathcal{K}_1 \cap \mathcal{K}_2) \to 0$$

and

$$0 \to C^*((\boldsymbol{Y},\boldsymbol{B})^{\mathcal{K}_1\cup\mathcal{K}_2}) \to C^*((\boldsymbol{Y},\boldsymbol{B})^{\mathcal{K}_1}) \oplus C^*((\boldsymbol{Y},\boldsymbol{B})^{\mathcal{K}_2}) \to C^*((\boldsymbol{Y},\boldsymbol{B})^{\mathcal{K}_1\cap\mathcal{K}_2}) \to 0$$

The zig-zags between the middle and right nonzero terms are quasi-isomorphisms by induction. Then the zig-zag on the left is also a quasi-isomorphism by the cohomology long exact sequence and five lemma.

It may be more illuminating to realise the dgas in question as the limits of dgas correponding to simplices $I \in \mathcal{K}$. Namely, given a subset $J \subset V$, let $\varDelta(J)$ denote a simplex on J, viewed as a simplicial complex on V (with ghost vertices $V - J$). Then

$$R_I(\varDelta(J)) = \Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[v_j\colon j \in J]\big/(u_jv_j = v_j^2 = 0,\ j \in J - I),$$

Consider the $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$-diagram

$$\mathcal{R}_{I,\mathcal{K}}\colon \mathrm{CAT}^{\mathrm{op}}(\mathcal{K}) \longrightarrow \mathrm{DGA}, \quad J \longmapsto R_I(\varDelta(J)),$$

sending a morphism $J_1 \subset J_2$ of $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$ to the surjection of dgas $R_I(\varDelta(J_2)) \to R_I(\varDelta(J_1))$. Then

$$R_I(\mathcal{K}) = \lim \mathcal{R}_{I,\mathcal{K}} = \lim_{J\in\mathcal{K}} R_I(\varDelta(J))$$

Similarly, we have a $\mathrm{CAT}^{\mathrm{op}}(\mathcal{K})$-diagram

$$\mathcal{C}_{I,\mathcal{K}}\colon \mathrm{CAT}^{\mathrm{op}}(\mathcal{K}) \longrightarrow \mathrm{DGA}, \quad J \longmapsto C^*((\boldsymbol{Y},\boldsymbol{B})^J).$$

The zig-zag of quasi-isomorphisms (3.2) induces an objectwise weak equivalence of diagrams $\mathcal{R}_{I,\mathcal{K}} \simeq \mathcal{C}_{I,\mathcal{K}}$. The canonical maps $\mathcal{R}_{I,\mathcal{K}}(J) \to \lim \mathcal{R}_{I,\mathcal{K}}|_{\mathrm{CAT}^{\mathrm{op}}(\partial\varDelta(J))}$ and $\mathcal{C}_{I,\mathcal{K}}(J) \to \lim \mathcal{C}_{I,\mathcal{K}}|_{\mathrm{CAT}^{\mathrm{op}}(\partial\varDelta(J))}$ are fibrations (surjections of dgas). Therefore, both diagrams $\mathcal{R}_{I,\mathcal{K}}$ and $\mathcal{C}_{I,\mathcal{K}}$ are Reedy fibrant (see [3, Appendix C.1]). Their limits are therefore quasi-isomorphic. Thus, we obtain the required zig-zag of quasi-isomorphisms of dgas

$$R_I(\mathcal{K}) = \lim_{J\in\mathcal{K}} R_I(\varDelta(J)) \simeq \lim_{J\in\mathcal{K}} C^*((\boldsymbol{Y},\boldsymbol{B})^J) \xleftarrow{\simeq} C^*(\operatorname*{colim}_{J\in\mathcal{K}}(\boldsymbol{Y},\boldsymbol{B})^J) = C^*((\boldsymbol{Y},\boldsymbol{B})^{\mathcal{K}}),$$

where the second-to-last map is a quasi-isomorphism by excision (or by Mayer–Vietoris). □

For the equivariant cohomology, we obtain

Theorem 3.3. *There are isomorphisms of rings*

$$\begin{aligned} H^*_{T_I}(\mathcal{Z}_{\mathcal{K}}) &\cong H^*\bigl(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}], d\bigr) \cong H^*\bigl(R_I(\mathcal{K}), d\bigr) \\ &\cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\bigl(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}]\bigr), \end{aligned}$$

where $\mathbb{Z}[v_i\colon i \in I]$ is the $\mathbb{Z}[v_1,\ldots,v_m]$-module via the homomorphism sending v_i to 0 for $i \notin I$.

Proof. The first two isomorphisms follow from Theorem 3.2. For the last one, consider the Koszul resolution $\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[v_i\colon i \notin I] \to \mathbb{Z}$ of the augmentation $\mathbb{Z}[v_i\colon i \notin I]$-module $\mathbb{Z}$. Tensoring it with $\mathbb{Z}[v_i\colon i \in I]$ we obtain a free resolution of the $\mathbb{Z}[v_1,\ldots,v_m]$-module $\mathbb{Z}[v_i\colon i \in I]$:

$$\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[v_1,\ldots,v_m] \to \mathbb{Z}[v_i\colon i \in I].$$

Then $\operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[v_i\colon i \in I], \mathbb{Z}[\mathcal{K}])$ is the cohomology of the complex obtained by applying $\otimes_{\mathbb{Z}[v_1,\ldots,v_m]}\mathbb{Z}[\mathcal{K}]$ to the resolution above, which gives $\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}]$. □

When $I=[m]$, we obtain that the singular cochain algebra of $ET^m\times_{T^m}\mathcal Z_{\mathcal K}\simeq(\mathbb CP^\infty,pt)^{\mathcal K}$ is quasi-isomorphic to $\mathbb Z[\mathcal K]$ with zero differential, which is the integral formality result of [11].

When $I=\varnothing$, we obtain the description of the ordinary integral cohomology of $\mathcal Z_{\mathcal K}$ of [2] and [5].

4. Equivariant formality

A T^k-space X is called *equivariantly formal* if $H^*_{T^k}(X)$ is free as a module over $H^*_{T^k}(pt)=H^*(BT^k)$. The latter condition implies that the spectral sequence of the bundle $ET^k\times_{T^k}X\to BT^k$ collapses at the E_2 page.

Using the results of the previous section, we obtain that $\mathcal Z_{\mathcal K}$ is equivariantly formal with respect to the action of T_I if $\operatorname{Tor}_{\mathbb Z[v_1,\dots,v_m]}(\mathbb Z[v_i\colon i\in I],\mathbb Z[\mathcal K])$ is free as a module over $H^*(BT_I)=\mathbb Z[v_i\colon i\in I]$.

Lemma 4.1. *Let $\mathcal K=\Delta[m]$. Then $H^*_{T_I}(\mathcal Z_{\mathcal K})$ is free as a $H^*(BT_I)$-module, for any $I\subset[m]$.*

Proof. For $\mathcal K=\Delta[m]$, we have $\mathcal Z_{\mathcal K}\cong D^{2m}$ is T_I-equivariantly contractible. Hence, $H^*_{T_I}(\mathcal Z_{\mathcal K})\cong H^*_{T_I}(pt)=H^*(BT_I)$ is a free $H^*(BT_I)$-module. □

Lemma 4.2. *Let $\mathcal K=\partial\Delta[m]$, the boundary of a simplex on $[m]$. Then $\mathcal Z_{\mathcal K}\cong S^{2m-1}$ and $H^*_{T_I}(\mathcal Z_{\mathcal K})$ is free as a $H^*(BT_I)$-module, for any $I\subsetneq[m]$.*

Proof. Consider the spectral sequence of the bundle $ET_I\times_{T_I}\mathcal Z_{\mathcal K}\to BT_I$ with fibre $\mathcal Z_{\mathcal K}\cong S^{2m-1}$. We claim that the homomorphism $H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\to H^*(\mathcal Z_{\mathcal K})$ induced by the inclusion of the fibre is surjective. Indeed, by the construction of the previous section, $H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\to H^*(\mathcal Z_{\mathcal K})$ is the cohomology homomorphism induced by the dga map

$$\bigl(\Lambda[u_i\colon i\notin I]\otimes\mathbb Z[\mathcal K],d\bigr)\to\bigl(\Lambda[u_1,\dots,u_m]\otimes\mathbb Z[\mathcal K],d\bigr).$$

We have $H^*(\mathcal Z_{\mathcal K})=\mathbb Z\langle 1,[u_iv_1\cdots\widehat{v}_i\cdots v_m]\rangle$, where $[u_iv_1\cdots\widehat{v}_i\cdots v_m]\in H^{2m-1}(\mathcal Z_{\mathcal K})$ denotes the cohomology class of the cocycle $u_iv_1\cdots\widehat{v}_i\cdots v_m$ with v_i omitted (note that $\mathbb Z[\mathcal K]=\mathbb Z[v_1,\dots,v_m]/(v_1\cdots v_m)$). Choosing $i\notin I$ we get that $[u_iv_1\cdots\widehat{v}_i\cdots v_m]$ also represents a nontrivial cohomology class in $H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})$ (here we use the fact that $I\ne[m]$). Hence, $H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\to H^*(\mathcal Z_{\mathcal K})$ is surjective.

Now $H^q(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\to H^q(\mathcal Z_{\mathcal K})$ is the edge homomorphism

$$H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\to E_\infty^{0,q}\to E_2^{0,q}=H^q(\mathcal Z_{\mathcal K})$$

of the spectral sequence. Its surjectivity implies $E_2^{0,q}=E_\infty^{0,q}$, that is, all the differentials from the first column are trivial. By the multiplicative structure in the spectral sequence, all other differentials are also trivial. We obtain $H^*(ET_I\times_{T_I}\mathcal Z_{\mathcal K})\cong E_\infty=E_2\cong H^*(BT_I)\otimes H^*(\mathcal Z_{\mathcal K})$, a free $H^*(BT_I)$-module. □

Let $\mathcal K_1$ and $\mathcal K_2$ be simplicial complexes on the sets V_1 and V_2, respectively. Their *join* is the simplicial complex on $V_1\sqcup V_2$ given by

$$\mathcal K_1*\mathcal K_2=\{I_1\sqcup I_2\subset V_1\sqcup V_2\colon I_1\in\mathcal K_1,\ I_2\in\mathcal K_2\}.$$

Lemma 4.3. *Let $I_1\subset V_1$, $I_2\subset V_2$, $V=V_1\sqcup V_2$, $I=I_1\sqcup I_2$ and $\mathcal K=\mathcal K_1*\mathcal K_2$. Suppose that $H^*_{T_{I_1}}(\mathcal Z_{\mathcal K_1})$ is free as a $H^*(BT_{I_1})$-module, and $H^*_{T_{I_2}}(\mathcal Z_{\mathcal K_2})$ is free as a $H^*(BT_{I_2})$-module. Then $H^*_{T_I}(\mathcal Z_{\mathcal K})$ is free as a $H^*(BT_I)$-module.*

Proof. We have $\mathcal Z_{\mathcal K}\cong\mathcal Z_{\mathcal K_1}\times\mathcal Z_{\mathcal K_2}$ by [3, Proposition 4.1.3]. Then

$$\begin{aligned}H^*_{T_I}(\mathcal Z_{\mathcal K})&\cong H^*_{T_I}(\mathcal Z_{\mathcal K_1}\times\mathcal Z_{\mathcal K_2})=H^*\bigl(ET_I\times_{T_I}(\mathcal Z_{\mathcal K_1}\times\mathcal Z_{\mathcal K_2})\bigr)\\&\cong H^*\bigl((ET_{I_1}\times_{T_{I_1}}\times\mathcal Z_{\mathcal K_1})\times(ET_{I_2}\times_{T_{I_2}}\times\mathcal Z_{\mathcal K_2})\bigr)\\&\cong H^*(ET_{I_1}\times_{T_{I_1}}\mathcal Z_{\mathcal K_1})\otimes H^*(ET_{I_2}\times_{T_{I_2}}\mathcal Z_{\mathcal K_2})=H^*_{T_{I_1}}(\mathcal Z_{\mathcal K_1})\otimes H^*_{T_{I_2}}(\mathcal Z_{\mathcal K_2}).\end{aligned}$$

The second-to-last isomorphism above follows by the Künneth formula, because $H^*(ET_{I_1} \times_{T_{I_1}} \mathcal{Z}_{\mathcal{K}_1})$ is a free $\mathbb{Z}$-module (as it is a free $\mathbb{Z}[v_i\colon i \in I_1]$-module). The claim follows, since $H^*(BT_I) = H^*(BT_{I_1}) \otimes H^*(BT_{I_2})$. □

Lemma 4.4. *Let $I \notin \mathcal{K}$. Then $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not free as a module over $H^*(BT_I)$.*

Proof. Take $v_I = \prod_{i\in I} v_i \in H^*(BT_I)$. Then $v_I \cdot 1 = [v_I] = 0$, because v_I represents zero in $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}}) = H(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}])$. Hence, 1 is a $H^*(BT_I)$-torsion element, and $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not free as a $H^*(BT_I)$-module. □

Let $\mathcal{K}$ be a simplicial complex on V and $V' \subset V$. The subcomplex $\mathcal{K}' = \{I \in \mathcal{K}\colon I \subset V'\}$ is called a *full subcomplex* (an *induced subcomplex* on V'). Equivalently, $\mathcal{K}' \subset \mathcal{K}$ is a full subcomplex if any missing face of $\mathcal{K}'$ is a missing face of $\mathcal{K}$.

Lemma 4.5. *If $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is a free $H^*(BT_I)$-module and $\mathcal{K}'$ is a full subcomplex of $\mathcal{K}$ such that $I \in \mathcal{K}'$, then $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}'})$ is also a free $H^*(BT_I)$-module.*

Proof. Since $\mathcal{K}'$ is a full subcomplex, there is a retraction $\mathcal{Z}_{\mathcal{K}'} \to \mathcal{Z}_{\mathcal{K}} \to \mathcal{Z}_{\mathcal{K}'}$ (see [13, Proposition 2.2] or [12, Lemma 4.2]), which is T_I-equivariant for any $I \subset V'$. It follows that $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}'})$ is a direct summand in the free $H^*(BT_I)$-module $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$. Hence, $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}'})$ is also free. □

The equivariant cohomology $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ may fail to be free as a $H^*(BT_I)$-module even when I is a simplex of $\mathcal{K}$:

Example 4.6. Let $\mathcal{K}$ be an m-cycle (the boundary of an m-gon), with vertices numbered counter-clockwise. Let $I = \{i\}$, so that T_I is the ith coordinate circle S^1_i. When $m = 3$ or $m = 4$, $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is free over $\mathbb{Z}[v_i]$ for all i by Lemma 4.2 and Lemma 4.3. Suppose that $m \geqslant 5$. Then the nonzero cohomology class in $H^3_{S^1_m}(\mathcal{Z}_{\mathcal{K}})$ represented by the cocycle $u_1v_3 \in \Lambda[u_1, \ldots, u_{m-1}] \otimes \mathbb{Z}[\mathcal{K}]$ is a $\mathbb{Z}[v_m]$-torsion element. Indeed, $v_m \cdot [u_1v_3] = [u_1v_3v_m] = 0$, since $v_3v_m = 0$ in $\mathbb{Z}[\mathcal{K}]$ for $m \geqslant 5$. Hence, $H^*_{S^1_m}(\mathcal{Z}_{\mathcal{K}})$ is not free as a $\mathbb{Z}[v_m]$-module.

Recall that a *missing face* (a *minimal non-face*) of a simplicial complex $\mathcal{K}$ on V is a subset $I \subset V$ such that $I \notin \mathcal{K}$ but every proper subset of I is in $\mathcal{K}$. In other words, I is a missing face if $\partial\Delta(I)$ is a subcomplex of $\mathcal{K}$, but $\Delta(I)$ is not. We denote by $\mathrm{MF}(\mathcal{K})$ the set of missing faces of $\mathcal{K}$.

Generalising Example 4.6, we have

Lemma 4.7. *Let I_1 and I_2 be missing faces of $\mathcal{K}$, and suppose that $I = I_1 - I_2$ is nonempty. Then $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not free as a module over $H^*(BT_I)$.*

Proof. Since I_1 and I_2 are distinct missing faces, we have $I_2 \not\subset I_1$. Take $j \in I_2 - I_1$. Then $j \notin I$. The cocycle $u_jv_{I_2-j}$ represents a nontrivial cohomology class in $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}}) = H(\Lambda[u_i\colon i \notin I] \otimes \mathbb{Z}[\mathcal{K}])$.

We claim that the cohomology class $[u_jv_{I_2-j}] \in H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is a $H^*(BT_I)$-torsion element. Indeed, take $v_I = \prod_{i\in I} v_i \in H^*(BT_I)$. Then

$$v_I \cdot [u_jv_{I_2-j}] = [u_jv_Iv_{I_2-j}] = [u_jv_{I_1}v_{I_2-I_1-j}] = 0,$$

since $v_{I_1} = 0$ in $\mathbb{Z}[\mathcal{K}]$. Hence, $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not free as a $H^*(BT_I)$-module. □

Theorem 4.8. *Let $\mathcal{K}$ be a simplicial complex on a finite set V. The following conditions are equivalent:*

(a) *For any $I \in \mathcal{K}$, the equivariant cohomology $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is a free module over $H^*(BT_I)$.*

(b) *There is a partition* $V = V_1 \sqcup \cdots \sqcup V_p \sqcup U$ *such that*

$$\mathcal{K} = \partial\Delta(V_1) * \cdots * \partial\Delta(V_p) * \Delta(U),$$

where $\Delta(U)$ *denotes a full simplex on* U*, and* $\partial\Delta(V_i)$ *denotes the boundary of a simplex on* V_i*.*

(c) *The rational face ring* $\mathbb{Q}[\mathcal{K}]$ *is a complete intersection ring (the quotient of the polynomial ring by an ideal generated by a regular sequence).*

Proof. (a)⇒(b) We have $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1,\ldots,v_m]/(t_1,\ldots,t_p)$ where $t_k = \prod_{i\in V_k} v_i$ is a square-free monomial and V_k is a missing face of $\mathcal{K}$, for $k = 1,\ldots,p$. Suppose some of these missing faces intersect nontrivially, say, $V_1 \cap V_2 \neq \varnothing$. Then $I = V_1 - V_2$ is nonempty, and $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not a free $H^*(BT_I)$-module by Lemma 4.7. A contradiction. Hence, $V_1,\ldots,V_p$ are pairwise non-intersecting, so $\mathcal{K}$ is as described in (b).

(b)⇒(a) Write $I = I_1 \sqcup \cdots \sqcup I_p \sqcup J$, where $I_k \subsetneq V_k$, $J \subset U$. Then $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is a free $H^*(BT_I)$-module by Lemmas 4.1, 4.2 and 4.3.

(b)⇒(c) Recall [3, §A.3] that a sequence of homogeneous elements $(t_1,\ldots,t_k)$ of positive degree in $\mathbb{Q}[v_1,\ldots,v_m]$ is a *regular sequence* if t_{i+1} is not a zero divisor in $\mathbb{Q}[v_1,\ldots,v_m]/(t_1,\ldots,t_i)$ for $0 \leqslant i < k$. If $\mathcal{K}$ is as in (b), then $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1,\ldots,v_m]/(t_1,\ldots,t_p)$, where $m = |V|$ and $t_k = \prod_{i\in V_k} v_i$ for $k = 1,\ldots,p$. Then $(t_1,\ldots,t_p)$ is a regular sequence, so $\mathbb{Q}[\mathcal{K}]$ is a complete intersection ring.

(c)⇒(b) Suppose $\mathbb{Q}[\mathcal{K}] = \mathbb{Q}[v_1,\ldots,v_m]/(t_1,\ldots,t_p)$ where $(t_1,\ldots,t_p)$ is a regular sequence. We can assume that $t_k = \prod_{i\in V_k} v_i$ where V_k is a missing face of $\mathcal{K}$, for $k = 1,\ldots,p$. Suppose some of these missing faces intersect nontrivially, say, $V_1 \cap V_2 \neq \varnothing$. Then $t_2 \cdot \prod_{i\in V_1-V_2} v_i = t_1 \cdot \prod_{j\in V_2-V_1} v_j$, so t_2 is a zero divisor in $\mathbb{Q}[v_1,\ldots,v_m]/(t_1)$. A contradiction. Hence, $V_1,\ldots,V_p$ are pairwise non-intersecting, so $\mathcal{K}$ is as described in (b). □

The equivalence (b)⇔(c) of Theorem 4.8 was noted in [11, §5].

Recall that $\mathcal{K}$ is called a *flag complex* if each of its missing faces has two vertices. A simplicial complex $\mathcal{K}$ is flag if and only if it has no ghost vertices and any set of vertices of $\mathcal{K}$ which are pairwise connected by edges spans a simplex. In the case of flag complexes we have the following specification of the criterion in Theorem 4.8.

Theorem 4.9. *Let* $\mathcal{K}$ *be a flag complex on* V*. Then the following conditions are equivalent:*

(a) $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ *is a free module over* $\mathbb{Z}[v_i]$ *for all* i*.*

(b) $\mathcal{K} = \partial\Delta(V_1) * \cdots * \partial\Delta(V_p) * \Delta(U)$ *where* $|V_k| = 2$ *for* $k = 1,\ldots,p$*.*

Proof. Implication (b)⇒(a) follows from Theorem 4.8, so we only need to prove (a)⇒(b). Let V_1, V_2 be missing faces. Then $|V_1| = |V_2| = 2$. If $V_1 \cap V_2 \neq \varnothing$, then $V_1 - V_2 = \{i\}$ for some $i \in V$. Then $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is not free as a $\mathbb{Z}[v_i]$-module by Lemma 4.7. This contradiction shows that all missing faces of $\mathcal{K}$ are pairwise non-intersecting, so $\mathcal{K}$ is as in (b). □

Here is an example showing that the equivalence of Theorem 4.9 does not hold in the non-flag case.

Example 4.10. Let $\mathcal{K}$ be the simplicial complex on 5 vertices with $\mathrm{MF}(\mathcal{K}) = \{I_1, I_2\}$, where $I_1 = \{1,2,3\}$ and $I_2 = \{3,4,5\}$. Then $H^*_{T_I}(\mathcal{Z}_{\mathcal{K}})$ is not free as a $H^*(BT_I)$-module for $I = \{1,2\}$ (or for $I = \{4,5\}$) due to Lemma 4.7. However, $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is a free $H^*(BS^1_i)$-module for all i. Indeed, it can be shown by the methods of [7, §8] or [1] that $\mathcal{Z}_{\mathcal{K}} \cong S^5 \vee S^5 \vee S^8$. The ordinary cohomology is generated by the following monomials in the Koszul algebra $\Lambda[u_1, \dots, u_5] \otimes \mathbb{Z}[\mathcal{K}]$:

$$H^*(\mathcal{Z}_{\mathcal{K}}) = \mathbb{Z}\langle 1, [u_k v_{I_1-k}], [u_j v_{I_2-j}], [u_{i_1} u_{i_2} v_{[5]-i_1-i_2}]\rangle,$$

where $k \in I_1$, $j \in I_2$, $[u_k v_{I_1-k}], [u_j v_{I_2-j}] \in H^5(\mathcal{Z}_{\mathcal{K}})$, $i_1 \in I_1 - I_2$, $i_2 \in I_2 - I_1$ and $[u_{i_1} u_{i_2} v_{[5]-i_1-i_2}] \in H^8(\mathcal{Z}_{\mathcal{K}})$.

Since both $I_1 - I_2$ and $I_2 - I_1$ contain two elements, we can choose k, j, i_1, i_2 such that $i \notin \{k, j, i_1, i_2\}$. Then the monomials $u_k v_{I_1-k}$, $u_j v_{I_2-j}$, $u_{i_1} u_{i_2} v_{[5]-i_1-i_2}$ represent nontrivial cohomology classes in $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$. This implies that the homomorphism $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}}) \to H^*(\mathcal{Z}_{\mathcal{K}})$ is surjective. Therefore, the spectral sequence of the bundle $ES^1_i \times_{S^1_i} \mathcal{Z}_{\mathcal{K}} \to BS^1_i$ collapses at the E_2 page, as in the proof of Lemma 4.2. It follows that $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}}) \cong H^*(BS^1_i) \otimes H^*(\mathcal{Z}_{\mathcal{K}})$, a free $H^*(BS^1_i)$-module.

The equivalence similar to that of Theorem 4.9 also holds when $\mathcal{K}$ is one-dimensional.

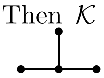
Theorem 4.11. *Let $\mathcal{K}$ be a one-dimensional complex (a simple graph). Then the following conditions are equivalent:*

(a) *$H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is a free module over $\mathbb{Z}[v_i]$ for any i.*

(b) *$\mathcal{K}$ is the one of the following: $\partial\Delta^2$, $\partial\Delta^1 * \partial\Delta^1$, $\partial\Delta^1$, Δ^1, $\partial\Delta^1 * \Delta^0$, Δ^0.*

Proof. Implication (b)⇒(a) follows from Theorem 4.8, so we only need to prove (a)⇒(b). We consider several cases.

Case 1: $\mathcal{K}$ is a tree. If it has no more than three vertices, then $\mathcal{K}$ is Δ^1, $\partial\Delta^1 * \Delta^0$ or Δ^0. In each of these cases $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is a free $\mathbb{Z}[v_i]$-module by Theorem 4.8.

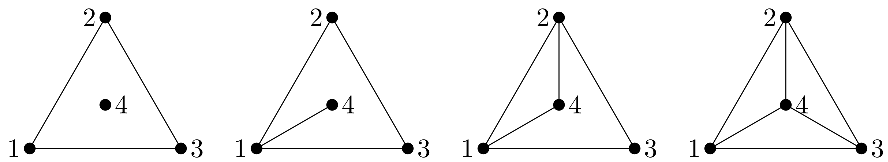
Suppose $\mathcal{K}$ has more than three vertices. Then $\mathcal{K}$ has a connected induced subgraph $\mathcal{K}_1$ on 4 vertices, which has the form  or . In both cases, there are $I_1, I_2 \in \mathrm{MF}(\mathcal{K}_1)$ such that $I_1 - I_2 = \{i\}$ for some i. Then $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}_1})$ is not free over $\mathbb{Z}[v_i]$ by Lemma 4.7, and $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is also not free by Lemma 4.5. A contradiction.

Case 2: $\mathcal{K}$ is a disjoint union of trees. If $\mathcal{K}$ has two vertices, then $\mathcal{K} = \partial\Delta^1$.

Suppose $\mathcal{K}$ has more than two vertices. Write $\mathcal{K} = \mathcal{K}_1 \sqcup \dots \sqcup \mathcal{K}_s$ where each $\mathcal{K}_j$ is a tree. Then each $\mathcal{K}_j$ has at most three vertices by Case 1. Take $I_1 = \{i, j\}$, $I_2 = \{k, j\}$, where $i \in \mathcal{K}_1$, $j \in \mathcal{K}_2$ and $\{k, j\} \notin \mathcal{K}$. Then $I_1, I_2 \in \mathrm{MF}(\mathcal{K})$ and $I_1 - I_2 = \{i\}$. Hence, $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is not a free $\mathbb{Z}[v_i]$-module by Lemma 4.7. A contradiction.

Case 3: $\mathcal{K}$ has a 3-cycle. If $\mathcal{K}$ is a 3-cycle, then $\mathcal{K} = \partial\Delta^2$.

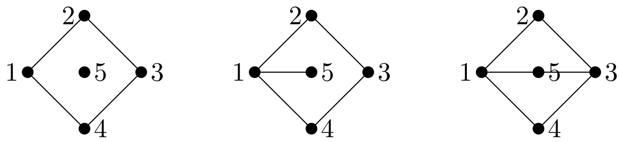
Suppose $\mathcal{K}$ has at least 4 vertices. Consider the induced subgraph on 4 vertices containing a 3-cycle. There are four cases:



In the first two cases, take $I_1 = \{3, 4\}$, $I_2 = \{2, 4\}$ in $\mathrm{MF}(\mathcal{K})$. In the last two cases, take $I_1 = \{1, 2, 3\}$ and $I_2 = \{1, 2, 4\}$ in $\mathrm{MF}(\mathcal{K})$. Then $I_1 - I_2 = \{3\}$ and $H^*_{S^1_3}(\mathcal{Z}_{\mathcal{K}})$ is not a free $\mathbb{Z}[v_3]$-module by Lemma 4.7. A contradiction.

Case 4: $\mathcal{K}$ has no 3-cycles and has a 4-cycle. If $\mathcal{K}$ is a 4-cycle, then $\mathcal{K} = \partial\Delta^1 * \partial\Delta^1$.

Suppose $\mathcal{K}$ has more than 4 vertices. Consider the induced subgraph on 5 vertices containing a 4-cycle. Since there are no 3-cycles, there are three cases:



In all cases take $I_1 = \{2,4\}$, $I_2 = \{4,5\}$ in $\mathrm{MF}(\mathcal{K})$, then $I_1 - I_2 = \{2\}$ and $H^*_{S^1_2}(\mathcal{Z}_{\mathcal{K}})$ is not a free $\mathbb{Z}[v_2]$-module by Lemma 4.7. A contradiction again.

Case 5: each minimal cycle in $\mathcal{K}$ has length at least 5. Then $\mathcal{K}$ has an induced subgraph $\mathcal{K}_1$ which is an m-cycle with $m \geqslant 5$. As in Example 4.6, we have that $H^*_{S^1_m}(\mathcal{Z}_{\mathcal{K}_1})$ is not free as a $\mathbb{Z}[v_m]$-module. So $H^*_{S^1_m}(\mathcal{Z}_{\mathcal{K}})$ is also not free by Lemma 4.5. A contradiction. $\square$

References

[1] Abramyan, Semyon; Panov, Taras. *Higher Whitehead products in moment-angle complexes and substitution of simplicial complexes.* Tr. Mat. Inst. Steklova 305 (2019), 7–28 (Russian); Proc. Steklov Inst. Math. 305 (2019), no. 1, 1–21 (English translation).

[2] Baskakov, Ilia; Buchstaber, Victor; Panov, Taras. *Cellular cochain algebras and torus actions.* Uspekhi Mat. Nauk 59 (2004), no. 3, 159–160 (Russian); Russian Math. Surveys 59 (2004), no. 3, 562–563 (English translation).

[3] Buchstaber, Victor; Panov, Taras. *Toric topology.* Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015.

[4] Eilenberg, Samuel; Moore, John C. *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors.* Comment. Math. Helv. 40 (1966), 199–236.

[5] Franz, Matthias. *On the integral cohomology of smooth toric varieties.* Proc. Steklov Inst. Math. 252 (2006), 53–62.

[6] Franz, Matthias. *Dga models for moment-angle complexes.* Preprint (2020); arXiv:2006.01571.

[7] Grbić, Jelena; Theriault, Stephen. *Homotopy theory in toric topology.* Uspekhi Mat. Nauk 71 (2016), no. 2, 3–80; Russian Math. Surveys 71 (2016), no. 2, 185–251 (English).

[8] Luo, Shisen; Matsumura, Tomoo; Moore, W. Frank. *Moment angle complexes and big Cohen–Macaulayness.* Algebr. Geom. Topol. 14 (2014), no. 1, 379–406.

[9] Mac Lane, Saunders. *Homology.* Springer, Berlin, 1963.

[10] McCleary, John. *A user's guide to spectral sequences.* Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

[11] Notbohm, Dietrich; Ray, Nigel. *On Davis–Januszkiewicz homotopy types. I. Formality and rationalisation.* Algebr. Geom. Topol. 5 (2005), 31–51.

[12] Panov, Taras; Theriault, Stephen. *The homotopy theory of polyhedral products associated with flag complexes.* Compositio Math. 155 (2019), no. 1, 206–228.

[13] Panov, Taras; Veryovkin, Yakov. *Polyhedral products and commutator subgroups of right-angled Artin and Coxeter groups.* Mat. Sbornik 207 (2016), no. 11, 105–126 (Russian); Sbornik Math. 207 (2016), no. 11, 1582–1600 (English translation).

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