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Almost uniform domains and Poincaré inequalities

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Abstract

Here we show existence of many subsets of Euclidean spaces that, despite having empty interior, still support Poincaré inequalities with respect to the restricted Lebesgue measure. Most importantly, despite the explicit constructions in our proofs, our methods do not depend on any rectilinear or self-similar structure of the underlying space. We instead employ the uniform domain condition of Martio and Sarvas. This condition relies on the measure density of such subsets, as well as the regularity and relative separation of their boundary components.

In doing so, our results hold true for metric spaces equipped with doubling measures and Poincaré inequalities in general, and for the Heisenberg groups in particular. To our knowledge, these are the first examples of such subsets on any (nonabelian) Carnot group. Such subsets also give new examples of Sobolev extension domains, also in the general setting of doubling metric measure spaces.

In the Euclidean case, our construction also includes the non-self-similar Sierpiński carpets of Mackay, Tyson and Wildrick, as well as higher dimensional analogues not treated in the literature. When specialized to the plane, our results lead to new, general sufficient conditions for a planar subset to be 2-Ahlfors regular and to satisfy the Loewner condition. Two of these conditions, uniform separation and regularity of boundary components, are also necessary. The sufficiency is obtained with an additional measure density condition.

Contents

1. *Introduction*

1.1. *Poincaré inequalities and Sierpiński sponges*

Let (X, d) be a complete metric space that supports a doubling measure μ. We wish to understand the following question:

If X supports a $(1, p)$-Poincaré inequality, then when does a subset Y of X, equipped with its restricted measure and metric, support a $(1, q)$-Poincaré inequality, and for which exponents $q \in [1, \infty)$?

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This question is motivated by the desire to construct a new, general class of examples that include so-called uniform domains and more generally, Sobolev extension domains. Below, our main results will give criteria to guarantee such examples, in both the Euclidean and the general metric space setting. To this end, we begin with some definitions.

DEFINITION 1.1. Let $r_0 > 0$. A proper metric measure space (X, d, μ) with a Radon measure μ is said to be D-DOUBLING AT SCALE r_0 — or (D, r_0)-DOUBLING for short — if for all $r \in (0, r_0)$ and any $x \in X$, we have

$$0 < \mu(B(x, 2r)) \leqslant D\mu(B(x, r)).$$

If (X, d, μ) is D-doubling at scale r_0 for all $r_0 > 0$, then X is said to be D-DOUBLING.

We will assume that the support of the measure equals the space, $\operatorname{supp}(\mu) = X$.

DEFINITION 1.2. Let $r_0 > 0$ and $1 \leqslant p < \infty$. A proper metric measure space (X, d, μ) with a Radon measure μ is said to satisfy a $(1, p)$-POINCARÉ INEQUALITY AT SCALE r_0 (with constant $C \geqslant 1$) if for all Lipschitz functions $f\colon X \to \mathbb{R}$ and all $x \in X$ and $r \in (0, r_0)$ we have for $B := B(x, r)$ and $CB := B(x, Cr)$

$$\fint_B |f - f_B| \, d\mu \leqslant Cr \left(\fint_{CB} \operatorname{Lip}[f]^p \, d\mu \right)^{\frac{1}{p}}. \tag{1.3}$$

If $r_0 = \infty$, then say that X satisfies a (GLOBAL) $(1, p)$-POINCARÉ INEQUALITY (with the same constants).

A space satisfying a Poincaré inequality and the doubling property is called a *PI-space*.

Here, for any measurable and locally integrable $f\colon X \to \mathbb{R}$ its average value on a ball is

$$f_B := \fint_B f \, d\mu := \frac{1}{\mu(B)} \int_B f \, d\mu,$$

and its pointwise Lipschitz constant is

$$\operatorname{Lip}[f](x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

In the literature, there are different definitions of Poincaré inequalities, all of which coincide with our definition in the case of complete metric spaces. For a detailed discussion of these issues, we refer to [**19, 22, 26**].

Poincaré inequalities play a profound role in analysis and the regularity of functions. In the general setting of metric measure spaces, they are crucial hypotheses for nontrivial definitions of generalized Sobolev spaces [**10, 18, 39**] and differentiability of Lipschitz functions [**10**]. Moreover, open subsets $\Omega \subset X$ supporting a $(1, p)$-Poincaré inequality and with a lower bound on their measure density are important examples of sets admitting extensions of Sobolev spaces. See [**7, 20, 25**] and below for more related historical discussion and references. We remark that applying our work there requires some care, as our constructions lead to closed sets. However, one can also consider Sobolev extension problems with other gradients which make sense also for closed sets.

Poincaré inequalities also play a profound role in the study of geometry of metric spaces, specifically in regards to quasi-conformal mappings between them [**23**]. Planar metric spaces that are Ahlfors 2-regular and that support a (1,2)-Poincaré inequality are examples of sets which admit uniformization by slit carpets, see [**31**, Section 7]. Such inequalities are also important in determining the so-called *conformal dimension* of a space [**30**]. In general, conformal dimension measures the extent to which Hausdorff dimension can be lowered by

quasi-symmetric maps, and it is known that any Ahlfors regular space satisfying a Poincaré inequality has conformal dimension equal to its Hausdorff dimension.

However, a good understanding of the geometric conditions that guarantee such inequalities, in particular for subsets, has remained a challenge. Particular examples of subsets in the plane satisfying Poincaré inequalities were given by Mackay, Tyson and Wildrick [31]. We briefly discuss a construction here that includes theirs.

Let $\mathbf{n} = (n_i)_{i=1}^{\infty}$ be a sequence of odd positive integers with $n_i \geqslant 3$. As a convention, put

$$\mathbf{n}^{-1} = \left(\frac{1}{n_i}\right)_{i=1}^{\infty}.$$

Fix a dimension $d \geqslant 2$. We define the Sierpiński sponge associated to $\mathbf{n}$ in $\mathbb{R}^d$ as follows.

(1) At the first stage, put $S_{0,\mathbf{n}} = [0,1]^d$ and $T^1_{0,\mathbf{n}} = [0,1]^d$ and $\mathcal{T}_{0,\mathbf{n}} = \{T^1_{0,\mathbf{n}}\}$.

(2) Assuming that we have defined sets $S_{k,\mathbf{n}}$ and $T^j_{k,\mathbf{n}}$ and collections of cubes $\mathcal{T}_{k,\mathbf{n}}$ at the kth stage, for $k \in \mathbb{N}$:

- subdivide each $T \in \mathcal{T}_{k,\mathbf{n}}$ into $(n_{k+1})^d$ equal subcubes;
- excluding the central subcube in T, index the remaining subcubes in any fashion as $T^j_{k+1,\mathbf{n}}$ and let $\mathcal{T}_{k+1,\mathbf{n}} = \{T^j_{k+1,\mathbf{n}}\}$ be the collection of all such subcubes. We note that for $k \in \mathbb{N}$, the side length of each subcube $T^j_{k,\mathbf{n}}$ is therefore

$$s_k = \prod_{i=1}^{k} \frac{1}{n_i};$$

(For consistency, let $s_0 = 1$.)

- define the $k+1$'th order pre -sponge as the set

$$S_{k+1,\mathbf{n}} = \bigcup_{T \in \mathcal{T}_{k+1,\mathbf{n}}} T.$$

(3) For technical purposes later, let $k \geqslant 1$ and define $\mathcal{R}_{\mathbf{n},k}$ to be the sub-collection of central cubes removed from cubes $T \in \mathcal{T}_{k-1,\mathbf{n}}$ at the k'th stage and put

$$\overline{\mathcal{R}}_{\mathbf{n},k} = \bigcup_{l=1}^{k} \mathcal{R}_{\mathbf{n},l}.$$

The Sierpiński sponge associated to the sequence $\mathbf{n}$ is then defined as

$$S_{\mathbf{n}} = \bigcap_{k \geqslant 0} S_{k,\mathbf{n}}. \tag{1.4}$$

When $d = 2$, we also refer to these sets as Sierpiński carpets, and the constant sequence $\mathbf{n} = (3,3,3\ldots)$ yields the usual 'middle-thirds' Sierpiński carpet, which is denoted by S_3.

The main result by Mackay, Tyson and Wildrick [31] states that Sierpiński carpets with positive Lebesgue measure satisfy Poincaré inequalities. Their proof was a *tour de force* in constructing so-called Semmes families of (rectifiable) curves and then applying a characterization of Poincaré inequalities from Keith [26]. (For precise definitions and a further discussion, see [40].)

However, even slight variations of their construction, such as removing a 'nearly central' square instead of a central one, would require a new construction of a curve family with new, equally technical details to check. Our motivation was therefore to find more general and robust methods that apply to *all dimensions*, as well as to *non-Euclidean geometries* too.

First of all, our methods lead to the following higher dimensional analogue of their result.

THEOREM 1.5. *Let $\mathbf{n} = (n_i)$ be a sequence of odd integers with $n_i \geqslant 3$, and let $d \geqslant 2$. Then the following conditions are equivalent for the Sierpiński sponge $S_{\mathbf{n}}$ in $\mathbb{R}^d$.*

(1) $\mathbf{n}^{-1} \in \ell^d(\mathbb{N})$.
(2) *The space $(S_{\mathbf{n}}, |\cdot|, \lambda)$ satisfies a $(1,p)$-Poincaré inequality for all $p > 1$.*
(3) *The space $(S_{\mathbf{n}}, |\cdot|, \lambda)$ satisfies a $(1,p)$-Poincaré inequality for some $p > 1$.*

In addition, we have the following complementary case.

(4) *If $S_{\mathbf{n}}$ has zero λ-measure, then there is no D-doubling measure μ, for any $D \in [1,\infty)$, such that $(S_{\mathbf{n}}, |\cdot|, \mu)$ satisfies a $(1,p)$-Poincaré inequality with $p \in [1,\infty)$.*

The borderline case of $p = 1$ can also be fully characterized in terms of $\mathbf{n}$. The case of $d = 2$ appeared before in [31]. The general borderline case for all $d \geqslant 2$ is presented in a separate paper by the authors (Eriksson-Bique and Gong, in a forthcoming), and the approach involves substantially different methods.

A crucial aspect of our theorem is the sharp characterization of the exponents p. In what follows, we also obtain essentially sharp characterizations for the given ranges of exponents in more general Euclidean constructions, and even in the general metric space context!

1.2. *The planar Loewner problem*

Motivated by this result, we consider general sets of the form $Y = \mathbb{R}^d \setminus \bigcup_{R\in\mathcal{R}} R$, for some countable collection $\mathcal{R}$ of open subsets and study when Y inherits a Poincaré inequality. (Bear in mind that the elements of $\mathcal{R}$ will still have good geometric properties, but are not necessarily polyhedral, or even Lipschitz.)

The case of $d = 2$ is particularly interesting, due to the connections with quasi-conformal geometry. In particular, for $d = 2$ the conditions given for $\mathcal{R}$ are not only sufficient, but also close to necessary. This also gives a partial answer to the following question.

QUESTION 1.6 ('Planar Loewner problem'). Classify all closed subsets of the plane which are Ahlfors 2-regular and 2-Loewner.

Although we will not explicitly define the Loewner condition here, we recall that a closed Ahlfors 2-regular subset is 2-Loewner if and only if it satisfies a (1,2)-Poincaré inequality; for a more general definition and further discussion, see [23].

Although natural to pose, this question has not been extensively studied in the literature. Prior results exist only for some specific cases. We now give a new, general, and sufficient condition for an affirmative answer to this problem. To formulate it, consider collections of removed sets $\mathcal{R}$ and subcollections of sets that meet a given ball $B(x,r)$,

$$\mathcal{R}(x,r) \;=\; \{R \in \mathcal{R} : R \cap B(x,r) \neq \emptyset\},$$

and consider further, for $N \in \mathbb{N}$, an 'N-fold density function' for $\mathcal{R}$ relative to balls:

$$s_N(x,r) \;=\; \inf\left\{ \sum_{R\in\mathcal{R}(x,r)\setminus I} \frac{\lambda(R)}{r^2} : I \subset \mathcal{R}, |I| \leqslant N \right\}, \tag{1.7}$$

where $\lambda(R)$ denotes the usual area, or Lebesgue measure, of R.

THEOREM 1.8. *A closed subset Y of $\mathbb{R}^2$ satisfies a $(1,p)$-Poincaré inequality for every $p \in (1,\infty)$ if it is of the form*

$$Y = \Omega \setminus \bigcup_{R\in\mathcal{R}} R,$$

where the following conditions hold for Ω and for $\mathcal{R}$, for some constants $K \geqslant 1$ and $s > 0$:

(1) *the set Ω is closed, each $R \in \mathcal{R}$ is open, and each boundary $\partial\Omega$ and ∂R for R is a K-quasi-circle;*
(2) *$\mathcal{R}$ is uniformly relatively s-separated, that is, for all $R, R' \in \mathcal{R} \cup \{\partial\Omega\}$,*

$$\Delta(R, R') := \frac{d(R, R')}{\min(\operatorname{diam}(R), \operatorname{diam}(R'))} \geqslant s;$$

(3) *there exists $N \in \mathbb{N}$ such that*

$$\limsup_{r \to 0} \sup_{x \in Y} s_N(x, r) = 0.$$

Indeed, Condition (3) requires the density of $\mathcal{R}$ at any $x \in Y$ to vanish, but allowing at each scale r for the N largest 'obstacles' in $\mathcal{R}$ to be excluded. A slightly stronger statement, which allows for the density only becoming sufficiently small, is given in Theorem 4.46.

Theorem 1.8 is new even when the collection of obstacles R and Ω have simple geometry, such as when Ω and every R are disks. It is known from [**31**, Corollary 1.9] that *there exist* subsets of this form with empty interior and which satisfy a (1,2)-Poincaré inequality. Such sets, called CIRCLE CARPETS, are constructed implicitly via uniformization and can therefore only be approximated numerically. In contrast, here we give a procedure that yields *explicit* circle carpets satisfying Poincaré inequalities, with a sharp characterization of the range of exponents. This flexibility extends to other shapes and higher dimensions, as described in Corollary 4.31.

To reiterate, the conditions for the sets $R \in \mathcal{R}$ come in three forms: the regularity of their boundaries, their separation, and their density. The first two conditions in the statement are necessary for a subset to be Loewner, as given in Theorem 4.40. These conditions also appear elsewhere in the literature; for instance, they are the relevant conditions in Bonk's work on uniformization of planar subsets [**8**]. Moreover, the conditions on summability also bear close resemblance to the summability conditions arising in other work on uniformizing planar metric spaces [**21, 34**].

1.3. *Metric spaces and Carnot groups*

In the proof of Theorem 1.8, the most crucial feature about the collection $\mathcal{R}$ is that $\mathbb{R}^2 \setminus R$ is a UNIFORM domain, for each $R \in \mathcal{R}$. Such sets were first studied in [**32, 42**]; see Definition 4.12. Roughly speaking, these correspond to domains Ω without 'outer cusps'. Domains in Euclidean space with Lipschitz boundaries are uniform domains, for example, in all dimensions.

In fact, uniformity is a purely metric property. A crucial result of Björn and Shanmugalingam asserts that uniform domains Ω in a doubling metric measure space X inherit a Poincaré inequality from X; see [**7**]. Motivated by this, we therefore formulate a more general theorem for metric spaces.

To this end, call a domain CO-UNIFORM if its complement is uniform and its boundary is connected. The uniform sparseness condition, mentioned below, combines Conditions (2) and (3) in Theorem 1.8 above; for precise statements, see Definitions 4.21 and 4.20. Note that the sequence $\mathbf{n}$ plays an analogous role as the one in Theorem 1.5, in that it handles the density of the omitted subsets.

THEOREM 1.9. *Let X be an Ahlfors Q-regular complete metric measure space admitting a $(1, p)$-Poincaré inequality, and let $\mathbf{n}$ be a sequence of positive integers with $\mathbf{n}^{-1} \in \ell^Q(\mathbb{N})$.*

If Ω is a bounded, A-uniform subset of X and if $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ is a uniformly $\mathbf{n}-$sparse collection of co-uniform subsets of Ω, then the set

$$S_{\mathbf{n}} = \Omega \setminus \bigcup_{k} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}} R,$$

with its restricted measure and metric, is Ahlfors Q-regular and satisfies a $(1,q)$-Poincaré inequality for each $q > p$. Moreover:

- *if $p > 1$, then it also satisfies a $(1,p)$-Poincaré inequality;*
- *if the union of all sets from $\mathcal{R}_{\mathbf{n},k}$, over all $k \in \mathbb{N}$, is dense in Ω, then $S_{\mathbf{n}}$ has empty interior.*

The ranges of the exponents in Theorem 1.9 are sharp. In particular, only for $p = 1$ do such removals of sets lead to a loss in range, namely the loss of the (1,1)-Poincaré inequality; see [**31**] for an example. For $p > 1$, no such loss occurs, due to the seminal self-improvement result of Keith and Zhong [**28**].

For some spaces, such as the Heisenberg group in particular and step-2 Carnot groups in general, the existence of uniform domains is well known, at all scales and locations within these spaces. In such cases, Theorem 1.9 can be used to give new examples of subsets with Poincaré inequalities and empty interior; see Subsection 4.3 for these examples, as well as some of the definitions relevant to these geometries. Due to a recent result by Rajala [**38**], it is likely that the result applies to any Carnot group.

1.4. *Sobolev extension domains*

As a corollary of our theorems, we obtain many new examples of Sobolev extension domains, both in Euclidean and non-Euclidean spaces. To wit, an open subset $\Omega \subset X$ is called a (Sobolev) extension domain if there exists a bounded extension operator $E : N^{1,p}(\Omega) \to N^{1,p}(X)$; in the case where Ω is open in $X = \mathbb{R}^d$ the Newtonian Sobolev space $N^{1,p}(\Omega)$, as introduced in [**39**], coincides with the classical Sobolev space $W^{1,p}(\Omega)$. This definition, when employing $N^{1,p}(\Omega)$, makes sense even for closed subsets Ω, while classically the interest has been mostly for open domains. However, the case of closed sets, as well as the relationship between open and closed extension domains is subtle.

The first examples of extension domains were given by Jones [**25**]. In general, a sufficient condition for Ω to be an extension domain is if Ω supports a $(1,q)$-Poincaré inequality for $q < p$. This condition, however, is not necessary unless p is sufficiently large, as discussed in [**7**].

It remains a difficult problem to give both necessary and sufficient conditions for a domain to be an extension domain. In fact, this has essentially been solved only for simply connected domains in the plane [**46**]. Our examples give flexible constructions of infinitely connected domains in $\mathbb{R}^d$ for $d \geqslant 2$, as well as in step-2 Carnot groups and in general metric spaces, that are Sobolev extension domains. These examples are new even in the planar setting. See [**7, 20**] for more related discussion and references, as well as the PhD thesis [**46**].

1.5. *Methodology: removing subsets versus 'fillings' of spaces*

Thus far, the results in this article apply to subsets Y obtained by removing, from an initial set, infinite collections $\mathcal{R}$ of well-behaved subsets at all locations and scales. As we will see later, these results are special cases of Theorem 2.7 and Corollary 4.19, where such sets Y are viewed from a different perspective. In particular, we view the intermediate sets Ω_r, each obtained by removing a finite sub-collection of subsets in $\mathcal{R}$ up to a given scale $r > 0$, as good approximations (or 'fillings') of Y; in particular, each Ω_r is doubling and supports a Poincaré inequality, both at scale r, and Ω_r also contains Y with small complement.

In fact, these three properties *alone* are sufficient for Y to support a Poincaré inequality, provided that the associated constants are uniform in r. No explicit removals of sets are actually needed for our proofs; the fillings Ω_r only need to satisfy these properties axiomatically, and

they need not be defined, *a priori*, in terms of any removed set. Similarly as for Sobolev extension domains [**20**], it is the *measure density* of the sets Ω_r that is crucial. (In fact, the smallness of $\Omega_r \setminus Y$ is given in terms of measure density; see Definition 2.6.)

The sufficiency of these properties in turn relies crucially on a new characterization of Poincaré inequalities, as studied by the first author [**14, 15**]. Roughly speaking, spaces supporting a Poincaré inequality cannot 'see' sets of small density: points that have small measure density, relative to a given set, can be connected by a quasi-geodesic that meets that set in correspondingly small length. This correspondence, moreover, depends quantitatively but nontrivially on the exponent p. Since we formulate density in terms of maximal functions, we refer to this characterization as 'maximal p-connectivity'.

Intuitively, Ω_r provides improved behavior for Y without adding much density. Once such fillings are available, pairs of points in Y that are at most a distance r apart can be joined by rectifiable curves inside Ω_r. Such curves may not lie entirely in Y, but as the measure density of $\Omega_r \setminus Y$ is small, by maximal connectivity there must be curves which spend little time in this set. The 'bad' portions of these curves can then be removed and replaced by 'good' portions, via a delicate iteration argument.

This filling process is subtle, and the dependence of the exponent p on the quality of the filling is nontrivial. This will be illustrated in the examples below in Subsection 2.2.

Interestingly, we avoid throughout this paper any discussion about the modulus of curve families, and we do not construct any curve families to estimate such moduli. However, in recent work it is shown that such curve families always exist on spaces satisfying Poincaré inequalities. Thus, our tools can be considered to implicitly construct Semmes families of curves. See [**1, 13**].

1.6. *General structure of paper*

In Section 2, we first recall basic notions and relevant notation, and then give precise definitions for fillings of subsets. The section concludes with the statement of our main result, Theorem 2.7, as well as auxiliary results and the strategy of the proof.

In Section 3, we prove Theorem 2.7; it states that subsets admitting such fillings, or 'fillable subsets', must also satisfy Poincaré inequalities. The proof requires Theorems 2.18 and 2.19, which are characterizations of $(1, p)$-Poincaré inequalities and will be proven later.

In Section 4, we apply Theorem 2.7 first to Sierpiński sponges, and then to general metric measure spaces with co-uniform domains removed. We conclude this section with new examples of subsets of the Heisenberg group that satisfy Poincaré inequalities, as well as a discussion of our sufficient condition for planar Loewner subsets. All of these applications use the results in Section 2, but readers may choose to see how these results are applied first, before reading those technical proofs. (To preserve the flow of discussion, the proofs of certain technical results, such as Theorem 4.22, are postponed to the Appendix.)

Lastly, in Section 5 we prove Theorems 2.18 and 2.19 by introducing a certain 'path-connectivity' function associated to metric measure spaces. (Readers who are primarily interested in the classification of Poincaré inequalities may opt to read Section 5 independently of the other sections.) In the Appendix, we prove Theorem 4.22, as well as other auxiliary results about uniform domains.

2. *Intermediate results*

2.1. *Notation and basic notions*

Throughout the paper, we will work on complete and proper metric measure spaces X equipped with some Radon measure μ. Consistently, Y refers to a closed subset of X which will be shown

to support Poincaré inequalities. In the Euclidean case $X = \mathbb{R}^n$, we will also denote such subsets by S, suggestively for 'sponge'.

Remark 2.1 (Types of constants). As a convention, we refer to certain constants as structural constants if they describe fixed parameters for standard hypotheses or conditions. These include the doubling constant $D \geqslant 1$, the constant $C \geqslant 1$ in the Poincaré inequality (as well as uniformity constants $A > 0$ that imply such inequalities), the choice of exponent $p \geqslant 1$, and the scale parameter $r_0 > 0$.

Moreover, conditions on a metric space X that depend on the scale parameter — that is, an upper (distance) bound between points on X — are referred to as local conditions. In particular, a locally D-doubling metric measure space refers to a (D, r_0)-doubling metric measure space for some $r_0 > 0$ and a local $(1,p)$-Poincaré inequality refers to a $(1,p)$-Poincaré inequality that is valid at scale r_0, for some $r_0 > 0$.

The same convention will apply to other conditions in the sequel. Note, in this convention, the scale r_0 is assumed to be uniform throughout the space. Our convention is therefore slightly different from others, such as in [6], where the scale can vary with the point.

Open balls in a metric space are denoted by $B = B(x,r)$, and their inflations by $CB = B(x, Cr)$, despite the ambiguity that balls may not be uniquely defined by their radii. If multiple metrics are used, we indicate the one used with a subscript, for example, $B_d(x,r)$ to mean the ball with respect to the metric d.

By a curve γ in a metric space X, we mean a Lipschitz map $\gamma\colon I \to X$, where $I \subset \mathbb{R}$ is a bounded closed interval. As a convention, we assume that all rectifiable curves are *parametrised by arc-length* unless otherwise specified, in which case it satisfies $\mathrm{Lip}\,(\gamma)(t) := \limsup_{s\to t} \frac{d(\gamma(t),\gamma(s))}{|s-t|} \leqslant 1$, $t \in I$.

A metric space X is called Λ-quasi-convex if for every $x, y \in X$ there exists a curve γ connecting x to y with $\mathrm{Len}(\gamma) \leqslant \Lambda d(x,y)$. Such a curve γ, when it exists, is called a Λ-quasi-geodesic. A space X is called Λ-quasi-convex at scale $r_0 > 0$, if the same holds for every $x, y \in X$ with $d(x,y) \leqslant r_0$.

Frequently, we restrict the metric and measure onto some subset $A \subset X$. On A the measure is denoted $\mu|_A$, and $d|_{A\times A}$, but we will often avoid this cumbersome notation. Also, metric balls in A are simply intersections $B_{d|_{A\times A}}(x,r) = B(x,r) \cap A$, and they are denoted occasionally by $B_A(x,r)$.

Related to Definition 1.1, a metric space X is said to be N-metric doubling, for some $N \in \mathbb{N}$, if for every ball $B(x,r)$ there exist $x_1, \ldots, x_m \in X$ for some $m \leqslant N$ such that

$$B(x,r) \subset \bigcup_{i=1}^{m} B(x_i, r/2).$$

Clearly, every metric space equipped with a D-doubling measure is D^4-metric doubling. Later we will specialize to doubling measures with certain quantitative growth, as below.

Definition 2.2. A proper metric measure space (X, d, μ) is said to be Ahlfors Q-regular with constant $C > 0$ if for all $0 < r < \mathrm{diam}(X)$ and any $x \in X$ we have

$$\frac{1}{C} r^Q \leqslant \mu(B(x,r)) \leqslant C r^Q.$$

The space is said to be Ahlfors Q-regular up to scale r_0 if the same holds for $r \in (0, r_0)$.

We define the centered Hardy-Littlewood maximal functions as

$$Mf(x) := \sup_{0<r} \fint_{B(x,r)} f \, d\mu. \tag{2.3}$$

$$M_s f(x) := \sup_{r\in(0,s)} \fint_{B(x,r)} f \, d\mu.$$

Here and in what follows, we will use a localized version of the Maximal Function Theorem, see [33, Theorem 2.19]. The proof below, given for completeness, is a slight modification of the classical argument.

LEMMA 2.4. *If $X = (X, d, \mu)$ is a D-doubling metric measure space at scale $8R$, then*

$$\mu(\{M_R f > \lambda\} \cap B(x,r)) \leqslant \frac{D^3 \|f|_{B(x,r+R)}\|_{L^1}}{\lambda}$$

for all $x \in X$, all $f \in L^1(X)$, and all $r, R, \lambda > 0$.

Proof. Put $E_\lambda := \{M_R f > \lambda\} \cap B(x,r)$. For each $y \in E_\lambda$, there exists $r_y \in (0,R)$ so that

$$\int_{B(y,r_y)} |f| \, d\mu > \lambda \mu(B(y,r_y)), \tag{2.5}$$

so $\{B(y,r_y)\}_{y\in E_\lambda}$ clearly covers E_λ. A standard 5-covering theorem [33, Theorem 2.1] (or alternatively [17, Theorems 2.8.4–2.8.6]) then asserts that there is a countable, pairwise-disjoint subcollection of balls $B_i := B(y_i, r_{y_i})$ for $i \in I$ with each $y_i \in E_\lambda$ and so that

$$\bigcup_{y\in E_\lambda} B(y,r_y) \subset \bigcup_{i\in I} B(y_i, 5r_{y_i}).$$

Using the fact that $\bigcup_{i\in I} B_i \subset B(x, r+R)$, we then obtain

$$\mu(E_\lambda) \leqslant \sum_i \mu(B(y_i, 5r_{y_i})) \leqslant D^3 \sum_i \mu(B_i) \leqslant \frac{D^3}{\lambda} \sum_i \int_{B_i} |f| \, d\mu \leqslant \frac{D^3}{\lambda} \int_{B(x,r+R)} |f| \, d\mu$$

as desired. □

2.2. *Poincaré inequalities via fillings*

In this subsection, we make precise the notion of filling and 'fillable set', the main tools in proving our results. One useful property of fillings Ω_r is that they satisfy a Poincaré inequality *a priori* only at scales comparable to r. For our applications, this property will be easy to check, in that the geometry of the filling at scale r will be kept simple.

DEFINITION 2.6. Let $\epsilon \in (0,1)$, $p \in [1,\infty)$, and $C, D \geqslant 1$. Given a closed subset Y of a complete space X, a closed subset $\Omega_r \subseteq X$ is called an ϵ-FILLING OF Y at scale $r > 0$ with constants (D, C, p) if the following conditions hold.

(1) $Y \subset \Omega_r$.

(2) For every $x \in Y$, the density condition $\frac{\mu(\Omega_r \cap B(x,r)\setminus Y)}{\mu(\Omega_r \cap B(x,r))} < \epsilon$ holds.

(3) The restricted space $(\Omega_r, d|_{\Omega_r\times\Omega_r}, \mu|_{\Omega_r})$ is D-doubling and satisfies a $(1,p)$-Poincaré inequality at scale $2r$,

$$\fint_B |f - f_B| d\mu \leqslant Cs \left(\fint_{CB} \text{Lip}\,(f)^p d\mu \right)^{1/p},$$

where $B = B_{\Omega_r}(x,s)$ is any ball in Ω_r with $s \leqslant 2r$.

Then, Y is called p-POINCARÉ ϵ-FILLABLE UP TO SCALE r_0, with constants (D, C) — or (ϵ, D, C, p)-PI FILLABLE UP TO SCALE r_0, for short — if there exists an ϵ-filling at scale r of Y with constants (D, C, p) and any $r \in (0, r_0)$.

We say that Y is ASYMPTOTICALLY p-POINCARÉ FILLABLE if for some fixed constants (D, C) and for any $\epsilon > 0$ there exists $r_0 > 0$ such that Y is (ϵ, D, C, p)-PI fillable up to scale r_0.

In terms of these sets, we can now give sufficient conditions for a subset to satisfy a Poincaré inequality.

THEOREM 2.7. *Fix structural constants (p, D, C, r_0) and let X be a D-doubling metric measure space. Then, for every $q > p$, there exist $\epsilon_q, C_q, C_r > 0$ with the following properties.*

(a) *If Y is a p-Poincaré, ϵ_q-fillable subset of X up to scale r_0 with constants (D, C), then it satisfies a $(1, q)$-Poincaré inequality with constant C_q at scale r_0/C_r.*

(b) *Further, if Y is an asymptotically p-Poincaré fillable subset of X, then it satisfies a local $(1, q)$-Poincaré inequality for every $q > p$.*

Here the constants ϵ_q and C_q, C' are independent of the original scale r_0, but depend on the other structural constants and on the exponent q.

REMARK 2.8. Note that X is not assumed, *a priori*, to support a Poincaré inequality; only the fillings Ω_r from Definition 2.6 do. In many cases, including our applications in Section 4, we will assume that X is a p-PI space, in which case good choices of Ω_r will inherit Poincaré inequalities from X.

Note that the local Poincaré inequality could be improved to a semi-local one [**6**] (that is, (1.3) holds at every scale, with constant depending on the scale and location only), if the space is proper and connected. In the case of bounded metric spaces, like non-self-similar Sierpiński carpets, this semi-local property further improves to the usual global type.

REMARK 2.9. It is crucial in Part (a) of the previous theorem that the density parameter ϵ_q be allowed to depend on the structural constants D, C, p.

Here we give some examples involving fillings of subsets and how the exponent of the Poincaré inequality can depend subtly on how the set is filled. In each case, we construct a filling with arbitrarily good Poincaré inequalities, namely local (1,1)-Poincaré inequalities. The subset, however, only inherits the Poincaré inequality if the density parameter is sufficiently small, relative to a controlled constant in the Poincaré inequality of the filling.

EXAMPLE 2.10. Let $X = [-1, 1]^2$, which is a (1,1)-PI-space, while the subset

$$Y = [-1, 0] \times [0, 1] \cup [0, 1] \times [-1, 0]$$

is a $(1, p)$-PI-space only for $p > 2$. However, if we 'thicken' Y at the origin, then the filling

$$\overline{Y}_r^h = Y \cup \overline{B(0, hr)}$$

satisfies a $(1, q)$-Poincaré inequality at scale r with constant C_q^h, where

$$C_q^h \approx_q \begin{cases} h^{\frac{q-2}{q}}, & \text{if } 1 \leqslant q < 2, \\ \log(1/h), & \text{if } q = 2. \end{cases}$$

and where C_q^h can be bounded independent of h for $q > 2$. Here, the ratio implied in $\approx_q$ depends on q, but not on h, and could be made explicit.

FIGURE 1. *An approximant of the space Y with the squares removed at the first three levels. The image is rotated by $90°$.*

For every $r > 0$, we can set $\Omega_r = \overline{Y}^h_r$, and see that Y is q-Poincaré h^2-fillable up to scale 1 with constants (D, C^h_q), for some uniform doubling constant D. By Theorem 2.7 then Y satisfies a $(1, q)$-Poincaré inequality for $q > 2$, as expected. However, for $q \in [1, 2]$, the Poincaré constant C^h_q blows up as $h \to 0$, so the subset Y need not, and does not, satisfy a $(1, q)$-Poincaré inequality for $q \in [1, 2]$.

The following example is closely related to the discussion of fat Sierpiński carpets and sponges in Section 4.1.

EXAMPLE 2.11. Let $X = [0, 1]^2$, and let $C_{1/3}$ be the usual 'middle thirds' Cantor set in [0,1] and denote by $\mathcal{I}_k$ the open removed intervals of length 3^{-k} in the construction of $C_{1/3}$. Now define the set of squares

$$\mathcal{R} = \left\{ I \times \left(\frac{1 - 3^{-k}}{2}, \frac{1 + 3^{-k}}{2} \right) \middle| I \in \mathcal{I}_k, k \in \mathbb{N} \right\}$$

and denote the complement of their union as

$$Y = [0, 1]^2 \setminus \bigcup_{R \in \mathcal{R}} R.$$

Unlike the standard 'middle-ninths' Sierpiński carpet, only the squares intersecting the line $y = \frac{1}{2}$ are removed. (See Figure 1.)

Putting $\alpha = \frac{\log(2)}{\log(3)}$ for the Hausdorff dimension of $C_{1/3}$, we now claim that Y with the restricted Lebesgue measure and Euclidean distance satisfies a $(1, p)$-Poincaré inequality if and only if $p > 2 - \alpha$. To see why, both of the sets

$$Y_+ = Y \cap [0, 1] \times \left[0, \frac{1}{2}\right] \text{ and } Y_- = Y \cap [0, 1] \times \left[\frac{1}{2}, 1\right]$$

are uniform domains (see Definition 4.12) and therefore satisfy (1,1)-Poincaré inequalities (see Theorem 4.14). Moreover, we have

$$Y = Y_+ \cup Y_- \text{ and } Y_+ \cap Y_- = C_{1/3} \times \{\frac{1}{2}\},$$

so Y arises from gluing $Y_\pm$ along a α-dimensional set and by [**23**, Theorem 6.15], it satisfies a $(1,p)$-Poincaré inequality for $p > 2-\alpha$. On the other hand, Y does not satisfy a Poincaré inequality for $p \in [1, 2-\alpha]$; indeed, consider the function

$$u(x,y) = \max\left\{\min\left\{\frac{1}{h}\left(y - \frac{1}{2}\right), 1\right\}, 0\right\}.$$

On $[0,1] \times (1/2, 1/2+h]$, we have Lip $(u) = \frac{1}{h}$, so if $q < 2-\alpha$, then for all $h < \frac{1}{3}$, we have

$$\fint_{[0,1]^2} \left|u - u_{[0,1]^2}\right| \, d\lambda \;\geqslant\; \frac{1}{6} \;\geqslant\; h^{\frac{2-\alpha-q}{q}} \approx_q \left(\fint_{[0,1]^2} \text{Lip }(u)^q \, d\lambda\right)^{1/q},$$

which contradicts the $(1,q)$-Poincaré inequality as $h \to 0$. The case $q = 2-\alpha$ is similar, but we consider the function

$$u(x,y) = \begin{cases} 1, & \text{if } y \leqslant \frac{1}{2}, \\ \min\left\{\max\left\{\log\left(\frac{h}{y-\frac{1}{2}}\right), 0\right\}, 1\right\}, & \text{if } y > \frac{1}{2}. \end{cases}$$

Again Y has certain good fillings that consist of

$$\Omega_r = [0,1]^2 \setminus \bigcup_{R \in \mathcal{R}, \operatorname{diam}(R) \geqslant r/9} R.$$

At scale r, only finitely many sets R with diameters larger than $r/9$ are near points in Ω_r. It follows that Ω_r satisfies (1,1)-Poincaré inequalities at scales comparable to r with constants (D, C) independent of r.

However, for balls centered on $y = 1/2$ the density of $\Omega_r \setminus Y$ is bounded from below, say by some constant $\delta > 0$. Thus, these are only $(\delta, D, C, 1)$-PI-fillable and not asymptotically 1-Poincaré fillable. This corresponds to the fact that we obtain only a $(1,p)$-Poincaré inequality for $p > 2-\lambda$, instead of for all $p > 1$.

2.3. *Poincaré inequalities via 'maximal' connectivity*

The proof of Theorem 2.7 is based on general techniques that reduce the Poincaré inequality to a certain connectivity property at all scales and with sets (or 'obstacles') of prescribed densities. These densities are in turn measured in terms of maximal functions.

The starting point is this very notion of connectivity: roughly speaking, 'if a set E has small measure density (in a scale invariant way), then there are curves of unit speed that spend only a short time within E'.

Definition 2.12. Let $\delta > 0$ and $C, p \geqslant 1$. We say that a pair of points $x, y \in X$ for a metric measure space (X, d, μ) is (C, δ, p)-max connected, if for every $\tau > 0$ with $r := d(x,y)$, and every Borel-measurable set E such that

$$M_{Cr}(1_E)(x) < \tau \text{ and } M_{Cr}(1_E)(y) < \tau, \tag{2.13}$$

there exists a 1-Lipschitz curve $\gamma\colon [0, L] \to X$, for some $L > 0$, such that:

(1) $\gamma(0) = x$;
(2) $\gamma(L) = y$;
(3) $\text{Len}(\gamma) \leqslant Cr$;

(4) the following integral inequality holds:

$$\int_\gamma 1_E \, ds \leqslant \delta \tau^{\frac{1}{p}} r. \tag{2.14}$$

We say that a space (X, d, μ) is p-MAXIMALLY CONNECTED AT SCALE r_0 with constants (C, δ) — or (C, δ, p)-MAX CONNECTED AT SCALE r_0, for short — if every pair $x, y \in X$ with $d(x, y) < r_0$ is (C, δ, p)-max connected.

REMARK 2.15. Since the measure is assumed to be Borel regular, it is enough to verify Definition 2.12 for all open (or all closed) 'obstacles' E. Indeed, if $\epsilon > 0$ and $E \subset X$ is any Borel set, we can find using Borel regularity an open set E' so that $E \subset E'$ and $M_{Cr}(1_{E' \setminus E})(x), M_{Cr}(1_{E' \setminus E})(y) < \epsilon$. The case of closed sets is only slightly harder, and, as we do not use it anywhere, we only sketch the details. One can for each open set E exhaust it with closed sets $E_k = \{x : d(x, X \setminus E) \geqslant \frac{1}{k}\}$. One then finds a sequence of curves γ_k for each E_k, and since $E_{k-1} \subset \mathrm{int}(E_k)$, then after passing to a subsequence and using monotone convergence, we can find a curve γ which satisfies (1)–(4) for E.

A technical issue with checking for maximal connectivity is that the desired maximal function estimates for X are not directly related to those for the filling Ω_r. Furthermore, it can be challenging to prove the property for all density 'levels' $\tau > 0$. This is dealt with the following variants of this connectivity.

DEFINITION 2.16. We say that a metric measure space (X, d, μ) is p-MAXIMALLY CONNECTED AT LEVEL τ_0 AND SCALE r_0 (with constants (C, δ)) — or (C, δ, τ_0, p)-MAX CONNECTED AT SCALE r_0, for short — if the p-maximal connectivity conditions of Definition 2.12 hold for only $\tau = \tau_0$, instead of for all τ.

This condition may seem technical at first. The core point, however, is that it allows for characterizing Poincaré inequalities in terms of sufficiently good avoidance of obstacles of a fixed level, so one need not consider obstacles of every level. Further, this 'fixed-level' property is inherited by sufficiently dense subsets.

LEMMA 2.17. *Suppose X is D-doubling and (C, δ, τ_0, p)-max connected at scale r_0 and that Y is a closed, Λ-quasi-convex subset of X. If $x, y \in Y$ satisfy $d(x, y) < r_0$, as well as*

$$M_{Cr} 1_{X \setminus Y}(x) < \frac{\tau_0}{2} \text{ and } M_{Cr} 1_{X \setminus Y}(y) < \frac{\tau_0}{2},$$

then the pair (x, y) is $(\Lambda C, \Lambda\delta, \frac{\tau_0}{2}, p)$-max connected relative to Y with its restricted measure and distance.

We will only sketch the main form of the argument, since the lemma will not be used directly and a variant appears later. The main idea, however, is replacing bad portions of an initial curve with better ones, as depicted in Figure 2.

Proof. By Remark 2.15, it suffices to consider open sets. Let $E \subset Y$ be a relatively open arbitrary open set with $M^Y_{Cr} 1_E(z) < \tau_0/2$ for $z = x, y$ but where the maximal function is computed relative to Y; for $F = E \cup (X \setminus Y)$, it then follows that $M_{Cr} 1_F(z) < \tau_0$, where the maximal function is once again relative to X.

Thus the definition of max-connectivity gives a curve γ that spends at most $\delta \tau_0^{1/p} r$ in the complement of Y and the set E. The set $\gamma^{-1}(X \setminus Y)$ consists of countably many disjoint maximal open intervals (a_i, b_i), so we can replace each $\gamma|_{(a_i, b_i)}$ by a Λ-quasi-geodesic in Y

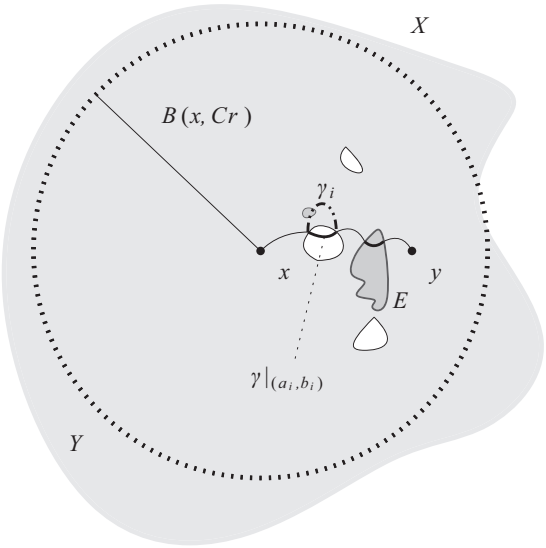

FIGURE 2. *Proof of Lemma 2.17. Connectivity involves constructing a curve in the gray subset Y between a pair of points x, y while avoiding the dark gray subset E as well as possible. The connectivity of X is used to give a 'proto-curve', whose portions $\gamma|_{(a_i,b_i)}$ in the complement $X \setminus Y$ are replaced by detours γ_i constructed using quasi-convexity (the dash-dotted line segment).*

that joins $\gamma(a_i)$ and $\gamma(b_i)$. This produces a new curve γ' which lies entirely in Y, is at most $\Lambda C d(x, y)$ long, and spends at most $\Lambda\delta\tau_0^{1/p} r$ time in E, as desired. □

Our connectivity property is related to the $(1, p)$-Poincaré inequality via the following two theorems. We discuss their applications first in the next section, and their proofs will appear later in Section 5.

THEOREM 2.18. *Fix structural constants (p, D, C, r_0). If X is D-doubling at scale r_0 and satisfies a $(1, p)$-Poincaré inequality at scale r_0 with constant C, then X is (C_0, Δ, p)-max connected at scale $r_0/2$, where C_0 and Δ depend solely on the structural constants.*

The converse also holds true, but requires a *sufficiently small* value for δ.

THEOREM 2.19. *Fix structural constants (p, D, C, r_0). There exists $\delta_{p,D} > 0$ such that if X is D-doubling at scale r_0 and (C, δ, τ_0, p)-max connected at scale r_0 for some $\tau_0 \in (0, 1)$ and some $\delta \in (0, \delta_{p,D})$, then it also satisfies a $(1, p)$-Poincaré inequality at scale r_0/C_r with constant C_p, where C_p, C_r are independent of scale r_0 but depends quantitatively on all the other structural constants, as well as δ and τ_0.*

As emphasized in the notation, the above constant $\delta_{p,D}$ depends only on p and D and no other structural constants.

However, a small parameter value for δ is not serious; the next result assures that such values for δ always occur at some density level τ but for slightly larger exponents than p.

LEMMA 2.20. *With the same constants as in Theorem 2.18, let X be a D-doubling metric measure space that is (C, Δ, p)-max connected at scale r, and let $q > p$. For each $\delta \in (0, 1)$, there exists $\tau_0 = \tau_0(q, \delta) \in (0, 1)$ so that X is (C, δ, τ', q)-max connected at scale r for any $\tau' \in (0, \tau_0)$.*

Proof. Choose $\tau_0(q,\delta) = \min\{1, (\frac{\delta}{\Delta})^{\frac{pq}{q-p}}\}$ and $\tau' \in (0, \tau_0(q,\delta))$. We will show (C,δ,τ',q)-max connectivity. Let x, y, E be as in the Definitions 2.12 and 2.16 at scale r, that is, $d(x,y) < r$ and

$$M_{Cd(x,y)}(1_E)(x) < \tau'$$
$$M_{Cd(x,y)}(1_E)(y) < \tau'.$$

By (C,Δ,p)-max connectivity, there is a curve γ connecting x to y with length at most $Cd(x,y)$ and with

$$\int_\gamma 1_E \, ds \leqslant \Delta \tau'^{\frac{1}{p}} d(x,y).$$

By our choice of τ_0, we have $\Delta\tau'^{1/p} = (\Delta\tau'^{1/p-1/q})\tau'^{1/q} \leqslant \delta\tau'^{1/q}$, and thus we also have

$$\int_\gamma 1_E \, ds \leqslant \delta \tau'^{\frac{1}{q}} d(x,y),$$

and in particular, γ already verifies the (C,δ,τ',q)-max connectivity condition. □

To reiterate, to prove that a p-fillable subset Y satisfies a $(1,p)$-Poincaré inequality, by Theorem 2.19 it is sufficient to prove the maximal connectivity property for Y at a certain level and for fixed choices C and $\delta < \delta_{p,D}$. Similarly as in Lemma 2.17, this property will be 'inherited' from a filling Ω_r at a comparable scale.

With these general statements at hand, we will employ the following strategy for the proof of Theorem 2.7.

(1) Theorem 2.18 guarantees that any filling Ω_r of Y will satisfy maximal connectivity properties with exponent p and *some* initial parameter Δ.

(2) From Lemma 2.20, we obtain (C,δ,τ_0,q)-maximal connectivity for Ω_r at scale r for *arbitrarily small* parameters δ, but at the expense of a slightly larger exponent q.

(3) Similarly to Lemma 2.17, due to quasi-convexity (see Lemma 3.2) Y inherits the maximal connectivity property from its filling Ω_r, but with δ' slightly larger than δ. This parameter δ' can be ensured to be less than the given threshold $\delta_{p,D}$, however, by an initially small choice of δ in the previous step.

(4) Using maximal connectivity and quasi-convexity (again), we show Y satisfies a $(1,q)$-Poincaré inequality via Theorem 2.19.

Here $q > p$ is needed to apply the argument from Lemma 2.20. If $p > 1$, this could be avoided via Keith–Zhong [**28**], since we could first improve the Poincaré inequality for each Ω_r to an exponent $p' < p$.

3. *Proof that 'fillable' sets satisfy Poincaré inequalities*

3.1. *Initial geometric considerations*

Now, we show that the underlying (restricted) measure of a fillable subset is well behaved. More precisely, we show that a fillable subset Y inherits the doubling property from its fillings Ω_r. Recall that throughout this paper, $Y \subset \Omega_r \subset X$, where Ω_r will be the relevant fillings.

LEMMA 3.1. *Fix structural constants (p, D, C, r_0). If Y is (ϵ, D, C, p)-PI fillable up to scale r_0 for some $\epsilon \in (0,1)$, then Y is $(\frac{D}{1-\epsilon}, r_0)$-doubling.*

Proof. Let $r \in (0, r_0)$ and $x \in Y$. From item (2) of Definition 2.6, we have

$$\begin{aligned}\mu(\Omega_r \cap B(x,r)) &= \mu(\Omega_r \cap B(x,r) \cap Y) + \mu(\Omega_r \cap B(x,r) \setminus Y)\\ &< \mu(Y \cap B(x,r)) + \epsilon\, \mu(\Omega_r \cap B(x,r))\\ \therefore (1-\epsilon)\, \mu(\Omega_r \cap B(x,r)) &< \mu(Y \cap B(x,r)).\end{aligned}$$

Since Ω_r is assumed D-doubling with respect to the restricted measure $\mu|_{\Omega_r}$ and since Y is a subset of Ω_r, it follows that

$$\mu(Y \cap B(x,2r)) \leqslant \mu(\Omega_r \cap B(x,2r)) \leqslant D\, \mu(\Omega_r \cap B(x,r)) \leqslant \frac{D}{1-\epsilon}\mu(Y \cap B(x,r)).$$

So the claim follows with doubling constant $\frac{D}{1-\epsilon}$. □

We next show that PI-fillable subsets Y are quasi-convex. This connectivity property is derived from stronger ones, that is, the Poincaré inequalities of the fillings Ω_r. For clarity later, given $f \in L^1(X)$ and $R > 0$ we specify the choice of metric space for maximal functions by using the shorthand

$$M_R^r f(x) := \sup_{\rho \in (0,R)} \fint_{B(x,\rho)\cap\Omega_r} |f|\, d\mu$$

$$M_R^0 f(x) := \sup_{\rho \in (0,R)} \fint_{B(x,\rho)\cap Y} |f|\, d\mu,$$

where Ω_r is as in Definition 2.12.

LEMMA 3.2. *Fix structural constants (p, D, C, r_0). There exist $\epsilon_0, \Lambda, r_1 > 0$, depending solely on the structural constants, so that if Y is a (ϵ, D, C, p)-PI fillable subset of a metric space X at scale r_0, for some $\epsilon \in (0, \epsilon_0)$, then it is Λ-quasi-convex at scale r_1.*

Proof. By hypothesis, Y is (ϵ, D, C, p)-fillable up to scale r_0, for some $\epsilon \in (0, \epsilon_0)$, so there exist fillings Ω_r for every $r \in (0, r_0)$ with $Y \subset \Omega_r \subset X$ that are D-doubling at scale $2r$, that support a $(1, p)$-Poincaré inequality at scale r with constant C, and so that

$$\frac{\mu(\Omega_r \cap B(z,r) \setminus Y)}{\mu(\Omega_r \cap B(z,r))} < \epsilon < \epsilon_0$$

holds for all $z \in Y$. From Theorem 2.18, we conclude that the fillings Ω_r are (C_0, Δ, p)-max connected for some C_0 and Δ at scale $r/2$. Choose $\tau_0 = \frac{1}{\Delta^p 4^p}$ so that $\Delta \tau_0^{1/p} r \leqslant r/4$ and fix

$$\epsilon_0 \;=\; D^{-(10+\lceil \log_2(C_0))\rceil)}\tau_0 \text{ and } \Lambda = 2C_0 \text{ and } r_1 = \frac{r_0}{2^5 C_0}.$$

Since C_0 and Δ depend only on the structural constants, by Theorem 2.18, the same is true of ϵ_0, Λ, and r_1.

We now show that Y is Λ-quasi-convex at scale r_1. For every $x, y \in Y$ with $r = d(x,y) < r_1$, we will construct a Λ-quasi-geodesic joining x and y, using a recursive argument.

Base case(s). Fix $R = 2^5 C_0 r$. The initial curve will be constructed in Ω_R and will lie almost entirely in Y. To begin, define an obstacle

$$E \;:=\; X \setminus (Y \cup \overline{B(x, r/16)} \cup \overline{B(y, r/16)}).$$

In particular, this implies for $\rho \in (\frac{1}{16}r, R)$ that

$$\frac{\mu(\Omega_R \cap B(x,\rho) \cap E)}{\mu(\Omega_R \cap B(x,\rho))} \;\leqslant\; \frac{\mu(\Omega_R \cap B(x,R) \setminus Y)}{\mu\big(\Omega_R \cap B(x, \frac{1}{16}r)\big)} \;\leqslant\; D^{10+\lceil \log_2(C_0)\rceil} \frac{\mu(\Omega_R \cap B(x,R) \setminus Y)}{\mu(\Omega_R \cap B(x,R))}$$

and since $\mu(\Omega_R \cap B(x,\rho) \cap E) = 0$ holds whenever $\rho \in (0, \frac{1}{16}r)$, it follows that

$$M^R_{C_0 r} 1_E(z) < D^{10+\lceil \log_2(C_0)\rceil}\epsilon < \tau_0 \text{ for } z = x, y. \tag{3.3}$$

For future consistency of notation, put $x_{1,1} := x$ and $y_{1,1} := y$ and $x_{i,1} := y_{i,1} := y$ for $i \geqslant 2$. Also define $r_{i,1} = d(x_{i,1}, y_{i,1})$, in which case

$$\sum_i r_{i,1} = r_{1,1} \ \leqslant \ r.$$

Recall that Ω_R is (C_0, Δ, p)-max connected at scale $R/2 > r$. By applying Definition 2.12 to E, equation (3.3) guarantees the existence of a C_0-quasi-geodesic $\gamma_1 : [0, L_1] \to \Omega_R$, for some length[†] $L_1 \in (0, C_0 r)$, joining x and y in $\Omega_R \subset X$, and so that

$$\int_{\gamma_1} 1_E \, ds \leqslant \Delta \tau_0^{1/p} r \leqslant r/4. \tag{3.4}$$

Consider the exit times

$$t_{1,1} := \sup\left\{t \in [0, L_1] \mid d(\gamma_1(t), x) \leqslant r/8\right\}$$

$$T_{1,1} := \inf\left\{t \in [0, L_1] \mid d(\gamma_1(t), y) \leqslant r/8\right\}.$$

Since Y is closed, the set E is open and it follows that $\gamma_1^{-1}(E) \cup (0, t_{1,1}) \cup (T_{1,1}, L_1)$ is open in $\mathbb{R}$, so it is a countable union of open intervals

$$(0, t_{1,1}) \cup (T_{1,1}, L_1) \cup \gamma_1^{-1}(E) \ = \ \bigcup_{i=1}^{\infty} (a^i, b^i)$$

with $a^i \leqslant b^i$ and where each pair $x_{i,2} := \gamma_1(a^i)$ and $y_{i,2} := \gamma_1(b^i)$, of distance $r_{i,2} := d(x_{i,2}, y_{i,2})$ apart, also lie in Y. (If the union is finite, then there exists $N \in \mathbb{N}$ so that $a^n = b^n$ for $n \geqslant N$.) Also,

$$\gamma_1^{-1}(X \setminus Y) \subset \bigcup_{i=1}^{\infty} (a^i, b^i). \tag{3.5}$$

Equation (3.4) thus implies that

$$\sum_i r_{i,2} \leqslant \operatorname{len}(\gamma_1 \cap \Omega_r \setminus Y) + \frac{r}{4} \ \leqslant \ \left(\Delta \tau_0^{1/p} + \frac{1}{4}\right) r \ \leqslant \ \frac{r}{2}.$$

Since γ_1 is parametrized by length, and $\operatorname{Len}(\gamma_1) \leqslant \Lambda r = \Lambda r_{1,1}$, it trivially holds that

$$\gamma_1 \setminus Y \subset \bigcup_{i=1}^{\infty} B(x_{i,1}, \Lambda r_{i,1}). \tag{3.6}$$

Recursive step. Let $k \in \mathbb{N}$ be given, with $k \geqslant 2$, and suppose the sequence $((x_{j,k}, y_{j,k}))_{j=1}^{\infty}$ in $Y \times Y$ has already been defined, with $r_{j,k} := d(x_{j,k}, y_{j,k}) < r_0$ and with the property that

$$\sum_j r_{j,k} \ \leqslant \ 2^{1-k} r. \tag{3.7}$$

Assume further that a C_k-quasi-geodesic $\gamma_{k-1} : [0, L_{k-1}] \to X$ joining x and y has already been defined for some $L_{k-1} \in (0, C_{k-1} r)$, where

$$C_{k-1} \ := \ 2(1 - 2^{-(k-1)}) C_0 \tag{3.8}$$

[†]Recall the convention that all rectifiable curves are assumed to be parametrized with respect to arc-length, unless otherwise specified. The only time below that we will need this we will indicate such curves by an asterisk.

and with the property that there exist $\{a^j_{k-1}, b^j_{k-1}\}_{j=1}^{\infty} \subset [0, L_{k-1}]$ with

$$x_{j,k} \;=\; \gamma_{k-1}(a^j_{k-1}) \text{ and } y_{j,k} \;=\; \gamma_{k-1}(b^j_{k-1}) \text{ and } r_{j,k} = d(x_{j,k}, y_{j,k})$$

and which satisfies the avoidance properties

$$\gamma^{-1}_{k-1}(X \setminus Y) \;\subset\; \bigcup_{j=1}^{\infty} (a^j_{k-1}, b^j_{k-1}), \tag{3.9}$$

$$\gamma_{k-1} \setminus Y \;\subset\; \bigcup_{j=1}^{\infty} B(x_{j,k-1}, \Lambda r_{j,k-1}). \tag{3.10}$$

By applying the same argument as in the base case, with $x_{j,k}$ and $y_{j,k}$ and $r_{j,k}$ in place of x and y and r, take fillings $\Omega_{j,k} := \Omega_{2^5 C_0 r_{j,k}}$ of Y that are (C_0, Δ, p)-max connected at scales $2^4 C_0 r_{j,k}$. Using obstacles

$$E_{j,k} \;:=\; X \setminus (Y \cup \overline{B(x_{j,k}, r_{j,k}/16)} \cup \overline{B(y_{j,k}, r_{j,k}/16)}),$$

and estimating similarly as (3.3), there exist C_0-quasi-geodesics $\gamma_{j,k} : [0, L_{j,k}] \to \Omega_{j,k} \subset X$ joining $x_{j,k}$ to $y_{j,k}$ in $\Omega_{j,k}$, so that

$$\sum_{j=1}^{\infty} \int_{\gamma_{j,k}} 1_{E_{j,k}}\, ds \;\leqslant\; \sum_{j=1}^{\infty} \Delta \tau_0^{1/p} r_{j,k} \;\overset{(3.7)}{\leqslant}\; 2^{-1-k} r \tag{3.11}$$

and whose lengths $L_{j,k} \leqslant C_0 r_{j,k}$ satisfy

$$\mathcal{H}^1\left(\bigcup_j \gamma_{j,k}\right) \;\leqslant\; \sum_j L_{j,k} \;\leqslant\; \sum_j C_0 r_{j,k} \;\overset{(3.7)}{\leqslant}\; 2^{1-k} C_0 r. \tag{3.12}$$

As before, for each $j \in \mathbb{N}$, set exit times

$$t_{j,k} := \sup\left\{t \in [0, L_{j,k}] \mid d(\gamma_{j,k}(t), x_{j,k}) \leqslant r_{j,k}/8\right\}$$

$$T_{j,k} := \inf\left\{t \in [0, L_{j,k}] \mid d(\gamma_{j,k}(t), y_{j,k}) \leqslant r_{j,k}/8\right\}.$$

The preimage $\gamma^{-1}_{j,k}(X \setminus Y)$ is open in $\mathbb{R}$ and satisfies

$$(0, t_{j,k}) \cup (T_{j,k}, L_{j,k}) \cup \gamma^{-1}_{j,k}(X \setminus Y) \;=\; \bigcup_{l=1}^{\infty} (a^{l*}_{j,k}, b^{l*}_{j,k})$$

for sequences of pairs $a^{l*}_{j,k} \leqslant b^{l*}_{j,k}$. Reindexing $i = i(j,l)$ as needed, put

$$x_{i,k+1} := \gamma_{j,k}(a^{i*}_k), \text{ where } a^{i*}_k \;:=\; a^{l*}_{j,k}$$

$$y_{i,k+1} := \gamma_{j,k}(b^{i*}_k), \text{ where } b^{i*}_k \;:=\; b^{l*}_{j,k}$$

$$r_{i,k+1} := d(x_{i,k+1}, y_{i,k+1}).$$

Based on (3.7) and (3.11) and our choice of $t_{j,k}$ and $T_{j,k}$, it holds that

$$\begin{aligned}\sum_{i=1}^{\infty} r_{i,k+1} &= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} d(\gamma_{j,k}(a^{l*}_{j,k}), \gamma_{j,k}(b^{l*}_{j,k})) \\ &\leqslant \sum_{j=1}^{\infty}\left(\frac{r_{j,k}}{4} + \int_{\gamma_{j,k}} 1_{E_{j,k}}\, ds\right) \;\leqslant\; \frac{1}{4}\cdot 2^{-(k-1)} r + 2^{-1-k} r \;\leqslant\; 2^{-k} r.\end{aligned}$$

Toward a new curve, consider sub-curve lengths

$$L^j_{k-1} := \mathrm{len}(\gamma_{k-1}\Big([a^j_{k-1}, b^j_{k-1}]\Big))$$

$$L^*_k := \mathrm{len}(\gamma_{k-1}) + \sum_{j=1}^{\infty}(L_{j,k} - L^j_{k-1}).$$

for all j and k. We further define a parametrization for a curve of length L^*_k, and where each $\gamma_{j,k}$ replaces $\gamma_{k-1}|_{[a^j_{k-1}, b^j_{k-1}]}$, as follows:

$$\gamma^*_k(t) := \begin{cases} \gamma_{j,k}\left(\frac{L_{j,k}}{b^j_{k-1}-a^j_{k-1}}\Big(t - a^j_{k-1}\Big)\right), & \text{if } t \in [a^j_{k-1}, b^j_{k-1}] \text{ for some } j \in \mathbb{N}, \\ \gamma_{k-1}(t), & \text{otherwise.} \end{cases}$$

Let γ_k be the arclength parametrisation of γ^*_k. Let a^j_k, b^j_k correspond to a^{j*}_k, b^{j*}_k under this reparametrization. By equation (3.9), $\gamma_{k-1}(t)$ can only lie in $X \setminus Y$ whenever $t \in [a^j_{k-1}, b^j_{k-1}]$ for some $j \in \mathbb{N}$, that is, where the images of $\gamma_{j,k}$ and γ_k agree. With the same reindexing $i = i(j,l)$, this gives the avoidance property

$$\gamma_k^{-1}(X \setminus Y) \subset \bigcup_{i=1}^{\infty}(a^i_k, b^i_k) \tag{3.13}$$

and since the $\gamma_{j,k}$ have length at most $\Lambda r_{j,k}$, the other avoidance property follows:

$$\gamma_k \setminus Y \subset \bigcup_{i=1}^{\infty} B(x_{i,k}, \Lambda r_{i,k}). \tag{3.14}$$

From (3.7) and (3.8), it follows that

$$\begin{aligned} \mathrm{len}(\gamma_k) \leqslant L^*_k \ &\leqslant\ \mathrm{len}(\gamma_{k-1}) + \sum_{j=1}^{\infty} L_{j,k} \\ &\leqslant 2(1-2^{-(k-1)})C_0 r + C_0 \sum_{j=1}^{\infty} r_{j,k} \\ &\leqslant \frac{1-2^{-(k-1)}}{1-\frac{1}{2}} C_0 r + 2^{-(k-1)} C_0 r \ = \ \frac{1-2^{-k}}{1-\frac{1}{2}} C_0 r \ = \ 2(1-2^{-k})C_0 r. \end{aligned} \tag{3.15}$$

By construction, for each $k \in \mathbb{N}$ there exists $j_1, j_2 \in \mathbb{N}$ so that $x = a^{j_1}_{k-1}$ and $y = b^{j_2}_{k-1}$, so γ_k therefore joins x and y. By the previous estimate, it is therefore a C_k-quasi-geodesic with

$$C_k \ := \ 2(1-2^{-k})C_0,$$

which completes the induction step.

A limiting curve. Putting $\gamma'_k(t) := \gamma_k(\frac{\Lambda r}{L_k} t)$, it follows that $\{\gamma'_k\}_{k=1}^{\infty}$ is a family of 1-Lipschitz functions on $[0, \Lambda r]$, each joining x to y. By the Arzelá–Ascoli theorem, there therefore exists a sublimit function $\gamma : [0, \Lambda r] \to X$ that is 1-Lipschitz and joins x and y. Since γ is 1-Lipschitz, we obtain

$$\mathrm{Len}(\gamma) \leqslant \Lambda r \leqslant \Lambda d(x,y),$$

and γ is the desired Λ-quasi-geodesic connecting x to y.

We lastly claim that $\gamma([0,L]) \subset Y$. From the inclusion (3.10) and the estimate (3.7), the Hausdorff 1-content of $\gamma_k \setminus Y$ satisfies

$$\mathcal{H}^1_\infty(\gamma_k \setminus Y) \leqslant \mathcal{H}^1_\infty(\gamma_k \cap \bigcup_{j=1}^{\infty} B(x_{j,k}, \Lambda r_{j,k})) \leqslant 2^{1-k}\Lambda r$$

and therefore vanishes, as $k \to \infty$; we therefore conclude $\mathcal{H}^1(\gamma \setminus Y) = 0$ since γ is continuous and Y is closed. Indeed, if γ spent any time in the complement of Y, then by continuity, the Hausdorff content of $\gamma_k \setminus Y$ would have a definite lower bound for large k, contradicting the previous limit calculation. □

3.2. *Proof of Theorem 2.7, Part (a)*

In light of Theorem 2.19, it suffices to prove the following statement instead of the original statement of Theorem 2.7:

THEOREM 3.16. *Let X be a metric measure space, fix structural constants (p, D, C, r_0), and let $\delta > 0$ be arbitrary. For every $q > p$, there exist $\epsilon_q, \tau \in (0,1)$, $C' \geqslant 1$ and $r_1 \in (0, r_0)$, such that if $\epsilon \in (0, \epsilon_q)$, then every (ϵ, D, C, p)-fillable subset Y of X up to scale r_0 is:*

(1) *$2D$-doubling at scale r_1; and*
(2) *(C', δ, τ, q)-max connected at scale r_1.*

REMARK 3.17 (Dependence on parameters). Here r_1 is the only constant that depends on the original scale r_0. In fact, it suffices that $r_1 = r_0/(20C')$; see the end of Step 1 of the proof. As for ϵ_q, τ, and C', they all depend on the remaining structural constants but ϵ_q and τ depend additionally on δ and q.

Proof. We proceed in three steps: (1) fixing parameters for definiteness, (2) passing the density conditions (2.13) from points in Y to points in the fillings Ω_r, and then (3) constructing the quasi-geodesics explicitly.

Step 1: Fixing parameters and their dependencies. Let $\delta \in (0,1)$ and $q > p$ be given. Let $\Lambda = \Lambda(D, C, p)$ be the constant from Lemma 3.2, and let ϵ_0 be the filling threshold for Λ-quasi-convexity to be guaranteed for Y. Each filling Ω_r satisfies $(1,p)$-Poincaré inequalities at scale r, so by Theorem 2.18 and Lemma 2.20 there exists a constant $C_0 \geqslant 1$ such that for any $\delta' > 0$ there is some $\tau_0 \in (0,1)$ such that Ω_r is (C_0, δ', τ', q)-max connected at scale $r/2$ — that is, it is q-maximally connected at scale $r/2$ and level τ' with constants (C_0, δ') for each $\tau' \in (0, \tau_0)$.

Now choose $\delta' \in (0,1)$ sufficiently small so that both conditions below hold:

$$\Lambda(6(2D)^{\frac{4}{q}} + 2)\delta' < \delta, \tag{3.18}$$

$$(2\Lambda + 2C_0 + 4\Lambda(2D)^{\frac{4}{q}})\delta' \leqslant C_0. \tag{3.19}$$

In particular, (3.18) implies $\delta' \leqslant \frac{1}{4}$. This fixes $\tau_0 \in (0,1)$ with dependence on data $\tau_0 = \tau_0(C, D, \delta', q)$ as from Theorem 2.18 and Lemma 2.20, in which case the fillings Ω_r are $(C_0, \delta', \tau_0, q)$-max connected at scale $r/2$. In particular, we may assume $\tau_0 < C_0^{-q}$.

Next, choose $\tau \in (0,1)$ with analogous dependence $\tau = \tau(C, D, \delta, q)$ so that

$$(2D)^4 \tau \leqslant \frac{\tau_0}{2},$$

and let $m = m(C_0) \in \mathbb{N}$ and $n = n(\tau, \delta, p) \in \mathbb{N}$ satisfy

$$2^{m-1} < C_0 + 1 < 2^m \text{ and } \frac{1}{2}\delta'\tau^{1/p} \leqslant 2^{-n} < \delta'\tau^{1/p} < \delta'\tau^{1/q}. \tag{3.20}$$

Letting $\epsilon_q := \min\{\frac{1}{4}D^{-(5+n+m)}\tau, \epsilon_0\}$, it follows that $\epsilon_q < \frac{1}{2}$ and each $\epsilon \in (0, \epsilon_q)$ satisfies

$$((2D)^4\tau + 4D^{5+n+m}\epsilon)^{1/q} < 2(2D)^{4/q}\tau^{1/q} \tag{3.21}$$

and in particular, that

$$(2D)^4\tau + 4D^{5+n+m}\epsilon < \tau_0. \tag{3.22}$$

Now let $\epsilon \in (0, \epsilon_q)$ and $r_0 > 0$ be given, and assume that Y is a (ϵ, D, C, p)-PI fillable subset of X, up to scale r_0. Since $\epsilon_q \leqslant \frac{1}{2}$, it follows from Lemma 3.1 that Y is $(2D, r_0)$-doubling.

Fix $C' = 2C_0$. We now show Y is (C', δ, τ, q)-max connected at scale $r_1 = r_0/(20C')$.

Step 2: Finding nearby dense points. To verify (C', δ, τ, q)-max connectivity at scale $\frac{1}{20C'}r_0$, take an arbitrary pair $x, y \in Y$ satisfying $r := d(x, y) \in (0, \frac{1}{20C'}r_0)$ and an arbitrary Borel set E such that

$$M^0_{C'r}1_E(x) < \tau \text{ and } M^0_{C'r}1_E(y) < \tau. \tag{3.23}$$

Our goal is to construct a curve γ in Y with length at most $C'r$ which connects x and y with

$$\int_\gamma 1_E \, ds \leqslant \delta\tau^{1/q}r.$$

Let $\Omega_{2C'r}$ be a filling of Y from Definition 2.6, so

$$\frac{\mu(\Omega_{2C'r} \cap B(x, 2C'r) \setminus Y)}{\mu(\Omega_{2C'r} \cap B(x, 2C'r))} < \epsilon \tag{3.24}$$

and as a shorthand, for $\rho > 0$ put

$$B_{2C'r}(x, \rho) = B(x, \rho) \cap \Omega_{2C'r}.$$

Computing first with (3.24) and the D-doubling property of $\Omega_{2C'r}$ yields

$$\begin{aligned}\mu(B_{2C'r}(x, 2C'r) \setminus Y) &\overset{(3.24)}{\leqslant} \epsilon\mu(B_{2C'r}(x, 2C'r)) \\ &\overset{(3.20)}{\leqslant} D^{m+1}\epsilon\mu(B_{2C'r}(x, r)) \overset{(3.20)}{\leqslant} D^{m+n+1}\epsilon\mu(B_{2C'r}(x, \delta'\tau^{1/q}r))\end{aligned}$$

as well as the estimate below, where B_Y is the ball in Y:

$$\begin{aligned}\mu(B_Y(x, \delta'\tau^{1/q}r)) &\geqslant \mu(B_{2C'r}(x, \delta'\tau^{1/q}r)) - \mu(B_{2C'r}(x, 2C'r) \setminus Y) \\ &\geqslant (1 - D^{m+n+1}\epsilon)\mu(B_{2C'r}(x, \delta'\tau^{1/q}r)) \overset{(3.22)}{\geqslant} \frac{3}{4}\mu(B_{2C'r}(x, \delta'\tau^{1/q}r)).\end{aligned} \tag{3.25}$$

Putting $R := (1 + 2\delta'\tau^{1/q})r$, for $l = 4D^{n+m+5}\epsilon$ consider the set

$$\mathcal{D} := \left\{x' \in B_{2C'r}(x, \delta'\tau^{1/q}r) : \; M^{2C'r}_{C_0R}1_{\Omega_{2C'r}\setminus Y}(x') > l\right\},$$

and note that $C_0(1 + 3\delta'\tau^{1/q})r \leqslant C'r$, so Lemma 2.4 implies

$$\begin{aligned}\mu(\mathcal{D}) &\leqslant \frac{D^3}{l}\mu(B_{2C'r}(x, C_0(1 + 3\delta'\tau^{1/q})r) \setminus Y) \\ &\leqslant \frac{D^3}{l}\mu(B_{2C'r}(x, 2C'r) \setminus Y) \\ &\overset{(3.24)}{\leqslant} \frac{D^3\epsilon}{l}\mu(B_{2C'r}(x, 2C'r)) \\ &\overset{(3.20)}{\leqslant} \frac{D^{n+m+5}\epsilon}{l}\mu(B_{2C'r}(x, \delta'\tau^{1/q}r)) = \frac{\mu(B_{2C'r}(x, \delta'\tau^{1/q}r))}{4} \overset{(3.25)}{\leqslant} \frac{\mu(B_Y(x, \delta'\tau^{\frac{1}{q}}r))}{3}.\end{aligned}$$

A similar argument with $l = (2D)^4\tau$ yields

$$\begin{aligned}\mu(\{x' \in B_Y(x, \delta'\tau^{1/q}r); M^0_{\delta'\tau^{1/q}r}1_E(x') > l\}) &\leqslant \frac{(2D)^3\mu(E \cap B(x, 2\delta'\tau^{1/q}r))}{l} \\ &\overset{(3.20)}{\leqslant} \frac{(4D)^4\mu(B_Y(x, \delta'\tau^{1/q}r))}{l} M^0_{C'r}1_E(x) \\ &\leqslant \frac{(4D)^4\mu(B_Y(x, \delta'\tau^{1/q}r))}{l}\tau \\ &< \frac{\mu(B_Y(x, \delta'\tau^{\frac{1}{q}}r))}{2}.\end{aligned}$$

As a result of the previous estimates, there exist $x' \in B(x, \delta'\tau^{1/q}r) \cap Y \subset \Omega_{2C'r}$ so that

$$4D^{n+m+5}\epsilon > M^{2C'r}_{C_0R}1_{\Omega_r \setminus Y}(x') \tag{3.26}$$

as well as

$$(2D)^4\tau > M^0_{\delta'\tau^{1/q}r}1_E(x'). \tag{3.27}$$

With R as before, note that any $s \in (\delta'\tau^{1/q}r, C_0R)$ and $x' \in B(x, \delta'\tau^{1/q}r)$ satisfy

$$B(x', s) \subset B(x, s + \delta'\tau^{1/q}r) \subset B(x, 2s) \subset B(x, C'r).$$

Then, doubling and our previous assumption (3.23) on x yield

$$\begin{aligned}\fint_{B_{2C'r}(x',s)} 1_E \, d\mu = \frac{\mu(E \cap B(x', s))}{\mu(B_{2C'r}(x', s))} &\leqslant \frac{\mu(E \cap B(x, s + \delta'\tau^{1/q}r))}{\mu(B_Y(x', s))} \\ &\leqslant \tau\frac{\mu(B_Y(x, s + \delta'\tau^{1/q}r))}{\mu(B_Y(x', s))} \leqslant \tau\frac{\mu(B_Y(x, 2s))}{\mu(B_Y(x', s))} \\ &\leqslant \tau\frac{\mu(B_Y(x', 4s))}{\mu(B_Y(x', s))} \leqslant (2D)^2\tau.\end{aligned}$$

As for $s \in (0, \delta'\tau^{1/q}r)$ and for x' satisfying (3.27), we have

$$\fint_{B_{2C'r}(x',s)} 1_E \, d\mu = \frac{\mu(E \cap B(x', s))}{\mu(B_{2C'r}(x', s))} \leqslant \frac{\mu(E \cap B(x', s))}{\mu(B_Y(x', s))} \leqslant (2D)^4\tau,$$

so the previous two estimates combine to yield

$$M^{2C'r}_{C'R}(1_E)(x') \leqslant (2D)^4\tau. \tag{3.28}$$

Put $F_r = E \cup \Omega_{2C'r} \setminus Y$. Subadditivity of the maximal function and equations (3.26) and (3.28) further yield

$$M^{2C'r}_{C'R}(1_{F_r})(x') \leqslant M^{2C'r}_{C'R}(1_E)(x') + M^{2C'r}_{C'R}(1_{\Omega_{2C'R}\setminus X})(x') \leqslant (2D)^4\tau + 4D^{5+n+m}\epsilon \overset{(3.22)}{<} \tau_0.$$

Similarly, since $M^0_{C'r}1_E(y) < \tau$, there exists $y' \in B(y, \delta'\tau^{\frac{1}{q}}) \cap Y \subset \Omega_{2C'r}$ so that

$$M^{2C'r}_{C'R}1_{F_r}(y') < \tau_0.$$

Step 3: Arranging quasi-geodesics. The space $\Omega_{2C'r}$ is $(C_0, \delta', (2D)^4\tau + 4D^{5+n+m}\epsilon, q)$-max connected at scale $C'r$. Since

$$d(x', y') \leqslant d(x', x) + d(x, y) + d(y, y') \leqslant \delta'\tau^{1/q}r + r + \delta'\tau^{1/q}r \leqslant R \leqslant 2r < C'r,$$

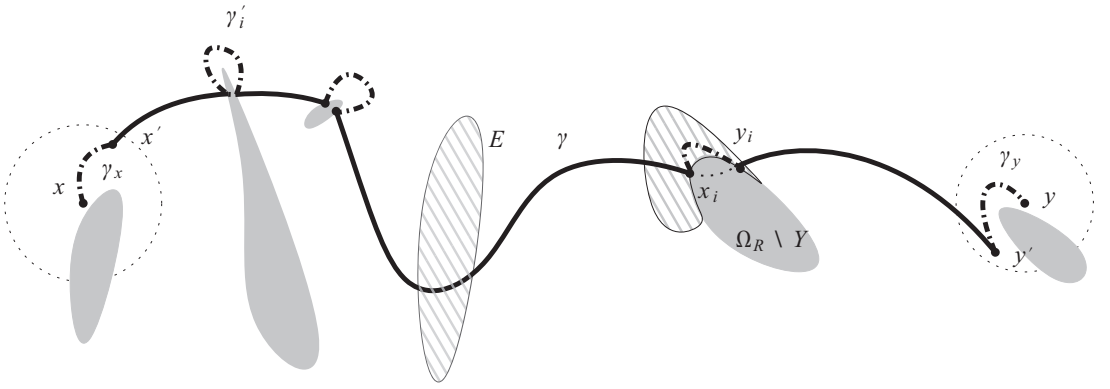

FIGURE 3. *Connectivity involves constructing a curve γ that almost avoids a prescribed obstacle E with small density. In the proof of Theorem 3.16, this requires first finding a curve in the filling Ω_R from nearby points x', y', and then patching the curve with 'detours' γ_i' to fully avoid $\Omega_R \setminus Y$, and γ_x, γ_y to connect x and y. In the figure, the solid black curve indicates γ in the filling Ω_R, with the dotted parts indicating the parts replaced by the dash-dotted detours.*

there thus exists $L > 0$ and a rectifiable curve $\gamma_1 : [0, L] \to \Omega_{2C'r}$ of length at most $C_0 R$ and so that $\gamma_1(0) = x'$ and $\gamma_1(L) = y'$ and

$$\int_{\gamma_1} 1_E \, ds \leqslant \int_{\gamma_1} 1_{F_r} \, ds \leqslant \delta'\big((2D)^4\tau + 4D^{5+n+m}\epsilon\big)^{1/q} R \overset{(3.21)}{\leqslant} 2\delta'(2D)^{4/q}\tau^{1/q} r. \tag{3.29}$$

We now modify γ_1 so that it lies entirely in Y and joins x and y. This is done by replacing portions of the curve with curves in Y, and appending two segments on each end. (See Figure 3.) This uses the Λ-quasi-convexity of Y at scale $r_1 = \frac{r_0}{2C_0}$ from Lemma 3.2.

First, the set $\gamma_1^{-1}(\Omega_{2C'r} \setminus Y)$ is open and can be expressed as a (possibly finite) union of countably many open disjoint intervals:

$$\gamma_1^{-1}(\Omega_{2C'r} \setminus Y) = \bigcup_i (a_i, b_i).$$

Let $x_i = \gamma_1(a_i)$ and $y_i = \gamma_1(b_i)$. Since Y is Λ-quasi-convex, we can find curves $\gamma_i' \colon [0, L_i] \to Y$ connecting x_i to y_i, which are parametrized by length and satisfy

$$\sum_i L_i \leqslant \sum_i \Lambda d(x_i, y_i) \leqslant \Lambda \int_{\gamma_1} 1_{F_r} \, ds \leqslant 2\Lambda\delta'(2D)^{4/q}\tau^{1/q} r.$$

Similarly as in the proof of Lemma 3.2, define a curve by patching the intervals (a_i, b_i) with the curves γ_i', that is,

$$\gamma_2^*(t) := \begin{cases} \gamma_i'\Big(\frac{L_i}{b_i - a_i}(t - a_i)\Big), & \text{if } t \in [a_i, b_i] \text{ for some } i \in \mathbb{N}, \\ \gamma_1(t), & \text{otherwise,} \end{cases}$$

and let γ_2 be its arclength parametrization. Now, γ_2 lies entirely in Y, since γ_1 only lies outside of Y in the intervals (a_i, b_i). Further,

$$\begin{cases} \displaystyle\int_{\gamma_2} 1_E \, ds \leqslant \int_{\gamma_1} 1_E \, ds + \sum_i L_i \\ \qquad \leqslant 2\delta'(2D)^{4/q}\tau^{1/q} r + 2\Lambda\delta'(2D)^{4/q}\tau^{1/q} r \leqslant 6\Lambda\delta'(2D)^{4/q}\tau^{1/q} r, \end{cases} \tag{3.30}$$

and

$$\mathrm{Len}(\gamma_2) \leqslant \mathrm{Len}(\gamma_1) + \sum_i L_i \leqslant C_0 R + 4\Lambda\delta'(2D)^{4/q}\tau^{1/q} r.$$

Next, the pairs of points x, x' and y, y', can be joined by Λ-quasi-geodesics γ_x and γ_y, respectively. Taking the concatenated curve

$$\gamma := \gamma_x \cup \gamma_2 \cup \gamma_y,$$

it follows from (3.29) that the required avoidance holds

$$\begin{aligned}\int_\gamma 1_E \, ds &\leqslant \mathrm{len}(\gamma_x) + \int_{\gamma_2} 1_E \, ds + \mathrm{len}(\gamma_y) \\ &\overset{(3.30)}{<} \Lambda\delta'\tau^{1/q} r + 6\Lambda\delta'(2D)^{4/q}\tau^{1/q} r + \Lambda\delta'\tau^{1/q} r \overset{(3.18)}{<} \delta\tau^{1/q} r,\end{aligned}$$

as well as

$$\begin{aligned}\mathrm{Len}(\gamma) &\leqslant \mathrm{Len}(\gamma_x) + \mathrm{Len}(\gamma_2) + \mathrm{Len}(\gamma_y) \\ &\leqslant \Lambda\delta'\tau^{1/q} r + C_0 R + 4\Lambda\delta'(2D)^{4/q}\tau^{1/q} r + \Lambda\delta'\tau^{1/q} r \\ &\leqslant \Big(C_0 + (2\Lambda + 2C_0 + 4\Lambda(2D)^{4/q})\delta'\tau^{1/q}\Big) r \overset{(3.19)}{\leqslant} 2C_0 r = C' r.\end{aligned}$$

This curve satisfies the desired estimates, and shows (C', δ, τ, q)-max connectivity. □

We now apply the previous theorem to obtain Poincaré inequalities for fillable sets.

Proof of Theorem 2.7, *Part* (a). Fix structural constants (p, D, C, r_0), which in turn fix the constant $C' = C'(D, C, p)$ in Theorem 3.16. Next, let $q > p$ be given and let $\delta_{q,2D} \in (0, 1)$ be as in Theorem 2.19 under the choice of structural constants $(q, 2D, C', r_0)$.

Applying now Theorem 3.16 and Remark 3.17, there exists $\epsilon_q > 0$ such that if $\epsilon \in (0, \epsilon_q)$ and if Y is (ϵ, D, C, p)-PI fillable, then Y is also $2D$-doubling and $(C', \delta_{q,2D}, \tau, q)$-max connected for some τ, both at scale $r_1 = r_0/(20C')$.

Since $\delta_{q,2D}$ was chosen as in Theorem 2.19, the space Y satisfies a $(1, q)$-Poincaré inequality with constant $C_q = C_q(q, D, C', \tau)$ at scale $r_1/C'_r = r_0/C_r$ for some constants C_r and C'_r. □

Proof of Theorem 2.7, *Part* (b). By Part (a), there is a density parameter ϵ_q such that the $(1, q)$-Poincaré inequality holds. Now, if Y is asymptotically p-Poincaré fillable, then there exists for any $\epsilon > 0$ a scale $r_\epsilon > 0$ where Y is (ϵ, D, C, p)-PI fillable. Choosing $\epsilon \in (0, \epsilon_q)$ for any fixed $q > p$, the local $(1, q)$-Poincaré inequality follows. □

4. *Application: Generalized Sierpiński sponges and uniform domains*

Here we apply the general filling theorem to prove Poincaré inequalities in various new contexts.

4.1. *Sierpiński sponges*

In this subsection, we prove Theorem 1.5 for sponges $S_{\mathbf{n}}$. A crucial property is the following separation condition, given below, for sub-cubes $R \in \overline{\mathcal{R}}_{\mathbf{n},k}$ removed through stages 1 through k in the construction of $S_{\mathbf{n}}$.

LEMMA 4.1. *If $R, R' \in \overline{\mathcal{R}}_{\mathbf{n},k}$ with $R \neq R'$, then*

$$d(R, R') \geqslant \frac{1}{3} s_{k-1} \text{ and } d(R, \partial[0,1]^d) \geqslant \frac{1}{3} s_{k-1}.$$

In particular, the removed sub-cubes are uniformly $\frac{1}{3\sqrt{d}}$-separated.

Proof. Without loss of generality, let $R \in \mathcal{R}_{\mathbf{n},l}$ and $R' \in \mathcal{R}_{\mathbf{n},l'}$ with $k \geqslant l \geqslant l'$. Let T be the unique cube in $\mathcal{T}_{l-1,\mathbf{n}}$ that contains R. Clearly $R' \cap T \subset \partial T$ and $n_l \geqslant 3$, so

$$d(R, R') \geqslant d(R, \partial T) \ \geqslant \ \frac{1}{3} s_{l-1} \ \geqslant \ \frac{1}{3} s_{k-1}$$

and moreover

$$\frac{1}{3} s_{l-1} \ \geqslant \ \frac{1}{3} s_l \ \geqslant \ \frac{1}{3} \min\left\{ \frac{\operatorname{diam}(R)}{\sqrt{d}}, \frac{\operatorname{diam}(R')}{\sqrt{d}} \right\}.$$

The same argument works for $\partial[0,1]^d$. □

To clarify the relationship between Case (4) in Theorem 1.5 and the other cases below, we note that the set $S_{\mathbf{n}}$ has positive Lebesgue measure if and only if $\mathbf{n}^{-1} \in \ell^d(\mathbb{N})$, that is

$$\sum_{i=1}^{\infty} \frac{1}{n_i^d} < \infty,$$

and this follows directly from Lemma 4.3.

The proof of Theorem 1.5 will be given in separate lemmas. First, Case (4) is proven directly from certain consequences of Poincaré inequalities, namely Cheeger's Rademacher Theorem [**10**]. To keep the discussion self-contained, we introduce the relevant notions in context, below.

Proof of Case (4) *of Theorem* 1.5. If $S_{\mathbf{n}}$ supports a $(1,p)$-Poincaré inequality for some $p \geqslant 1$ with respect to some doubling measure μ, then Cheeger's theorem [**10**] holds. In particular, there exist a partition $\{S_{\mathbf{n}}^j\}$ of $S_{\mathbf{n}}$ and Lipschitz maps $\varphi^j : S_{\mathbf{n}}^j \to \mathbb{R}^{m_j}$ so that for every Lipschitz function $f : S_{\mathbf{n}} \to \mathbb{R}$ there exists a unique L^∞-vectorfield $D^j f : S_{\mathbf{n}}^j \to \mathbb{R}^{m_j}$ so that, for μ-a.e. $x \in S_{\mathbf{n}}^j$, it holds that

$$\frac{f(y) - f(x) - D^j f(x) \cdot (\varphi^j(y) - \varphi^j(x))}{|x - y|} \to 0$$

as $y \to x$. By a result of Keith [**27**, Theorem 2.7], the components φ_k^j of each φ^j can be chosen to be distance functions of the form

$$\varphi_k^j(x) = |x - x_k^j|$$

for some $x_k^j \in S_{\mathbf{n}}^j$. Each is (classically) differentiable everywhere except at x_k^j, so each $D^j f(x)$ can be replaced with the vectorfield

$$\nabla \varphi^j(x) D^j f(x) : S_{\mathbf{n}}^j \to \mathbb{R}^d,$$

where $\nabla \varphi^j$ is the $d \times m_j$ matrix whose columns are the gradients of the components. In other words, each f is μ-a.e. differentiable with respect to the linear coordinate functions x_j as well as the generalized 'coordinates' φ^j. Thus, for every U_i the chart ϕ^j can be chosen using a subset of the coordinates. Since on every positive μ-measured subset of $S_{\mathbf{n}}$ the coordinates x_j are linearly independent on $S_{\mathbf{n}}$, then we need all the coordinates and we can choose the charts as $\phi^j(x) = x$. The result of De Philippis, Rindler and Marchese [**12**], which proves a conjecture of Cheeger, ensures that $\phi^j(S_{\mathbf{n}}^j) = S_{\mathbf{n}}^j$ has positive Lebesgue measure. □

As we will see, the equivalence of Conditions (1)–(3) is a special case of Theorem 2.7. We begin with checking properties of the Lebesgue measure λ restricted to $S_{\mathbf{n}}$.

LEMMA 4.2 (Basic volume estimate). *Let $T \in \mathcal{T}_{\mathbf{n},k}$, then*

$$\exp\left(-2\sum_{i=k+1}^{\infty}\frac{1}{n_j^d}\right) \leqslant \frac{\lambda(T\cap S_{\mathbf{n}})}{\lambda(T)} = \prod_{i=k+1}^{\infty}\left(1-\frac{1}{n_i^d}\right) \leqslant \exp\left(-\sum_{i=k+1}^{\infty}\frac{1}{n_j^d}\right).$$

Proof. It is easy to show inductively that

$$\lambda(T\cap S_{\mathbf{n}}) = \lambda(T)\prod_{i=k+1}^{\infty}\left(1-\frac{1}{n_i^d}\right),$$

from which the estimate follows, since $e^{-2x} \leqslant 1-x \leqslant e^{-x}$ for $x = \frac{1}{n_j^d} \in [0, \frac{1}{2}]$. □

LEMMA 4.3. *If $\mathbf{n}$ is a sequence of odd positive integers with $\mathbf{n}^{-1} \in \ell^d(\mathbb{N})$, then $S_{\mathbf{n}}$ is Ahlfors d-regular for some constant $C_{AR} = C_{AR}(\mathbf{n}, d)$. In particular, $S_{\mathbf{n}}$ is $2^d C_{AR}$-doubling.*

Proof. Given $x \in S_{\mathbf{n}}$, $r \in (0, \operatorname{diam}(S_{\mathbf{n}})) = (0, \sqrt{d})$, and $\rho \in (0, r]$, let $Q(x,\rho)$ be the cube with center x and edges parallel to the coordinate axes and of length $\rho/\sqrt{d}$, so $Q(x,r) \subset B(x,r)$. Choose $k \geqslant 1$ so that

$$8\sqrt{d}s_k \;\leqslant\; r \;<\; 8\sqrt{d}s_{k-1} \tag{4.4}$$

and let $T_{x,r} \in \mathcal{T}_{k-1,\mathbf{n}}$ be such that $x \in T_{x,r}$ and define

$$\mathcal{T}_{x,r} := \{T \in \mathcal{T}_{k,\mathbf{n}} \mid T \subset Q(x,r)\cap T_{x,r}\}.$$

Let $R \in \mathcal{R}_{k,\mathbf{n}}$ be the central square of $T_{x,r}$. Then $\mathcal{T}_{x,r}$ covers $Q(x,\frac{r}{2})\cap T_{x,r}\setminus R$. Moreover

$$\lambda\Big(Q\Big(x,\frac{r}{2}\Big)\cap T_{x,r}\Big) \leqslant \lambda\Big(Q\Big(x,\frac{r}{2}\Big)\cap T_{x,r}\setminus R\Big) + \lambda(R).$$

Thus,

$$2|\mathcal{T}_{x,r}|s_k^d \geqslant |\mathcal{T}_{x,r}|s_k^d + \lambda(R) \geqslant \lambda\Big(Q\Big(x,\frac{r}{2}\Big)\cap T_{x,r}\setminus R\Big) + \lambda(R) \geqslant \lambda\Big(Q\Big(x,\frac{r}{2}\Big)\cap T_{x,r}\Big),$$

since $|\mathcal{T}_{x,r}| \geqslant 2$, and $\lambda(R) = s_k^d$. The estimate

$$|\mathcal{T}_{x,r}|s_k^d \geqslant \frac{1}{2}\lambda(Q(x,\frac{r}{2})\cap T_{x,r}) \geqslant \frac{1}{2}\min\{r/(2\sqrt{d}), s_{k-1}/2\}^d \geqslant \frac{r^d}{2\Big(2^4\sqrt{d}\Big)^d} \tag{4.5}$$

follows easily from (4.4), because $Q(x,\frac{r}{2})\cap T_{x,r}$ is a rectangle with side lengths at least $\min\{r/(2\sqrt{d}), s_{k-1}/2\}$. Thus, using the fact that for any k and any $T \in \mathcal{T}_{\mathbf{n},k}$,

$$\lambda(T\cap S_{\mathbf{n}}) = c_{\mathbf{n},k}\lambda(T), \text{ where } c_{\mathbf{n},k} = \prod_{j=k+1}^{\infty}\left(1-\frac{1}{n_j^d}\right). \tag{4.6}$$

Lemma 4.2 implies

$$\begin{aligned}\lambda(B(x,r)\cap S_{\mathbf{n}}) &\geqslant \lambda(Q(x,r)\cap S_{\mathbf{n}}\cap T_{x,r})\\ &\geqslant \sum_{T\in\mathcal{T}_{x,r}}\lambda(T\cap S_{\mathbf{n}}) \overset{(4.6)}{=} c_{\mathbf{n},k}\sum_{T\in\mathcal{T}_{x,r}}\lambda(T) \;\geqslant\; c_{\mathbf{n},k}|\mathcal{T}_{x,r}|s_k^d \overset{(4.5)}{\geqslant} \frac{c_{\mathbf{n},0}}{2}\frac{r^d}{(2^4\sqrt{d})^d}.\end{aligned}$$

The result then follows with constant $C_{AR} = \frac{2(2^4\sqrt{d})^d}{c_{\mathbf{n},0}}$. Note that the upper bound for Ahlfors regularity is trivial. □

LEMMA 4.7. *The set $S_{\mathbf{n}}$ is an asymptotically 1-Poincaré fillable subset of $\mathbb{R}^d$.*

Proof. Let $D = 2^d C_{AR}$ be the doubling constant from Lemma 4.3. Now, consider the domains $Y_1 = \mathbb{R}^d$ and $Y_2 = [0,1]^d$ and $Y_3 = \mathbb{R}^d \setminus R$, for $R \in \overline{\mathcal{R}}_{\mathbf{n},k}$. Each of these satisfies a Poincaré inequality with inflation factor 1, that is, $CB \cap Y_i = B \cap Y_i$; see equation (4.8); this follows, for example, from [**19**] and the chained ball condition which is easy to verify in this case. In particular, for each $i = 1,2,3$ and for any ball $B := B(x,s)$ and any Lipschitz function f on Y_i, we have

$$\fint_{B\cap Y_i} |f - f_{B\cap Y_i}|\, d\lambda \leqslant C_{PI}\, s \fint_{B\cap Y_i} \mathrm{Lip}\,[f]\, d\lambda, \tag{4.8}$$

where the constant C_{PI} is independent of i, B and f. This holds, *a priori*, for any Lipschitz function in $\mathbb{R}^d$ and taking extensions as necessary, for any Lipschitz function defined on $Y_i \cap B$.

For each $\epsilon \in (0,1)$, choose $\delta \in (0, \epsilon/4)$ so that

$$1 - (1-\delta)^d < \frac{\epsilon}{4^{d+1}\sqrt{d}^d \lambda(B(0,1))}$$

in which case it holds, for all $r > 0$, that

$$\lambda(B(x,r) \setminus B(x, r(1-\delta))) = \lambda(B(0,1)) \cdot \big(1 - (1-\delta)^d\big) r^d < \frac{\epsilon r^d}{4^{d+1}\sqrt{d}^d}. \tag{4.9}$$

Next, choose $j_0 \in \mathbb{N}$ so that both $\sum_{i=j_0}^{\infty} \frac{1}{n_i^d} < \frac{\epsilon}{4}$ and $n_i \geqslant 2^5\sqrt{d}\delta^{-1}$ for all $i \geqslant j_0$. We now claim that $S_{\mathbf{n}}$ is 1-Poincaré ϵ-fillable (Definition 2.6) at scale

$$r_0 = s_{j_0+1} = \prod_{i=1}^{j_0+1} \frac{1}{n_i}$$

with the above constants (C_{PI}, D).

To see why, let $r \in (0, r_0)$ and $x \in S_{\mathbf{n}}$ be given. Since $d, n_{j_0+1} \in \mathbb{N}$, it follows that $\frac{2\sqrt{d}}{\delta} s_{j_0} \geqslant \frac{1}{n_{j_0+1}} s_{j_0} = r_0$, so choose $k \geqslant j_0$ so that

$$\frac{2\sqrt{d}}{\delta} s_{k+1} \;\leqslant\; r \;<\; \frac{2\sqrt{d}}{\delta} s_k.$$

Now let $\Omega_r := S_{k,\mathbf{n}}$. To show fillability, we need to show **(i)** doubling, **(ii)** a local Poincaré inequality and **(iii)** an ϵ-density bound. By Lemma 4.3, the set Ω_r, which contains $S_{\mathbf{n}}$ and is contained in $[0,1]^2$, is Ahlfors 2-regular when equipped with the (restricted) Lebesgue measure and hence doubling.

With **(i)** now settled, we show the local Poincaré inequality **(ii)**. Based on our choice of j_0 and k, we have

$$s_{k-1} \;=\; n_k s_k \;>\; \frac{2^5\sqrt{d}}{\delta} \frac{\delta}{2\sqrt{d}} r \;\geqslant\; 2^4 r$$

in which case Lemma 4.1 implies

$$d(R, R') \geqslant \frac{1}{3} s_{k-1} > 4r \tag{4.10}$$

for all $R, R' \in \overline{\mathcal{R}}_{\mathbf{n},k}$ with $R \neq R'$. Thus for each $x \in S_{\mathbf{n}}$ there is at most one $R \in \overline{\mathcal{R}}_{\mathbf{n},k}$ that meets $B(x, 2r)$. Also, if such a cube R exists, then similarly from Lemma 4.1, it follows that

$$d(R, \partial[0,1]^d) \geqslant 2r$$

so $B(x, 2r)$ would not intersect $\partial[0,1]^d$.

Now, for arbitrary $x \in S_{\mathbf{n}}$, fix a ball $B(x,s) \cap \Omega_r$ with $s \leqslant 2r$. As before, at most one R can meet $B(x,s)$, so

$$B(x,s) \cap \Omega_r = B(x,s) \cap Y_i$$

holds for some $i = 1, 2, 3$ as above, and equation (4.8) is precisely the local Poincaré inequality for Ω_r at scale s, as desired.

Finally, we show the density bound **(iii)**; that is, condition (2) in Definition 2.6. First observe that $B(x,r) \cap \Omega_r$ contains a cube with side length $r/(4\sqrt{d})$, in which case it holds that

$$\lambda(B(x,r) \cap \Omega_r) \geqslant \frac{r^d}{4^d \sqrt{d}^d}. \tag{4.11}$$

Now, consider all remaining $(k+1)$'th order subcubes that are sufficiently near x, that is,

$$\mathcal{T}_{x,r} = \{T \in \mathcal{T}_{k+1,\mathbf{n}} \mid T \cap B(x, (1-\delta)r) \neq \emptyset\}.$$

From our previous choice of k, we have for all $T \in \mathcal{T}_{x,r}$ that

$$\operatorname{diam}(T) \leqslant 2\sqrt{d} s_{k+1} < \delta r,$$

and thus $T \subset B(x,r)$. The cubes in $\mathcal{T}_{k+1,\mathbf{n}}$ that are contained in $\Omega_r \cap B(x,r)$ thus cover $\Omega_r \cap B(x,r)$ except for a portion of the annulus $B(x,r) \setminus B(x,(1-\delta)r)$ as well as the removed cubes in $\mathcal{R}_{k+1,\mathbf{n}}$ which intersect $B(x,r)$. Let $\mathbf{R}$ be the union of such removed cubes. These extra portions have small volume, as we will see.

Each cube in $\mathcal{R}_{k+1,\mathbf{n}}$ that intersects $B(x,r)$ is contained in a cube in $\mathcal{T}_{k,\mathbf{n}}$ of side length s_k, and such larger cubes have pairwise-disjoint interiors. If $r \leqslant s_k$, then there are at most 3^d such cubes, so for dimensions $d \geqslant 2$ we have

$$\lambda(\mathbf{R}) \leqslant 3^d s_{k+1}^d \leqslant 3^d \left(\frac{\delta r}{2\sqrt{d}}\right)^d \leqslant \left(\frac{3\delta}{2\sqrt{2}}\right)^d r^d \leqslant (\sqrt{2}\delta)^d r^d.$$

If $r \geqslant s_k$, then there are at most $(\frac{2r}{s_k} + 2)^d$ such cubes. Recalling that $s_k = n_{k+1} s_{k+1}$, our previous choices of j_0 and k now yield

$$\begin{aligned}\lambda(\mathbf{R}) \leqslant \left(\frac{2r}{s_k} + 2\right)^d s_{k+1}^d &= \frac{2^d s_{k+1}^d}{s_k^d}(r + n_{k+1}s_{k+1})^d \leqslant \frac{2^d}{n_{k+1}^d}\left(1 + \frac{n_{k+1}\delta}{2\sqrt{d}}\right)^d r^d \\ &= 2^d \left(\frac{1}{n_{k+1}} + \frac{\delta}{2\sqrt{d}}\right)^d r^d \leqslant \frac{2^d \delta^d}{\sqrt{d}^d} r^d \leqslant (\sqrt{2}\delta)^d r^d.\end{aligned}$$

Note that $\delta < \frac{\epsilon}{4} < \frac{1}{4}$ from before implies that $2 - \delta > \sqrt{2}$ as well as

$$\sqrt{2}\delta \leqslant (1 - (1-\delta))(2-\delta) \leqslant (1 - (1-\delta)) \sum_{m=0}^{d-1} (1-\delta)^m = 1 - (1-\delta)^d,$$

so the previous paragraph, the choice of δ from before, and (4.9)–(4.11) imply

$$\begin{aligned}\sum_{T\in\mathcal{T}_{x,r}} \lambda(T) &\geqslant \lambda(B(x,r)\cap\Omega_r) - \lambda(B(x,r)\setminus B(x,(1-\delta)r)) - \lambda(\mathbf{R})\\ &\geqslant \lambda(B(x,r)\cap\Omega_r) - \frac{\epsilon r^d}{4^{d+1}\sqrt{d}^d} - (\sqrt{2}\delta)^d r^d\\ &\geqslant \Big(1-\frac{\epsilon}{2}\Big)\lambda(B(x,r)\cap\Omega_r).\end{aligned}$$

Also, from Lemma 4.2 for every $T\in\mathcal{T}_{x,r}$ we get

$$\lambda(T\cap S_{\mathbf{n}}) \;\geqslant\; \exp\left(-2\sum_{i=k}^{\infty}\frac{1}{n_i^d}\right)\lambda(T) \;\geqslant\; \left(1-\sum_{i=j_0}^{\infty}\frac{2}{n_j^d}\right)\lambda(T) \;\geqslant\; \left(1-\frac{\epsilon}{4^d\sqrt{d}^d}\right)\lambda(T)$$

and as a result,

$$\begin{aligned}\lambda(B(x,r)\cap S_{\mathbf{n}}) &\geqslant \sum_{T\in\mathcal{T}_{x,r}} \lambda(T\cap S_{\mathbf{n}})\\ &\geqslant \Big(1-\frac{\epsilon}{2}\Big)\sum_{T\in\mathcal{T}_{x,r}}\lambda(T)\\ &\geqslant \Big(1-\frac{\epsilon}{2}\Big)^2\lambda(B(x,r)\cap\Omega_r) \geqslant (1-\epsilon)\lambda(B(x,r)\cap\Omega_r).\end{aligned}$$

Thus subtracting $\lambda(B(x,r)\cap\Omega_r)$ from both sides yields the result. □

The equivalence of Conditions (1) through (3) in Theorem 1.5 is now easy to see.

Proof of (1) ⇔ (2) ⇔ (3) *in Theorem* 1.5. The statement (2) ⇒ (3) is trivial. Note that the contrapositive of (4) also proves that (3) ⇒ (1).

As for (1) ⇒ (2), Lemma 4.3 shows that $S_{\mathbf{n}}$ is in fact Ahlfors d-regular. Then Lemma 4.7 shows that $S_{\mathbf{n}}$ is asymptotically 1-Poincaré fillable, and thus by Theorem 2.7 it satisfies a local $(1,p)$-Poincaré inequality at scale $r_0 = r_0(p,d,\mathbf{n})$ for any $p>1$. However, since $S_{\mathbf{n}}$ is connected and uniformly doubling, then as a consequence of [**6**, Theorem 1.3] the entire space $S_{\mathbf{n}}$ satisfies a (global) $(1,p)$-Poincaré inequality. Note that, while the reference [**6**] deals with so-called 'semi-local" inequalities, in our case of bounded diameter these suffice for a global inequality. □

4.2. *General metric carpets*

In this section, we extend the proof of the previous section to give examples of Sierpiński sponges in general metric spaces. In particular, we prove Theorem 1.9.

The crucial role here is played by uniform domains. We note that conventionally, uniform domains are assumed to be open sets. Our definition, however, will allow for closed sets as well. Indeed, one can show that if a closed set Ω is uniform, then its interior $\mathrm{int}(\Omega)$ is uniform. The converse holds, at least in doubling metric spaces, if Ω is the closure of its interior. It is worth noting that, on the other hand, a closure of a nonuniform domain may be uniform, such as in the case of a slit disk. However, our starting point will always be closed sets.

Definition 4.12 (Uniform Domains). Given a metric space $X=(X,d)$, $A>0$, a subset $\Omega\subset X$, and points $x,y\in\Omega$, a continuous curve $\gamma\colon [0,1]\to\Omega$ is called an A-uniform curve (with respect to x, y, and Ω) if it connects x and y with $\mathrm{diam}\gamma \leqslant Ad(x,y)$ and

$$d(\gamma(t),\Omega^c) \geqslant A^{-1}\min(\mathrm{diam}(\gamma|_{[0,t]}),\mathrm{diam}(\gamma|_{[t,1]})). \tag{4.13}$$

We say that Ω is A-UNIFORM UP TO SCALE r if for all $x, y \in \Omega$ with $d(x, y) < r$ there exists an A-uniform curve with respect to x, y, and Ω.

Lastly, Ω is A-UNIFORM if it is A-uniform up to scale r, for all $r > 0$.

Alternative definitions, and their mutual equivalence, are discussed in [**32, 42**]. For example, if the space is doubling and quasi-convex, then γ could be assumed to be a rectifiable curve and diameter could be replaced with length in the definition. So in the context of uniformity (and only in this context), by a 'curve' we allow for curves to be continuous only, and not necessarily Lipschitz.

We remark, that in the case $\Omega = X$, the condition is vacuously satisfied if X is quasi-convex, as the distance to an empty set is interpreted to be infinity.

For us, uniform domains are quite flexible to construct, and they inherit good geometric properties from the spaces containing them. In particular, there is the following version of [**7**, Theorem 4.4].

THEOREM 4.14 (Björn-Shanmugalingam). *Let $1 \leqslant p < \infty$. If (X, d, μ) is D-doubling and satisfies a $(1, p)$-Poincaré inequality with constant C, and if Ω is a closed, A-uniform domain up to scale r_0 in X, then, with its restricted measure and metric, Ω is also $\overline{D}$-doubling and satisfies a $(1, p)$-Poincaré inequality at scale $r_0/2$ with constants $\overline{D} = \overline{D}(D, A)$ and $\overline{C} = \overline{C}(D, C, A, p)$.*

REMARK 4.15. To be clear, in [**7**, Theorem 4.4] only the global case of $r_0 = \infty$ and an open set Ω is explicitly discussed. Next, we briefly indicate the required modifications. Indeed, uniformity implies that $\partial\Omega$ is porous, and thus has measure zero. See, for example, [**9**, Lemma 3.2] for a result on and definition of porosity. Then, as remarked before Definition 4.12, $\tilde{\Omega} = \operatorname{int}(\Omega)$ is an open uniform domain, and satisfies the Poincaré inequality at scale r_0/C by the argument in [**7**, Theorem 4.4]. Since $\partial\Omega$ has measure zero, and $\tilde{\Omega}$ is dense in Ω, the Poincaré inequality and doubling also hold for Ω. Following their proof, these properties hold initially at some scale r_0/C with a constant C.

However, following the proof of [**6**, Theorem 4.4] and under the additional hypothesis that Ω is metric doubling and A-uniform up to scale r_0, we may upgrade the scale to r_0 with a uniform constant. In [**6**], the proof uses properness and connectivity to get nonquantitative bounds on the number of balls involved and that need to be chained. However, the only modification needed is a quantitative bound on the number of such balls needed, which follows here from doubling and uniformity. We refer the reader to the proof of [**6**, Theorem 4.4] for more details.

REMARK 4.16. There are many examples of uniform domains.

(1) Bounded convex subsets of $\mathbb{R}^d$ are uniform, where the uniformity constant A depends on the eccentricity of the convex subset.

(2) Compact domains with Lipschitz-regular boundaries in $\mathbb{R}^n$ are uniform, as well as their complements. The constants depend quantitatively on the Lipschitz constants of the local representations and the sizes of the charts covering the boundary.

(3) $C^{1,1}$-compact domains and their complements in any step-2 Carnot group, including the (first) Heisenberg group, are uniform with respect to their Carnot–Carathéodory metrics [**37**]. Here, $C^{1,1}$-regularity is with respect to the Euclidean smooth structure. For an introduction to Carnot groups, we refer the reader to [**37**]. See also Section 4.3 for a discussion of the Heisenberg group (from a purely metric space perspective).

(4) Let $f\colon X \to Y$ be a quasi-symmetric map between metric spaces (X, d) and (Y, d'), that is, that there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ with necessarily $\eta(0) = 0$ and $\eta(t) \to \infty$ as $t \to \infty$ so that

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right) \text{ for all } x,y,z \in X.$$

If Ω is a uniform domain in X, then $f(\Omega)$ is also uniform in Y. The constants are quantitative with respect to the uniformity of Ω and the distortion function η.
In particular, if $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a K-quasi-conformal map, then it is η-quasi-symmetric [**43**], and so $f(B(0,1))$ and $f(\mathbb{R}^d \setminus B(0,1))$ are uniform.

(5) Recently, Rajala [**38**] has proven that in any quasi-convex doubling space there exists an abundance of uniform domains. In fact, every bounded domain can be approximated by uniform domains in the Hausdorff metric. (The dependence on constants is not given explicitly there, but can likely be made explicit in some cases.)

Our main theorem has an immediate consequence for uniform domains, or more generally, what we call 'almost-uniform' domains.

Definition 4.17. A subset Y of X is called (ϵ, A)-almost uniform at scale r_0 if for every $r \in (0, r_0)$ there is a connected, closed subset Ω_r of X that is A-uniform up to scale $4r$, and so that $Y \subset \Omega_r$ and for every $x \in Y$ it holds that

$$\frac{\mu(\Omega_r \cap B(x,r) \setminus Y)}{\mu(\Omega_r \cap B(x,r))} < \epsilon. \tag{4.18}$$

Corollary 4.19. *Let (p, D, C, A) be structural constants and $r_0 > 0$.*

If (X, d, μ) is a D-doubling space that satisfies a $(1,p)$-Poincaré inequality with constant C, then for any $q > p$ there exists $\epsilon > 0$, depending on the structural constants, such that if $Y \subset X$ is (ϵ, A)-almost uniform at scale $r_0 > 0$, then Y with its restricted metric and measure satisfies a $(1,q)$-Poincaré inequality at scale $r_1 = r_1(D, C, A, r_0)$.

Moreover, if Y is (ϵ, A)-almost uniform for all $\epsilon \in (0, \frac{1}{2})$, then it satisfies a $(1,q)$-Poincaré inequality for every $q > p$.

Proof. By applying Definition 4.17 and Theorem 4.14 to Y, for each $r \in (0, r_0)$ the filling Ω_r with its restricted measure is $\overline{D}$-doubling at scale $2r$ and satisfies a $(1,p)$-Poincaré inequality at scale $2r$ with constant $\overline{C} = \overline{C}(D, C, A, p)$ independent of r. Thus, together with $Y \subset \Omega_r$ we see that for each $r > 0$ the filling Ω_r satisfies Definition 2.6 and thus the claim follows from Theorem 2.7. □

Instead of prescribing *a priori* 'fillings' to subsets in the sense of Theorem 2.7, we now return to the perspective in the Introduction (Subsection 1.3) and consider constructions on general PI-spaces akin to Sierpiński sponges. In this original but opposite viewpoint, we first consider *complements* of certain domains.

Definition 4.20. Let $A > 0$. An open, bounded subset Ω of a metric space X is called A-co-uniform if $X \setminus \Omega$ is A-uniform and $\partial\Omega$ is connected.

To define 'metric sponges' in terms of dyadic decompositions is nontrivial, as compared with Sierpiński sponges in $\mathbb{R}^d$. In general, metric measure spaces need not admit dyadic decompositions; even in the case of doubling measures, the cells of a Christ dyadic decomposition do not necessarily form a collection of uniform domains with a uniform constant.

We therefore define a construction in terms of removed sets (or 'obstacles') instead. As there is no guarantee of self-similarity in an arbitrary metric space, these sets are given in terms of a strengthening of item (2) of Theorem 1.8, the uniform relative separation property applied to co-uniform domains instead of quasi-disks; see item (5) below.

Definition 4.21. Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers, and consider scales, given inductively as $s_0 = 1$ and

$$s_k = \frac{1}{n_k} s_{k-1}$$

for $k \in \mathbb{N}$. A sequence of collections of domains $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ in Ω forms a uniformly $\mathbf{n}$-sparse collection of co-uniform sets in Ω if there exist constants $\delta, L > 0$ and $A \geqslant 1$ so that for each $R \in \mathcal{R}_{\mathbf{n},k}$:

(1) $R \subset \Omega$;
(2) R is A-co-uniform and Ω is A-uniform;
(3) $\operatorname{diam}(R) \leqslant L s_k \operatorname{diam}(\Omega)$;
(4) $d(R, \Omega^c) \geqslant \delta s_{k-1} \operatorname{diam}(\Omega)$;
(5) if moreover $R' \in \mathcal{R}_{\mathbf{n},k'}$ with $k \geqslant k'$, then $d(R, R') \geqslant \delta s_{k-1} \operatorname{diam}(\Omega)$.

Moreover, $\{\mathcal{R}_{\mathbf{n},k}\}$ is called dense in Ω whenever $\bigcup_{k\in\mathbb{N}} \bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} R$ is dense in Ω. We lastly define

$$S_{\mathbf{n}} := \Omega \setminus \bigcup_{k} \bigcup_{R\in\mathcal{R}_{\mathbf{n},k}} R.$$

It is worth mentioning here that Condition (5) appears as equation (4.10) and was crucial in the proof for Sierpiński sponges. It will be similarly useful in the sequel.

Recall that Theorem 1.9 asserts that:

- *On an Ahlfors-regular p-PI space, the complement of a uniformly sparse collection of co-uniform sets is also an Ahlfors-regular p-PI space.*

As an initial, geometric idea of the proof, we now state our main technical tool.

Theorem 4.22. *Fix structural constants $A_1, A_2, C, D \geqslant 1$. Let X be a C-quasi-convex, D-metric doubling metric space, let Ω be an A_1-uniform subset of X, and let S be a bounded, A_2-co-uniform subset of X. If*

$$\overline{S} \subset \operatorname{int}(\Omega),$$

then $\Omega \setminus S$ is A'-uniform in X, with dependence $A' = A'(A_1, A_2, C, D, \frac{d(S,\Omega^c)}{\operatorname{diam}(S)})$.

For clarity, we postpone its proof to the Appendix. Applying it to an induction argument, however, yields the following useful result: cutting out a finite collection of co-uniform domains preserves uniformity. For simplicity, it is formulated in terms of the relative distance, from item (2) of Theorem 1.8:

$$\Delta(E, F) := \frac{d(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}.$$

Corollary 4.23. *Fix structural constants $A_1, A_2, C, D \geqslant 1$. Let X be a D-metric doubling, C-quasi-convex metric space, let Ω be a A_1-uniform domain in X and for $i = 1, \ldots, N$ let S_i be a A_2-co- uniform domain in X such that $\Delta(S_i, S_j) \geqslant \epsilon$ for $i \neq j$ and $d(S_i, \Omega^c) \geqslant \epsilon \operatorname{diam}(S_i)$. Then $\Omega \setminus \bigcup_{i=1}^{N} S_i$ is also uniform in X.*

Proof. Order the elements S_i so that $\operatorname{diam}(S_i) \leqslant \operatorname{diam}(S_j)$ for $i \geqslant j$ and define recursively

$$\Omega_i = \begin{cases} \Omega \setminus S_1, & \text{if } i = 1 \\ \Omega_{i-1} \setminus S_i, & \text{if } 2 \leqslant i \leqslant N. \end{cases}$$

Put $A_0' = A_1$. By Theorem 4.22, we have that Ω_1 is A_1'-uniform with $A_1' = A'(A_0', A_2, C, D, \epsilon)$, where A' is now treated as a function of the given parameters.

Proceed by induction and assume now that Ω_n is A_n'-uniform with dependence $A_n' = A'(A_{n-1}', A_2, C, D, \epsilon)$. By the separation condition, we know that

$$d(S_{n+1}, \Omega_n^c) \geqslant \epsilon \operatorname{diam}(S_{n+1}).$$

Therefore, again by Theorem 4.22, we have that Ω_{n+1} is A_{n+1}-uniform with dependence $A_{n+1}' = A'(A_n', A_2, C, D, \epsilon)$. □

As in the proof of Theorem 1.5, we need analogues of Lemmas 4.2 and 4.3, but for uniformly sparse collections of co-uniform sets instead of Sierpiński sponges. Their proofs being similarly straightforward, we postpone them to the Appendix and focus on how they imply Theorem 1.9 instead.

LEMMA 4.24. *Let $\Omega \subset X$ be an A-uniform subset, and assume that (X, d, μ) is Ahlfors Q-regular with constant C_{AR}. Then Ω is Ahlfors Q-regular with constant $C_{AR,\Omega} = (4A)^Q C_{AR}$ when equipped with the restricted measure and metric.*

LEMMA 4.25. *Under the hypotheses of Theorem 1.9, if $r \geqslant s_k \operatorname{diam}(\Omega)$, then*

$$\mu\left(B(x,r) \cap \bigcup_{l=k+1}^{\infty} \bigcup_{R \in \mathcal{R}_{\mathbf{n},l}} R\right) \leqslant C_\delta r^Q \sum_{i=k+1}^{\infty} \frac{1}{n_i^Q},$$

holds for each $x \in S_{\mathbf{n}}$, where C_δ depends quantitatively on C_{AR} and Q, as well as on δ and L from Definition 4.21.

We are now ready to verify the Poincaré inequality, for metric space sponges formed from uniformly sparse collections of co-uniform sets.

Proof of Theorem 1.9. Scale the statement so that $\operatorname{diam}(\Omega) = 1$. The domains $Y_1 = X$ and $Y_2 = \Omega$ and $Y_3 = X \setminus R$ are uniform domains with some constant A by definition, for any $R \in \bigcup_{k=1}^{\infty} \mathcal{R}_{\mathbf{n},k}$. So, each Y_i is uniformly Ahlfors Q-regular with constant $C_{AR,Y}$ by Lemma 4.24. Let C be the constant of the Poincaré inequality of X, and D be the doubling constant of X. These fix the structural constants (p, D, C, A) in Corollary 4.19. Applying this corollary yields an $\epsilon > 0$.

Local doubling and Poincaré inequalities will follow once we show that $S_{\mathbf{n}}$ is almost uniform.

Let C_δ be the constant from Lemma 4.25. Choose first $K_\epsilon \in \mathbb{N}$ so large that

$$\sum_{i=K_\epsilon}^{\infty} \frac{1}{n_i^Q} \leqslant \frac{\epsilon}{C_\delta C_{AR,Y}}$$

and so that $n_i \geqslant \frac{2^5 A}{\delta}$ for every $i \geqslant K_\epsilon$. Then, define $r_0 = \delta s_{K_\epsilon+1}/(2^4 AL)$. Now, we show that $S_{\mathbf{n}}$ is (ϵ, A)-almost uniform at level r_0, with the aforementioned fixed structural constants. To that avail, let $x \in S_{\mathbf{n}}$ and $r \in (0, r_0)$ be arbitrary. Choose $k \geqslant K_\epsilon$ so that

$$\frac{\delta s_k}{2^4 A} < r \leqslant \frac{\delta s_{k-1}}{2^4 A}.$$

Analogously as for Sierpiński sponges, put

$$\overline{\mathcal{R}}_{\mathbf{n},k} = \bigcup_{l=1}^{k} \mathcal{R}_{\mathbf{n},l} \text{ and } S_{\mathbf{n},l} = \Omega \setminus \bigcup_{R \in \overline{\mathcal{R}}_{\mathbf{n},l}} R$$

and just as in the proof of Lemma 4.7, define the filling $\Omega_r := S_{\mathbf{n},l}$.

Since $8Ar \leqslant \delta s_{k-1}/2$, there is at most one $R \in \overline{\mathcal{R}}_{\mathbf{n},k}$ which intersects $B(x, 8Ar)$, so

$$\Omega_r \cap B(x, 8Ar) = Y_i \cap B(x, 8Ar) \tag{4.26}$$

for some $i = 1, 2, 3$. Since Y_i is A-uniform, any $y \in B(x, 4r)$ can be connected to x with an A-uniform curve with respect to Y_i, so by (4.26) that same curve is an A-uniform curve with respect to Ω_r. That is, Ω_r is A-uniform at scale $4r$.

So to satisfy Definition 4.17 we only need to check the density condition (4.18). But, by the choice of K_ϵ, we have $s_{k+1} \leqslant r$, and thus by Lemma 4.25

$$\mu\left(B(x,r) \cap \bigcup_{l=k+1}^{\infty} \bigcup_{R \in \mathcal{R}_{\mathbf{n},l}} R\right) \leqslant C_\delta r^Q \sum_{i=k+1}^{\infty} \frac{1}{n_i^Q} \leqslant \frac{\epsilon}{C_{AR,Y}} r^Q.$$

Since $\Omega_r \setminus S_{\mathbf{n}}$ lies in $\bigcup_{l=k+1}^{\infty} \bigcup_{R \in \mathcal{R}_{\mathbf{n},l}} R$, we estimate its density in $B(x,r)$ to be

$$\frac{\mu(\Omega_r \setminus S_{\mathbf{n}} \cap B(x,r))}{\mu(\Omega_r \cap B(x,r))} \leqslant \frac{\mu\left(B(x,r) \cap \bigcup_{l=k+1}^{\infty} \bigcup_{R \in \mathcal{R}_{\mathbf{n},l}} R\right)}{\mu(\Omega_r \cap B(x,r))} \leqslant \frac{\frac{\epsilon}{C_{AR,Y}} r^Q}{\frac{1}{C_{AR,Y}} r^Q} \leqslant \epsilon.$$

Here, we again used (4.26) and that Y_i are Ahlfors $C_{AR,Y}$-regular, for some $i = 1, 2, 3$.

This verifies all the conditions in Definition 4.17, in which case the conclusion of the Theorem follows by Corollary 4.19. Finally, the remark on density is trivial, and the remark on the exponent p follows from Keith–Zhong [**28**], since our spaces are complete. To be more specific, Keith–Zhong is applied first to X to improve its Poincaré inequality, and then the first part is applied to obtain a better inequality for the fillable set Y. The density is also explained in more detail in the context of the Heisenberg group below.

Finally, an estimate as above using Lemma 4.25 gives the Ahlfors regularity of $S_{\mathbf{n}}$ for balls of size $r < r_0$. Since Ω is bounded, the Ahlfors regularity then follows immediately. Indeed, the upper bound in Ahlfors regularity follows from that of X, and the lower bound from $\mu(B(x,r)) \geqslant \mu(B(x,r_0))$ if $r \geqslant r_0$. Further, the local Poincaré inequality upgrades to a Poincaré inequality (since Ω is bounded) from [**6**, Theorem 7.3] once we see that $S_{\mathbf{n}}$ is connected. To see this, let $x, y \in S_{\mathbf{n}}$ be arbitrary, and let γ be any continuous curve in Ω connecting x, y. Let

$$E = (\gamma \cap S_{\mathbf{n}}) \cup \bigcup_{k=1}^{\infty} \bigcup_{R \in \mathcal{R}_{\mathbf{n},k}, R \cap \gamma \neq \emptyset} \partial R.$$

The set E is easily seen to be a connected compact subset of $S_{\mathbf{n}}$ (since ∂R are connected by assumption), and thus $S_{\mathbf{n}}$ is connected. □

4.3. *Non-Euclidean examples: Heisenberg meets Sierpiński*

We briefly discuss the (first) Heisenberg group $\mathbb{H}$, which is a nilpotent Lie group of step 2 and in particular, a topological 3-manifold. Though the same results apply to all step-2 Carnot groups, we restrict our discussion to this case, for ease of exposition.

When equipped with the so-called Carnot–Carathéodory metric d_{CC} induced from its Lie algebra of vector fields, $\mathbb{H}$ becomes a highly non-Euclidean metric space. In particular, recent theorems of Cheeger and Kleiner [**11**] imply that $(\mathbb{H}, d_{CC})$ admits no isometric (or even bi-Lipschitz) embedding into any Hilbert space. Their proof uses the fact that $\mathbb{H}$ satisfies a (1,1)-Poincaré inequality and therefore a Rademacher-type theorem for Lipschitz functions.

As for specific properties, topologically we have $\mathbb{H} = \mathbb{R}^3$ but the group law

$$(x, y, t) \times (u, v, w) = (x + u, y + v, t + w + \frac{1}{2}(xv - uy))$$

induces a Lie group structure on $\mathbb{H}$ with an associated nilpotent Lie algebra. For simplicity, instead of the Carnot–Carathéodory distance d_{CC} on $\mathbb{H}$, as discussed say in Montgomery's book [35], we introduce the Koranyí norm

$$N(x,y,t) = \left((x^2+y^2)^2+t^2\right)^{\frac{1}{4}},$$

which induces another distance $d(p,q) = N(q^{-1}p)$, between points $p,q \in \mathbb{H}$, that is bi-Lipschitz equivalent to d_{CC}. Moreover, $N(x,y,t) \leqslant \sqrt{\|(x,y,t)\|_2}$ if $\|(x,y,t)\|_2 \leqslant 1$.

It is known that the Haar measure on $\mathbb{H}$ is the usual Lebesgue measure λ on $\mathbb{R}^3$ and that $\mathbb{H}$ is Ahlfors 4-regular with respect to it. Somewhat surprisingly, $(\mathbb{H}, d_{CC}, \lambda)$ satisfies a $(1,p)$-Poincaré inequality. The $p=2$ case was first observed by Jerison [24]; for the optimal exponent $p=1$, see the proof of Lanconelli and Morbidelli [29]. (For more discussion about the geometry of these spaces, as well as the general theory of Carnot groups, we refer the reader to [4], [35], or [44].)

In the spirit of the prior subsection, we now show the existence of metric sponges in the Heisenberg group, so it suffices to show the existence and uniform sparsity of co- uniform domains in $\mathbb{H}$. To this end, we proceed in two steps:

(1) *Geometric preliminaries.* Recall that on $\mathbb{H}$ there are natural dilations

$$\delta_s(x,y,t) = (s^{-1}x, s^{-1}y, s^{-2}t)$$

that are also Lie group automorphisms. Moreover, for any $g \in \mathbb{H}$, the left-translation

$$L_g(x) = g \times x$$

is an isometry in both the Lie group and the metric space senses, so consider the 'conformal mappings'

$$A_{\lambda,g} = L_g \circ \delta_\lambda.$$

Now if E, Ω are fixed, bounded subsets of $\mathbb{H}$ with $C^{1,1}$-boundary, then a result of Morbidelli [37] implies that Ω and $\mathbb{H} \setminus E$ are A-uniform domains for some $A > 0$. (As an example, the Euclidean unit ball $B_{\mathrm{eucl}}(0,1)$ as a subset of $\mathbb{H}$ has boundary $\partial E = \partial B_{\mathrm{eucl}}(0,1)$ with this regularity.)

Further, since $A_{\lambda,g}$ act by an isometry and a scaling map, the domains

$$A_{\lambda,g}(\mathbb{H} \setminus E) = \mathbb{H} \setminus A_{\lambda,g}(E)$$

remain A-uniform as $\lambda \in (0,\infty)$ and $g \in \mathbb{H}$ vary.

(2) *The iterative construction.* Fix a sequence $\mathbf{n} = \{n_i\}_{i=1}^\infty$ in $\mathbb{N}$ such that $\mathbf{n}^{-1} \in \ell^4(\mathbb{N})$ and $n_i \geqslant 3$ for all $i \in \mathbb{N}$, and define scales $\{s_k\}_{k=0}^\infty$ exactly as in Definition 4.21. We will define inductively our obstacles by first choosing center points at every scale, and then choosing collections of scaled and translated copies of the Euclidean unit ball with these centers as the obstacles. (In what follows, all the metric notions will be with respect to the distance on $\mathbb{H}$ defined above.)

First, let $\Omega = \overline{B}_{\mathrm{eucl}}(0,1)$, so $\mathrm{diam}(\Omega) \leqslant 2$. Now define $G_1 = \{0\}$ and

$$\mathcal{R}_{1,\mathbf{n}} = \{A_{s_1,0}(B_{\mathrm{eucl}}(0,1))\}$$

and let $S_{1,\mathbf{n}} = \Omega \setminus B_{\mathrm{eucl}}(0,s_1)$ be the 'pre-sponge' at the first stage.

Assuming $G_k, \mathcal{R}_{k,\mathbf{n}}, S_{k,\mathbf{n}}$ have already been defined at some stage $k \in \mathbb{N}$, we next define $G_{k+1}, \mathcal{R}_{k+1,\mathbf{n}}, S_{k+1,\mathbf{n}}$ at the next stage as follows. Let G_{k+1} be a collection of points such that each $g \in G_{k+1}$ satisfies

$$d(g, \partial S_{k,\mathbf{n}}) \geqslant s_k \text{ and } d(g,g') \geqslant s_k \tag{4.27}$$

for each $g' \in G_{k+1}$. (Such a collection could be empty.) Moreover, call G_{k+1} *maximal* if no other collection of points G' satisfying (4.27) strictly contains G_{k+1}. Putting

$$\mathcal{R}_{k+1,\mathbf{n}} = \{A_{s_{k+1},g}(B_{\mathrm{eucl}}(0,1)) \mid g \in G_{k+1}\},$$

the $(k+1)$-stage pre-sponge is

$$S_{k+1,\mathbf{n}} \;=\; S_{k,\mathbf{n}} \setminus \bigcup_{R\in\mathcal{R}_{k+1,\mathbf{n}}} R \;=\; \Omega \setminus \bigcup_{l=1}^{k+1} \bigcup_{R\in\mathcal{R}_{l,\mathbf{n}}} R.$$

Finally, define

$$S_{\mathbf{n}} = \bigcap_{k=1}^{\infty} S_{k,\mathbf{n}}.$$

LEMMA 4.28. *Let $\mathbf{n}, G_k, \mathcal{R}_{k,\mathbf{n}}, S_{\mathbf{n}}, A$ be as above. Then, the sets $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ in Ω form a uniformly $\mathbf{n}$-sparse collection of co-uniform subsets in Ω.*

Moreover, if each G_{k+1} is chosen to be maximal, relative to $\{G_i\}_{i=1}^{k}$, then $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ is dense in Ω and $S_{\mathbf{n}}$ has empty interior.

Proof. First, let $R_k \in \mathcal{R}_{k,\mathbf{n}}$ and $R_l \in \mathcal{R}_{l,\mathbf{n}}$ be arbitrary with $k \geqslant l$, so $R_k = A_{s_k,g_k}(B_{\mathrm{eucl}}(0,1))$ and $R_l = A_{s_l,g_l}(B_{\mathrm{eucl}}(0,1))$ for some $g_k \in G_k$ and $g_l \in G_l$.

To show the separation property, as a first case let $k > l$, so (4.27) implies that

$$d(g_k, R_l) \geqslant d(g_k, \partial S_{l,\mathbf{n}}) \geqslant d(g_k, \partial S_{k-1,\mathbf{n}}) \geqslant s_{k-1}, \tag{4.29}$$

in which case the Triangle inequality further implies

$$d(R_k, R_l) \geqslant d(g_k, R_l) - s_k \geqslant s_{k-1} - s_k \geqslant \frac{s_k}{2}.$$

As for $k = l$, applying (4.29) with $l-1 = k-1$ in place of k, as well as (4.27), yields

$$d(R_k, R_l) \geqslant d(g_k, g_l) - d(g_k, \partial R_k) - d(g_l, \partial R_l) \geqslant s_{k-1} - 2s_k \geqslant s_{k-1} - \frac{2s_{k-1}}{3} \geqslant \frac{1}{6}s_{k-1}\mathrm{diam}(\Omega).$$

Similarly if $k \geqslant l$, then (4.27) implies

$$d(R_k, \Omega^c) \geqslant d(R_k, \partial S_{k-1,\mathbf{n}}) \geqslant d(g_k, \partial S_{k-1,\mathbf{n}}) - s_k \geqslant s_{k-1} - \frac{s_{k-1}}{2} = \frac{1}{2}s_{k-1} \geqslant \frac{1}{6}s_{k-1}\mathrm{diam}(\Omega),$$

so $\delta = \frac{1}{6}$ yields the desired separation. Moreover, $\mathrm{diam}(R_k) \leqslant 2s_k$ follows from construction, so the diameter bound follows with $L = 2$.

As in (1) before the statement of the Lemma, each R_k has $C^{1,1}$-boundary, so each $X \setminus R_k$ is A-uniform with A independent of k; the same is true of Ω. It follows that the collection $\{\mathcal{R}_{\mathbf{n},k}\}_{k=1}^{\infty}$ is uniformly $\mathbf{n}$-sparse.

As for density, let $x \in \Omega$ be arbitrary, let $r \in (0, \frac{1}{3}s_1)$, and choose $k \geqslant 1$ so that

$$s_{k+1} < r \leqslant s_k.$$

Now, $B_{\mathrm{eucl}}(x, s_{k+1})$ and hence $B_{\mathrm{eucl}}(x, r)$ must intersect some $R_l \in \mathcal{R}_{l,\mathbf{n}}$ for some $l \leqslant k+2$, otherwise $G_{k+2} \cup \{x\}$ would form a larger collection of points satisfying the desired separation bounds; this, however, would contradict maximality of G_{k+2}.

Finally, we can apply Lemma 4.28 and Theorem 1.9 to conclude the following result.

COROLLARY 4.30. *Let $G_k, n_k, \mathcal{R}_{k,\mathbf{n}}, S_{\mathbf{n}}, \Omega, A$ be defined as above. Then $S_{\mathbf{n}}$ is a compact subset of $\mathbb{H}$ which has empty interior, is Ahlfors 4-regular, and satisfies a $(1,p)$-Poincaré inequality for any $p > 1$.*

In conclusion, we note that the above construction applies to all step-2 Carnot groups, such as higher dimensional Heisenberg groups, or for that matter, any Carnot group where uniform domains exist at all scales and locations. Moreover, replacing the left-translations L_g with Euclidean translations $x \mapsto x + g$ and the anisotropic dilations δ_s with Euclidean dilations, the analogous construction still works for Euclidean spaces $\mathbb{R}^d$. In this case, this gives new examples of Sierpiński carpets and sponges supporting Poincaré inequalities, where the complementary domains are self-similar copies of E, with $\mathbb{R}^d \setminus E$ uniform.

Corollary 4.31. *Let $d \in \mathbb{N}$ with $d \geqslant 2$, let Ω be a uniform domain in $\mathbb{R}^d$, and let E be a bounded open subset of Ω that is co-uniform in $\mathbb{R}^d$ with $0 \in E$ and $\operatorname{diam}(E) \leqslant 1$. Given a sequence $\mathbf{n} = (n_i)_{i=1}^{\infty}$ in $\mathbb{N}$ with each $n_i \geqslant 3$ and with $\mathbf{n}^{-1} \in \ell^d(\mathbb{N})$, if $\{G_k\}_{k=1}^{\infty}$ is a sequence of uniformly $\mathbf{n}$-sparse collections of points in Ω, defined analogously as above, then the set*

$$S = \Omega \setminus \bigcup_{k=1}^{\infty} \bigcup_{g \in G_k} (s_k E + g)$$

is Ahlfors d-regular and satisfies a $(1,p)$-Poincaré inequality for each $p > 1$. Moreover, S can be chosen to have empty interior.

4.4. *The problem of classifying Loewner carpets*

The previous subsections gave a general construction for 'sponges' that satisfy Poincaré inequalities, including on Euclidean spaces.

By varying the choice for subsets E in Corollary 4.31, we obtain many new possibilities beyond those in [**31**]. Instead of symmetry considerations, it is enough to impose regularity and sparsity conditions on E. For example, permissible subsets include E convex, E with connected and smooth boundary, or E any quasi-ball — that is, $E = f(B(0,1))$ where $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is any quasi-conformal map. Moreover, rescaled translates $s_k E + g$ of a single subset E can be replaced by collections of uniformly co-uniform subsets $\{E_{gk}\}$, provided that each E_{gk} contains the origin and has at most unit diameter.

Motivated by Corollary 4.31, we return to the planar case and study whether such examples of carpets are generic. In this context, we can make stronger conclusions.

We begin with the following theorem from [**45**], which gives topological criteria for carpets. Recall that a point x on a connected metric space X is called a cut point if $X \setminus \{x\}$ is disconnected and it is called a local cut point if there exists $r > 0$ so that x is a cut point of $B(x,r)$. Also, S_3 will be the usual 1/3-Sierpiński carpet, which in our notation from the introduction corresponds with $S_{\mathbf{n}}$ with $\mathbf{n} = (1/3, 1/3, \dots)$.

Theorem 4.32 (Whyburn). *Let S be a compact, connected, and locally connected subset of $\mathbb{R}^2$ with empty interior. If S has no cut points, then it is homeomorphic to S_3.*

In what follows we refer to such sets S as topological carpets, which must satisfy

$$\mathbb{R}^2 \setminus S = D_0 \cup \bigcup_{i=1}^{\infty} D_i,$$

where $\{D_i\}_{i=0}^{\infty}$ is a dense collection of open, pairwise-disjoint Jordan domains, with D_i bounded for $i \geqslant 1$ and with D_0 unbounded. (To be clear, a connected open subset $D \subset \mathbb{R}^2$ is called a Jordan domain if ∂D coincides with a Jordan curve.)

In fact, the Loewner condition for planar carpets implies being a topological carpet. Formulated below as Corollary 4.34, it is an easy consequence of the following result [**23**, Theorem 3.3].

THEOREM 4.33 (Heinonen–Koskela). *Let S be a Ahlfors Q-regular metric measure space that satisfies a $(1,Q)$-Poincaré inequality. Then, there is a constant $C \geqslant 1$ such that it is C-quasi-convex as well as C-annularly quasi-convex, that is for every $z \in S$ and any $r > 0$, if $x, y \in S \setminus B(z,r)$, then there exists a curve γ in $X \setminus B(z, r/C)$ connecting x to y with* $\mathrm{Len}(\gamma) \leqslant Cd(x,y)$.

COROLLARY 4.34. *If a compact subset S of $\mathbb{R}^2$ is Loewner — that is, it satisfies a (1,2)-Poincaré inequality and is Ahlfors 2-regular — and has empty interior, then S is a topological carpet.*

Proof. It is well known from [**10, 40**] that p-PI spaces are quasi-convex, and are therefore both connected and locally connected. Moreover, Loewner spaces lack local cut points, by Theorem 4.33. Thus the conditions of Theorem 4.32 are met, and we know that S is a topological carpet. □

This motivates the following definition.

DEFINITION 4.35. A compact subset $S \subset \mathbb{R}^n$ is called a p-POINCARÉ SPONGE if it has empty interior, is Ahlfors n-regular, and satisfies a $(1,p)$-Poincaré inequality. If $n = 2$, then S is also called a p-POINCARÉ CARPET.

In particular, if $n \geqslant 3$ and $p \leqslant n$, then S is called a LOEWNER SPONGE. Also, if instead $p \leqslant n = 2$, then S is called a LOEWNER CARPET.

It is now natural to reformulate the Planar Loewner problem (Question 1.6):

QUESTION 4.36. Can one classify Loewner carpets, or even p-Poincaré carpets, in terms of the construction from Corollary 4.31 ?

There are few techniques available to treat the case of sponges in dimensions $d \geqslant 3$, but for $d = 2$ techniques such as uniformization (see, for example, [**8**]) provide more possibilities for carpets.

In this subsection, we give a partial answer to Question 4.36. In particular, we give sufficient conditions for a topological carpet to be a p-Poincaré carpet, or even Loewner. In fact, two of these conditions are also necessary.

To formulate our result, we proceed with a well-known characterization of quasi-disks (that is, quasi-balls in dimension $d = 2$) from the literature [**5, 41**]. This first requires a few geometric definitions. A Jordan curve $\gamma\colon S^1 \to \mathbb{R}^2$ is of C-BOUNDED TURNING, for some $C \geqslant 1$, if for every $s, t \in S^1$ it holds that

$$\min\{\mathrm{diam}(\gamma(J_1)), \mathrm{diam}(\gamma(J_2))\} \leqslant Cd(\gamma(s), \gamma(t)), \tag{4.37}$$

where J_1, J_2 are the two open arcs in S^1 that satisfy $J_1 \cup J_2 = S^1 \setminus \{s,t\}$.

A Jordan curve $\gamma\colon S^1 \to \mathbb{R}^2$ is called a η-QUASI-CIRCLE, if there exists $\gamma'\colon S^1 \to \mathbb{R}^2$ with the same image as γ, and which is η-quasi-symmetric, as given in Item (4) of Remark 4.16. A QUASI-DISK is a domain of the form $D = f(B(0,1))$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasi-symmetric.

THEOREM 4.38 (Beurling–Ahlfors). *A bounded Jordan domain D is a quasi-disk if and only if ∂D is a quasi-circle.*

THEOREM 4.39 (Tukia–Väisälä). *A Jordan curve γ is a quasi-circle if and only if it of bounded turning.*

Now recall the notion of relative distance from item (2) of Theorem 1.8: a collection of sets $\mathcal{R}$ is called UNIFORMLY RELATIVELY s-SEPARATED if $\Delta(E,F) \geqslant s$ for every disjoint pair $E, F \in \mathcal{R}$.

THEOREM 4.40. *If S is a Loewner carpet, then there are countably many pairwise disjoint, Jordan domains D_i, Ω such that*

$$S = \Omega \setminus \bigcup_{i=1}^{\infty} D_i.$$

and where each ∂D_i and $\partial\Omega$ form a uniformly relatively s-separated collection of uniformly η-quasi-circles for some $s > 0$ and some distortion function $\eta : [0,\infty) \to [0,\infty)$.

Proof. As S is closed, we decompose the complement into open components

$$\mathbb{R}^2 \setminus S = \bigcup_{i=0}^{\infty} D_i$$

where at most one component, say D_0, is unbounded. Define $\Omega = \mathbb{R}^2 \setminus D_0$. Since S is Loewner, by [**23**, Theorem 3.3], it lacks local cut points. Further, by Theorem 4.33 we obtain that S is C-quasi-convex and C-annularly quasi-convex, with $C \geqslant 1$. It then follows from Theorems 4.32 and 4.33 that the D_i are Jordan domains with pairwise disjoint closures.

Put $C_b = 2C^2 + 1$. We now show that each ∂D_i is of C_b-bounded turning, for all $i \in \mathbb{N}$. (For $i = 0$, the argument is similar and we omit it here.)

Let $\gamma\colon S^1 \to \partial D_i$ be a parametrization of the boundary as a Jordan curve. Let $s, t \in S^1$ be arbitrary and distinct and let J_1, J_2 be the arcs in S^1 defined by these points. Now, if $\gamma(J_1)$ or $\gamma(J_2)$ is contained in the ball $B(\gamma(s), C_b R_{st})$, where

$$R_{s,t} = |\gamma(s) - \gamma(t)|,$$

then (4.37) clearly follows. So assume instead that

$$\gamma(J_j) \nsubseteq B(\gamma(s), C_b R_{s,t})$$

for both $j = 1, 2$, so there are points $x_j \in \gamma(J_j) \setminus B(\gamma(s), 2C^2 R_{s,t})$ for both $j = 1, 2$.

Since S is C-quasi-convex, there is a rectifiable curve σ_S joining $\gamma(s)$ and $\gamma(t)$ of length at most $CR_{s,t}$ within S. It is well known, say by Moore's work [**36**, Theorem 1], that there exists a simple subcurve σ'_L in σ_S that also joins $\gamma(s)$ and $\gamma(t)$. Also, since D_i is a Jordan domain, there is a simple curve σ_D joining $\gamma(s)$ and $\gamma(t)$ while intersecting ∂D_i only at those two points. Form the Jordan curve σ by concatenating the two simple arcs σ'_L and σ_D. Since $\sigma \subset D_i \cup B(\gamma(s), CR_{st})$, we know that $x_1, x_2 \notin \sigma$.

The curve σ divides $\mathbb{R}^2$ into two components U, V so that $\partial U = \sigma = \partial V$. Since D_i is an open set containing a point of ∂U and ∂V, we must have that D_i intersects both U and V. However, since D_i is Jordan, every point in $D_i \setminus \sigma$ can be connected either to x_1 or x_2 while avoiding σ. Now, if $x_1, x_2 \in U$, then every point of $D_i \setminus \sigma$ would belong to U, which is not possible. Similarly for V, and thus x_i must lie in separate components of $\mathbb{R}^2 \setminus \sigma$, that is, one belongs to U and another to V. In particular, σ separates the points x_1, x_2.

However, $x_j \in S$, and by annular quasi-convexity there exists a curve connecting x_1 and x_2, within S and contained in $\mathbb{R}^2 \setminus B(\gamma(s), 2CR_{s,t})$ and thus avoiding σ. Thus x_1 and x_2 belong to the same component of $\mathbb{R}^2 \setminus \sigma$, which is a contradiction.

We now show uniform s-separation for $s = \frac{1}{2^4 C^2 + 2}$; that is, for all D_i, D_j with $D_i \neq D_j$ that

$$d(D_i, D_j) \geqslant s \min\{\operatorname{diam}(D_i), \operatorname{diam}(D_j)\}. \tag{4.41}$$

Supposing otherwise, there would exist a pair, say D_i, D_j, where (4.41) fails. Choose a pair of points $a \in \partial D_i, b \in \partial D_j$ with $|a-b| = d(D_i, D_j)$. Next, let ℓ be the line segment joining a and b, which is contained in $\mathbb{R}^2 \setminus (D_i \cup D_j)$. Choose two points $x_1 \in D_i, x_2 \in D_j$ with

$$d(x_1, a) \geqslant \operatorname{diam}(D_i)/2 \geqslant 8C^2 d(D_i, D_j) \text{ and } d(x_2, b) \geqslant 8C^2 d(D_i, D_j).$$

The points x_1, a divide ∂D_i into two arcs J_1, J_2. Next, since J_i are connected, we can find points $s_i \in J_i$ with $d(s_i, a) = 2Cd(D_i, D_j)$. Thus $d(s_1, s_2) \leqslant 4Cd(D_i, D_j)$. By the annular quasi-convexity condition, and combined with [**36**, Theorem 1], we can find a curve σ_L connecting s_1 to s_2 within $B(a, 4C^2 d(D_i, D_j)) \setminus B(a, 2d(D_j, D_j))$. Again find a curve σ_D within D_i connecting s_i, and form the Jordan curve σ by concatenation of σ_L and σ_D. As above, this curve will separate x_1 and a. However, since σ cannot intersect ℓ, and x_2 can be connected to ℓ while lying strictly within D_j, we see that x_2 lies in the same component defined by σ as a. Hence, x_2 lies in a different component of $\mathbb{R}^2 \setminus \sigma$ than x_1. But this contradicts the annular quasi-convexity condition, just as before. □

The assumptions of uniform separation and uniform quasi-disks have appeared before in [**8**, Theorem 1.1].

THEOREM 4.42 (Bonk). *If $S = \Omega \setminus \bigcup_{i \in I} D_i$, where D_i and Ω, for $i \in I$ are an at most countable collection of uniformly η-quasi-disks, with $\{\partial\Omega\} \cup \{\partial D_i\}_i$ uniformly relatively separated, then there exists a quasi-symmetry $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, such that*

$$f(S) = B(0,1) \setminus \bigcup_{i \in I} B(x_i, r_i).$$

In other words, every such set S is quasi-symmetric to a similar set with circle boundaries. One can also find quasi-symmetric maps with images with square boundaries, or any other self-similar shapes. The proof follows from identical arguments to [**8**, Theorem 1.6].

As a corollary, we obtain a result, which is known to many specialists.

COROLLARY 4.43. *If S is a Loewner carpet, then there exist quasi-symmetries $f : S \to S'$ and $g\colon S \to S''$ so that*

$$S' = B(0,1) \setminus \bigcup_{i \in I} B(x_i, r_i) \text{ and } S'' = [0,1]^2 \setminus \bigcup_{i \in I} Q_i,$$

where $\{\bar{B}(x_i, r_i)\}_{i \in I}$ is a pairwise disjoint collection of disks in $B(0,1)$ and $\{Q_i\}_{i \in I}$ is a collection of open squares in $[0,1]^2$ with pairwise disjoint closures.

This reduces the classification of Loewner carpets to the problem of classifying square carpets. As of now, though, no such classification exists, even with such explicit boundaries. However, we give instead a sufficient condition in terms of an assumption on density. Let $\mathcal{R} := \{D_i\}_{i \in I}$ be a countable collection of connected open sets in $\mathbb{R}^2$, consider the indices of those sets near a fixed ball, denoted as

$$I(x,r) := \{i \in I : D_i \cap B(x,r) \neq \emptyset\}, \tag{4.44}$$

and for $N \in \mathbb{N}$, consider a variant of the 'N-fold density function' from (1.7), given as

$$s_N(x,r) := \inf \left\{ \sum_{i \in I(x,r) \setminus J} \frac{\lambda(D_i)}{r^2} : J \subset I, |J| \leqslant N \right\}. \tag{4.45}$$

Note that if D_i are uniform quasi-disks, then $\operatorname{diam}(D_i)^2 \sim \lambda(D_i)$.

The following is a more quantitative version of Theorem 1.8, which can be considered its corollary.

THEOREM 4.46. *Let Ω, D_i, for $i \in I$, be a countable collection of uniform η-quasi-disks such that $D_i \subset \Omega$ and that $\{\partial\Omega\} \cup \{\partial D_i\}_i$ are uniformly relatively s-separated. Fix $N \in \mathbb{N}$. For every $p \in (1, \infty)$, there exists $\epsilon_{p,N} > 0$, depending on s, η, such that if*

$$\limsup_{r\to 0} \sup_{x\in X} s_N(x, r) < \epsilon_{p,N},$$

then $S = \Omega \setminus \bigcup_{i\in I} D_i$ is a p-Poincaré carpet. In particular, if there exists $N \in \mathbb{N}$ such that

$$\lim_{r\to 0} \sup_{x\in X} s_N(x, r) = 0,$$

then S is a Loewner carpet.

We remark, that for self-similar Sierpiński carpets $S_{\mathbf{n}}$ it follows from the proof in Theorem 1.5 that

$$\lim_{r\to 0} \sup_{x\in X} s_1(x, r) = 0.$$

Proof. It is sufficient to show the first claim.

Firstly, as a consequence of Theorem 4.38, the set $\mathbb{R}^2 \setminus D_i$ is a quasi-symmetric image of $\mathbb{R}^2 \setminus B(0, 1)$. Then, since uniformity is preserved under quasi-symmetries [**32**], we see that the D_i are co-uniform domains in the sense of Definition 4.20 with the same uniform constant. Similarly, the D_i are all uniform domains and there is a constant C_d, independent of i, so that $\operatorname{diam}(D_i)^2 \leqslant C_d\lambda(D_i)$. Similarly Ω is a uniform domain. Let $D \leqslant 9$ be the metric doubling constant of $\mathbb{R}^2$.

Now fix N and define for any subset $J \subset I$ the set

$$\Omega_J := \Omega \setminus \bigcup_{i\in J} D_i.$$

By Corollary 4.23, each Ω_J, with $|J| \leqslant D^8 N$, is an A-uniform domain with constant A depending only on N, s, η and in particular, independent of J, so by Lemma 4.24 it is also Ahlfors 2-regular with constant C_λ depending only on N, s, η.

With A, C_λ, and C_d now fixed, let $\epsilon > 0$ be the constant from Corollary 4.19 such that any (ϵ, A)-almost uniform subset of $\mathbb{R}^2$ necessarily satisfies a $(1, p)$-Poincaré inequality. Define

$$\epsilon_{p,N} = 2^{-3}A^{-2}C_\lambda^{-1}C_d^{-1}D^{-8}\epsilon. \tag{4.47}$$

Now, by assumption there exists $r_0 > 0$ such that

$$\sup_{x\in X} s_N(x, 2Ar) < \epsilon_{p,N}$$

for all $r \in (0, r_0)$. Fix such an $r \in (0, r_0)$.

To construct the filling, take an Ar-net[†] $\mathcal{N} = \{x_i\}$ of S and define a covering of S by balls $\mathcal{B} = \{B(x_i, 2Ar)\}$. By the D-metric doubling condition, for $x \in S$, each $B(x, 2^4Ar)$ intersects at most D^8 many balls in $\mathcal{B}$. Let $\mathcal{N}_{x,r}$ be the collection of the indices i so that $B(x, 2^4Ar) \cap B(x_i, 2Ar)$ is not empty. In other words, we have $|\mathcal{N}_{x,r}| \leqslant D^8$ for any $x \in S$.

[†]A set $\mathcal{N}$ is a ϵ-NET, if it is maximal subject to the condition that for each $x_i, x_j \in \mathcal{N}$ distinct it holds that $d(x_i, x_j) \geqslant \epsilon$.

Now for each $B(x_i, 2Ar) \in \mathcal{B}$, let $I(x_i, 2Ar)$ be the set of indices as in (4.44), and choose a subset $J_i \subset I(x_i, 2Ar)$ with $|J_i| \leqslant N$ so that

$$\sum_{j \in I(x_i,2Ar)\setminus J_i} \frac{\lambda(D_j)}{(2Ar)^2} = s_N(x_i, 2Ar) < \epsilon_{p,N}.$$

By choice of $\epsilon_{p,N}$, we have that if $j \in I(x_i, 2Ar) \setminus J_i$, then $\operatorname{diam}(D_j) \leqslant r$, as otherwise

$$\lambda(D_j) \geqslant C_d^{-1} \operatorname{diam}(D_j)^2 \geqslant (2^3 A^2 \epsilon_{p,N}) r^2$$

would be a contradiction. In particular, if D_j is such that $D_j \cap B(x_i, 2Ar) \neq \emptyset$ and $\operatorname{diam}(D_j) \geqslant r$, then $j \in J_i$.

Now let $\mathcal{J} = \bigcup_{x_i \in \mathcal{N}} J_i$, and define $\Omega_r := \Omega_{\mathcal{J}} = \Omega \setminus \bigcup_{i \in \mathcal{J}} D_i$. We will show that Ω_r is our desired filling.

We first show the local uniformity at scale $4r$. Take $x, y \in \Omega_r$ with $d(x, y) \leqslant 4r$. Define

$$J = \bigcup_{i \in \mathcal{N}_{x,r}} J_i.$$

Since $|\mathcal{N}_{x,r}| \leqslant D^8$, we have $|J| \leqslant D^8 N$. Consider now some $j \in \mathcal{J}$ with $D_j \cap B(x, 8Ar) \neq \emptyset$. If $\operatorname{diam}(D_j) \geqslant r$, then we have an i so that $B(x_i, 2Ar) \cap D_j \cap B(x, 2^4 Ar) \neq \emptyset$ and we must have $i \in J_i \subset J$ by the choice of $\epsilon_{p,N}$ and the previous two paragraphs. If instead $\operatorname{diam}(D_j) \leqslant r$, we can take any $B(x_i, 2Ar)$ which intersects D_j and thus $B(x, 2^4 Ar)$ with $j \in J_i \subset J$. Either way, any $j \in \mathcal{J}$ such that $D_j \cap B(x, 8Ar) \neq \emptyset$ will satisfy $j \in J$. It follows that, for each $\rho \in (0, 8Ar]$,

$$\Omega_r \cap B(x, \rho) = \Omega_{\mathcal{J}} \cap B(x, \rho) = \Omega_J \cap B(x, \rho).$$

Since Ω_J is A-uniform, we have that x, y can be connected by an A-uniform curve within $\mathrm{g}\Omega_J$, which will also automatically be an A-uniform curve within Ω_r. Similarly, we obtain that Ω_r is Ahlfors 2-regular with constant C_λ up to scale $2r$.

Next, we show the desired density bound. We have that

$$\Omega_r \setminus S \cap B(x, r) = \Omega_J \setminus S \cap B(x, r) \subset \bigcup_{i \in \mathcal{N}_{x,r}} \bigcup_{j \in I(x_i,2Ar)\setminus J_i} D_j. \tag{4.48}$$

Then the choice in equation (4.47), Inclusion (4.48) and Ahlfors regularity of Ω_J lead to

$$\begin{aligned}\frac{\lambda(\Omega_r \setminus S \cap B(x,r))}{\lambda(B(x,r) \cap \Omega_r)} = \frac{\lambda(\Omega_J \setminus S \cap B(x,r))}{\lambda(B(x,r) \cap \Omega_J)} &\leqslant \frac{\displaystyle\sum_{i \in \mathcal{N}_{x,r}} \sum_{j \in I(x_i,2Ar)\setminus J_i} \lambda(D_i)}{\frac{1}{C_\lambda} r^2} \\ &= 4A^2 C_\lambda \sum_{i \in \mathcal{N}_{x,r}} s_N(x_i, 2Ar) \ < \ 8A^2 D^8 C_\lambda \epsilon_{p,N} \ < \ \epsilon,\end{aligned}$$

which is the desired density condition; the Poincaré inequality follows. □

5. *General Poincaré results*

We begin with some basic definitions. In what follows, $X = (X, d)$ always refers to a metric space.

Definition 5.1. A Lipschitz map $\gamma\colon K \to X$ from a compact subset K of $\mathbb{R}$ is called a curve fragment in X. The domain K is also denoted by $\operatorname{Dom}(\gamma)$.

Length for curve fragments is defined analogously as for curves, that is

$$\mathrm{Len}(\gamma) := \sup_{n\in\mathbb{N}} \sup_{t_1,\dots t_n \in K} \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where we further assume $t_i \leqslant t_j$ for $i \leqslant j$. Furthermore, the set

$$\mathrm{Undef}(\gamma) = (\min(K), \max(K)) \setminus K,$$

is always a countable union of disjoint open intervals, called GAPS, as follows:

$$\mathrm{Undef}(\gamma) = \bigcup_i (a_i, b_i). \tag{5.2}$$

From this, we define the TOTAL GAP SIZE as

$$\mathrm{Gap}(\gamma) := \sum_i d(\gamma(a_i), \gamma(b_i)).$$

The PATH INTEGRAL of a Lipschitz function $f : X \to \mathbb{R}$ over a curve fragment γ is canonically defined as

$$\int_\gamma f \, ds = \int_K f(\gamma(t)) d_\gamma(t) \, dt,$$

where $d_\gamma(t)$ is the metric derivative of γ, that is,

$$d_\gamma(t) := \lim_{h\to 0} \frac{d(\gamma(t), \gamma(t+h))}{h},$$

which exists for almost every $t \in K$. This coincides with the definition of Ambrosio [2] for curves, when first embedding the metric space X into a Banach space, such as L^∞, and filling in the gaps of γ with line segments to construct a curve. This enlarged curve has a well-defined metric derivative and integral, and the ones for curve fragments are obtained by restriction. For a similar discussion, see [3, 14].

We will employ the proof of the characterization of (global) Poincaré inequalities from [23, Lemma 5.1], in order to prove new characterizations.

DEFINITION 5.3. Let $1 \leqslant p < \infty$. A proper metric measure space (X, d, μ) is said to satisfy a POINTWISE $(1,p)$-POINCARÉ INEQUALITY at scale $r_0 > 0$ with constant $C \geqslant 1$, if for all locally Lipschitz functions $f : X \to \mathbb{R}$ and all $x, y \in X$ with $r := d(x,y) \in (0, r_0)$, we have

$$|f(x) - f(y)| \leqslant Cr\Big(M_{Cr}\mathrm{Lip}\,[f]^p(x)^{\frac{1}{p}} + M_{Cr}\mathrm{Lip}\,[f]^p(y)^{\frac{1}{p}}\Big). \tag{5.4}$$

By [23, Lemma 5.15], this is equivalent to a Poincaré inequality. The proof in [23] covers global Poincaré inequalities, but the same argument applies to the local version as well. For completeness, we state the result and show the modifications, which only involve tracking the scales of the balls/pairs of points used.

THEOREM 5.5. *Let $D \geqslant 1$. For a proper space X, the following conditions are equivalent.*

(1) *X is (D, r_0)-doubling and satisfies a $(1,p)$-Poincaré inequality with constant $C_1 \geqslant 1$ at some scale $r_0 > 0$.*

(2) *X is (D, r_2)-doubling and satisfies a $(1,p)$-pointwise Poincaré inequality with constant $C_2 \geqslant 1$ at scale $r_2 > 0$.*

Here, the constants in Items (1) *and* (2) *depend quantitatively on one another, with* $r_2 = r_0/2$ *when going from* (1) $\Longrightarrow$ (2) *and* $r_0 = r_2/(2C_2)$ *when going* (2) $\Longrightarrow$ (1). *Also, in either direction,*

$$\frac{1}{C} \leqslant \frac{C_1}{C_2} \leqslant C$$

for some universal constant $C = C(D, p)$.

Proof. Assume throughout that f is an arbitrary Lipschitz function.

We first prove (1) $\Rightarrow$ (2). Choose $r_2 = r_0/2$ and let $x, y \in X$ satisfy $r := d(x, y) < r_2$. Consider balls $B_i = B(x, 2^{1+i}r)$ for $i \leqslant 0$ and $B_i = B(y, 2^{1-i}r)$ for $i > 0$, all of which have radius less than r_0 and thus the local Poincaré inequality can be applied to them. Then for $i \leqslant -1$, we obtain $B_{i+1} = 2B_i$, as well as

$$\begin{aligned} |f_{B_i} - f_{B_{i+1}}| &\leqslant \fint_{B_i} |f - f_{B_{i+1}}| \, d\mu \\ &\leqslant D^2 \fint_{B_{i+1}} |f - f_{B_{i+1}}| \, d\mu \leqslant D^2 C_1 2^{2+i} r \left(\fint_{C_1 B_{i+1}} \mathrm{Lip}\,[f]^p \, d\mu \right)^{\frac{1}{p}} \end{aligned}$$

while for $i \geqslant 0$, we have $B_{i+1} \subset B_i \subset 4B_{i+1}$ and

$$\begin{aligned} |f_{B_i} - f_{B_{i+1}}| &\leqslant \fint_{B_{i+1}} |f - f_{B_i}| \, d\mu \\ &\leqslant \frac{\mu(B_i)}{\mu(B_{i+1})} \fint_{B_i} |f - f_{B_i}| \, d\mu \leqslant D^2 C_1 2^{1-i} r \left(\fint_{C_1 B_i} \mathrm{Lip}\,[f]^p \, d\mu \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, we get by a telescoping sum argument that

$$|f(x) - f(y)| \leqslant \sum_{i \in \mathbb{Z}} |f_{B_i} - f_{B_{i+1}}| \leqslant 4D^2 C_1 r \Big(M_{2C_1 r}(\mathrm{Lip}\,[f](x)^p)^{\frac{1}{p}} + M_{2C_1 r}(\mathrm{Lip}\,[f](y)^p)^{\frac{1}{p}} \Big).$$

Next, we prove (2) $\Rightarrow$ (1). Let $r_0 = r_2/(2C_2)$ and fix $B = B(x, r)$ with $r < r_0$. By subtracting the median from f, we can assume that

$$\min\left(\mu(\{f \leqslant 0\} \cap B), \mu(\{f \geqslant 0\} \cap B)\right) \geqslant \frac{1}{2}\mu(B).$$

Now define $E_k^{\pm} = \{\pm f \geqslant 2^k\} \cap B$. We first prove a weak type bound using a covering argument. Now if $z \in E_k^{\pm}$ and $y \in \{\pm f \leqslant 0\} \cap B$, then

$$d(z, y) \leqslant 2r < 2r_0 < r_2,$$

so by the pointwise Poincaré inequality, there exist $w \in X$ and $r_w \leqslant C_2 r$ such that

$$\fint_{B(w, r_w)} \mathrm{Lip}\,[f]^p \, d\mu \geqslant \frac{2^{kp-1}}{r^p C_2^p}, \tag{5.6}$$

and either $z \in B(w, r_w)$ or $y \in B(w, r_w)$.

Suppose first that $r_w \leqslant r_0/8$ for each w so arising. Now by an easy argument such as in [**23**, Lemma 5.1], the collection of balls $B(w, r_w)$ cover either $E_k^{\pm}$ or $\{\pm f \leqslant 0\} \cap B$. In the latter case then we get a cover of $\{\pm f \leqslant 0\} \cap B$, and thus using the 5B-Covering Lemma [**33**] (since we have doubling at scale $2r_0$), we get

$$\mu(E_k^{\pm}) \leqslant \frac{1}{2} \leqslant \mu(\{\pm f \leqslant 0\} \cap B) \leqslant \frac{D^{3p} C_2^p r^p}{2^{kp-1}} \int_{2C_2 B} \mathrm{Lip}\,[f](x)^p \, d\mu. \tag{5.7}$$

In the case that they cover $E_k^{\pm}$, we obtain the same estimate by covering $E_k^{\pm}$ directly.

If instead $r_w > r_0 2^{-3}$ for some w, then the claim follows easily from doubling and using a single ball. By applying Maz'ya's trick, that is, applying the above argument with the truncated function

$$u_k^{\pm}(x) = \pm(\min(\max(\pm f, 2^{k-1}), 2^k) - 2^{k-1})$$

in place of f and at level 2^{k-1} in place of 2^k, and since

$$\text{Lip } u_k^{\pm} = 1_{E_{k-1}^{\pm} \setminus E_k^{\pm}} \text{Lip } f$$

almost everywhere (see, for example, [**3**, Lemma 2.6]), then analogously as (5.7) we obtain

$$\mu(E_k^{\pm}) \leqslant \frac{2^{p+1} D^{3p} C_2^p r^p}{2^{kp}} \int_{2C_2 B \cap (E_{k-1}^{\pm} \setminus E_k^{\pm})} \text{Lip } [f](x)^p \, d\mu, \tag{5.8}$$

which when multiplied by 2^{kp} and summed over k gives

$$\fint_B |f|^p \, d\mu \leqslant 2^{p+1} D^{3p} C_2^p r^p \frac{\mu(2C_2 B)}{\mu(B)} \fint_{2C_2 B} \text{Lip } [f](x)^p \, d\mu. \tag{5.9}$$

Then, via Hölder's inequality, doubling and the triangle inequality, we obtain

$$\begin{aligned} \fint_B |f - f_B| \, d\mu &\leqslant 2 \fint_B |f| \, d\mu \\ &\leqslant 2 \left(\fint_B |f|^p \, d\mu \right)^{\frac{1}{p}} \leqslant 2^3 D^{5+\log_2(C_2)} C_2 r \left(\fint_{2C_2 B} \text{Lip } [f](x)^p \, d\mu \right)^{\frac{1}{p}}, \end{aligned}$$

which concludes the proof. □

The proofs of Theorems 2.18 and 2.19 can be more succinctly formulated with a certain function that measures the connectivity of a space by rectifiable curves. Let $p \in [1, \infty)$ be fixed. Since we consider a local notion of connectivity, we include the scale $r_0 > 0$ used.

First define $\Gamma_{x,y}(L)$ to be the set of Lipschitz curve fragments connecting x to y and with length at most $Ld(x, y)$, let $LSC_{0,1}(X)$ be the collection of lower semi-continuous functions from X to [0,1], and let $\mathcal{E}^p_{x,y,C}(\tau)$ be the class of τ-admissible functions

$$\mathcal{E}^p_{x,y,C}(\tau) := \{g \in LSC_{0,1}(X) \mid \left(M_{Cd(x,y)} g^p(x)\right)^{\frac{1}{p}} < \tau, \left(M_{Cd(x,y)} g^p(y)\right)^{\frac{1}{p}} < \tau\}.$$

Finally, define the connectivity function as follows:

$$\alpha^p_{r_0,C}(L, \tau) := \sup_{x \in X} \sup_{y \in \bar{B}(x, r_0)} \sup_{g \in \mathcal{E}^p_{x,y,C}(\tau)} \inf_{\gamma \in \Gamma_{x,y}(L)} \frac{\int_\gamma g \, ds + \text{Gap}(\gamma)}{d(x, y)}.$$

Clearly $\alpha^p_{r_0,C}(L, \tau) \leqslant 1$ always holds, since the trivial curve fragment $\gamma \colon \{0, d(x, y)\} \to X$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$ attains the bound 1. For every $c \geqslant 1$, it is also clear that

$$\alpha^p_{r_0,C}(L, c\tau) \leqslant c\alpha^p_{r_0,C}(L, \tau), \tag{5.10}$$

whereas nontrivial consequences occur for X when (5.10) holds for all $c > 0$.

Lemma 5.11. *Let $1 \leqslant p < \infty$, let $D \geqslant 1$, let $r_0 > 0$, and let X be a (D, r_0)-doubling metric measure space. If for some $C, C', L \geqslant 1$ with $C \leqslant 2C'$, we have*

$$\alpha^p_{r_0,C}(L, \tau) \leqslant C'\tau$$

for all $\tau \in (0, 1]$, then X satisfies a pointwise $(1, p)$-Poincaré inequality with constant $2C'$ at scale r_0, and moreover a $(1, p)$-Poincaré inequality at scale $r_0/(2C')$.

Proof. Let $x, y \in X$ with $r := d(x,y) \in (0, r_0)$ be arbitrary and let $f : X \to \mathbb{R}$ be any Lipschitz function. By scale invariance of the Poincaré inequality, it suffices to assume that f is 1/2-Lipschitz, so by defining

$$\tau := \max\left((M_{Cr}(\text{Lip } f)^p(x))^{\frac{1}{p}}, (M_{Cr}(\text{Lip } f)^p(y))^{\frac{1}{p}}\right) \leqslant \frac{1}{2},$$

then, by a variant of the Vitali–Caratheodory theorem (see [**16**, Lemma 2.5] for details) for any small $\epsilon \in (0, \frac{1}{2})$, there exists a lower semi-continuous $g : X \to \mathbb{R}$ so that Lip $f \leqslant g < 1$ (except possibly at x, y) and so that

$$\max\left((M_{Cr}g^p(x))^{\frac{1}{p}}, (M_{Cr}g^p(y))^{\frac{1}{p}}\right) \leqslant \tau + \epsilon \leqslant 1.$$

Since f is assumed 1/2-Lipschitz, every curve fragment $\gamma \in \Gamma_{x,y}(L)$ satisfies

$$|f(x) - f(y)| \leqslant \int_\gamma g \, ds + \text{Gap}(\gamma)$$

so by infimizing over $\gamma \in \Gamma_{x,y}(L)$, letting $\epsilon < \tau$ and by the definition of τ above, we have also

$$\begin{aligned} |f(x) - f(y)| &\leqslant r\alpha^p_{r_0,C}(L, 2\tau) \\ &\leqslant 2C'r\tau \leqslant 2C'r((M_{Cr}\text{Lip } [f]^p(x))^{\frac{1}{p}} + (M_{Cr}\text{Lip } [f]^p(y))^{\frac{1}{p}}). \end{aligned}$$

This is the desired pointwise estimate at scale r_0. Here, we use $Cr \leqslant 2C'r$, which is needed for the precise constants in our pointwise estimates[†]. Finally by Lemma 5.5, we also have a $(1,p)$-Poincaré inequality at scale $r_0/(2C')$. □

The crucial part of the proof of Theorem 2.19 is the following estimate.

LEMMA 5.12. *Let $1 \leqslant p < \infty$, let $D \geqslant 1$, and let X be a (D, r_0)-doubling metric measure space. If $\tau_0 \in (0,1)$ and $\delta \in (0, \frac{1}{2}D^{-5/p})$ are such that X is (C, δ, τ_0, p)-max connected at scale r_0, then*

$$\alpha^p_{r_1,2C}(L, \tau) \leqslant C'\tau \tag{5.13}$$

for every $\tau \in (0,1)$ and for the choice of parameters

$$L = \frac{C}{1 - \delta\tau_0^{1/p}} \text{ and } r_1 = \frac{r_0}{5C} \text{ and } C' = \frac{2D^{5/p}C}{\tau_0^{1/p}(1 - 2\delta D^{5/p})}. \tag{5.14}$$

Proof. Fix $\tau, \delta, r_1 > 0$ as in the statement, and let $\Lambda = 2D^{5/p}\tau_0^{-1/p}$. Let x, y be arbitrary with $r := d(x,y) \in (0, r_1)$, and let $g \in \mathcal{E}^p_{x,y,2C}(\tau)$. Define

$$E = \{\ z \mid (M_{Cr}g^p(z))^{\frac{1}{p}} > \Lambda\tau\ \}.$$

We first prove that E has a desired maximal function bound at x and y.

Let $s \in (0, Cr)$ be arbitrary. We first show that, for every $z \in E \cap B(x,s)$, we have

$$M_{Cr}g^p(z) \leqslant M_{2s}g^p(z). \tag{5.15}$$

[†]We remark, that one could also, alternatively, deal with two constants, that is an estimate of the form $|f(x) - f(y)| \leqslant Cd(x,y)((M_{\Lambda r}\text{Lip } [f]^p(x))^{\frac{1}{p}} + (M_{\Lambda r}\text{Lip } [f]^p(y))^{\frac{1}{p}})$, where C, Λ would be constants and not necessarily equal. As we already have many constants to keep track of, we simplify these as equal with the slightly unfortunate restriction of $C \leqslant 2C'$. However, as C' can always be made larger, this is not significant for us.

This is trivial when $2s \geqslant Cr$. Then consider $2s < Cr$; for the same reasons, the averages of g at scales $R \in (2s, Cr)$ are strictly smaller than the left-hand side of equation (5.15). Since $g \in \mathcal{E}^p_{x,y,2C}(\tau)$, for such R our choice of Λ implies

$$\fint_{B(z,R)} g^p \, d\mu \leqslant D \fint_{B(x,2R)} g^p \, d\mu \leqslant D\tau^p < \frac{\Lambda^p \tau^p}{2} < \frac{M_{Cr} g^p(z)}{2}.$$

Thus the supremum of $M_{Cr} g^p(z)$ must already be attained for radii $R \in (0, 2s)$.

Then, from equation (5.15), we have $E \cap B(x,s) = \{z \in B(x,s) | (M_{\min\{2s,Cr\}} g^p)^{\frac{1}{p}} > \Lambda\tau\}$. Noting first that

$$\min\{2s, Cr\} + s \leqslant Cr + s \leqslant 2Cr \text{ and } \min\{2s, Cr\} + s \leqslant 4s,$$

by Lemma 2.4 applied to the scale $s < r_0/4$, and the maximal function bound for g and by local doubling, we get

$$\begin{aligned}\fint_{B(x,s)} 1_E \, d\mu &\leqslant \frac{\mu(\{M_{\min\{2s,Cr\}} g^p > \Lambda^p \tau^p\} \cap B(x,s))}{\mu(B(x,s))} \\ &\leqslant \frac{D^3 \int_{B(x,\min\{2s,Cr\}+s)} g^p \, d\mu}{\mu(B(x,s))\Lambda^p \tau^p} < \frac{D^5}{\Lambda^p} < \tau_0.\end{aligned}$$

In this application of Lemma 2.4 we need the doubling at a larger scale. Taking the supremum over s, we get $M_{Cr} 1_E(x) < \tau_0$ and symmetrically $M_{Cr} 1_E(y) < \tau_0$. Let $\epsilon > 0$ be arbitrary. By Definition 2.16, there exists a curve $\gamma\colon I \to X$, with

$$\int_\gamma 1_E \, ds \leqslant \delta \tau_0^{\frac{1}{p}} r.$$

Let $O = \gamma^{-1}(E)$, which is open since the Hardy–Littlewood maximal function is lower semi-continuous, and define $K = (I \setminus O) \cup \{\min(I), \max(I)\}$. Then, defining $\gamma' = \gamma|_K$, we obtain a curve fragment $\gamma'\colon K \to X$ with

$$\mathrm{Len}(\gamma') \leqslant \mathrm{Len}(\gamma) \leqslant Cr.$$

Now let $\mathrm{Undef}(\gamma') = \bigcup_i (a_i, b_i)$ as in (5.2) and note that for every gap (a_i, b_i) of γ', we have $\gamma((a_i, b_i)) \subset E$ and

$$d_i := d(\gamma(a_i), \gamma(b_i)) \leqslant \mathrm{Len}(\gamma|_{[a_i,b_i]\cap K}) \leqslant \int_{\gamma|_{[a_i,b_i]}} 1 \, ds = \int_{\gamma|_{[a_i,b_i]}} 1_E \, ds.$$

Thus summing over i gives

$$\mathrm{Gap}(\gamma') \leqslant \int_\gamma 1_E \, ds \leqslant \delta \tau_0^{\frac{1}{p}} r.$$

Now, clearly γ' avoids E except possibly at x, y. Thus, by the lower semi-continuity of g, we also have $g(\gamma'(t)) \leqslant \Lambda\tau$ for every $t \in K$. In particular,

$$\int_{\gamma'} g \, ds \leqslant \Lambda\tau \mathrm{Len}(\gamma') \leqslant \Lambda\tau Cr. \tag{5.16}$$

By the assumption, $\delta\tau_0^{1/p} < \frac{1}{2}$, so each of these gaps is of size less than r_1. By our prior estimates, we obtain

$$\sum_i d_i = \mathrm{Gap}(\gamma') \leqslant \delta \tau_0^{\frac{1}{p}} r.$$

Now let $\epsilon > 0$ be given. We have $M_{2Cd_i}(g^p(\gamma'(t))^{1/p} < \Lambda\tau$ for $t = a_i, b_i$, so by the definition of $\alpha^p_{r_1,2C}(L, \Lambda\tau)$ there are curve fragments γ_i of length at most Ld_i connecting $\gamma'(a_i)$ and $\gamma'(b_i)$ and

$$\int_{\gamma_i} g\ ds + \mathrm{Gap}(\gamma_i) \leqslant \alpha^p_{r_1,2C}(L, \Lambda\tau)d_i + 2^{-i}\epsilon.$$

Now, by a dilation and translation, we can assume that the domains of γ_i are $[a_i, b_i]$, and that the curves are uniformly Lipschitz. Thus, we can define a new curve γ'' by the choices $\gamma''(t) = \gamma'(t)$ for $t \in K$ and $\gamma''(t) = \gamma_i(t)$ for $t \in [a'_i, b'_i]$. This is clearly Lipschitz and

$$\mathrm{Len}(\gamma'') \leqslant \mathrm{Len}(\gamma') + \sum_i \mathrm{Len}(\gamma_i) \leqslant (C + \delta\tau_0^{1/p}L)r \leqslant Lr.$$

Further, using the above estimates and estimate (5.16)

$$\inf_{\overline{\gamma}\in\Gamma_{x,y}(L)} \int_{\overline{\gamma}} g\ ds + \mathrm{Gap}(\overline{\gamma}) \leqslant \int_{\gamma''} g\ ds + \mathrm{Gap}(\gamma'') \leqslant \int_{\gamma'} g\ ds + \sum_i \int_{\gamma_i} g\ ds + \mathrm{Gap}(\gamma_i)$$

$$\leqslant C\Lambda\tau r + \delta\tau_0^{1/p} r\alpha^p_{r_1,2C}(L, \Lambda\tau) + \epsilon.$$

Letting first $\epsilon \to 0$, taking suprema over g and y and x, and dividing by r, we obtain

$$\alpha^p_{r_1,2C}(L, \tau) \leqslant C\Lambda\tau + \delta\tau_0^{1/p}\alpha^p_{r_1,2C}(L, \Lambda\tau).$$

Finally combining this with equation (5.10), our initial choice of Λ yields

$$\alpha^p_{r_1,2C}(L, \tau) \leqslant \frac{2D^{5/p}C}{\tau_0^{1/p}}\tau + 2\delta D^{5/p}\alpha^p_{r_1,2C}(L, \tau),$$

and solving for $\alpha^p_{r_1,2C}(L, \tau)$ gives

$$\alpha^p_{r_1,2C}(L, \tau) \leqslant \frac{2D^{5/p}C}{\tau_0^{1/p}(1 - 2\delta D^{5/p})}\tau = C'\tau$$

as desired. □

We now have all the tools to prove Theorems 2.18 and 2.19. The argument for the first result is similar to the one presented in [**14**], so we only sketch the details.

Proof of Theorem 2.18. Assume that the space satisfies a $(1, p)$-Poincaré inequality at scale r_0 with constant $C_1 = C$, so by Theorem 5.5 it also satisfies a pointwise $(1, p)$-Poincaré inequality at scale $r_0/2$ with constant C_2. To prove the maximal connectivity condition, fix $x, y \in X$, put $r = d(x, y)$, fix $\tau \in (0, 1)$, and fix a Borel set E with $M_{C_2 r}1_E(z) < \tau$ for $z = x, y$. By Remark 2.15, it is sufficient to assume E open. We will construct a curve γ with controlled length and which almost avoids the set E. Define

$$\mathcal{F}_x(z) = \inf_{\gamma} \int_{\gamma} (1_E + \tau)\ ds.$$

The infimum is taken over rectifiable curves γ connecting x to y.

Since the space is Λ-quasi-convex at scale $r_0/2$ with Λ depending only on C and D (see, for example, [**10**]), this infimum is finite.[†] It is easy to see that $\mathrm{Lip}\,[\mathcal{F}_x] \leqslant \Lambda(1_E + \tau)$. Thus, by the pointwise Poincaré inequality, we have

$$\mathcal{F}_x(y) = \mathcal{F}_x(y) - \mathcal{F}_x(x) \leqslant C_2\Lambda r(M_{C_2 r}1_E(x) + M_{C_2 r}1_E(y) + 2\tau).$$

[†]This step requires a proof using a local Poincaré inequality which is a fairly straightforward modification of the previous one. See, for example, [**6**, Proposition 4.8].

Thus, there must be some curve γ such that

$$\int_\gamma (1_E + \tau)\, ds \leqslant C_2\Lambda r(M_{C_2 r}1_E(x) + M_{C_2 r}1_E(y) + 3\tau) < C_2\Lambda(2\tau + 3\tau)r < 6C_2\Lambda\tau r.$$

In particular, $\mathrm{Len}(\gamma) \leqslant 6C_2\Lambda r$. The same inequality also verifies the (C_0, Δ, p)-maximal connectivity condition (2.14) for γ with constants $C_0 = 6C_2\Lambda$ and $\Delta = 6C_2\Lambda$. □

Proof of Theorem 2.19. Let $\delta_{p,D} = \frac{1}{2}D^{-5/p}$. If the space is (C, δ, τ_0, p)-max connected and $\delta \in (0, \frac{1}{2}D^{-5/p})$, then by Lemma 5.12 we have

$$\alpha^p_{r_1,2C}(L, \tau) \leqslant C'\tau$$

for $r_1 = r_0/5C$, with $2C \leqslant C'$. So by Lemma 5.11, the space satisfies a $(1, p)$-Poincaré inequality at scale $r_1/(2C') = r_0/(10CC')$ with constant C_p, where C_p depends quantitatively on C' and hence on δ, D, C, τ_0, and p. □

Appendix. *On preserving uniformity by removal processes*

Here we give a proof of Theorem 4.22, our main technical tool in the construction of metric sponges. This requires some preliminary lemmas for uniform domains.

A.1. *Initial properties of the measure.* One useful property of a uniform domain Ω corresponds roughly to the boundary $\partial\Omega$ being porous (see, for example, [**9**] for a definition). We recall a variant of [**7**, Lemma 4.2] first, and sketch the proof.

Lemma A.1 [**7**]. *If Ω is an A-uniform subset of X then it satisfies the following corkscrew condition: for all $x \in \Omega$ and $r \in (0, \mathrm{diam}(\Omega))$, there exists $y \in B_\Omega(x, r)$ so that*

$$B\Big(y, \frac{r}{4A}\Big) \subset \Omega \cap B(x, r).$$

Proof. Let $x \in \Omega$ and $r \in (0, \mathrm{diam}(\Omega))$ be arbitrary. Choose $y \in \Omega$ so that

$$d(x, y) \geqslant \frac{\mathrm{diam}(\Omega)}{2}.$$

Then, let γ be the A-uniform curve connecting x to y. By continuity, there is a t such that $d(\gamma(t), x) = r/4$, and thus also $d(\gamma(t), y) \geqslant r/4$. Therefore,

$$d(\gamma(t), \Omega^c) \geqslant \frac{1}{A}\min\{\mathrm{diam}(\gamma|_{[0,t]}), \mathrm{diam}(\gamma|_{[t,1]})\} \geqslant \frac{r}{4A},$$

and thus $B(\gamma(t), \frac{r}{4A}) \subset \Omega$ and

$$B\Big(\gamma(t), \frac{r}{4A}\Big) \subset B\Big(\gamma(t), \frac{r}{2}\Big) \subset B(x, r),$$

which completes the proof. □

From this, we conclude useful properties of the restricted measure on Ω, such as Ahlfors regularity and a basic volume (or measure) estimate for removed ‘ obstacles’.

Proof of Lemma 4.24. Let $x \in X, r \in (0, \mathrm{diam}(\Omega))$ and let $C_{AR,\Omega} = (4A)^Q C_{AR}$. Firstly, the upper bound in the Ahlfors Q-regularity condition is trivial:

$$\mu(B(x, r) \cap \Omega)) \leqslant \mu(B(x, r)) \leqslant C_{AR}r^Q \leqslant C_{AR,\Omega}r^Q.$$

Now, by Lemma A.1, there is a $y \in B(x,r) \cap \Omega$ such that $B(y, \frac{r}{4A}) \subset \Omega$, in which case

$$\mu(B(x,r) \cap \Omega)) \geqslant \mu\Big(B\Big(y, \frac{r}{4A}\Big)\Big) \geqslant \frac{r^Q}{(4A)^Q C_{AR}} \geqslant \frac{r^Q}{C_{AR,\Omega}}$$

and the result follows. □

Proof of Lemma 4.25. Scale the statement so that $\operatorname{diam}(\Omega) = 1$. Fix $C_\delta = C_{AR}^3(2L(1+\delta))^Q \delta^{-Q}$ and, for $l > k$, let $\mathcal{R}^l_{x,r}$ be the set of all $R \in \mathcal{R}_{\mathbf{n},l}$ so that $R \cap B(x,r) \neq \emptyset$. It is sufficient to prove that

$$\mu\left(\bigcup_{R \in \mathcal{R}^l_{x,r}} R\right) \leqslant \frac{C_\delta}{n_l^Q} r^Q$$

for every $l > k$; the desired estimate follows from summation over l.

Given $R \in \mathcal{R}^l_{x,r}$ let $x_R \in R \cap B(x,r)$, so $R \subset B(x_R, Ls_l)$ follows from Definition 4.21. Since $r \geqslant s_k > s_l$, we have

$$B(x_R, \delta s_{l-1}/2) \subset B(x, r + \delta s_k) \subset B(x, (1+\delta)r).$$

By separation, the balls $B(x_R, \delta s_{l-1})$ are disjoint for distinct R. We then estimate using Ahlfors regularity

$$\begin{aligned}
\mu\left(B(x,r) \cap \bigcup_{R \in \mathcal{R}_{\mathbf{n},l}} R\right) \leqslant \sum_{R \in \mathcal{R}^l_{x,r}} \mu(R) &\leqslant \sum_{R \in \mathcal{R}^l_{x,r}} \mu(B(x_R, Ls_l)) \\
&\leqslant \frac{C_{AR}^2 (2Ls_l)^Q}{\delta^Q s_{l-1}^Q} \sum_{R \in \mathcal{R}^l_{x,r}} \mu(B(x_R, \delta s_{l-1}/2)) \\
&\leqslant \frac{C_{AR}^2 2^Q L^Q}{\delta^Q n_l^Q} \mu(B(x, (1+\delta)r)) \\
&\leqslant \frac{C_{AR}^3 (2L(1+\delta))^Q}{\delta^Q n_l^Q} r^Q = \frac{C_\delta}{n_l^Q} r^Q.
\end{aligned}$$

as desired. □

A.2. *Preserving uniformity.* One of the forthcoming technical issues in removing a set R is that an arbitrary uniform curve relative to a pair of points in $X \setminus R$ may travel 'too far away' from R. To resolve this, we verify the following result, in whose proof we use the argument from [**42**, Theorem 4.1].

To fix notation, for a metric space $X = (X, d)$ and for $\epsilon > 0$ we denote ϵ-neighborhoods of subsets Y of X by

$$N_\epsilon(Y) := \bigcup_{x \in Y} B(x, \epsilon).$$

Lemma A.2. *Fix $D, C, A \geqslant 1$. Let X be a C-quasi-convex, D-metric doubling metric space. If S is a bounded, A-co-uniform domain in X, then for every $\epsilon > 0$ there is a constant $L_\epsilon = L_\epsilon(C, D, A)$ such that for every $x, y \in N_{\epsilon \operatorname{diam}(S)}(S) \setminus S$, there exists a L_ϵ-uniform curve γ with respect to x, y, and $X \setminus S$ with $\gamma \subset N_{4(C+A^2)\epsilon \operatorname{diam}(S)}(S)$.*

Proof. The statement is scale invariant, so assume $\operatorname{diam}(S) = 1$. Fix $\epsilon > 0$. Let $x, y \in N_\epsilon(S) \setminus S$ be arbitrary. If $d(x,y) \leqslant \epsilon$, the result follows simply by choosing the A-uniform curve with

respect to x, y, and $X \setminus S$. Thus assume $d(x,y) > \epsilon$, in which case

$$d(x,y) \leqslant 2\epsilon + \operatorname{diam}(S) \leqslant 2\epsilon + 1.$$

Let S_ϵ be a maximally ϵ-separated subset of $N_{C\epsilon}(S) \setminus S$, that is for each distinct $a, b \in S_\epsilon$ we have $d(a,b) \geqslant \epsilon$. The union $\bigcup_{s \in S_\epsilon} B(s, 2\epsilon)$ covers $N_{C\epsilon}(S) \setminus S$, so by quasi-convexity, connectivity of ∂S, and doubling, there exists $M_0 \in \mathbb{N}$ with dependence $M_0 = M_0(\epsilon, C, D)$ as well as a chain of points $\{x_i\}_{i=1}^M$ in $S_\epsilon \cup \{x,y\} \subset N_{C\epsilon}(S) \setminus S$ satisfying $x_1 = x$, $x_M = y$, $3 \leqslant M \leqslant M_0$, and

$$\frac{\epsilon}{2} \;\leqslant\; d(x_i, x_{i+1}) \;<\; 2\epsilon.$$

Note, quasi-convexity is used simply to ensure that the points x, y can be connected to ∂S. For $i = 1, \dots, M-1$, let $\gamma_i \colon [0,1] \to X$ be the A-uniform curve with respect to x_i, x_{i+1}, and $X \setminus S$, so $\operatorname{diam}(\gamma_i) \leqslant 2A\epsilon$. By continuity, there exists $t_i \in [0,1]$ such that

$$\frac{\epsilon}{4} \leqslant \min\{\operatorname{diam}(\gamma_i|_{[0,t_i]}), \operatorname{diam}(\gamma_i|_{[t_i,1]})\}.$$

Then for $i = 1, \dots, M-2$, let γ_i' be the A-uniform curve with respect to $\gamma_i(t_i)$, $\gamma_{i+1}(t_{i+1})$, and $X \setminus S$. Define γ to be the concatenation of $\gamma_1|_{[0,t_1]}$ with $\gamma_M|_{[t_{M-1},1]}$ and all the γ_i'. Direct calculation and Definition 4.12 imply that

$$\begin{aligned}\operatorname{diam}(\gamma_i') \;&\leqslant Ad(\gamma_i(t_i), \gamma_{i+1}(t_{i+1}))\\ &\leqslant A(d(\gamma_i(t_i), x_{i+1}) + d(x_{i+1}, \gamma_{i+1}(t_{i+1}))) \leqslant A(\operatorname{diam}(\gamma_i) + \operatorname{diam}(\gamma_{i+1})) \leqslant 4A^2\epsilon,\end{aligned}$$

and $d(\gamma_i'(t), S) \geqslant \frac{\epsilon}{8A^2}$ for $t \in [0,1]$. Now,

$$\operatorname{diam}(\gamma) \leqslant 4MA^2\epsilon \leqslant 4MA^2 d(x,y).$$

Also, if $\gamma(t)$ intersects with γ_i', then

$$d(\gamma(t), S) \geqslant \frac{\epsilon}{8A^2} \geqslant \frac{\operatorname{diam}(\gamma)}{32MA^4} \geqslant \frac{1}{32MA^2} \min\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\}.$$

As for the cases when $\gamma(t)$ coincides with a point on $\gamma_1(s)$ or $\gamma_M(s)$, the estimate follows from the A-uniformity of γ_1 and γ_M. To clarify, this involves some case checking. We expand only the case of $\gamma(t)$ coinciding with $\gamma_1(s)$, when we have $d(\gamma(t), \Omega^c) = d(\gamma_1(s), \Omega^c) \geqslant \frac{1}{A} \min\{\operatorname{diam}(\gamma_1|_{[0,s]}), \operatorname{diam}(\gamma_1|_{[s,1]})\}$. We also have $\operatorname{diam}(\gamma_1|_{[0,s]}) = \operatorname{diam}(\gamma|_{[0,t]})$, so if the minimum is attained with $\operatorname{diam}(\gamma_1|_{[0,s]})$ the inequality is immediate. If the minimum is attained by the second option, then we have $\operatorname{diam}(\gamma_1|_{[s,1]}) \geqslant \epsilon/4 \geqslant \frac{1}{4A}\operatorname{diam}(\gamma|_{[0,t]})$ by the choice of t_1. In combination, we get that γ is an $32MA^4$-uniform curve contained in $N_{2(C+A)\epsilon\operatorname{diam}(S)}(S)$. The containment follows since $\gamma_i' \subset N_{2(C+A)\epsilon\operatorname{diam}(S)}(S)$. □

We will need the following simple lemma on uniform domains.

Lemma A.3. *Let Ω be an open domain and let $x, y \in \Omega$. If $\gamma \colon [0,1] \to \Omega$ is an A-uniform curve with respect to x, y, and Ω, then for every $t \in [0,1]$ it holds that*

$$d(\gamma(t), \Omega^c) \geqslant \frac{1}{4A} \min\{d(x, \Omega^c) + \operatorname{diam}(\gamma|_{[0,t]}), d(y, \Omega^c) + \operatorname{diam}(\gamma|_{[t,1]})\}.$$

Proof. Up to symmetry, assume $\operatorname{diam}(\gamma|_{[0,t]}) \leqslant \operatorname{diam}(\gamma|_{[t,1]})$. If

$$\operatorname{diam}(\gamma|_{[0,t]}) \geqslant \frac{d(x, \Omega^c)}{2},$$

then the claim follows from A-uniformity. If, on the other hand,

$$\operatorname{diam}(\gamma|_{[0,t]}) \leqslant \frac{d(x,\Omega^c)}{2},$$

then, by the Triangle inequality,

$$\begin{aligned} d(\gamma(t),\Omega^c) &\geqslant d(x,\Omega^c) - d(\gamma(t),x) \\ &\geqslant d(x,\Omega^c) - \operatorname{diam}(\gamma|_{[0,t]}) \geqslant \frac{d(x,\Omega^c)}{2} \geqslant \frac{1}{4}d(x,\Omega^c) + \frac{1}{4}\operatorname{diam}(\gamma|_{[0,t]}) \end{aligned}$$

which, with $A \geqslant 1$, is the desired result. □

We are now ready to show that for co-uniform subsets S of uniform domains Ω, their relative complements $\Omega \setminus S$ are also uniform.

Proof of Theorem 4.22. Let $A_\Omega = A_1 > 0$ and $A_S = A_2 > 0$ be the uniformity constants of Ω and $X \setminus S$, respectively. Fix $\epsilon = \frac{d(S,\Omega^c)}{\operatorname{diam}(S)}$. Without loss of generality, assume $\operatorname{diam}(S) = 1$. Letting $\delta_0 \in (0, \min\{1/A_\Omega, 1/A_S\})$ to be determined later, we show that $\Omega \setminus S$ is A'-uniform for some $A' \geqslant 1/\delta_0$, that is, that for each $x, y \in \Omega \setminus S$, there is a curve γ so that

$$d(\gamma(t),\Omega^c \cup S) \;\geqslant\; \frac{1}{A'}\min\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\} \tag{A.4}$$

and where $\operatorname{diam}(\gamma) \leqslant A' d(x,y)$.

Let $x, y \in \Omega \setminus S$ be arbitrary. If $d(x,y) < \frac{\epsilon}{3(A_S + A_\Omega)}$, the claim follows by either using the uniformity of $X \setminus S$ or the uniformity of Ω, depending on which of S or Ω^c is closer to x or y. Thus, without loss of generality assume $d(x,y) \geqslant \frac{\epsilon}{3(A_S + A_\Omega)}$. Also, without loss of generality, assume $x, y \notin \partial S$. The case of either $x, y \in \partial S$ can be obtained by using the uniformity of Ω to connect points $x', y' \in \Omega \setminus \bar{S}$ to x, y, respectively, with

$$\max\{d(x,x'), d(y,y')\} \;\leqslant\; \frac{1}{A_\Omega^2} d(S,\Omega^c).$$

By uniformity of Ω, there is an A_Ω-uniform curve $\gamma_0\colon [0,1] \to X$ with respect to x, y, and Ω, so define the set

$$\mathcal{B} = \big\{t \in [0,1] \mid d(\gamma_0(t), S) < \delta_0 \min\{\operatorname{diam}(\gamma_0|_{[0,t]}), \operatorname{diam}(\gamma_0|_{[t,1]})\}\big\}.$$

If $\mathcal{B} = \emptyset$, then γ_0 satisfies (A.4) with δ_0 in place of $\frac{1}{A'}$, and thus $\gamma = \gamma_0$ would be the desired curve. Otherwise, $\mathcal{B}$ is open, and hence a countable union of disjoint open intervals,

$$\mathcal{B} = \bigcup_{i \in J} I_i$$

for some possibly finite subset $J \subset \mathbb{N}$ with $I_i = (a_i, b_i)$. Note that for each $z = a_i, b_i$ we have equality in the above condition, that is

$$d(\gamma_0(z), S) \;=\; \delta_0 \min\{\operatorname{diam}(\gamma_0|_{[0,z]}), \operatorname{diam}(\gamma_0|_{[z,1]})\}. \tag{A.5}$$

Let $C' = 4(C + A_S^2 + \epsilon)$ and let $L = L_{\epsilon/(3C')} = L_{\epsilon/(3C')}(C, D, A_S)$ be the constant from Lemma A.2. We now replace each $\gamma_0|_{I_i}$ with a new curve γ_i so that the concatenation satisfies (A.4); in particular, we claim that we can choose γ_i to have

$$d(\gamma_i, \Omega^c) \geqslant \frac{\max\{d(\gamma_0(a_i), S), d(\gamma_0(b_i), S)\}}{4} + \frac{\epsilon}{12C'} d(\gamma_0(a_i), \gamma_0(b_i)) \tag{A.6}$$

and with the constant $C'' = \max\{L, A_S\}$,

$$\operatorname{diam}(\gamma_i) \leqslant C'' d(\gamma_0(a_i), \gamma_0(b_i)) \tag{A.7}$$

holds for each $i \in \mathbb{N}$.

We proceed by cases, as follows. Suppose first that

$$\frac{\epsilon}{3C'} > \max\{d(\gamma_0(a_i), S), d(\gamma_0(b_i), S)\} \tag{A.8}$$

is true. So by Lemma A.2 with $\epsilon/(3C')$ in place of ϵ, there is a curve γ_i in $N_{\epsilon/3}(S)$ that joins $\gamma_0(a_i)$ and $\gamma_0(b_i)$ and which is L-uniform with respect to $X \setminus S$. In particular, (A.7) holds with $C'' = L$ and our choice of ϵ yields

$$d(\gamma_i(t), \Omega^c) \;\geqslant\; d(S, \Omega^c) - d(\gamma_i(t), S) \;\geqslant\; \epsilon - \frac{\epsilon}{3}$$

so (A.6) follows from (A.8) and

$$d(\gamma_0(a_i), \gamma_0(b_i)) \leqslant \operatorname{diam}(S) + d(\gamma_0(a_i), S) + d(\gamma_0(b_i), S) \leqslant 1 + \frac{2\epsilon}{C'}.$$

If (A.8) is false, then instead by co-uniformity, there is a A_S-uniform curve γ_i with respect to $\gamma_0(a_i)$, $\gamma_0(b_i)$, and $X \setminus S$. We now claim that the distance estimates (A.6) and (A.7) hold for these curves γ_i. To this end, by symmetry we may assume that

$$d(\gamma_0(a_i), S) \;\geqslant\; \max\left\{\frac{\epsilon}{3C'}, d(\gamma_0(b_i), S)\right\}.$$

Introduce the short-hand notation $x_i := \gamma_0(a_i), y_i := \gamma_0(b_i)$. Assume now that $\delta_0 < \frac{\epsilon}{32 A_\Omega A_S C'}$, which with (A.5) implies

$$d(x_i, \Omega^c) \;\geqslant\; \frac{1}{A_\Omega} \min\{\operatorname{diam}(\gamma_0|_{[0,a_i]}), \operatorname{diam}(\gamma_0|_{[a_i,1]})\} = \frac{1}{\delta_0 A_\Omega} d(x_i, S) \geqslant \frac{1}{\delta_0 A_\Omega} \frac{\epsilon}{3C'}.$$

Then combining the previous estimates and the choice of δ_0 yields

$$d(x_i, y_i) \leqslant \operatorname{diam}(S) + 2d(x_i, S) \leqslant \frac{6C'\delta_0 A_\Omega}{\epsilon} d(x_i, \Omega^c) \leqslant \frac{1}{8A_S} d(x_i, \Omega^c),$$

and

$$d(x_i, y_i) \leqslant \operatorname{diam}(S) + 2d(x_i, S) \leqslant \frac{6C'}{\epsilon} d(x_i, S).$$

We have (A.7) and therefore

$$\begin{aligned} d(\gamma_i, \Omega^c) \geqslant d(x_i, \Omega^c) - \operatorname{diam}(\gamma_i) &\geqslant d(x_i, \Omega^c) - A_S d(x_i, y_i) \\ &\geqslant \frac{d(x_i, S)}{2} = \frac{\max\{d(x_i, S), d(y_i, S)\}}{2}. \end{aligned}$$

In particular, (A.6) holds in both cases for the γ_i as constructed.

In either case, C''-uniformity of γ_i with respect to $X \setminus S$ and Lemma A.3 imply that for all t in the domain of γ_i

$$d(\gamma_i(t), S) \geqslant \frac{1}{C''} \frac{\min\{d(x_i, S) + \operatorname{diam}(\gamma_i|_{[a_i,t]}), d(y_i, S) + \operatorname{diam}(\gamma_i|_{[t,b_i]})\}}{4}. \tag{A.9}$$

Now, similarly to the proof of Lemma 3.2 reparametrize each γ_i to have domain $I_i = [a_i, b_i]$ and define the concatenation $\gamma\colon [0,1] \to X$ by $\gamma(t) = \gamma_i(t)$ if $t \in I_i$, and $\gamma(t) = \gamma_0(t)$ for all other $t \in [0,1]$. This concatenated curve is the desired uniform curve and we will proceed to estimate its diameter and distance to $S \cup \Omega^c$.

The diameter bounds for γ_i in (A.7) give rather directly that γ is continuous. By (A.7), each γ_i has diameter at most

$$\operatorname{diam}(\gamma_i) \leqslant C'' d(x_i, y_i) \leqslant C'' \operatorname{diam}(\gamma_0),$$

so it follows that the concatenation γ has diameter at most

$$\operatorname{diam}(\gamma) \leqslant \operatorname{diam}(\gamma_0) + 2 \max_i \operatorname{diam}(\gamma_i) \leqslant (1 + 2C'')\operatorname{diam}(\gamma_0) \leqslant A_\Omega(1 + 2C'')d(x, y).$$

To check the uniformity condition (4.13), we again proceed by cases. Supposing first that $t \notin I_i$ for any index i, put $U_0 = [0, t]$ and $U_1 = [t, 1]$. For $k = 0, 1$ we have from (A.7)

$$\operatorname{diam}(\gamma|_{U_k}) \leqslant \operatorname{diam}(\gamma_0|_{U_k}) + 2 \max_{i, I_i \subset U_k} \operatorname{diam}(\gamma_i|_{I_i}) \leqslant (1 + 2C'')\operatorname{diam}(\gamma_0|_{U_k}) \tag{A.10}$$

Then, we get

$$d(\gamma(t), \Omega^c) \geqslant \frac{1}{A_\Omega} \min_{k=0,1} \operatorname{diam}(\gamma_0|_{U_k}) \geqslant \frac{1}{A_\Omega(1 + 2C'')} \min_{k=0,1} \operatorname{diam}(\gamma_0|_{U_k}), \tag{A.11}$$

and (from the definition of $\mathcal{B}$)

$$d(\gamma(t), S) \geqslant \delta_0 \min_{k=0,1} \operatorname{diam}(\gamma_0|_{U_k}) \geqslant \frac{\delta_0}{(1 + 2C'')} \min_{k=0,1} \operatorname{diam}(\gamma_0|_{U_k}). \tag{A.12}$$

Now consider the remaining case where $t \in I_i$ for some $i \in J$, in which case $U_k \cup I_i$ and $U_k \cap I_i$ and $U_k \setminus I_i$ are all intervals for $k = 0, 1$. Similarly as above,

$$\operatorname{diam}(\gamma|_{U_k}) \leqslant \operatorname{diam}(\gamma_0|_{U_k \setminus I_i}) + \operatorname{diam}(\gamma_i|_{U_k \cap I_i}) \leqslant (1 + 2C'')(\operatorname{diam}(\gamma_0|_{U_k \setminus I_i}) + d(x_i, y_i)). \tag{A.13}$$

Taking a minimum over $k = 0, 1$ in (A.13) gives

$$\min_{k=0,1} \operatorname{diam}(\gamma|_{U_k}) \leqslant (1 + 2C'')(\min_{k=0,1} \operatorname{diam}(\gamma_0|_{U_k \setminus I_i}) + d(x_i, y_i)). \tag{A.14}$$

Combining our work with $\frac{\epsilon}{12C'} \geqslant \delta_0$ gives the following.

$$\begin{aligned}
d(\gamma(t), \Omega^c) &\overset{(A.6)}{\geqslant} \frac{\max\{d(x_i, S), d(y_i, S)\}}{4} + \frac{\epsilon}{12C'} d(x_i, y_i) \\
&\overset{\substack{(A.5)\\(A.14)}}{\geqslant} \frac{\delta_0 \min_{k=0,1} \operatorname{diam}(\gamma|_{U_k})}{1 + 2C''} \\
d(\gamma(t), S) &\overset{(A.9)}{\geqslant} \frac{1}{C''} \frac{\min\{d(x_i, S) + \operatorname{diam}(\gamma_i|_{[a_i,t]}), d(y_i, S) + \operatorname{diam}(\gamma_i|_{[t,b_i]})\}}{4} \\
&\overset{(A.5)}{\geqslant} \frac{\delta_0}{4C''} \min_{k=0,1} \{\operatorname{diam}(\gamma_0|_{U_k \setminus I_i}) + \operatorname{diam}(\gamma_i|_{U_k \cap I_i}), \operatorname{diam}(\gamma_0|_{U_k \cup I_i}) + \operatorname{diam}(\gamma_i|_{I_i \setminus U_k})\} \\
&\overset{(A.13)}{\geqslant} \frac{\delta_0}{4C''(1 + 2C'')} \min_{k=0,1} \operatorname{diam}(\gamma|_{U_k}).
\end{aligned}$$

In the ultimate inequality, we bound each of the terms in the minimum first, and then combine the bound. Now, the previous two estimates give for $t \in (a_i, b_i)$ that

$$d(\gamma(t), S \cup \Omega^c) \geqslant \frac{\delta_0}{4C''(1 + 2C'')} \min\Big\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\Big\}. \tag{A.15}$$

The estimates (A.15) together with the diameter bound show that the curve γ is A'–uniform for

$$A' = \max\left\{\frac{4C''(1 + 2C'')}{\delta_0}, (1 + 2C'')A_\Omega\right\}. \qquad \square$$

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