# Reliable and Secure Multishot Network Coding using Linearized Reed-Solomon Codes 

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#### Abstract

Multishot network coding is considered in a worstcase adversarial setting in which an omniscient adversary with unbounded computational resources may inject erroneous packets in up to $t$ links, erase up to $\rho$ packets, and wire-tap up to $\mu$ links, all throughout $\ell$ shots of a linearly-coded network. Assuming no knowledge of the underlying linear network code (in particular, the network topology and underlying linear code may be random and change with time), a coding scheme achieving zero-error communication and perfect secrecy is obtained based on linearized Reed-Solomon codes. The scheme achieves the maximum possible secret message size of $\ell n^{\prime}-2 t-\rho-\mu$ packets for coherent communication, where $n^{\prime}$ is the number of outgoing links at the source, for any packet length $m \geq n^{\prime}$ (largest possible range). By lifting this construction, coding schemes for non-coherent communication are obtained with information rates close to optimal for practical instances. The required field size is $q^{m}$, where $q>\ell$, thus $q^{m} \approx \ell^{n^{\prime}}$, which is always smaller than that of a Gabidulin code tailored for $\ell$ shots, which would be at least $2^{\ell n^{\prime}}$. A Welch-Berlekamp sum-rank decoding algorithm for linearized Reed-Solomon codes is provided, having quadratic complexity in the total length $n=\ell n^{\prime}$, and which can be adapted to handle not only errors, but also erasures, wiretap observations and non-coherent communication. Combined with the obtained field size, the given decoding complexity is of $\mathcal{O}\left(n^{\prime 4} \ell^{2} \log (\ell)^{2}\right)$ operations in $\mathbb{F}_{2}$, whereas the most efficient known decoding algorithm for a Gabidulin code has a complexity of $\mathcal{O}\left(n^{\prime 3.69} \ell^{3.69} \log (\ell)^{2}\right)$ operations in $\mathbb{F}_{2}$, assuming a multiplication in a finite field $\mathbb{F}$ costs about $\log (|\mathbb{F}|)^{2}$ operations in $\mathbb{F}_{2}$.


Index Terms-Linearized Reed-Solomon codes, multishot network coding, network error-correction, sum-rank metric, sumsubspace codes, wire-tap channel.

## I. Introduction

LINEAR network coding over a finite field $\mathbb{F}_{q_{0}}$, introduced in [1]-[3], permits maximum information flow from a source to several sinks simultaneously in one shot (multicast). Moreover, for sufficiently large field size $q_{0}$, the maximum information flow can be achieved with high probability by a random choice of coding coefficients at each node, without knowledge of the network topology [4].

Correction of link errors was considered in [5]-[10], and secrecy against link wire-tapping was studied in [11]-[15]. Some of these works assume probabilistic error correction, and some require knowledge or modification of the linear

[^0]network code for the outer code construction. Error-correcting codes under an adversarial model without such requirements (thus compatible with random linear network coding) were first given in [16], [17] for non-coherent communication (in which the sink has no knowledge of the coding coefficients of the incoming links), and in [18] for coherent communication. Coding schemes that provide perfect secrecy and zero-error communication, without knowledge or modification of the underlying linear network code, were first given in [19]. The coherent-case construction in that work (similarly in [16][18]) is based on Gabidulin codes [20], [21]. If the packets sent through the links of the network in one shot are vectors in $\mathbb{F}_{q_{0}}^{m}$, then coding schemes based on Gabidulin codes achieve the maximum secret message size of $n^{\prime}-2 t-\rho-\mu$ packets, where $n^{\prime}$ is the number of outgoing links at the source, and for $t$ (link) errors, $\rho$ erasures and $\mu$ wire-tapped links. Moreover, the packet length $m$ is only restricted to $m \geq n^{\prime}$, which is the maximum possible range for $m$, where $n^{\prime}$ is a constraint given by the channel.
All of the works noted above make use of only one shot of the linearly-coded network. Correction of link errors in multishot network coding (see Fig. 1) was first investigated in [24], [25]. As noted in these works, the $\ell$-shot case can be treated as a 1 -shot case with number of outgoing links at the source $n=\ell n^{\prime}$. An $\ell$-shot Gabidulin code, that is, a Gabidulin code tailored for $\ell n^{\prime}$ outgoing links at the source, yields the maximum message size of $\ell n^{\prime}-2 t-\rho$ packets, but would require packet lengths $m \geq n=\ell n^{\prime}$ instead of $m \geq n^{\prime}$, which may be impractically large. More importantly, decoding an $\ell$-shot Gabidulin code using [26] would require $\mathcal{O}\left(n^{2}\right)$ operations over a field of size $q_{0}^{\ell n^{\prime}}$ (even faster decoders [27], [28] would be quite expensive for such field sizes, see Table I). To circumvent these issues, the authors of [24], [25] provide a multilevel construction that improves the error-correction capability of trivial concatenations of 1 -shot Gabidulin codes, with lower decoding complexity than an $\ell$-shot Gabidulin code, although without achieving the maximum secret message size.

Later, convolutional rank-metric codes were studied in [29]-[33], and a concatenation of an outer Hamming-metric convolutional code with an inner rank-metric block code was given in [34]. See [35] for a survey. Convolutional techniques yield codes achieving the streaming capacity for the burst rank loss networks described in [32], where recovery of a given packet is required before a certain delay. By truncating the convolutional codes in [31], [32] up to their memory, one may obtain block codes whose message size is close to the upper bound $\ell n^{\prime}-2 t-\rho$, but their field sizes are in general still


Fig. 1. A pattern of link errors in multishot linear network coding. Similarly for link observations by a wire-tapper. Dashed lines denote links directly affected by the adversary, and dotted lines denote links that suffer the effect indirectly. The network code may change with time. In this example, a rank deficiency results in an erasure for the first sink in the second shot. Although not depicted here, the network topology may also change with time. If the linear network code is defined over $\mathbb{F}_{q_{0}}=\mathbb{F}_{2}$, linearized Reed-Solomon codes in this case are defined over $\mathbb{F}_{q^{n^{\prime}}}=\mathbb{F}_{q^{2}}, 2 \mid q$, and allow us to use $\ell<q$ shots of the network, thus $\left|\mathbb{F}_{q^{2}}\right| \approx \ell^{2}$, which is quadratic in $\ell$. Observe that Gabidulin codes would require the field $\mathbb{F}_{2^{2 \ell}}$, which is exponential in $\ell$.


Fig. 2. Reliable and secure multishot linear network coding naturally extends reliable and secure multicast discrete memoryless channels. In this figure, the latter is depicted. Network coding is not necessary, which means that we may take the network base field $\mathbb{F}_{q_{0}}=\mathbb{F}_{2}$. Since $n^{\prime}=1$, linearized Reed-Solomon codes coincide in this case with classical and generalized Reed-Solomon codes [22], [23], which are defined over $\mathbb{F}_{q}$ and allow us to use $\ell<q$ shots, thus $\left|\mathbb{F}_{q}\right| \approx \ell$, which is linear in $\ell$. In this case, Gabidulin codes would require the field $\mathbb{F}_{2^{\ell} \ell}$, which is again exponential in $\ell$.
exponential in $n^{\prime}$ and $\ell$.
To the best of our knowledge, schemes that achieve zeroerror communication and perfect secrecy without knowledge of the underlying network code as in [19] have not yet been investigated in the multishot case.

In this work, we provide such a family of coding schemes for coherent communication with maximum secret message size of $\ell n^{\prime}-2 t-\rho-\mu$ packets, whose packet length is only restricted to $m \geq n^{\prime}$ (maximum possible range) in contrast to $m \geq \ell n^{\prime}$. The original base field of the network is embedded $\mathbb{F}_{q_{0}} \subseteq \mathbb{F}_{q}$ in a field of size $q=q_{0}^{s}$, for suitable $s$ satisfying $q>\ell$, and the obtained coding schemes are defined over the field $\mathbb{F}_{q^{m}}$, whose size is approximately $\max \left\{\ell, q_{0}\right\}^{n^{\prime}}$, and in most real scenarios $\left(\ell>q_{0}\right)$, it is $\ell^{n^{\prime}}$. We do not know if the range of values $q>\ell$, for $q$ given $\ell$, is maximum, but we conjecture that $q$ must be $\Omega(\ell)$ (equivalently, $\ell=\mathcal{O}(q))$ as is the case for MDS codes (the case $n^{\prime}=1$ ). By lifting our construction, we adapt it to non-coherent communication with nearly optimal information rates for practical instances.

Our coding schemes are based on linearized Reed-Solomon codes, introduced in [36] and closely connected to skew Reed-Solomon codes [37]. Linearized Reed-Solomon codes are maximum sum-rank distance (MSRD) codes, which is precisely the property that gives the optimality of our coding schemes. We provide a Welch-Berlekamp sum-rank decoding algorithm for these codes that requires $\mathcal{O}\left(n^{2}\right)$ operations over the field of size $q^{m} \approx \ell^{n^{\prime}}$. Hence we obtain a reduction in the decoding complexity in operations over $\mathbb{F}_{2}$ of more than a degree in the number of shots compared to the existing decoding algorithms and codes. See Table $\square$ We remark that our algorithm is related to a skew-metric decoding algorithm
for skew Reed-Solomon codes recently given in [38]. As we will see, such an algorithm can be translated to a sumrank decoding algorithm for linearized Reed-Solomon codes, although this is not done in [38]. In addition, the algorithm in [38] has cubic complexity over the field of size $q^{m} \approx \ell^{n^{\prime}}$ and does not handle erasures, wire-tap observations and noncoherent communication, in contrast to our algorithm.

Linearized Reed-Solomon codes are natural hybrids between Reed-Solomon codes [22], [23] and Gabidulin codes [20], [21], [39], in the same way that the sum-rank metric is a hybrid between the Hamming and rank metrics, and adversarial multishot network coding is a hybrid between adversarial network coding and adversarial discrete memoryless channels (e.g., wire-tap channel of type II [40] or secret sharing [41]). See and compare Figs. 1 and 2 Furthermore, our WelchBerlekamp decoding algorithm includes the classical one [42] and its rank-metric version [26] as particular cases. What happens is that rank-metric and Hamming-metric codes are two extremal particular cases of sum-rank-metric codes that correspond to $\ell=1$ and $n^{\prime}=1$, respectively. With this point of view, the use of linearized Reed-Solomon codes instead of Gabidulin codes as MSRD codes extends the idea of using Reed-Solomon codes rather than Gabidulin codes as MDS codes (see Figs. 1 and 2): in both cases, information rates are equal but Gabidulin codes require a field size exponentiated to the $\ell$ th power, thus providing no advantage unless $\ell=1$ (the rank-metric case). See Table 【I in Subsection V-A for a summarized classification of Reed-Solomon-like evaluation codes with maximum distance for certain metrics.

We conclude by giving two other interpretations of reliable and secure $\ell$-shot network coding. First, $\ell$-shot network coding
can be thought of as 1 -shot network coding where we have knowledge of at least $\ell$ connected components in the underlying graph after removing the source and sinks. Observe that here we may have a different number $n_{i}$ of outgoing links in each component. If each connected component has a single outgoing link, we recover the (multicast) discrete memoryless channel (see Fig. 2). Second, we may view $\ell$-shot network coding as 1 -shot network coding where packets are divided into $\ell$ sub-packets. In this scenario, errors, erasures and wiretapper observations occur in certain sub-packets (rather than over whole packets) in certain links. If the sub-packets are considered as symbols, this gives another interpretation of multishot network coding as hybrid between network coding and a discrete memoryless channel. The work [43] treats a similar hybrid of symbol and link errors, although they consider that if a symbol error occurs in one link, it occurs in all links (i.e., all transmitted subspaces). This may make error-correction impossible if the adversary spreads the errors in pessimistic patterns.

## Summary and Significance of our Main Contributions

Our coding schemes achieve the maximum secret message size of $n^{\prime} \ell-2 t-\rho-\mu$ packets for coherent communication (Theorem 6) and nearly optimal information rates for noncoherent communication (Theorem 8), as stated above, and satisfy the following:

1) Their field size is $q^{m}$, where $q>\ell, q=q_{0}^{s}$ for some $s \in \mathbb{N}$, and $m \geq n^{\prime}$. Thus their field size is approximately

$$
\begin{equation*}
\max \left\{\ell, q_{0}\right\}^{n^{\prime}} \tag{1}
\end{equation*}
$$

whereas an $\ell$-shot Gabidulin code always requires the field size $q_{0}^{\ell n^{\prime}}$, which is always considerably larger, since: a) $\ell^{n^{\prime}} \ll\left(q_{0}^{\ell}\right)^{n^{\prime}}$, if $q_{0}<\ell$, and b) $q_{0}^{n^{\prime}} \ll\left(q_{0}^{n^{\prime}}\right)^{\ell}$, if $q_{0} \geq \ell$. Notice, moreover, that $\ell^{n^{\prime}}$ is polynomial in $\ell$, whereas $q_{0}^{\ell n^{\prime}}$ is exponential in $\ell$. See also Figs. 1 and 2 for instances with $n^{\prime}=2$ and $n^{\prime}=1$, respectively, and $q_{0}=2$.

TABLE I
DECODING COMPLEXITY IN NO. OF OPERATIONS OVER $\mathbb{F}_{2}$

| Algorithm | Code | Complexity |
| :---: | :---: | :---: |
| Subsec. V-C] | Linearized RS [36] | $\mathcal{O}\left(n^{\prime 4} \ell^{2} \log (\ell)^{2}\right)$ |
| $[38]$ | Skew RS [37] | $\mathcal{O}\left(n^{\prime 5} \ell^{3} \log (\ell)^{2}\right)$ |
| $[27]$ | Gabidulin [20] | $\mathcal{O}\left(n^{\prime 3.69} \ell^{3.69} \log (\ell)^{2}\right)$ |
| $[26]$ | Gabidulin [20] | $\mathcal{O}\left(n^{\prime 4} \ell^{4}\right)$ |

2) We provide a decoding algorithm with complexity of $\mathcal{O}\left(n^{2}\right)$ operations over $\mathbb{F}_{q^{m}}$ (recall that $n=\ell n^{\prime}$ ). Assuming that one multiplication in a finite field $\mathbb{F}$ of characteristic 2 costs about $\log (|\mathbb{F}|)^{2}$ operations in $\mathbb{F}_{2}$, our decoding algorithm has complexity of

$$
\begin{equation*}
\mathcal{O}\left(n^{\prime 4} \ell^{2} \log (\ell)^{2}\right) \tag{2}
\end{equation*}
$$

operations over $\mathbb{F}_{2}$, as a function of $\ell$ and $n^{\prime}$. Observe also that in most practical situations, $\ell \gg n^{\prime}$ and $n^{\prime}$ is a constant of the channel. Table $\square$ shows a comparison between this complexity and those of other algorithms in the literature,
for coding schemes that are optimal in the coherent case or near optimal in the noncoherent case. We always assume that a multiplication in $\mathbb{F}$ costs $\log (|\mathbb{F}|)^{2}$ operations in $\mathbb{F}_{2}$. Note that, setting $n^{\prime}=1$, linearized Reed-Solomon codes become classical Reed-Solomon codes, and the obtained complexity in (2) is of $\mathcal{O}\left(\ell^{2} \log (\ell)^{2}\right)$ operations in $\mathbb{F}_{2}$, which is that of the quadratic-complexity Welch-Berlekamp decoding algorithm.
3) By multiplying the message vector by a generator matrix, our coding scheme has encoding complexity of $\mathcal{O}(k n)$ operations over $\mathbb{F}_{q^{m}}$. Since encoding with a linear code has in general complexity of $\mathcal{O}(k n)$ operations over the corresponding field, we obtain reductions in encoding complexity similar to those shown in Table $\rrbracket$ in operations over $\mathbb{F}_{2}$.

## Organization

The remainder of this paper is organized as follows. In Section (II we formulate zero-error communication and perfect secrecy in adversarial multishot network coding. This is a natural extension of the formulation in [19] to the multishot case. In Section III, we give sufficient and necessary conditions for coding schemes to correct a given number of errors and erasures, and for perfect secrecy under a given number of wiretapped links. This again is a natural extension of [18], [19], and the sufficient condition for error correction was already given in [25]. In Section IV] we provide the above-mentioned constructions based on the linearized Reed-Solomon codes introduced in [36]. We then give Singleton-type bounds, and prove that our schemes attain them for coherent communication and are close to them in the non-coherent case. Finally, we give in Section $\nabla$ a Welch-Berlekamp sum-rank decoding algorithm for linearized Reed-Solomon codes.

## Notation and Preliminaries

Throughout this paper, we will fix a prime power $q$ and positive integers $m, \ell, n=n_{1}+n_{2}+\cdots+n_{\ell}$, and $N=$ $N_{1}+N_{2}+\cdots+N_{\ell}$. The number $m$ corresponds to the packet length in each shot, $\ell$ is the number of shots, and $n_{i}$ and $N_{i}$ are the number of outgoing links at the source and incoming links to the sink, respectively, in the $i$ th shot.

We will denote by $\mathbb{F}_{q}$ the finite field with $q$ elements. For a field $\mathbb{F}$, we will denote by $\mathbb{F}^{m \times n}$ the set of $m \times n$ matrices with entries in $\mathbb{F}$, and we denote $\mathbb{F}^{n}=\mathbb{F}^{1 \times n}$. The field over which we consider linearity, ranks, and dimensions will be clear from the context. We will implicitly consider "erased" matrices, which may be denoted by $*$. We will define $\mathbb{F}^{m \times 0}=$ $\{*\}$ and $\mathbb{F}^{0 \times n}=\{*\}$. Operations with matrices are assumed to be extended to $*$ in the obvious way. For instance, $\operatorname{Rk}(*)=0$, $A *=*$ for $A \in \mathbb{F}^{n \times m}$, etc.

For matrices $A_{i} \in \mathbb{F}^{N_{i} \times n_{i}}$, for $i=1,2, \ldots, \ell$, we define the block-diagonal matrix
$\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right)=\left(\begin{array}{cccc}A_{1} & 0 & \ldots & 0 \\ 0 & A_{2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{\ell}\end{array}\right) \in \mathbb{F}^{N \times n}$.
To define $\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right)$, for each $i$ such that $A_{i}=* \in$ $\mathbb{F}^{0 \times n_{i}}$, we only add $n_{i}$ zero columns between the $(i-1)$ th
and the $(i+1)$ th blocks. For example, if $A \in \mathbb{F}^{N_{1} \times n_{1}}$, $* \in \mathbb{F}^{0 \times n_{2}}$ and $B \in \mathbb{F}^{N_{3} \times n_{3}}$, we define $\operatorname{diag}(A, *, B) \in$ $\mathbb{F}^{\left(N_{1}+N_{3}\right) \times\left(n_{1}+n_{2}+n_{3}\right)}$ as putting $n_{2}$ zero columns between the first $n_{1}$ and the last $n_{3}$ columns in $\operatorname{diag}(A, B) \in$ $\mathbb{F}^{\left(N_{1}+N_{3}\right) \times\left(n_{1}+n_{3}\right)}$.

Fix an ordered basis $\mathcal{A}=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right\}$ of $\mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$. For any non-negative integer $s$, we denote by $M_{\mathcal{A}}: \mathbb{F}_{q^{m}}^{s} \longrightarrow$ $\mathbb{F}_{q}^{m \times s}$ the corresponding matrix representation map, given by

$$
M_{\mathcal{A}}(\mathbf{c})=\left(\begin{array}{cccc}
c_{11} & c_{12} & \ldots & c_{1 s}  \tag{3}\\
c_{21} & c_{22} & \ldots & c_{2 s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m 1} & c_{m 2} & \ldots & c_{m s}
\end{array}\right)
$$

for $\mathbf{c}=\left(c_{1}, c_{2}, \ldots, c_{s}\right) \in \mathbb{F}_{q^{m}}^{s}$, where $c_{i, 1}, c_{i, 2}, \ldots, c_{i, m} \in \mathbb{F}_{q}$ are the unique scalars such that $c_{i}=\sum_{j=1}^{m} \alpha_{j} c_{i, j} \in \mathbb{F}_{q^{m}}$, for $i=1,2, \ldots, s$.

Given $X \in \mathbb{F}_{q}^{m \times n}$, we denote by $\operatorname{Row}(X) \subseteq \mathbb{F}_{q}^{n}$ and $\operatorname{Col}(X) \subseteq \mathbb{F}_{q}^{m}$ the vector spaces generated by the rows and the columns of $X$, respectively. For $\mathbf{c} \in \mathbb{F}_{q^{m}}^{n}$, we denote $\operatorname{Row}(\mathbf{c})=\operatorname{Row}\left(M_{\mathcal{A}}(\mathbf{c})\right) \subseteq \mathbb{F}_{q}^{n}$ and $\operatorname{Col}(\mathbf{c})=\operatorname{Col}\left(M_{\mathcal{A}}(\mathbf{c})\right) \subseteq$ $\mathbb{F}_{q}^{m}$. The latter depends on $\mathcal{A}$, but we omit this for simplicity.
Throughout the paper, we will use the short notation $\left(x_{i}\right)_{i=1}^{\ell}=\left(x_{1}, x_{2}, \ldots, x_{\ell}\right)$ for tuples, where $x_{1}, x_{2}, \ldots, x_{\ell}$ could be any type of element (integers, elements of finite fields, sets, subspaces...).

## II. Reliability and Security in Adversarial Multishot Network Coding

In this section, we will formulate reliability and security in multishot network coding under a worst-case adversarial model. Our analysis naturally extends that in [19] to the multishot case (see also [16]-[18]).

We will consider $\ell$ shots of a network where linear network coding over the finite field $\mathbb{F}_{q}$ is implemented [1]-[3], where we have no knowledge of or control over the network topology or the encoding coefficients (thus random linear network coding [4] may be implemented). The network topology may change with time, the encoding coefficients at each node are independent for different shots, and we assume no delays. The packets sent in each shot through the links of the network and linearly combined at each node are vectors in $\mathbb{F}_{q}^{m}$. We assume that an adversary is able to inject error packets in up to $t$ links, modify transfer matrices to erase up to $\rho$ encoded packets, and wire-tap up to $\mu$ links, all distributed over the $\ell$ shots of the network as the adversary wishes (see Fig. 11. Other than these restrictions, the adversary is assumed to be omniscient (he or she knows the coding scheme used at the source, the network topology and all encoding coefficients in the network) and have unlimited computational power.

Fix $i=1,2, \ldots, \ell$. Assume that the $i$ th shot has as input a matrix $X_{i} \in \mathbb{F}_{q}^{m \times n_{i}}$. Define $t_{i} \geq 0$ and $\rho_{i} \geq 0$ as the number of link errors and erasures, respectively, that the adversary introduces, and $\mu_{i} \geq 0$ as the number of links that the adversary wire-taps, all in the $i$ th shot. These numbers are thus arbitrary with sums $t, \rho$ and $\mu$, respectively, and only these sums are known or estimated by the source or sinks.

Following the model in [19. Sec. III], the output to the receiver is a matrix

$$
Y_{i}=X_{i} A_{i}^{T}+E_{i} \in \mathbb{F}_{q}^{m \times N_{i}}
$$

for a transfer matrix $A_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$ with $\operatorname{Rk}\left(A_{i}\right) \geq n_{i}-\rho_{i}$, and for an error matrix $E_{i} \in \mathbb{F}_{q}^{m \times N_{i}}$ with $\operatorname{Rk}\left(E_{i}\right) \leq t_{i}$, where equalities can be achieved. The transfer matrix $A_{i}$ may be known by the receiver (coherent communication) or not (non-coherent communication). The non-coherent case is more realistic if random linear network coding is applied. However, we include both since, as we will see, optimal solutions for the coherent case, which is simpler to solve, will give near optimal solutions for the non-coherent case. Following the same model, the adversary obtains the matrix

$$
W_{i}=X_{i} B_{i}^{T} \in \mathbb{F}_{q}^{m \times \mu_{i}}
$$

for a wire-tap transfer matrix $B_{i} \in \mathbb{F}_{q}^{\mu_{i} \times n_{i}}$.
Define now the set of input codewords, the set of output words to the receiver, and the set of output messages to the wire-tapper, respectively, as follows:

$$
\begin{aligned}
\mathcal{X} & =\mathbb{F}_{q}^{m \times n_{1}} \times \mathbb{F}_{q}^{m \times n_{2}} \times \cdots \times \mathbb{F}_{q}^{m \times n_{\ell}} \\
\mathcal{Y} & =\mathbb{F}_{q}^{m \times N_{1}} \times \mathbb{F}_{q}^{m \times N_{2}} \times \cdots \times \mathbb{F}_{q}^{m \times N_{\ell}} \\
\mathcal{W} & =\bigcup_{\substack{0 \leq \mu_{i} \leq n_{i} \\
1 \leq i \leq \ell}} \mathbb{F}_{q}^{m \times \mu_{1}} \times \mathbb{F}_{q}^{m \times \mu_{2}} \times \cdots \times \mathbb{F}_{q}^{m \times \mu_{\ell}}
\end{aligned}
$$

Assume that the sent codeword is $X \in \mathcal{X}$. With the previous restrictions on the adversary, in the case of coherent communication with transfer matrices $A_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$, for $i=1,2, \ldots, \ell$, such that

$$
\sum_{i=1}^{\ell} \operatorname{Rk}\left(A_{i}\right) \geq n-\rho
$$

the output of the $\ell$-shot network is restricted to the subset

$$
\begin{aligned}
\mathcal{Y}_{X}^{A_{1}, A_{2}, \ldots, A_{\ell}}(t)= & \left\{Y \in \mathcal{Y} \mid Y_{i}=X_{i} A_{i}^{T}+E_{i}\right. \\
& \left.E_{i} \in \mathbb{F}_{q}^{m \times N_{i}}, \sum_{i=1}^{\ell} \operatorname{Rk}\left(E_{i}\right) \leq t\right\}
\end{aligned}
$$

In the case of non-coherent communication, the output is instead restricted to the subset

$$
\begin{aligned}
\mathcal{Y}_{X}(t, \rho)= & \left\{Y \in \mathcal{Y} \mid Y_{i}=X_{i} A_{i}^{T}+E_{i}\right. \\
& A_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}, \sum_{i=1}^{\ell} \operatorname{Rk}\left(A_{i}\right) \geq n-\rho, \\
& \left.E_{i} \in \mathbb{F}_{q}^{m \times N_{i}}, \sum_{i=1}^{\ell} \operatorname{Rk}\left(E_{i}\right) \leq t\right\}
\end{aligned}
$$

Finally, the output to the adversary is restricted to the subset

$$
\begin{aligned}
& \mathcal{W}_{X}(\mu)=\left\{W \in \mathcal{W} \mid W_{i}=X_{i} B_{i}^{T}\right. \\
&\left.B_{i} \in \mathbb{F}_{q}^{\mu_{i} \times n_{i}}, \sum_{i=1}^{\ell} \mu_{i}=\mu\right\}
\end{aligned}
$$

Remark 1. Observe that by the matrix representation map (3), we may consider $\mathcal{X}=\mathbb{F}_{q^{m}}^{n}$ and $\mathcal{Y}=\mathbb{F}_{q^{m}}^{N}$, which are vector spaces over $\mathbb{F}_{q^{m}}$. We may also consider $\mathcal{W}=\mathbb{F}_{q^{m}}^{\mu}$,
but the union symbol above expresses the fact that we do not know how many links the wire-tapper observes in each shot.

In this work, we will consider coding schemes as follows.
Definition 1 (Coding schemes). Let $\mathcal{S}$ be the set of secret messages and let $S \in \mathcal{S}$ be a random variable in $\mathcal{S}$. A coding scheme is a randomized function $F: \mathcal{S} \longrightarrow \mathcal{X}$. For unique decoding, we assume that $\mathcal{X}_{S} \cap \mathcal{X}_{T}=\varnothing$ if $S, T \in \mathcal{S}$ and $S \neq T$, where

$$
\mathcal{X}_{S}=\{X \in \mathcal{X} \mid P(X \mid S)>0\}
$$

for $S \in \mathcal{S}$. We define the support scheme of $F$ as the collection of disjoint sets $\operatorname{Supp}(F)=\left\{\mathcal{X}_{S}\right\}_{S \in \mathcal{S}}$.

As in [19], we will require zero-error communication and perfect secrecy. We can adapt the definition of universal reliable and secure coding schemes from [19] to multishot network coding as follows.

Definition 2. With notation as in the previous definition and for integers $t \geq 0, \rho \geq 0$ and $\mu \geq 0$, we say that a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$ is

1) $t$-error and $\rho$-erasure-correcting for coherent communication if for all transfer matrices $A_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$, for $i=1,2, \ldots, \ell$, with $\sum_{i=1}^{\ell} \operatorname{Rk}\left(A_{i}\right) \geq n-\rho$, there exists a decoding function $D: \mathcal{Y} \longrightarrow \mathcal{S}$, which may depend on $A_{1}, A_{2}, \ldots, A_{\ell}$, such that

$$
D(Y)=S
$$

for all $Y \in \bigcup_{X \in \mathcal{X}_{S}} \mathcal{Y}_{X}^{A_{1}, A_{2}, \ldots, A_{\ell}}(t)$ and all $S \in \mathcal{S}$.
2) $t$-error and $\rho$-erasure-correcting for non-coherent communication if there exists a decoding function $D: \mathcal{Y} \longrightarrow$ $\mathcal{S}$ such that

$$
D(Y)=S
$$

for all $Y \in \bigcup_{X \in \mathcal{X}_{S}} \mathcal{Y}_{X}(t, \rho)$ and all $S \in \mathcal{S}$.
3) Secure under $\mu$ observations if it holds that

$$
H(S \mid W)=H(S)
$$

or equivalently $I(S ; W)=0$, for all $W \in$ $\bigcup_{X \in \mathcal{X}_{S}} \mathcal{W}_{X}(\mu)$ and all $S \in \mathcal{S}$.
We conclude by recalling how to construct coding schemes using nested linear code pairs. The idea goes back to [40] for the wire-tap channel of type II.

Definition 3 (Nested coset coding schemes). Given nested linear codes $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq \mathbb{F}_{q^{m}}^{n}$ and $\mathcal{S}=\mathbb{F}_{q^{m}}^{k}$, where $k=\operatorname{dim}\left(\mathcal{C}_{1}\right)-\operatorname{dim}\left(\mathcal{C}_{2}\right)$, we define its corresponding coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$ as follows. Choose a vector space $\mathcal{V}$ such that $\mathcal{C}_{1}=\mathcal{C}_{2} \oplus \mathcal{V}$, where $\oplus$ denotes the direct sum of vector spaces, and a vector space isomorphism $\psi: \mathbb{F}_{q^{m}}^{k} \longrightarrow \mathcal{V}$. Then we define $F$ as any randomized function such that $\mathcal{X}_{S}=\psi(S)+\mathcal{C}_{2}$, for $S \in \mathbb{F}_{q^{m}}^{k}$.

Observe that $\mathcal{C}_{2}=\{\mathbf{0}\}$ corresponds to using a single linear code and a linear deterministic coding scheme. This is suitable for reliability but not for security. The other extreme case, $\mathcal{C}_{1}=\mathbb{F}_{q^{m}}^{n}$, is suitable for security but not for reliability.

## III. Measures of Reliability and Security

In this section, we will extend the studies in [18], [19] regarding what parameter of a coding scheme gives a necessary and sufficient condition for it to be $t$-error and $\rho$ -erasure-correcting, or secure under $\mu$ observations. In Subsection III-A, we study error and erasure correction for coherent communication. In Subsection III-B we study error and erasure correction for non-coherent communication. Finally, in Subsection III-C, we study security resistance against a wiretapper. Since the proofs are natural extensions of those in [18], [19], we have moved them to the appendix.

## A. The Sum-rank Metric and Coherent Communication

We start by defining the sum-rank metric in $\mathbb{F}_{q^{m}}^{n}$, which was introduced in [25]. It was implicitly considered earlier in the space-time coding literature (see [44, Sec. III]).
Definition 4 (Sum-rank metric [25]). Let $\mathbf{c}=\left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}\right.$, $\left.\ldots, \mathbf{c}^{(\ell)}\right) \in \mathbb{F}_{q^{m}}^{n}$, where $\mathbf{c}^{(i)} \in \mathbb{F}_{q^{m}}^{n_{i}}$, for $i=1,2, \ldots, \ell$. We define the sum-rank weight of $\mathbf{c}$ as

$$
\mathrm{wt}_{S R}(\mathbf{c})=\sum_{i=1}^{\ell} \operatorname{Rk}\left(M_{\mathcal{A}}\left(\mathbf{c}^{(i)}\right)\right)
$$

where $M_{\mathcal{A}}$ is as in (3), taking $s=n_{i}$ for $\mathbf{c}^{(i)}$. Finally, we define the sum-rank metric $\mathrm{d}_{S R}:\left(\mathbb{F}_{q^{m}}^{n}\right)^{2} \longrightarrow \mathbb{N}$ as

$$
\mathrm{d}_{S R}(\mathbf{c}, \mathbf{d})=\mathrm{wt}_{S R}(\mathbf{c}-\mathbf{d}),
$$

for all $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^{m}}^{n}$.
Observe that indeed the sum-rank metric is a metric, and in particular, it satisfies the triangle inequality. Observe also that sum-rank weights and metrics depend on the decomposition $n=n_{1}+n_{2}+\cdots+n_{\ell}$ and the subfield $\mathbb{F}_{q} \subseteq \mathbb{F}_{q^{m}}$. However, we do not write this dependency for brevity.

Next we define minimum sum-rank distances of supports of coding schemes.

Definition 5 (Minimum sum-rank distance). Given a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$, we define its minimum sum-rank distance as that of its support scheme (see Definition 1):
$\mathrm{d}_{S R}(\operatorname{Supp}(F))=\min \left\{\mathrm{d}_{S R}(\mathbf{c}, \mathbf{d}) \mid \mathbf{c} \in \mathcal{X}_{S}, \mathbf{d} \in \mathcal{X}_{T}, S \neq T\right\}$.
When $F$ is constructed from nested linear codes $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq$ $\mathbb{F}_{q^{m}}^{n}$, its minimum sum-rank distance is the relative minimum sum-rank distance of the codes:

$$
\mathrm{d}_{S R}\left(\mathcal{C}_{1}, \mathcal{C}_{2}\right)=\min \left\{\mathrm{wt}_{S R}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C}_{1} \backslash \mathcal{C}_{2}\right\}
$$

The minimum sum-rank distance of a single linear code $\mathcal{C} \subseteq$ $\mathbb{F}_{q^{m}}^{n}$ is $\mathrm{d}_{S R}(\mathcal{C})=\mathrm{d}_{S R}(\mathcal{C},\{\mathbf{0}\})$.

We now give a sufficient and necessary condition for coding schemes to be $t$-error and $\rho$-erasure-correcting for coherent communication, for any non-negative integers $t$ and $\rho$. The sufficient part has already been proven in [25] Th. 1] for a deterministic code. We will make use of both implications in the proof of Theorem 5 .

Theorem 1. Given integers $t \geq 0$ and $\rho \geq 0$, a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$ is t-error and $\rho$-erasure-correcting for coherent communication if, and only if,

$$
2 t+\rho<\mathrm{d}_{S R}(\operatorname{Supp}(F))
$$

Proof. See Appendix A

## B. The Sum-injection and Sum-subspace Distances and Noncoherent Communication

Let $\mathbf{n}=\left(n_{1}, n_{2}, \ldots, n_{\ell}\right)$. We will consider the Cartesian product lattice

$$
\mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{n}}\right)=\mathcal{P}\left(\mathbb{F}_{q}^{n_{1}}\right) \times \mathcal{P}\left(\mathbb{F}_{q}^{n_{2}}\right) \times \cdots \times \mathcal{P}\left(\mathbb{F}_{q}^{n_{\ell}}\right)
$$

where $\mathcal{P}\left(\mathbb{F}_{q}^{n_{i}}\right)$ is the lattice of vector subspaces of $\mathbb{F}_{q}^{n_{i}}$ (that is, the collection of all vector subspaces of $\mathbb{F}_{q}^{n_{i}}$ considered with sums and intersections of vector subspaces as operations), for $i=1,2, \ldots, \ell$. We also denote $\mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$ for $\mathbf{n}=(m, m, \ldots, m)$, of length $\ell$. These Cartesian products were first considered for multishot network coding in [24, Subsec. II-B].

As in the single shot case [16], the actual information preserved in a non-coherent multishot linear coded network is the list of column spaces of the transmitted matrices. We may justify this by extending the argument in [18, Subsec. VA]. We start by connecting codewords in $\mathbb{F}_{q^{m}}^{n}$ to elements in the lattice $\mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$.
Definition 6. Given $\mathbf{c}=\left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(\ell)}\right) \in \mathbb{F}_{q^{m}}^{n}$, where $\mathbf{c}^{(i)} \in \mathbb{F}_{q^{m}}^{n_{i}}$, for $i=1,2, \ldots, \ell$, we define

$$
\operatorname{Col}_{\Sigma}(\mathbf{c})=\left(\operatorname{Col}\left(\mathbf{c}^{(i)}\right)\right)_{i=1}^{\ell} \in \mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}
$$

For a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$, we define its sum-subspace support scheme as $\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))=\left\{\operatorname{Col}_{\Sigma}\left(\mathcal{X}_{S}\right)\right\}_{S \in \mathcal{S}}$, where

$$
\operatorname{Col}_{\Sigma}\left(\mathcal{X}_{S}\right)=\left\{\operatorname{Col}_{\Sigma}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{X}_{S}\right\}
$$

The error and erasure-correction capability of coding schemes will be measured by an extension of the injection distance [18, Def. 2] to the multishot scenario.
Definition 7 (Sum-injection distance). Given $\mathcal{U}=\left(\mathcal{U}_{1}, \mathcal{U}_{2}\right.$, $\left.\ldots, \mathcal{U}_{\ell}\right), \mathcal{V}=\left(\mathcal{V}_{1}, \mathcal{V}_{2}, \ldots, \mathcal{V}_{\ell}\right) \in \mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$, we define their sum-injection distance as

$$
\begin{aligned}
\mathrm{d}_{S I}(\mathcal{U}, \mathcal{V}) & =\sum_{i=1}^{\ell} \mathrm{d}_{I}\left(\mathcal{U}_{i}, \mathcal{V}_{i}\right) \\
& =\sum_{i=1}^{\ell}\left(\operatorname{dim}\left(\mathcal{U}_{i}+\mathcal{V}_{i}\right)-\min \left\{\operatorname{dim}\left(\mathcal{U}_{i}\right), \operatorname{dim}\left(\mathcal{V}_{i}\right)\right\}\right) \\
& =\sum_{i=1}^{\ell}\left(\max \left\{\operatorname{dim}\left(\mathcal{U}_{i}\right), \operatorname{dim}\left(\mathcal{V}_{i}\right)\right\}-\operatorname{dim}\left(\mathcal{U}_{i} \cap \mathcal{V}_{i}\right)\right) .
\end{aligned}
$$

For a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$, we define its minimum sum-injection distance as

$$
\begin{aligned}
\mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))\right)=\min \{ & \mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}(\mathbf{c}), \operatorname{Col}_{\Sigma}(\mathbf{d})\right) \\
& \left.\mid \mathbf{c} \in \mathcal{X}_{S}, \mathbf{d} \in \mathcal{X}_{T}, S \neq T\right\} .
\end{aligned}
$$

The following result extends [18, Th. 20] to the multishot scenario.

Theorem 2. Given integers $t \geq 0$ and $\rho \geq 0$, a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$ is t-error and $\rho$-erasure-correcting for noncoherent communication if, and only if,

$$
2 t+\rho<\mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))\right)
$$

In particular, it must hold that $\operatorname{Col}_{\Sigma}(\mathbf{c}) \neq \operatorname{Col}_{\Sigma}(\mathbf{d})$ if $\mathbf{c} \in \mathcal{X}_{S}$, $\mathbf{d} \in \mathcal{X}_{T}$ and $S \neq T$.

## Proof. See Appendix A.

As in the single shot case [16], the number of packets injected in the $i$ th shot by a codeword $\mathbf{c}$, with $\operatorname{Col}_{\Sigma}(\mathbf{c})=$ $\mathcal{U} \in \mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$, coincides with $\operatorname{dim}\left(\mathcal{U}_{i}\right)$. If all codewords inject the same number of packets in a given shot, we may say that the coding scheme is sum-constant-dimension. For such coding schemes, the sum-injection distance coincides with the sumsubspace distance introduced in [24, Eq. (2)].
Definition 8 (Sum-subspace distance [24]). Given $\mathcal{U}, \mathcal{V} \in$ $\mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$, we define their sum-subspace distance as

$$
\begin{aligned}
\mathrm{d}_{S S}(\mathcal{U}, \mathcal{V}) & =\sum_{i=1}^{\ell} \mathrm{d}_{S}\left(\mathcal{U}_{i}, \mathcal{V}_{i}\right) \\
& =\sum_{i=1}^{\ell}\left(\operatorname{dim}\left(\mathcal{U}_{i}+\mathcal{V}_{i}\right)-\operatorname{dim}\left(\mathcal{U}_{i} \cap \mathcal{V}_{i}\right)\right)
\end{aligned}
$$

For a coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$, we define its minimum sum-subspace distance, denoted by $\mathrm{d}_{S S}\left(\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))\right)$, analogously to their minimum sum-injection distance (Definition 7).

Observe that by [18, Eq. (28)], it holds that

$$
\begin{equation*}
\mathrm{d}_{S I}(\mathcal{U}, \mathcal{V})=\frac{1}{2} \mathrm{~d}_{S S}(\mathcal{U}, \boldsymbol{V})+\frac{1}{2} \sum_{i=1}^{\ell}\left|\operatorname{dim}\left(\mathcal{U}_{i}\right)-\operatorname{dim}\left(\mathcal{V}_{i}\right)\right| \tag{4}
\end{equation*}
$$

for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}\left(\mathbb{F}_{q}^{m}\right)^{\ell}$. Hence as explained earlier, for sum-constant-dimension codes we may simply consider its minimum sum-subspace distance. Then the sufficient part in Theorem 2 was already proven in [25, Th. 1].

## C. Measuring Security Resistance Against a Wire-tapper

In this subsection, we give a sufficient and necessary condition for coding schemes built from nested linear code pairs to be secure under a given number of observations. We start by estimating the information leakage to the wire-tapper. This is a natural extension of [19, Lemma 6]. We will however follow the steps in the proof of [45, Prop. 16] (see also [46, Lemma 7]).
Lemma 1. Let $F: \mathcal{S} \longrightarrow \mathcal{X}$ be a coding scheme built from nested linear codes $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq \mathbb{F}_{q^{m}}^{n}$ as in Definition 3 Assume that $X=F(S)$ is the uniform random variable in $\mathcal{X}_{S}$ given $S \in \mathcal{S}$. Let $B_{i} \in \mathbb{F}_{q}^{\mu_{i} \times n_{i}}$ and $\mathcal{L}_{i}=\operatorname{Row}\left(B_{i}\right) \subseteq \mathbb{F}_{q}^{n_{i}}$, for $i=1,2, \ldots, \ell$, define $B=\operatorname{diag}\left(B_{1}, B_{2}, \ldots, B_{\ell}\right) \in \mathbb{F}_{q}^{\mu \times n}$, $\mathcal{L}=\left(\mathcal{L}_{1}, \mathcal{L}_{2}, \ldots, \mathcal{L}_{\ell}\right) \in \mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{n}}\right)$ and

$$
\mathcal{V}_{\mathcal{L}}=\left\{\mathbf{c} \in \mathbb{F}_{q^{m}}^{n} \mid \operatorname{Row}\left(\mathbf{c}^{(i)}\right) \subseteq \mathcal{L}_{i}, 1 \leq i \leq \ell\right\}
$$

Taking logarithms with base $q^{m}$ in entropy and mutual information, it holds that

$$
\begin{equation*}
I\left(S ; X B^{T}\right) \leq \operatorname{dim}\left(\mathcal{C}_{2}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right)-\operatorname{dim}\left(\mathcal{C}_{1}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right) \tag{5}
\end{equation*}
$$

and equality holds if $S$ is the uniform random variable in $\mathcal{S}$.

## Proof. See Appendix A

Thus we may conclude the following.
Theorem 3. Let $F: \mathcal{S} \longrightarrow \mathcal{X}$ be a coding scheme built from nested linear codes $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq \mathbb{F}_{q^{m}}^{n}$ as in Definition 3 Assume that $F(S)$ is the uniform random variable in $\mathcal{X}_{S}$ given $S \in \mathcal{S}$. Given an integer $\mu \geq 0$, the coset coding scheme is secure under $\mu$ observations if

$$
\mu<\mathrm{d}_{S R}\left(\mathcal{C}_{2}^{\perp}, \mathcal{C}_{1}^{\perp}\right)
$$

The reversed implication also holds if $S$ is the uniform random variable in $\mathcal{S}$.

Proof. This follows by combining the previous lemma with the straightforward fact that $\mu<\mathrm{d}_{S R}\left(\mathcal{C}_{2}^{\perp}, \mathcal{C}_{1}^{\perp}\right)$ if, and only if, $\mathcal{C}_{2}^{\perp} \cap \mathcal{V}_{\mathcal{L}}=\mathcal{C}_{1}^{\perp} \cap \mathcal{V}_{\mathcal{L}}$ for all $\mathcal{L} \in \mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{n}}\right)$ such that $\sum_{i=1}^{\ell} \operatorname{dim}\left(\mathcal{L}_{i}\right) \leq \mu$.

## IV. Coding Schemes Based on Linearized Reed-Solomon Codes

In this section, we will introduce a family of coding schemes whose secret message entropy is the maximum possible for coherent communication, for any fixed bound on the number of errors, erasures and wire-tapped links, with the restrictions $1 \leq \ell \leq q-1$ and $1 \leq n_{i} \leq m$, for $i=1,2, \ldots, \ell$. In particular, $n \leq(q-1) m$. Assuming the secret message is a uniform variable, these coding schemes have maximum secret message size for the largest possible range of packet lengths.

By a process analogous to lifting [17] Subsec. IV-A], we will also provide coding schemes with nearly optimal information rate for non-coherent communication, for the same parameters, among sum-constant-dimension codes.

The building blocks of these coding schemes are linearized Reed-Solomon codes, introduced in [36, Def. 31]. We review these codes in Subsection IV-A, and we compute their duals. We then give upper bounds on the message entropy or size and show that the above-mentioned coding schemes attain them in the coherent case (Subsection IV-B), and are close in the noncoherent case (Subsection IV-C).

## A. Linearized Reed-Solomon Codes and their Duals

Throughout this subsection, we assume that $1 \leq \ell \leq q-1$ and $1 \leq n_{i} \leq m$, for $i=1,2, \ldots, \ell$. Therefore $n \leq(q-1) m$.

Let $\sigma: \mathbb{F}_{q^{m}} \longrightarrow \mathbb{F}_{q^{m}}$ be the field automorphism given by $\sigma(a)=a^{q^{r}}$, for all $a \in \mathbb{F}_{q^{m}}$, where $1 \leq r \leq m$ and $\operatorname{gcd}(r, m)=1$ (i.e., $\left\{a \in \mathbb{F}_{q^{m}} \mid \sigma(a)=a\right\}=\mathbb{F}_{q}$ ). We may set for simplicity $r=1$, but we need to consider the general case later to include dual codes. We need to define linear operators as in [36, Def. 20]. Over finite fields (our case), polynomials in these operators can be regarded as linearized polynomials [47, Ch. 3].

Definition 9 (Linear operators [36|). Fix $a \in \mathbb{F}_{q^{m}}$, and define its $i$ th norm as $N_{i}(a)=\sigma^{i-1}(a) \cdots \sigma(a) a$. Now define the $\mathbb{F}_{q}$-linear operator $\mathcal{D}_{a}^{i}: \mathbb{F}_{q^{m}} \longrightarrow \mathbb{F}_{q^{m}}$ by

$$
\begin{equation*}
\mathcal{D}_{a}^{i}(b)=\sigma^{i}(b) N_{i}(a) \tag{6}
\end{equation*}
$$

for all $b \in \mathbb{F}_{q^{m}}$, and all $i \in \mathbb{N}$. Define also $\mathcal{D}_{a}=\mathcal{D}_{a}^{1}$ and observe that $\mathcal{D}_{a}^{i+1}=\mathcal{D}_{a} \circ \mathcal{D}_{a}^{i}$, for $i \in \mathbb{N}$. We will write $N_{i}^{\sigma}$ and $\mathcal{D}_{a}^{\sigma}$ when it is not clear which automorphism $\sigma$ we are using.

We also need the concept of conjugacy in $\mathbb{F}_{q^{m}}$, which was given in [48] (see also [49, Eq. (2.5)]).

Definition 10 (Conjugacy [48]). Given $a, b \in \mathbb{F}_{q^{m}}$, we say that they are conjugates if there exists $c \in \mathbb{F}_{q^{m}}^{*}$ such that

$$
b=\sigma(c) c^{-1} a
$$

This defines an equivalence relation on $\mathbb{F}_{q^{m}}$, and thus a partition of $\mathbb{F}_{q^{m}}$ into conjugacy classes. Take now a primitive element $\gamma$ of $\mathbb{F}_{q^{m}}$, meaning that $\mathbb{F}_{q^{m}}^{*}=\left\{\gamma^{0}, \gamma^{1}, \gamma^{2} \ldots, \gamma^{q^{m}-2}\right\}$ (see [50, page 97]), and observe that

$$
\gamma^{j} \neq \sigma(c) c^{-1} \gamma^{i}
$$

for all $c \in \mathbb{F}_{q^{m}}^{*}$ and all $0 \leq i<j \leq q-2$. Hence $\gamma^{0}, \gamma^{1}, \gamma^{2}, \ldots, \gamma^{q-2}$ constitute representatives of disjoint conjugacy classes. Moreover, one can easily see that they represent all conjugacy classes except the trivial one $\{0\}$. Thus the following definition is a particular case of [36, Def. 31].

Definition 11 (Linearized Reed-Solomon codes [36]). Fix linearly independent sets $\mathcal{B}^{(i)}=\left\{\beta_{1}^{(i)}, \beta_{2}^{(i)}, \ldots, \beta_{n_{i}}^{(i)}\right\} \subseteq \mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$, for $i=1,2, \ldots, \ell$, and denote $\mathcal{B}=\left(\mathcal{B}^{(1)}, \mathcal{B}^{(2)}, \ldots\right.$, $\left.\mathcal{B}^{(\ell)}\right)$. Here, we emphasize that the elements in $\mathcal{B}^{(i)}$ are linearly independent, but there may be linear dependencies between elements in $\mathcal{B}^{(i)}$ and elements in $\bigcup_{j \neq i} \mathcal{B}^{(j)}$. In particular, it is possible to take $\mathcal{B}^{(1)}=\mathcal{B}^{(2)}=\ldots=\mathcal{B}^{(\ell)}$. Let $\gamma$ be a primitive element of $\mathbb{F}_{q^{m}}$. For $k=0,1,2, \ldots, n$, we define the linearized Reed-Solomon code of dimension $k$ as the linear code $\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma) \subseteq \mathbb{F}_{q^{m}}^{n}$ with generator matrix given by

$$
\begin{equation*}
D=\left(D_{1}\left|D_{2}\right| \ldots \mid D_{\ell}\right) \in \mathbb{F}_{q^{m}}^{k \times n} \tag{7}
\end{equation*}
$$

where
$D_{i}=\left(\begin{array}{cccc}\beta_{1}^{(i)} & \beta_{2}^{(i)} & \ldots & \beta_{n_{i}}^{(i)} \\ \mathcal{D}_{\gamma^{i-1}}^{(i)}\left(\beta_{1}^{(i)}\right) & \mathcal{D}_{\gamma^{i-1}}\left(\beta_{2}^{(i)}\right) & \ldots & \mathcal{D}_{\gamma^{i-1}}\left(\beta_{n_{i}}^{(i)}\right) \\ \mathcal{D}_{\gamma^{i-1}}^{2}\left(\beta_{1}^{(i)}\right) & \mathcal{D}_{\gamma^{i-1}}^{2}\left(\beta_{2}^{(i)}\right) & \ldots & \mathcal{D}_{\gamma^{i-1}}^{2}\left(\beta_{n_{i}}^{(i)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_{\gamma^{i-1}}^{k-1}\left(\beta_{1}^{(i)}\right) & \mathcal{D}_{\gamma^{i-1}}^{k-1}\left(\beta_{2}^{(i)}\right) & \ldots & \mathcal{D}_{\gamma^{i-1}}^{k-1}\left(\beta_{n_{i}}^{(i)}\right)\end{array}\right)$
for $i=1,2, \ldots, \ell$.
By [36, Prop. 33] (see also Subsection V-A), this code is isomorphic as a vector space to a $k$-dimensional skew Reed-Solomon code [37, Def. 7]. In particular, it is also $k$ dimensional and the generator matrix in (7) has full rank. Moreover, linearized Reed-Solomon codes are maximum sumrank distance codes, which was proven in [36]. The next result combines [36, Prop. 34] and [36, Th. 4].

Proposition 1 ([36]). For a linear code $\mathcal{C} \subseteq \mathbb{F}_{q^{m}}^{n}$ of dimension $k$, it holds that

$$
\mathrm{d}_{S R}(\mathcal{C}) \leq n-k+1
$$

Moreover, equality holds for $\mathcal{C}=\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)$ as in Definition 11 That is, linearized Reed-Solomon codes are maximum sum-rank distance (MSRD) codes.

Observe that Gabidulin codes [20], [21] and their extension [39] are obtained as particular cases by choosing $\ell=1$ (thus $\left.n=n_{1}\right)$ : In that case, we have that $N_{j}\left(\gamma^{0}\right)=N_{j}(1)=1$, hence

$$
\mathcal{D}_{\gamma^{0}}^{j}\left(\beta_{i}^{(1)}\right)=\sigma^{j}\left(\beta_{i}^{(1)}\right)
$$

for all $i=1,2, \ldots, n$ and $j=0,1, \ldots, k-1$.
One can also recover Reed-Solomon codes [22] and generalized Reed-Solomon codes [23] by setting $\sigma=\mathrm{Id}$ or $m=1$ (thus $n_{1}=n_{2}=\ldots=n_{\ell}=1$ ): In that case, we have that $N_{j}\left(\gamma^{i-1}\right)=\left(\gamma^{i-1}\right)^{j}$ and $\sigma\left(\beta_{1}^{(i)}\right)=\beta_{1}^{(i)}$, hence

$$
\mathcal{D}_{\gamma^{i-1}}^{j}\left(\beta_{1}^{(i)}\right)=\beta_{1}^{(i)}\left(\gamma^{i-1}\right)^{j},
$$

for $i=1,2, \ldots, \ell$ and $j=0,1, \ldots, k-1$. This explains the discussion in Section [

Intuitively, the conjugacy representative $\gamma^{i-1}$ and the linear operator $\mathcal{D}_{\gamma^{i-1}}$ are used in the $i$ th shot of the network as a Gabidulin code, whereas using different conjugacy classes allows to correct link errors globally, seeing the code blockwise as a Reed-Solomon code (observe the correspondence between Fig. 1 and the generator matrix (7)). See also Table II in Subsection V-A for a summary of maximum-distance evaluation codes of Reed-Solomon type for different metrics.

We now prove that the duals of these codes are again linearized Reed-Solomon codes, thus also maximum sum-rank distance, which was not proven in [36]. This result extends [23, Th. 1], [20, Th. 7] and [39, Th. 2]. We will need it to obtain optimal secure coding schemes in view of Theorem 3. It also has the advantage of providing explicit parity-check matrices for linearized Reed-Solomon codes in the form of (7).

Theorem 4. For $k=0,1,2, \ldots, n$, if $\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)$ is as in Definition 11] then there exist linearly independent sets $\mathcal{A}^{(i)}=$ $\left\{\alpha_{1}^{(i)}, \alpha_{2}^{(i)}, \ldots, \alpha_{n_{i}}^{(i)}\right\} \subseteq \mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$, for $i=1,2, \ldots, \ell$, such that

$$
\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)^{\perp}=\mathcal{C}_{L, n-k}^{\sigma^{-1}}\left(\mathcal{A}, \sigma^{-1}(\gamma)\right)
$$

where $\mathcal{A}=\left(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(\ell)}\right)$, and where $\sigma^{-1}(\gamma)$ is a primitive element of $\mathbb{F}_{q^{m}}$. In particular, $\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)^{\perp}$ is also an MSRD code.

Proof. First $\mathcal{C}_{L, n-1}^{\sigma}(\mathcal{B}, \gamma)^{\perp}$ is generated by one vector $\boldsymbol{\alpha}=$ $\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \ldots, \boldsymbol{\alpha}^{(\ell)}\right) \in \mathbb{F}_{q^{m}}^{n}$, where $\boldsymbol{\alpha}^{(i)} \in \mathbb{F}_{q^{m}}^{n_{i}}$, for $i=1,2, \ldots, \ell$. Assume that there exists $i$ such that $\alpha_{1}^{(i)}, \alpha_{2}^{(i)}, \ldots, \alpha_{n_{i}}^{(i)}$ are linearly dependent over $\mathbb{F}_{q}$. Then there exists a non-zero $\boldsymbol{\lambda} \in \mathbb{F}_{q}^{n}$ with sum-rank weight equal to 1 , such that $\boldsymbol{\alpha} \boldsymbol{\lambda}^{T}=0$. This implies that $\boldsymbol{\lambda} \in \mathcal{C}_{L, n-1}^{\sigma}(\mathcal{B}, \gamma)$, but $\mathrm{d}_{S R}\left(\mathcal{C}_{L, n-1}^{\sigma}(\mathcal{B}, \gamma)\right)=2$, which is a contradiction. Thus $\alpha_{1}^{(i)}, \alpha_{2}^{(i)}, \ldots, \alpha_{n_{i}}^{(i)}$ are linearly independent over $\mathbb{F}_{q}$, for $i=$
$1,2, \ldots, \ell$. Now by definition, and denoting $\mathcal{D}_{\gamma^{i-1}}=\mathcal{D}_{\gamma^{i-1}}^{\sigma}$, we have that

$$
\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}} \alpha_{j}^{(i)} \mathcal{D}_{\gamma^{i-1}}^{l}\left(\beta_{j}^{(i)}\right)=0
$$

for $l=0,1,2, \ldots, n-2$. Take $t=0,1,2, \ldots, n-k-1$. Applying the automorphism $\sigma^{-t}$, we see that

$$
\begin{aligned}
0 & =\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}} \sigma^{-t}\left(\alpha_{j}^{(i)}\right) \sigma^{-t}\left(\mathcal{D}_{\gamma^{i-1}}^{l}\left(\beta_{j}^{(i)}\right)\right) \\
& =\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}} \sigma^{-t}\left(\alpha_{j}^{(i)}\right) \sigma^{-t}\left(N_{t}^{\sigma}\left(\gamma^{i-1}\right)\right) \mathcal{D}_{\gamma^{i-1}}^{l-t}\left(\beta_{j}^{(i)}\right) \\
& =\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}} \sigma^{-t}\left(\alpha_{j}^{(i)}\right) N_{t}^{\sigma^{-1}}\left(\sigma^{-1}(\gamma)^{i-1}\right) \mathcal{D}_{\gamma^{i-1}}^{l-t}\left(\beta_{j}^{(i)}\right) \\
& =\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}}\left(\mathcal{D}_{\sigma^{-1}(\gamma)^{i-1}}^{\sigma^{-1}}\right)^{t}\left(\alpha_{j}^{(i)}\right) \mathcal{D}_{\gamma^{i-1}}^{l-t}\left(\beta_{j}^{(i)}\right)
\end{aligned}
$$

for $i=1,2, \ldots, \ell$. By considering $t=0,1,2, \ldots, n-k-1$ and $l=0,1,2, \ldots, n-2$, we may consider the number $l-t$ to run from 0 to $k-1$. In conclusion, we have proven that

$$
\mathcal{C}_{L, n-k}^{\sigma^{-1}}\left(\mathcal{A}, \sigma^{-1}(\gamma)\right) \subseteq \mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)^{\perp}
$$

and by computing dimensions, equality holds. Finally, note that $\sigma^{-1}(\gamma)$ is a primitive element since $\gamma$ is a primitive element and $\sigma$ is a field automorphism.

Observe that the linearly independent sets $\mathcal{A}$ reduce in the Gabidulin case $(\ell=1)$ to the single linearly independent set denoted by $\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right\}$ in [20, Eq. (24)]. The sets $\mathcal{A}$ also reduce in the Reed-Solomon case $\left(n_{1}=n_{2}=\ldots=n_{\ell}=\right.$ 1) to the non-zero column multipliers $a_{1}^{\prime}, a_{2}^{\prime}, \ldots, a_{\ell}^{\prime} \in \mathbb{F}_{q}^{*}(\ell$ linearly independent sets in $\mathbb{F}_{q}$ over $\mathbb{F}_{q}$ ) of the dual generalized Reed-Solomon codes in [23, Th. 1].

## B. Optimal Coding Schemes for Coherent Communication

To claim the optimality of coding schemes built from linearized Reed-Solomon codes, we will extend the upper bound on the entropy of the secret message from [19, Th. 12] to the multishot case.
Theorem 5. Let $t \geq 0, \rho \geq 0$ and $\mu \geq 0$ be integers such that $2 t+\rho+\mu<n$. For given random distributions (not necessarily uniform), if the coding scheme $F: \mathcal{S} \longrightarrow \mathcal{X}$ is t-error and $\rho$-erasure-correcting for coherent communication, and secure under $\mu$ observations, then it holds that

$$
H(S) \leq n-2 t-\rho-\mu
$$

where entropy is taken with logarithms with base $q^{m}$.
Proof. Define $\rho^{\prime}=2 t+\rho$. By Theorem 1] $F$ is $\rho^{\prime}$-erasurecorrecting. Decompose $\rho^{\prime}=\rho_{1}+\rho_{2}+\cdots+\rho_{\ell}$ and $\mu=$ $\mu_{1}+\mu_{2}+\cdots+\mu_{\ell}$, with $\rho_{i}, \mu_{i} \geq 0$ and $\rho_{i}+\mu_{i} \leq n_{i}$, for $i=1,2, \ldots, \ell$. Let $A_{i} \in \mathbb{F}_{q}^{\left(n_{i}-\rho_{i}\right) \times n_{i}}$ and $B_{i} \in \mathbb{F}_{q}^{\mu_{i} \times n_{i}}$ be the matrices formed by the first $n_{i}-\rho_{i}$ and $\mu_{i}$ rows of the $n_{i} \times n_{i}$ identity matrix, respectively, for $i=1,2, \ldots, \ell$.

Define $A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right) \in \mathbb{F}_{q}^{\left(n-\rho^{\prime}\right) \times n}$ and $B=$ $\operatorname{diag}\left(B_{1}, B_{2}, \ldots, B_{\ell}\right) \in \mathbb{F}_{q}^{\mu \times n}$. Observe that $B$ is a submatrix of $A$, and let $D \in \mathbb{F}_{q}^{\left(n-\rho^{\prime}-\mu\right) \times n}$ be the matrix formed by those rows in $A$ that are not in $B$.

Now let $X=F(S)$, as random variables. Then Items 1 and 3 in Definition 2 allow us to write

$$
\begin{aligned}
H(S) & =H\left(S \mid X B^{T}\right) \\
& \leq H\left(S, X A^{T} \mid X B^{T}\right) \\
& =H\left(S \mid X A^{T}, X B^{T}\right)+H\left(X A^{T} \mid X B^{T}\right) \\
& =H\left(X A^{T} \mid X B^{T}\right) \\
& \leq H\left(X D^{T}\right) \\
& \leq n-\rho^{\prime}-\mu=n-2 t-\rho-\mu,
\end{aligned}
$$

and we are done.
We may now claim the optimality of coding schemes built from nested pairs of linearized Reed-Solomon codes:

Theorem 6. Fix integers $t \geq 0, \rho \geq 0, \mu \geq 0$ with $2 t+\rho+\mu<n$, and assume that $1 \leq \ell \leq q-1$ and $1 \leq n_{i} \leq m$, for $i=1,2, \ldots, \ell$. Let $k_{1}=n-2 t-\rho$ and $k_{2}=\mu$ and define $\mathcal{C}_{1}=\mathcal{C}_{L, k_{1}}^{\sigma}(\mathcal{B}, \gamma)$ and $\mathcal{C}_{2}=\mathcal{C}_{L, k_{2}}^{\sigma}(\mathcal{B}, \gamma)$ as in Definition 11 The coset coding scheme in Definition [3] corresponding to $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq \mathbb{F}_{q^{m}}^{n}$ and choosing $S$ and $X=F(S)$ with uniform distributions, is t-error and $\rho$-erasure-correcting for coherent communication, and is secure under $\mu$ observations. In addition, the entropy of the secret message satisfies that

$$
H(S)=\log _{q^{m}}|\mathcal{S}|=k_{1}-k_{2}=n-2 t-\rho-\mu
$$

Hence the coding scheme is optimal according to Theorem 5
Proof. This follows by combining Theorem 1. Theorem 3 Proposition 11 and Theorem 4

Remark 2. In a similar way to the proof of [19] Th. 10], we may show that if a coding scheme has secret size achieving the upper bound in Theorem [5] then it must hold that $m \geq n_{i}$, for $i=1,2, \ldots, \ell$. Thus, the coding schemes in the previous theorem are defined over the largest possible range for the packet length $m$, which is the same as in the 1-shot case.

However, we do not yet know if the condition $\ell<q$ can be relaxed. As in the MDS case ( $n_{1}=n_{2}=\ldots=n_{\ell}=1$ ), we conjecture that $\ell$ must be $\mathcal{O}(q)$ (for fixed $m$ and $n_{i}$ ). We leave this bound as open problem. See also Section VI

## C. Nearly Optimal Coding schemes for Non-coherent Communication

In this subsection, we adapt the lifting construction from
[17, Def. 3] to linearized Reed-Solomon codes. In contrast with the coherent case, the information rate of such construction is no longer optimal. However, we will show that it is close to optimal in reasonable scenarios in practice, as shown in [17, Sec. IV] (see also [19, Sec. VII]). We will make a few simplifying assumptions.

First, in practical situations it is desirable to encode separately (in different layers) for secrecy and reliability, which can be done using nested coset coding schemes as in Definition 3
(see [19, Subsec. VII-B]). Since linearized Reed-Solomon codes give optimal security in view of Theorems 5 and 6, we will only consider deterministic coding schemes and error and erasure correction. Moreover, the lifting construction will only add packet headers that contain no information about the secret message, hence they do not affect the overall security performance [19, Subsec. VII-E].

Second, by the fact that only column spaces are of importance for reliability in the non-coherent case (Theorem 2), we will consider sum-subspace codes [24, Subsec. II-B]. Since a codeword can inject up to $n_{i}$ packets in the $i$ th shot, we will restrict our study to sum-constant-dimension codes, which are simply subsets of the Cartesian product
$\mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{M}}, \mathbf{n}\right)=\mathcal{P}\left(\mathbb{F}_{q}^{M_{1}}, n_{1}\right) \times \mathcal{P}\left(\mathbb{F}_{q}^{M_{2}}, n_{2}\right) \times \cdots \times \mathcal{P}\left(\mathbb{F}_{q}^{M_{\ell}}, n_{\ell}\right)$,
where $\mathbf{M}=\left(M_{1}, M_{2}, \ldots, M_{\ell}\right)$, and $\mathcal{P}\left(\mathbb{F}_{q}^{M_{i}}, n_{i}\right)$ is the family of vector subspaces of $\mathbb{F}_{q}^{M_{i}}$ of dimension $n_{i}$, for $i=1,2, \ldots, \ell$. In view of Subsection III-B we will consider the following parameters of sum-constant-dimension codes.
Definition 12 (Sum-constant-dimension codes). A sum-constant-dimension code of type $\left[\mathbf{M}, \mathbf{n}, \log _{q}|\mathcal{C}|, d\right]_{q}$ is any subset $\mathcal{C} \subseteq \mathcal{P}\left(\mathbb{F}_{q}^{\mathrm{M}}, \mathbf{n}\right)$ such that $\mathrm{d}_{S S}(\mathcal{C})=2 d$. We define its rate as

$$
R=\frac{\log _{q}|\mathcal{C}|}{\sum_{i=1}^{\ell} M_{i} n_{i}}
$$

For integers $0 \leq N \leq M$, recall that the $q$-ary Gaussian coefficients are given by

$$
\left[\begin{array}{c}
M \\
N
\end{array}\right]_{q}=\left|\mathcal{P}\left(\mathbb{F}_{q}^{M}, N\right)\right|=\prod_{j=0}^{N-1} \frac{q^{M}-q^{j}}{q^{N}-q^{j}}
$$

Therefore, we have that

$$
\left|\mathcal{P}\left(\mathbb{F}_{q}^{\mathrm{M}}, \mathbf{n}\right)\right|=\prod_{i=1}^{\ell}\left[\begin{array}{c}
M_{i}  \tag{8}\\
n_{i}
\end{array}\right]_{q}
$$

The following sum-subspace Singleton bound is a refinement of that in [24, Subsec. VI-B] (in the bound in [24], a single puncturing removes a whole factor $\left.\mathcal{P}\left(\mathbb{F}_{q}^{M_{i}}, n_{i}\right)\right)$.
Theorem 7. Let $\mathcal{C}$ be a sum-constant-dimension code of type $\left[\mathbf{M}, \mathbf{n}, \log _{q}|\mathcal{C}|, d\right]_{q}$. It holds that

$$
|\mathcal{C}| \leq \min \prod_{i=1}^{\ell}\left[\begin{array}{l}
M_{i}-\delta_{i}  \tag{9}\\
M_{i}-n_{i}
\end{array}\right]_{q}
$$

where the minimum is taken over numbers $0 \leq \delta_{i} \leq n_{i}$, for $i=1,2, \ldots, \ell$, such that $\delta_{1}+\delta_{2}+\cdots+\delta_{\ell}=d-1$.
Proof. Given $\mathcal{U}=\left(\mathcal{U}_{1}, \mathcal{U}_{2}, \ldots, \mathcal{U}_{\ell}\right) \in \mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{M}}, \mathbf{n}\right)$ and a hyperplane $\mathcal{W} \in \mathcal{P}\left(\mathbb{F}_{q}^{M_{i}}, M_{i}-1\right)$, we define the restricted list of subspaces $\mathcal{U}_{i, \mathcal{W}}$ as that obtained from $\mathcal{U}$ by substituting $\mathcal{U}_{i}$ by an $\left(n_{i}-1\right)$-dimensional subspace of $\mathcal{U}_{i} \cap \mathcal{W}$, then mapped by a vector space isomorphism (depending only on $\mathcal{W}$ ) to a subspace of $\mathbb{F}_{q}^{M_{i}-1}$.

Assuming that $d>2$, the restricted code $\mathcal{C}_{i, \mathcal{W}}=\left\{\mathcal{U}_{i, \mathcal{W}} \mid\right.$ $\mathcal{U} \in \mathcal{C}\}$ is a sum-constant-dimension code of type $\left[\mathbf{M}-\mathbf{e}_{i}, \mathbf{n}-\right.$ $\left.\mathbf{e}_{i},|\mathcal{C}|, d^{\prime}\right]_{q}$, where $d^{\prime} \geq d-2>0$ and $\mathbf{e}_{i}$ is the $i$ th vector of
the canonical basis. The proof of this claim is exactly as that of [16, Th. 8] and is left to the reader.

Finally, applying such restriction operations to $\mathcal{C}, \delta_{i}$ times in the $i$ th block, we obtain a sum-constant-dimension code $\mathcal{C}^{\prime}$ of type $\left[\mathbf{M}-\boldsymbol{\delta}, \mathbf{n}-\boldsymbol{\delta}, \log _{q}|\mathcal{C}|, d^{\prime}\right]_{q}$, where $d^{\prime}>0$ and $\boldsymbol{\delta}=\left(\delta_{1}, \delta_{2}, \ldots, \delta_{\ell}\right)$. The result follows now from (8) and the fact that

$$
|\mathcal{C}|=\left|\mathcal{C}^{\prime}\right| \leq\left|\mathcal{P}\left(\mathbb{F}_{q}^{\mathrm{M}-\boldsymbol{\delta}}, \mathbf{n}-\boldsymbol{\delta}\right)\right|
$$

We will not simplify further the minimum in (9), since it will not be necessary for our purposes in Theorem 8

We now recall the extension of the lifting procedure from [17, Def. 3] to the multishot scenario, which was first considered in [25, Subsec. III-D]. In the rest of the subsection, we will assume that $M_{i}=n_{i}+m$, for $i=1,2, \ldots, \ell$.

Definition 13 (Lifting [25]). We define the lifting map $\mathcal{I}_{\Sigma}: \mathbb{F}_{q^{m}}^{n} \longrightarrow \mathcal{P}\left(\mathbb{F}_{q}^{\mathbf{M}}, \mathbf{n}\right)$ as follows. For $\mathbf{c}=\left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}\right.$, $\left.\ldots, \mathbf{c}^{(\ell)}\right) \in \mathbb{F}_{q^{m}}^{n}$, where $\mathbf{c}^{(i)} \in \mathbb{F}_{q^{m}}^{n_{i}}$, for $i=1,2, \ldots, \ell$, we define

$$
\mathcal{I}_{\Sigma}(\mathbf{c})=\left(\operatorname{Col}\left(\frac{M_{\mathcal{A}}\left(\mathbf{c}^{(i)}\right)}{I_{n_{i}}}\right)\right)_{i=1}^{\ell}
$$

For a block code $\mathcal{C} \subseteq \mathbb{F}_{q^{m}}^{n}$, we define its lifting as the sum-constant-dimension code $\mathcal{I}_{\Sigma}(\mathcal{C}) \subseteq \mathcal{P}\left(\mathbb{F}_{q}^{\mathrm{M}}, \mathbf{n}\right)$.

We now observe that the sum-subspace distance between lifted codewords coincides with twice the sum-rank distance of the codewords themselves. This is a straightforward extension of [17, Prop. 4] and is left to the reader.
Proposition 2. Given $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^{m}}^{n}$ and a block code $\mathcal{C} \subseteq \mathbb{F}_{q^{m}}^{n}$, it holds that

$$
\begin{aligned}
\mathrm{d}_{S S}\left(\mathcal{I}_{\Sigma}(\mathbf{c}), \mathcal{I}_{\Sigma}(\mathbf{d})\right) & =2 \mathrm{~d}_{S R}(\mathbf{c}, \mathbf{d}) \\
\mathrm{d}_{S S}\left(\mathcal{I}_{\Sigma}(\mathcal{C})\right) & =2 \mathrm{~d}_{S R}(\mathcal{C})
\end{aligned}
$$

Equipped with all these tools, we may finally claim the near optimality of lifted linearized Reed-Solomon codes. We follow the lines of [17, Prop. 5]. For simplicity in the formulas, we assume that the lengths $n_{i}$ are all equal.
Theorem 8. Assume that $n^{\prime}=n_{1}=n_{2}=\ldots=n_{\ell}$ and recall that $M_{1}=M_{2}=\ldots=M_{\ell}=n^{\prime}+m$. Assume also that $1 \leq \ell \leq q-1$ and $1 \leq n^{\prime} \leq m$. Let $1 \leq k \leq n$ and define $d=n-k+1$.

Denote by $R_{1}$ the rate of the lifted linearized Reed-Solomon code $\mathcal{I}_{\Sigma}\left(\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma)\right)$ (Definitions 11 and 13) and by $R_{2}$, the maximum rate of a sum-constant-dimension code of type $[\mathbf{M}, \mathbf{n}, N, d]_{q}$, running over all possible $N \in \mathbb{R}$. It holds that

$$
\frac{R_{2}-R_{1}}{R_{2}}<\frac{\ell}{k} \cdot \frac{2}{m \log _{2}(q)}
$$

Proof. Let $\mathcal{C}$ be a sum-constant-dimension code of type $\left[\mathbf{M}, \mathbf{n}, \log _{q}|\mathcal{C}|, d\right]_{q}$ and with rate $R_{2}$. Let the notation be as in Theorem 7 for $\mathcal{C}$ and for an arbitrary partition $d-1=$ $\delta_{1}+\delta_{2}+\cdots+\delta_{\ell}$. By [16, Lemma 4], we have that

$$
\left[\begin{array}{l}
M_{i}-\delta_{i}  \tag{10}\\
M_{i}-n_{i}
\end{array}\right]_{q}=\left[\begin{array}{c}
n^{\prime}+m-\delta_{i} \\
m
\end{array}\right]_{q}<4 q^{m\left(n^{\prime}-\delta_{i}\right)}
$$

for $i=1,2, \ldots, \ell$. Therefore, by Theorem[7] we have that

$$
|\mathcal{C}|<\prod_{i=1}^{\ell}\left(4 q^{m\left(n^{\prime}-\delta_{i}\right)}\right)=4^{\ell} q^{m(n-d+1)}=4^{\ell} q^{m k}
$$

since $\ell n^{\prime}=n, \delta_{1}+\delta_{2}+\cdots+\delta_{\ell}=d-1$, and $n-d+1=k$ by the assumptions and Proposition 2 Thus, it holds that

$$
R_{2}<\frac{\log _{q}(4) \ell+m k}{\ell\left(m+n^{\prime}\right) n^{\prime}}
$$

On the other hand, we have that

$$
R_{2} \geq R_{1}=\frac{m k}{\ell\left(m+n^{\prime}\right) n^{\prime}}
$$

Hence $R_{2}-R_{1}<\log _{q}(4) /\left(\left(m+n^{\prime}\right) n^{\prime}\right)$, and we conclude that

$$
\frac{R_{2}-R_{1}}{R_{2}}<\frac{\log _{q}(4)}{\left(m+n^{\prime}\right) n^{\prime}} \cdot \frac{\ell}{k} \cdot \frac{\left(m+n^{\prime}\right) n^{\prime}}{m}
$$

from which the desired bound follows easily.
Observe that $k / \ell$ is the rate of information packets injected per shot. If we inject at least one packet of information per shot, then $\ell / k \leq 1$, and the previous upper bound is simply $2 /\left(m \log _{2}(q)\right)$, which is essentially that in [17, Prop. 5], and is independent of the number of shots. Moreover, as shown in the proof of [16, Lemma 4], the factor 4 in the upper bound (10) on $q$-ary Gaussian coefficients can be refined to

$$
h(q)=\prod_{j=1}^{\infty} \frac{1}{1-q^{-j}}<4
$$

which decreases quickly as $q$ increases. Thus if $k / \ell \geq 1$, and following the previous proof, we may also obtain the bound

$$
\frac{R_{2}-R_{1}}{R_{2}}<\frac{\log _{q}(h(q))}{m}
$$

which is very small even for moderate values of $q$ and $m$.

## V. A Welch-Berlekamp Sum-Rank Decoding Algorithm for Linearized Reed-Solomon Codes

In this section, we show how to adapt Loidreau's version [26] of the Welch-Berlekamp decoding algorithm [42] to a sum-rank decoding algorithm with quadratic complexity over $\mathbb{F}_{q^{m}}$ for linearized Reed-Solomon codes (Definition 11).

Since working with linearized polynomials in the operators (6) requires keeping track of both $a$ (the conjugacy class) and $b$ (basis vectors in that conjugacy class), we give the algorithm for the skew metric [36, Def. 9] and skew Reed-Solomon codes [37, Def. 7], where evaluation points need not be partitioned into classes. Using tools from [36, Sec. 3], we will see in Subsection $V-A$ that both algorithms can be translated into each other after $\mathcal{O}(n)$ multiplications in $\mathbb{F}_{q^{m}}$.

We note that a skew-metric decoding algorithm for skew Reed-Solomon codes was also recently given in [38]. However, this algorithm has cubic complexity. More importantly, it is not translated to the sum-rank metric, so it is not applicable for reliability and security in multishot network coding. In particular, it does not handle erasures, non-coherent communication or wire-tapper observations. Finally, observe that the
step from cubic complexity to quadratic complexity is a bigger jump than from the state of the art (previous to [38]) to a sumrank decoding algorithm of cubic complexity, as a function of the number of shots $\ell$ and in operations over $\mathbb{F}_{2}$ (see Table $\square$ ).

We recall also that the use of skew polynomials was proposed in [51] for secret sharing. This corresponds to using skew Reed-Solomon codes [37] as MDS codes for reliability and security. However, without transforming these codes into MSRD linearized Reed-Solomon codes [36], such schemes are not suitable for multishot network coding.

The notation throughout this section is as in Subsection IV-A

## A. Skew Metrics and Skew Reed-Solomon Codes

Define the skew polynomial ring $\mathbb{F}_{q^{m}}[x ; \sigma]$ as the vector space over $\mathbb{F}_{q^{m}}$ with basis $\left\{x^{i} \mid i \in \mathbb{N}\right\}$ and with product given by the rules $x^{i} x^{j}=x^{i+j}$ and

$$
\begin{equation*}
x a=\sigma(a) x \tag{11}
\end{equation*}
$$

for all $a \in \mathbb{F}_{q^{m}}$ and all $i, j \in \mathbb{N}$. Define the degree of a non-zero $F=\sum_{i \in \mathbb{N}} F_{i} x^{i} \in \mathbb{F}_{q^{m}}[x ; \sigma]$, denoted by $\operatorname{deg}(F)$, as the maximum $i \in \mathbb{N}$ such that $F_{i} \neq 0$. We also define $\operatorname{deg}(0)=\infty$.

This ring was introduced with more generality in [52]. It is non-commutative and both a left and right Euclidean domain. We will see that evaluations of linearized polynomials as in Definition 9 correspond to arithmetic evaluations of skew polynomials, as defined in [48], [49].

Definition 14 (Evaluation [48], [49]). Given $F \in \mathbb{F}_{q^{m}}[x ; \sigma]$, we define its evaluation in $a \in \mathbb{F}_{q^{m}}$ as the unique $F(a) \in \mathbb{F}_{q^{m}}$ such that there exists $G \in \mathbb{F}_{q^{m}}[x ; \sigma]$ with

$$
F=G(x-a)+F(a)
$$

Given a subset $\Omega \subseteq \mathbb{F}_{q^{m}}$, we denote by $\mathbb{F}_{q^{m}}^{\Omega}$ the set of functions $f: \Omega \longrightarrow \mathbb{F}_{q^{m}}$. We then define the evaluation map over $\Omega$ as the linear map

$$
\begin{equation*}
E_{\Omega}: \mathbb{F}_{q^{m}}[x ; \sigma] \longrightarrow \mathbb{F}_{q^{m}}^{\Omega} \tag{12}
\end{equation*}
$$

where $f=E_{\Omega}(F) \in \mathbb{F}_{q^{m}}^{\Omega}$ is given by $f(a)=F(a)$, for all $a \in \Omega$ and for $F \in \mathbb{F}_{q^{m}}[x ; \sigma]$. Again, we write $E_{\Omega}^{\sigma}$ when there can be confusion about $\sigma$.

We will need some basic concepts regarding the zero sets of skew polynomials.
Definition 15 (Zeros of skew polynomials). Given a set $A \subseteq$ $\mathbb{F}_{q^{m}}[x ; \sigma]$, we define its zero set as

$$
Z(A)=\left\{a \in \mathbb{F}_{q^{m}} \mid F(a)=0, \forall F \in A\right\}
$$

Given a subset $\Omega \subseteq \mathbb{F}_{q^{m}}$, we define its associated ideal as

$$
I(\Omega)=\left\{F \in \mathbb{F}_{q^{m}}[x ; \sigma] \mid F(a)=0, \forall a \in \Omega\right\}
$$

Observe that $I(\Omega)$ is a left ideal in $\mathbb{F}_{q^{m}}[x ; \sigma]$. Since $\mathbb{F}_{q^{m}}[x ; \sigma]$ is a right Euclidean domain, there exists a unique monic skew polynomial $F_{\Omega} \in I(\Omega)$ of minimal degree among those in $I(\Omega)$, which in turn generates $I(\Omega)$ as left ideal. Such a skew polynomial is called the minimal skew polynomial of
$\Omega$ [48], [49]. With this in mind, we may recall the concepts of $P$-closed sets, $P$-independence and P-bases from [53, Sec. 4] (see also [48]):
Definition 16 (P-bases [48], [53]). Given a subset $\Omega \subseteq \mathbb{F}_{q^{m}}$, we define its P-closure as $\bar{\Omega}=Z(I(\Omega))=Z\left(F_{\Omega}\right)$, and we say that it is P-closed if $\bar{\Omega}=\Omega$.

We say that $a \in \mathbb{F}_{q^{m}}$ is P-independent from $\Omega \subseteq \mathbb{F}_{q^{m}}$ if it does not belong to $\bar{\Omega}$. A set $\Omega \subseteq \mathbb{F}_{q^{m}}$ is called P-independent if every $a \in \Omega$ is P-independent from $\Omega \backslash\{a\}$.

Given a P-closed set $\Omega \subseteq \mathbb{F}_{q^{m}}$, we say that $\mathcal{B} \subseteq \Omega$ is a P-basis of $\Omega$ if it is P-independent and $\Omega=\overline{\mathcal{B}}$.

The following two lemmas give simple and useful connection between P-bases and minimal skew polynomials.

Lemma 2 ([48]). Given a finite set $\Omega \subseteq \mathbb{F}_{q^{m}}$, it holds that

$$
\operatorname{deg}\left(F_{\Omega}\right) \leq|\Omega|
$$

where equality holds if, and only if, $\Omega$ is $P$-independent.
Lemma 3 ([48], [53]). Given a $P$-closed set $\Omega \subseteq \mathbb{F}_{q^{m}}$, it admits a P-basis and any two of them have the same number of elements, which is

$$
\operatorname{Rk}(\Omega) \stackrel{\text { def }}{=} \operatorname{deg}\left(F_{\Omega}\right)
$$

From now on, we will fix a P-closed set $\Omega \subseteq \mathbb{F}_{q^{m}}$ with $n=\operatorname{Rk}(\Omega)$. We will also denote by $\mathbb{F}_{q^{m}}[x ; \sigma]_{n}$ the vector space of skew polynomials of degree less than $n$.

The following lemma, given more generally in [48, Th. 8], is the main idea behind skew weights and skew Reed-Solomon codes.

Lemma 4 (Lagrange interpolation [48]). The evaluation map (12) restricted to $\mathbb{F}_{q^{m}}[x ; \sigma]_{n}$, that is

$$
E_{\mathcal{B}}: \mathbb{F}_{q^{m}}[x ; \sigma]_{n} \longrightarrow \mathbb{F}_{q^{m}}^{\mathcal{B}}
$$

is a vector space isomorphism, for any P-basis $\mathcal{B}$ of $\Omega$.
The definitions of skew weights [36, Def. 9] and skew ReedSolomon codes [37, Def. 7] can now be given as follows:
Definition 17 (Skew weights [36]). Given $F \in \mathbb{F}_{q^{m}}[x ; \sigma]_{n}$ and $f=E_{\mathcal{B}}(F) \in \mathbb{F}_{q^{m}}^{\mathcal{B}}$, for a P-basis $\mathcal{B}$ of $\Omega$, we define their skew weight over $\Omega$ as

$$
\mathrm{wt}_{\mathcal{B}}(f)=\mathrm{wt}_{\Omega}(F)=n-\operatorname{Rk}\left(Z_{\Omega}(F)\right)
$$

where $Z_{\Omega}(F)=Z(F) \cap \Omega=Z\left(\left\{F, F_{\Omega}\right\}\right)$ is the P-closed set of zeros of $F$ in $\Omega$.

As shown in [36], these functions are indeed weights and define a corresponding metric, called the skew metric.
Definition 18 (Skew Reed-Solomon codes [37]). For each $k=0,1,2, \ldots, n$, we define the ( $k$-dimensional) skew ReedSolomon code over a P-basis $\mathcal{B}$ of $\Omega$ as the linear code

$$
\mathcal{C}_{\mathcal{B}, k}^{\sigma}=E_{\mathcal{B}}^{\sigma}\left(\mathbb{F}_{q^{m}}[x ; \sigma]_{k}\right) \subseteq \mathbb{F}_{q^{m}}^{\mathcal{B}}
$$

The exact connection between skew metrics and sumrank metrics, and between skew Reed-Solomon codes and linearized Reed-Solomon codes was given in [36, Sec. 3]. We

TABLE II
MAXIMUM-DISTANCE EVALUATION CODES USING SKEW AND LINEARIZED POLYNOMIALS

| Code | Metric | Type of polynomials evaluated | Evaluation points | Lengths | Field |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Reed-Solomon (RS) [22] | Hamming | $\mathbb{F}_{q}[x ; \mathrm{Id}]=\mathbb{F}_{q}[x]$ | Pair-wise distinct | $n=\ell$ | $\mathbb{F}_{q}$ |
| Generalized RS [23] | Hamming | $\left\{\mathbb{F}_{q}\left[\mathcal{D}_{\gamma^{i-1}}\right]\right\}_{i=1}^{\ell} \equiv$ Multipliers $+\mathbb{F}_{q}[x]$ | Non-zero + pair-wise dist. | $\ell \leq q\left(n^{\prime}=1\right)$ | (Base: $\mathbb{F}_{q}$ ) |
| Skew RS [37] | Skew | $\mathbb{F}_{q^{m}}[x ; \sigma]$ | P -independent | $n=\ell n^{\prime}$ | $\mathbb{F}_{q^{m}}$ |
| Linearized RS [36] | Sum-rank | $\left\{\mathbb{F}_{q^{m}}\left[\mathcal{D}_{\gamma^{i-1}}\right]\right\}_{i=1}^{\ell}$ | Lin. indep. + conjugacy | $\ell<q \& n^{\prime} \leq m$ | (Base: $\mathbb{F}_{q}$ ) |
| Skew RS in $C(1)$ | Skew in $C(1)$ | $\mathbb{F}_{q^{m}}[x ; \sigma] \approx\left\{\sum_{j} a_{j} x^{q^{j}-1} \mid a_{j} \in \mathbb{F}_{q^{m}}\right\}$ | Lin. indep. conjugants | $n=n^{\prime}$ | $\mathbb{F}_{q^{m}}$ |
| Gabidulin [20] | Rank | $\mathbb{F}_{q^{m}}\left[\mathcal{D}_{1}\right]=\left\{\sum_{j} a_{j} x^{q^{j}} \mid a_{j} \in \mathbb{F}_{q^{m}}\right\}$ | Lin. independent | $n^{\prime} \leq m(\ell=1)$ | $\text { (Base: } \mathbb{F}_{q} \text { ) }$ |

summarize it in the next theorem, where the first claim on $\mathcal{B}$ combines [48, Th. 23] and [49, Th. 4.5].

Theorem 9 ([36], [48], [49]). Assume that $1 \leq \ell \leq q-1$ and $1 \leq n_{i} \leq m$, for $i=1,2, \ldots, \ell$. Fix linearly independent sets $\mathcal{B}^{(i)}=\left\{\beta_{1}^{(i)}, \beta_{2}^{(i)}, \ldots, \beta_{n_{i}}^{(i)}\right\} \subseteq \mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$, and a primitive element $\gamma$ of $\mathbb{F}_{q^{m}}$. With notation as in Subsection IV-A define

$$
\begin{equation*}
b_{j}^{(i)}=\mathcal{D}_{\gamma^{i-1}}\left(\beta_{j}^{(i)}\right)\left(\beta_{j}^{(i)}\right)^{-1} \tag{13}
\end{equation*}
$$

for $j=1,2, \ldots, n_{i}$ and $i=1,2, \ldots, \ell$. Then $\mathcal{B}=\left\{b_{j}^{(i)} \mid 1 \leq\right.$ $\left.j \leq n_{i}, 1 \leq i \leq \ell\right\}$ is a P-basis of $\Omega=\overline{\mathcal{B}}$.

Next, if $F=\sum_{i \in \mathbb{N}} F_{i} x^{i} \in \mathbb{F}_{q^{m}}[x ; \sigma]$ and $F^{\mathcal{D}_{a}}=$ $\sum_{i \in \mathbb{N}} F_{i} \mathcal{D}_{a}^{i}$, for some $a \in \mathbb{F}_{q^{m}}$, then

$$
F\left(\mathcal{D}_{a}(\beta) \beta^{-1}\right)=F^{\mathcal{D}_{a}}(\beta) \beta^{-1}
$$

for all $\beta \in \mathbb{F}_{q^{m}}^{*}$. Define now the linear map $\psi_{\mathcal{B}}: \mathbb{F}_{q^{m}}^{n} \longrightarrow \mathbb{F}_{q^{m}}^{\mathcal{B}}$ by $\psi_{\mathcal{B}}\left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(\ell)}\right)=f$, where

$$
\begin{equation*}
f\left(b_{j}^{(i)}\right)=c_{j}^{(i)}\left(\beta_{j}^{(i)}\right)^{-1} \tag{14}
\end{equation*}
$$

for $j=1,2, \ldots, n_{i}$ and $i=1,2, \ldots, \ell$. It holds that

$$
\psi_{\mathcal{B}}\left(\mathcal{C}_{L, k}^{\sigma}\right)=\mathcal{C}_{\mathcal{B}, k}^{\sigma},
$$

for all $k=0,1, \ldots, n$. For $\mathbf{c} \in \mathbb{F}_{q^{m}}^{n}$, we also have that

$$
\mathrm{wt}_{\mathcal{B}}\left(\psi_{\mathcal{B}}(\mathbf{c})\right)=\mathrm{wt}_{S R}(\mathbf{c})
$$

What this theorem implies is that sum-rank metrics and linearized Reed-Solomon codes can be treated as skew metrics and skew Reed-Solomon codes. We only need to multiply the received word (a vector in $\mathbb{F}_{q^{m}}^{n}$ ) coordinate-wise by $n$ elements in $\mathbb{F}_{q^{m}}$ as in (14), and similarly compute the corresponding P-basis as in (13). This requires $\mathcal{O}(n)$ multiplications in $\mathbb{F}_{q^{m}}$. Thus, decoding algorithms can be directly translated from one scenario to the other at the expense of just $\mathcal{O}(n)$ multiplications in $\mathbb{F}_{q^{m}}$.

Theorem 9 also implies that linearized Reed-Solomon codes can be seen as generalized skew Reed-Solomon codes [54, Def. 9] for a special choice of column multipliers.

Table II provides a summary of skew and linearized ReedSolomon codes, and how they interpolate the intermediate cases between Reed-Solomon and Gabidulin codes. In that table, $C(1)$ stands for the conjugacy class of $1 \in \mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$. The skew Reed-Solomon codes corresponding to Gabidulin codes are not given a name or reference in the table, since they are only considered as a particular example in the literature.

## B. Key Equations

In this subsection, we will present the skew polynomial version of the key equations in the Welch-Berlekamp algorithm.

Fix a P-closed set $\Omega \subseteq \mathbb{F}_{q^{m}}$, one of its P-bases $\mathcal{B}=\left\{b_{1}\right.$, $\left.b_{2}, \ldots, b_{n}\right\}$ and $1 \leq k \leq n$, where $n=|\mathcal{B}|=\operatorname{Rk}(\Omega)$. We would like to decode up to

$$
t \stackrel{\text { def }}{=}\left\lfloor\frac{n-k}{2}\right\rfloor
$$

skew errors for the skew Reed-Solomon code $\mathcal{C}=\mathcal{C}_{\mathcal{B}, k}^{\sigma} \subseteq \mathbb{F}_{q^{m}}^{\mathcal{B}}$ (see Definitions 17 and 18). Assume then that $f \in \mathcal{C}, e \in \mathbb{F}_{q^{m}}^{\mathcal{B}}$ is such that $\operatorname{wt}_{\mathcal{B}}(e) \leq t$, and $r=f+e$ is the received word. By Lemma 4 we may instead consider

$$
R=F+G \in \mathbb{F}_{q^{m}}[x ; \sigma]_{n}
$$

where $F \in \mathbb{F}_{q^{m}}[x ; \sigma]_{k}, G \in \mathbb{F}_{q^{m}}[x ; \sigma]_{n}$ with $\mathrm{wt}_{\Omega}(G) \leq t$, $f=E_{\mathcal{B}}(F)$ and $e=E_{\mathcal{B}}(G)$. Following the original idea of the Welch-Berlekamp algorithm, we want to find a non-zero skew polynomial $L \in \mathbb{F}_{q^{m}}[x ; \sigma]$ of degree $\operatorname{deg}(L) \leq t$, such that

$$
\begin{equation*}
(L R)\left(b_{i}\right)=(L F)\left(b_{i}\right) \tag{15}
\end{equation*}
$$

for $i=1,2, \ldots, n$. But since we do not know $F$ (we want to find $F$ ), we look instead for skew polynomials $L, Q \in$ $\mathbb{F}_{q^{m}}[x ; \sigma]$ of degrees $\operatorname{deg}(L) \leq t$ and $\operatorname{deg}(Q) \leq t+k-1$, such that

$$
\begin{equation*}
(L R)\left(b_{i}\right)=Q\left(b_{i}\right) \tag{16}
\end{equation*}
$$

for $i=1,2, \ldots, n$. To solve these equations, the first problem that we encounter is evaluating a product of two skew polynomials. This is solved by considering the product rule 49 Th. 2.7].
Lemma 5 (Product rule [49]). Given $U, V \in \mathbb{F}_{q^{m}}[x ; \sigma]$ and $a \in \mathbb{F}_{q^{m}}$, let $c=V(a)$. If $c=0$, then $(U V)(a)=0$, and if $c \neq 0$, then

$$
(U V)(a)=U\left(a^{c}\right) V(a)
$$

where we use the notation $a^{c}=\sigma(c) c^{-1} a$.
Therefore, if $r_{i}=R\left(b_{i}\right)$, we need to find $L$ and $Q$ as before, satisfying $Q\left(b_{i}\right)=0$ for $i$ such that $r_{i}=0$, and

$$
\begin{equation*}
L\left(b_{i}^{r_{i}}\right) r_{i}=Q\left(b_{i}\right) \tag{17}
\end{equation*}
$$

for $i$ such that $r_{i} \neq 0$. Observe that $L\left(b_{i}^{r_{i}}\right) r_{i}=L^{\mathcal{D}}\left(r_{i}\right)$ by Theorem 9 for $\mathcal{D}=\mathcal{D}_{b_{i}}$, which is also defined for $r_{i}=0$.

Hence it makes sense to define " $L\left(b_{i}^{r_{i}}\right) r_{i}=0$ " when $r_{i}=0$, and we may consider (17), for all $i=1,2, \ldots, n$.

We start by checking that (15) can be solved (hence (17) can also be solved). The next two propositions can also be proven by results from [38] (see Remark 3).
Proposition 3. There exists a non-zero $L \in \mathbb{F}_{q^{m}}[x ; \sigma]$ of degree $\operatorname{deg}(L) \leq t$ satisfying (15), for $i=1,2, \ldots, n$ (recall that $\left.\mathrm{wt}_{\Omega}(G)=\mathrm{wt}_{\Omega}(R-F) \leq t\right)$.
Proof. Define $e_{i}=G\left(b_{i}\right)$, for $i=1,2, \ldots, n$. By Lemma 5 , we see that (15) is satisfied if

$$
\begin{equation*}
L\left(b_{i}^{e_{i}}\right)=0 \tag{18}
\end{equation*}
$$

for $i$ such that $e_{i} \neq 0$. The result [36, Prop. 14] says that there exists a P-basis $\mathcal{A}=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ of $\Omega$ such that

$$
\mathrm{wt}_{H}\left(E_{\mathcal{A}}(G)\right)=\mathrm{wt}_{\Omega}(G) \leq t
$$

where $\mathrm{wt}_{H}$ denotes Hamming weight. Let $d_{i}=G\left(a_{i}\right)$, for $i=1,2, \ldots, n$, and let $\Delta=\left\{a_{i}^{d_{i}} \mid d_{i} \neq 0\right\}$. Since $|\Delta| \leq t$, then $\operatorname{deg}\left(F_{\Delta}\right) \leq t$ by Lemma 2 Thus by choosing $L=F_{\Delta}$, we see that

$$
L\left(a_{i}^{d_{i}}\right)=0
$$

whenever $d_{i} \neq 0$. By Lemma 5, this is equivalent to

$$
(L G)\left(a_{i}\right)=0
$$

for $i=1,2, \ldots, n$. In other words, $E_{\mathcal{A}}(L G)=0$, which by Lemma 4 is equivalent to $E_{\mathcal{B}}(L G)=0$. Now again by Lemma 5 this means that (18) is satisfied, and we are done.

The second ingredient is checking that by solving (17), we may recover the message skew polynomial $F$ by a right Euclidean division.

Proposition 4. Assume that $L, Q \in \mathbb{F}_{q^{m}}[x ; \sigma]$ satisfy (17), with $\operatorname{deg}(L) \leq t$ and $\operatorname{deg}(Q) \leq t+k-1$. Then it holds that

$$
Q=L F
$$

Proof. First, define $c_{i}=(L F-Q)\left(b_{i}\right)$, for $i=1,2, \ldots, n$. Observe that if $a \in \Omega$ and $F(a)=R(a)$, then $(L F-Q)(a)=$ 0 by hypothesis and Lemma 5 Hence

$$
\mathrm{wt}_{\Omega}(L F-Q) \leq \mathrm{wt}_{\Omega}(F-R) \leq t
$$

Therefore applying Proposition 3 to $L F$ and $Q$ instead of $F$ and $R$, we see that there exists a non-zero $L_{0} \in \mathbb{F}_{q^{m}}[x ; \sigma]$ of degree $\operatorname{deg}\left(L_{0}\right) \leq t$ satisfying

$$
L_{0}\left(b_{i}^{c_{i}}\right)=0
$$

for $i$ such that $c_{i} \neq 0$. Define $P=L_{0}(L F-Q)$. If $c_{i}=0$, then by Lemma 5, we have that $P\left(b_{i}\right)=0$. Now if $c_{i} \neq 0$, then by Lemma 5, we have that

$$
P\left(b_{i}\right)=L_{0}\left(b_{i}^{c_{i}}\right)=0
$$

In other words, $E_{\mathcal{B}}(P)=0$, and since $\operatorname{deg}(P)<n$ (here is where we use that $t=\left\lfloor\frac{n-k}{2}\right\rfloor$ ), then $P=0$ by Lemma4. The result follows since $\mathbb{F}_{q^{m}}[x ; \sigma]$ is an integral domain.

Thus we will be able to efficiently decode if we can solve (17), since then we may just perform Euclidean division, whose complexity is of $t(t+k-1)=\mathcal{O}\left(n^{2}\right)$ multiplications in $\mathbb{F}_{q^{m}}$. Equations (17) form a system of $n$ linear equations whose unknowns are the coefficients of $L$ and $Q$. However, solving such a system by Gaussian elimination has complexity $\mathcal{O}\left(n^{3}\right)$ over $\mathbb{F}_{q^{m}}$. This is the approach in [38]. In the next subsection, we see how to reduce it to $\mathcal{O}\left(n^{2}\right)$.

Remark 3. Propositions 3 and 4 can also be derived by certain combination of results from [38]. However, the machinery developed in [38] rewrites skew weights [36] Def. 9] in terms of least common multiples instead of P-closed sets and $P$-independence (Subsec. $(V-A)$, being the latter essential to connect skew weights with sum-rank weights (see Theorem 9). Thus we have preferred to keep a self-contained proof based on the machinery in Subsec. V-A

## C. Algorithm with Quadratic Complexity

In this subsection, we follow the steps in [26, Subsec. 5.2] to solve (17) with overall complexity of $\mathcal{O}\left(n^{2}\right)$ multiplications in $\mathbb{F}_{q^{m}}$. The idea is to construct sequences of skew polynomials $L_{j}, Q_{j}, \widetilde{L}_{j}, \widetilde{Q}_{j} \in \mathbb{F}_{q^{m}}[x ; \sigma]$ such that

$$
\left.\begin{array}{ll}
L_{j}\left(b_{i}^{r_{i}}\right) r_{i}-Q_{j}\left(b_{i}\right) & =0  \tag{19}\\
\widetilde{L}_{j}\left(b_{i}^{r_{i}}\right) r_{i}-\widetilde{Q}_{j}\left(b_{i}\right) & =0
\end{array}\right\}
$$

for $i=1,2, \ldots, j$ and $j=k, k+1_{2} \ldots, n$, and where $\operatorname{deg}\left(L_{n}\right)$, $\operatorname{deg}\left(\widetilde{L}_{n}\right) \leq t$ and $\operatorname{deg}\left(Q_{n}\right), \operatorname{deg}\left(\widetilde{Q}_{n}\right) \leq t+k-1$. If we have constructed the $j$ th skew polynomials, we define

$$
\begin{align*}
s_{j} & =L_{j}\left(b_{j+1}^{r_{j+1}}\right) r_{j+1}-Q_{j}\left(b_{j+1}\right) \\
\widetilde{s}_{j} & =\widetilde{L}_{j}\left(b_{j+1}^{r_{j+1}}\right) r_{j+1}-\widetilde{Q}_{j}\left(b_{j+1}\right) \tag{20}
\end{align*}
$$

Next we define

$$
\begin{align*}
L_{j+1} & =\left(x-b_{j+1}^{s_{j}}\right) L_{j}  \tag{21}\\
Q_{j+1} & =\left(x-b_{j+1}^{s_{j}}\right) Q_{j}
\end{align*}
$$

if $s_{j} \neq 0$; and we also define $L_{j+1}=L_{j}$ and $Q_{j+1}=Q_{j}$, if $s_{j}=0$. We also define

$$
\begin{align*}
\widetilde{L}_{j+1} & =s_{j} \widetilde{L}_{j}-\widetilde{s}_{j} L_{j}  \tag{22}\\
\widetilde{Q}_{j+1} & =s_{j} \widetilde{Q}_{j}-\widetilde{s}_{j} Q_{j}
\end{align*}
$$

The important fact is that then the $(j+1)$ th skew polynomials satisfy (19), as we now show.

Proposition 5. If the $j$ th skew polynomials satisfy (19) for $i=1,2, \ldots, j$, then the $(j+1)$ th skew polynomials given in (21) and (22) satisfy (19) for $i=1,2, \ldots, j, j+1$.

Proof. In all cases, if $i \leq j$, then the result follows from Lemma 5, since (19) is satisfied for the $j$ th skew polynomials. Hence we only need to check the case $i=j+1$. It is straightforward to check it for the skew polynomials given in (22), since evaluation is a linear map. Hence we only need to prove it for the skew polynomials given in (21), and only in the case $s_{j} \neq 0$. Denote

$$
\begin{aligned}
& c=L_{j}\left(b_{j+1}^{r_{j+1}}\right) r_{j+1} \\
& d=Q_{j}\left(b_{j+1}\right)
\end{aligned}
$$

Observe that $s_{j}=c-d \neq 0$. We have that

$$
\begin{aligned}
& L_{j+1}\left(b_{j+1}^{r_{j+1}}\right) r_{j+1}-Q_{j+1}\left(b_{j+1}\right) \\
= & \left(b_{j+1}^{c}-b_{j+1}^{s_{j}}\right) c-\left(b_{j+1}^{d}-b_{j+1}^{s_{j}}\right) d \\
= & \left(b_{j+1}^{c} c-b_{j+1}^{d} d\right)-b_{j+1}^{c-d}(c-d)=0
\end{aligned}
$$

where the first equality follows from Lemma 5] and the last equality follows from the fact that the map $\lambda \mapsto F\left(b^{\lambda}\right) \lambda=$ $F^{\mathcal{D}_{b}}(\lambda)$ is additive, for $b, \lambda \in \mathbb{F}_{q^{m}}$ (see Theorem (9).

We now describe the actual steps of the algorithm, following the same scheme as in [26, Table 1]. First we need to precompute the minimum skew polynomial $F_{k}=F_{\left\{b_{1}, b_{2}, \ldots, b_{k}\right\}}$ and the unique Lagrange interpolating skew polynomial $G \in$ $\mathbb{F}_{q^{m}}[x ; \sigma]_{k}$ such that $G\left(b_{i}\right)=r_{i}$, for $i=1,2, \ldots, k$ (Lemma 4). Both can be computed with quadratic complexity by Newton interpolation (see Appendix B).

The algorithm is as follows, where the inputs are $k \in \mathbb{N}$, $\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{F}_{q^{m}}^{n}$ and $\left(r_{1}, r_{2}, \ldots, r_{n}\right) \in \mathbb{F}_{q^{m}}^{n}$.

1) Initialization and Newton interpolation:

- Compute $F_{k}$ and $G$.
- Set $L_{k}=0$ and $\widetilde{L}_{k}=x$.
- Set $Q_{k}=F_{k}$ and $\widetilde{Q}_{k}=G$.

2) Alternating step: For each $j=k, k+1, \ldots, n-1$, do

- Compute $s_{j}$ and $\widetilde{s}_{j}$ as in (20).
- Exchange the values $L_{j} \longleftrightarrow \widetilde{L}_{j}, Q_{j} \longleftrightarrow \widetilde{Q}_{j}$ and $s_{j} \longleftrightarrow \widetilde{s}_{j}$.
- Compute $L_{j+1}, Q_{j+1}, \widetilde{L}_{j+1}$ and $\widetilde{Q}_{j+1}$ as in (21) and (22).

3) Euclidean division:

- Set $L=\widetilde{L}_{n}$ and $Q=\widetilde{Q}_{n}$.
- Compute $F$ such that $Q=L F$ by Euclidean division.

4) Return the coefficients of $F:\left(F_{0}, F_{1}, \ldots, F_{k-1}\right) \in \mathbb{F}_{q^{m}}^{k}$.

As observed in [26], one can save memory by assigning each update of the sequences of skew polynomials to themselves.

The degrees of $L$ and $Q$ are upper bounded by $t$ and $t+k-1$, respectively. This can be shown exactly as in [26, Subsec. 5.2]. The idea is that we increase the degree at most by 1 for one of the two sequences in every step. By exchanging both sequences in each step and taking $n-k$ steps, at the end we have that

$$
\begin{aligned}
\operatorname{deg}(L) & \leq\left\lfloor\frac{n-k}{2}\right\rfloor \\
\operatorname{deg}(Q) & \leq\left\lfloor\frac{n-k}{2}\right\rfloor+k-1
\end{aligned}
$$

## D. Overall Complexity

There are mainly three basic operations in the arithmetic of skew and linearized polynomials, namely, multiplications and additions in $\mathbb{F}_{q^{m}}$, and applying $\sigma^{j}(a)=a^{q^{j}}$, for $a \in \mathbb{F}_{q^{m}}$ and $1 \leq j<m$. Multiplications are more expensive than additions, but we will count both. We anticipate that, in $\mathcal{O}$ notation, the amounts of multiplications and additions in our algorithm are roughly the same. We will neglect how many times we apply $\sigma^{j}$ since, when representing elements in $\mathbb{F}_{q^{m}}$ as vectors over $\mathbb{F}_{q}$ using a normal basis, applying $\sigma^{j}$ amounts to a cyclic shift of coordinates.

First, Newton interpolation in $k$ P-independent points requires $\mathcal{O}\left(k^{2}\right)$ multiplications and $\mathcal{O}\left(k^{2}\right)$ additions (see Appendix B). This is then the complexity of Step 1.

Second, evaluating a skew polynomial of degree $d$ by Horner's rule (i.e., Euclidean division by $x-a$ ) requires $\mathcal{O}(d)$ multiplications and $\mathcal{O}(d)$ additions. Hence Step 2 requires $\mathcal{O}((n-k)(t+k-1))$ multiplications and $\mathcal{O}((n-k)(t+k-1))$ additions since, in each of the $n-k$ iterations of Step 2, we compute 4 evaluations of skew polynomials of degree at most $t+k-1$ to compute $s_{j}$ and $\widetilde{s}_{j}$, plus another $\mathcal{O}(t+k-1)$ multiplications and $\mathcal{\sim} \mathcal{O}(t+k-1)$ additions to compute $L_{j+1}$, $Q_{j+1}, \widetilde{L}_{j+1}$ and $\widetilde{Q}_{j+1}$.

Third, Euclidean division of a skew polynomial of degree $t+k-1$ by another one of degree $t$ requires $\mathcal{O}(t t+k-$ $1))$ multiplications and $\mathcal{O}(t(t+k-1))$ additions. This is the complexity of Step 3.

Summing all these numbers, we see that the previous algorithm has complexity of $\mathcal{O}\left(t(t+k-1)+k^{2}\right)$ multiplications and $\mathcal{O}\left(t(t+k-1)+k^{2}\right)$ additions in $\mathbb{F}_{q^{m}}$, which can be simplified to $\mathcal{O}\left(n^{2}\right)$ since $2 t \leq n$ and $k \leq n$.

## E. Translating the Algorithm into Sum-rank Decoding of Linearized Reed-Solomon Codes

To transform this algorithm into a sum-rank decoding algorithm for linearized Reed-Solomon codes, we proceed as follows. First we build the P-basis $\mathcal{B}=\left\{b_{1}, b_{2}, \ldots, b_{n}\right\}$ by (13) from the linearly independent sets $\mathcal{B}^{(i)}=\left\{\beta_{1}^{(i)}, \beta_{2}^{(i)}\right.$, $\left.\ldots, \beta_{n_{i}}^{(i)}\right\} \subseteq \mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}, i=1,2, \ldots, \ell$, and the primitive element $\gamma \in \mathbb{F}_{q^{m}}^{*}$. In other words, we compute

$$
\begin{equation*}
b_{j}^{(i)}=\sigma\left(\beta_{j}^{(i)}\right)\left(\beta_{j}^{(i)}\right)^{-1} \gamma^{i-1} \tag{23}
\end{equation*}
$$

where $b_{j}^{(i)}=b_{l}$ and $l=n_{1}+n_{2}+\cdots+n_{i-1}+j$, for $j=$ $1,2, \ldots, n_{i}$ and $i=1,2, \ldots, \ell$. Then we map the received word to the vector $\left(r_{1}, r_{2}, \ldots, r_{n}\right) \in \mathbb{F}_{q^{m}}^{n}$ by the map $\psi_{\mathcal{B}}$ in Theorem 9 or equivalently, by (14). In other words, we compute

$$
\begin{equation*}
r_{j}^{(i)}=y_{j}^{(i)}\left(\beta_{j}^{(i)}\right)^{-1} \tag{24}
\end{equation*}
$$

where $r_{j}^{(i)}=r_{l}$ and $l=n_{1}+n_{2}+\cdots+n_{i-1}+j$, for $j=1$, $2, \ldots, n_{i}$, and where $\mathbf{y}^{(i)}=\left(y_{1}^{(i)}, y_{2}^{(i)}, \ldots, y_{n_{i}}^{(i)}\right) \in \mathbb{F}_{q^{m}}^{n_{i}}$ is the received word in the $i$ th shot, for $i=1,2, \ldots, \ell$.

These two processes require $3 n$ multiplications in $\mathbb{F}_{q^{m}}$. Finally we run the algorithm in Subsection V-C. By Theorem 9, the coefficients of $F$ coincide with those of $F^{\mathcal{D}_{a}}$, for any $a \in \mathbb{F}_{q^{m}}$, and also give the message in $\mathbb{F}_{q^{m}}^{k}$ for the corresponding linearized Reed-Solomon code.

## F. Including Erasures for Coherent Communication

Consider that we have used the linearized Reed-Solomon code $\mathcal{C}_{L, k}^{\sigma}(\mathcal{B}, \gamma) \subseteq \mathbb{F}_{q^{m}}^{n}$ as in Definition 11, and we receive

$$
\mathbf{y}=\mathbf{c} A^{T}+\mathbf{e} \in \mathbb{F}_{q^{m}}^{n-\rho}
$$

where $\mathrm{wt}_{S R}(\mathbf{e}) \leq t, A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{n}\right) \in \mathbb{F}_{q}^{(n-\rho) \times n}$, and $A_{i} \in \mathbb{F}_{q}^{\left(n_{i}-\rho_{i}\right) \times n_{i}}$, for $i=1,2, \ldots, \ell$. First, we may assume that all $A_{i}$ have full rank. Next compute
$\left(\alpha_{1}^{(i)}, \alpha_{2}^{(i)}, \ldots, \alpha_{n_{i}-\rho_{i}}^{(i)}\right)=\left(\beta_{1}^{(i)}, \beta_{2}^{(i)}, \ldots, \beta_{n_{i}}^{(i)}\right) A_{i}^{T} \in \mathbb{F}_{q^{m}}^{n_{i}-\rho_{i}}$,
where $\mathcal{B}^{(i)}=\left\{\beta_{1}^{(i)}, \beta_{2}^{(i)}, \ldots, \beta_{n_{i}}^{(i)}\right\}$, and define $\mathcal{A}^{(i)}=\left\{\alpha_{1}^{(i)}\right.$, $\left.\alpha_{2}^{(i)}, \ldots, \alpha_{n_{i}-\rho_{i}}^{(i)}\right\}$, for $i=1,2, \ldots, \ell$. It follows that the sets $\mathcal{A}_{i}$ are linearly independent over $\mathbb{F}_{q}$. Furthermore, by the linearity over $\mathbb{F}_{q}$ of the operators in Definition 9 we have that

$$
\begin{aligned}
& \left(\mathcal{D}_{\gamma^{i-1}}^{j}\left(\alpha_{1}^{(i)}\right), \mathcal{D}_{\gamma^{i-1}}^{j}\left(\alpha_{2}^{(i)}\right), \ldots, \mathcal{D}_{\gamma^{i-1}}^{j}\left(\alpha_{n_{i}-\rho_{i}}^{(i)}\right)\right) \\
= & \left(\mathcal{D}_{\gamma^{i-1}}^{j}\left(\beta_{1}^{(i)}\right), \mathcal{D}_{\gamma^{i-1}}^{j}\left(\beta_{2}^{(i)}\right), \ldots, \mathcal{D}_{\gamma^{i-1}}^{j}\left(\beta_{n_{i}}^{(i)}\right)\right) A_{i}^{T}
\end{aligned}
$$

for $j=0,1, \ldots, k-1$ and $i=1,2, \ldots, \ell$. Thus $\mathbf{c} A^{T}$ corresponds to the evaluation codeword in the linearized ReedSolomon code $\mathcal{C}_{L, k}^{\sigma}(\mathcal{A}, \gamma) \subseteq \mathbb{F}_{q^{m}}^{n-\rho}$, where $\mathcal{A}=\left(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\right.$, $\left.\ldots, \mathcal{A}^{(\ell)}\right)$. Moreover, the number of sum-rank errors is at most $t=\left\lfloor\frac{n-\rho-k}{2}\right\rfloor$, where $n-\rho$ is the length of the new code.

In conclusion, to recover the message we only need to compute $\mathcal{A}$ to find the new code $\mathcal{C}_{L, k}^{\sigma}(\mathcal{A}, \gamma)$, and then run the algorithm in Subsection V-C. Such a computation is equivalent to the multiplication of a vector in $\mathbb{F}_{q^{m}}^{n}$ with a matrix in $\mathbb{F}_{q}^{n \times(n-\rho)}$, thus about $\mathcal{O}(n(n-\rho))$ multiplications of an element in $\mathbb{F}_{q^{m}}$ with an element in $\mathbb{F}_{q}$. This complexity can be further reduced to about $\mathcal{O}(n(n-\rho) / \ell)$ such multiplications by the block diagonal form of $A$.

## G. Including Wire-tapper Observations for Coherent Communication

Consider a nested coset coding scheme, as in Definition 3 using linearized Reed-Solomon codes $\mathcal{C}_{2} \varsubsetneqq \mathcal{C}_{1} \subseteq \mathbb{F}_{q^{m}}^{n}$. If $k_{1}=\operatorname{dim}\left(\mathcal{C}_{1}\right)$ and $k_{2}=\operatorname{dim}\left(\mathcal{C}_{2}\right)$, we may choose $\mathcal{W}$ in Definition 3 as the vector space generated by the last $k_{1}-k_{2}$ rows of the generator matrix of $\mathcal{C}_{1}$ given as in (7).

The encoding is as follows. The message is $\mathbf{x}_{2} \in \mathbb{F}_{q^{m}}^{k_{1}-k_{2}}$. We generate uniformly at random $\mathbf{x}_{1} \in \mathbb{F}_{q^{m}}^{k_{2}}$, and we encode $\mathbf{x}=\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right) \in \mathbb{F}_{q^{m}}^{k_{1}}$ using the generator matrix of $\mathcal{C}_{1}$ in (7) to obtain the codeword $\mathbf{c} \in \mathbb{F}_{q^{m}}^{n}$.

Finally, the numbers of errors $t$ and erasures $\rho$ are constrained by $2 t+\rho \leq n-k_{1}$, by Theorem 6 Thus, we may apply the decoding algorithm in Subsection $\nabla$-C to the larger code $\mathcal{C}_{1}$ and we recover $\mathbf{x}=\left(\mathbf{x}_{1}, \mathbf{x}_{2}\right) \in \mathbb{F}_{q^{m}}^{k_{1}}$. Thus we recover the message plus the random keys, which we may simply discard (as usual for instance in the wire-tap channel of type II [40] or in secret sharing [41]).

## H. The Non-coherent Case

As seen in the previous subsection, the addition of the random keys for security against a wire-tapper influences the encoding, but not the decoding. Hence we assume $\mu=0$ wiretapper observations in this subsection.

We now argue as in [55, Sec. 4.4]. Let $\mathcal{C} \subseteq \mathbb{F}_{q^{m}}^{n}$ be a linearized Reed-Solomon code of dimension $k$. Assume that we transmit the codeword $\mathbf{c}=\left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(\ell)}\right) \in \mathcal{C}$ using the lifting process as in Definition 13. This means that the receiver obtains the matrices

$$
\begin{gathered}
Y_{i}=\left(\frac{M_{\mathcal{A}}\left(\mathbf{c}^{(i)}\right)}{I_{n_{i}}}\right) A_{i}^{T}+E_{i} \\
=\binom{M_{\mathcal{A}}\left(\mathbf{c}^{(i)} A_{i}^{T}+\mathbf{e}_{i}^{\prime}\right)}{\widehat{A}_{i}^{T}}=\binom{M_{\mathcal{A}}\left(\mathbf{y}^{(i)}\right)}{\widehat{A}_{i}^{T}},
\end{gathered}
$$

where $\operatorname{Rk}\left(E_{i}\right) \leq t_{i}$, and $A_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$, for $i=1,2, \ldots, \ell$. So now the receiver knows $\widehat{A}=\operatorname{diag}\left(\widehat{A}_{1}, \widehat{A}_{2}, \ldots, \widehat{A}_{\ell}\right)$ instead of $A$, and we may compute the linearized Reed-Solomon code $\mathcal{C}^{\prime}=\mathcal{C} \widehat{A}^{T}$ as in Subsection V-F We may also assume that $\operatorname{Rk}\left(Y_{i}\right)=N_{i}$, for $i=1,2, \ldots, \ell$, and $\widehat{A}$ also has full rank.

By Theorem 2 and Proposition 2, if $2 t+\rho<n-k$, then a minimum sum-injection distance decoder would give us the message. However, summing in $i$ Equation (4.54) in [55, Sec. 4.4], we have that

$$
\begin{gathered}
\mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}\left(\mathcal{I}_{\Sigma}(\mathbf{c})\right), \operatorname{Col}_{\Sigma}(Y)\right) \\
=\mathrm{wt}_{S R}\left(\mathbf{y}-\mathbf{c} \widehat{A}^{T}\right)+\sum_{i=1}^{\ell}\left[n_{i}-N_{i}\right]^{+},
\end{gathered}
$$

where $\mathbf{y}=\left(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(\ell)}\right)$. Since the sum of $\left[n_{i}-N_{i}\right]^{+}$ does not depend on $\mathbf{c}$, we may instead use a minimum sumrank distance decoder. Therefore, we have reduced the problem to finding the message corresponding to $\mathbf{c}^{\prime}=\mathbf{c} \widehat{A}^{T} \in \mathcal{C}^{\prime}=$ $\mathcal{C} \widehat{A}^{T}$ that minimizes $\mathrm{wt}_{S R}\left(\mathbf{y}-\mathbf{c}^{\prime}\right)$, and we can find a solution using the algorithm in Subsection V-C for $\mathcal{C}^{\prime}$.

## I. Iterative Encoding and Decoding

We now remark that encoding and decoding linearized Reed-Solomon codes can be done in an iterative manner.
First, by the block-wise structure of the generator matrix (7), we may send the packets corresponding to the $i$ th shot before computing those corresponding to the $(i+1)$ th shot. Second, note that Newton interpolation (Appendix B), the computation of the skew polynomials $L$ and $Q$, and the translation to sumrank decoding as in (23) and (24) are all of iterative nature. Therefore we may perform Steps 1 and 2 in the algorithm iteratively, first in $i$ then in $j$, as we receive the packets $y_{j}^{(i)} \in$ $\mathbb{F}_{q^{m}}$.

## VI. Conclusion and Open Problems

In this work, we have proposed the use of linearized Reed-Solomon codes for reliability and security in multishot random linear network coding under a worst-case adversarial model. We have shown that the corresponding coding schemes achieve the maximum secret message size in the coherent case, and close to maximum information rate in the noncoherent case. Moreover, the encoding and decoding can be performed iteratively and with overall quadratic complexity. Their advantage with respect to simply using Gabidulin codes is that their field size is roughly $\max \left\{\ell, q_{0}\right\}^{n^{\prime}}$ (polynomial in $\ell$ ), in contrast to $q_{0}^{\ell n^{\prime}}$ (exponential in $\ell$ ), where $n^{\prime}$ is the number of outgoing links at the source, $\ell$ is the number of shots, and $q_{0}$ is the base field size of the underlying linear network code. This is translated into a reduction of more than one degree in $\ell$ in the encoding and decoding complexity in number of operations over $\mathbb{F}_{2}$ (Table $\mathbb{\square}$ ).

We now list some open problems for future research:

1) The number of shots is restricted to $\ell<q$ when using linearized Reed-Solomon codes. In the Hamming-metric case ( $n^{\prime}=1$ ), it is well-known that an MDS code satisfies $\ell<2 q$ [56]. Since MSRD codes are MDS, we deduce that $\ell<2 q^{m} / n^{\prime}$. We conjecture, but leave as an open
problem, that MSRD codes must satisfy $\ell=\mathcal{O}(q)$ instead of $\ell=\mathcal{O}\left(q^{m} / n^{\prime}\right)=\mathcal{O}\left(q^{m}\right)$, for fixed values of $m$ and $n^{\prime}$.
2) Faster decoding algorithms exist for Gabidulin codes [27], [28], corresponding to the case $\ell=1$. We leave as an open problem finding analogous reductions of the decoding complexity of linearized Reed-Solomon codes.
3) Although unique decoding works analogously for all linearized Reed-Solomon codes, list-decoding seems to differ in the Hamming-metric $\left(n^{\prime}=1\right)$ and rank-metric $(\ell=1)$ cases (see [57], [58] and the references therein). This suggests, but we leave as an open problem, that the list-decoding of sum-rank codes and linearized Reed-Solomon codes behaves differently with respect to $n^{\prime}$ and $\ell$.
4) Both the rank-metric list-decodable codes from [58] and most maximum rank distance codes when $n^{\prime}>m$ are only linear over $\mathbb{F}_{q}$, instead of $\mathbb{F}_{q^{m}}$. We leave as an open problem the study of the security performance of $\mathbb{F}_{q}$-linear codes in multishot network coding, as done in [45].
5) Linearized Reed-Solomon codes require using the same base field $q$ and packet length $m$ in every shot of the network. We leave as an open problem the construction of MSRD codes for different base fields and packet lengths in different shots.
6) It seems natural for future research to study convolutional sum-rank-metric codes rather than convolutional rank-metric codes. That is, convolutional codes where we consider an $\ell$-shot sum-rank metric in each block rather than the rank metric. We conjecture that this should be similar to going from convolutional codes where each block is simply a field symbol to classical Hamming-metric convolutional codes.

## Appendix A

## Proofs for Section III

In this appendix, we give the proofs of the main results in Section III We start with the proof of Theorem 1 in Subsection III-A.

Proof of Theorem 17 First assume that the scheme is not $t$ error and $\rho$-erasure-correcting. Then there exist integers $0 \leq$ $\rho_{i} \leq n_{i}$ and full-rank matrices $A_{i} \in \mathbb{F}_{q}^{\left(n_{i}-\rho_{i}\right) \times n_{i}}$, for $i=$ $1,2, \ldots, \ell$, and there exist vectors $\mathbf{e}_{1}, \mathbf{e}_{2} \in \mathbb{F}_{q^{m}}^{n-\rho}, \mathbf{c}_{1} \in \mathcal{X}_{S}$ and $\mathbf{c}_{2} \in \mathcal{X}_{T}$, where $S \neq T$, such that

$$
\mathbf{c}_{1} A^{T}+\mathbf{e}_{1}=\mathbf{c}_{2} A^{T}+\mathbf{e}_{2}
$$

where $\rho=\rho_{1}+\rho_{2}+\cdots+\rho_{\ell}, A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right) \in$ $\mathbb{F}_{q}^{(n-\rho) \times n}$, and $\mathrm{wt}_{S R}\left(\mathbf{e}_{1}\right), \mathrm{wt}_{S R}\left(\mathbf{e}_{2}\right) \leq t$. By defining $\mathbf{c}=$ $\mathbf{c}_{1}-\mathbf{c}_{2}$, we see that

$$
\mathrm{wt}_{S R}\left(\mathbf{c} A^{T}\right)=\mathrm{wt}_{S R}\left(\mathbf{e}_{2}-\mathbf{e}_{1}\right) \leq 2 t
$$

Take now full-rank matrices $B_{i} \in \mathbb{F}_{q}^{\rho_{i} \times n_{i}}$ such that $\binom{A_{i}}{B_{i}} \in \mathbb{F}_{q}^{n_{i} \times n_{i}}$ is invertible, for $i=1,2, \ldots, \ell$. If $B=\operatorname{diag}\left(B_{1}, B_{2}, \ldots, B_{\ell}\right) \in \mathbb{F}_{q}^{\rho \times n}$ and

$$
D=\operatorname{diag}\left(\binom{A_{1}}{B_{1}},\binom{A_{2}}{B_{2}}, \ldots,\binom{A_{\ell}}{B_{\ell}}\right) \in \mathbb{F}_{q}^{n \times n}
$$

then $D$ is invertible and we conclude that

$$
\mathrm{d}_{S R}\left(\mathbf{c}_{1}, \mathbf{c}_{2}\right)=\mathrm{wt}_{S R}(\mathbf{c})=\mathrm{wt}_{S R}\left(\mathbf{c} D^{T}\right) \leq
$$

$$
\mathrm{wt}_{S R}\left(\mathbf{c} A^{T}\right)+\mathrm{wt}_{S R}\left(\mathbf{c} B^{T}\right) \leq 2 t+\rho,
$$

thus $2 t+\rho \geq \mathrm{d}_{S R}(\operatorname{Supp}(F))$.
Assume now that $2 t+\rho \geq \mathrm{d}_{S R}(\operatorname{Supp}(F))$. Take $\mathbf{c}=\mathbf{c}_{1}-\mathbf{c}_{2}$ such that $\mathbf{c}_{1} \in \mathcal{X}_{S}, \mathbf{c}_{2} \in \mathcal{X}_{T}, S \neq T$, and wt ${ }_{S R}(\mathbf{c}) \leq 2 t+\rho$. There exist full-rank matrices $B_{i} \in \mathbb{F}_{q}^{\left(2 t_{i}+\rho_{i}\right) \times n_{i}}$ and vectors $\mathbf{x}_{i} \in \mathbb{F}_{q^{m}}^{2 t_{i}+\rho_{i}}$ such that $\mathbf{c}^{(i)}=\mathbf{x}_{i} B_{i}$, for $i=1,2, \ldots, \ell$, where $t=t_{1}+t_{2}+\cdots+t_{\ell}$ and $\rho=\rho_{1}+\rho_{2}+\cdots+\rho_{\ell}$.

Now let $\mathbf{y}_{i} \in \mathbb{F}_{q^{m}}^{2 t_{i}}$ be the first $2 t_{i}$ components of $\mathbf{x}_{i} \in$ $\mathbb{F}_{q^{m}}^{2 t_{i}+\rho_{i}}$, and let $A_{i} \in \mathbb{F}_{q}^{\left(n_{i}-\rho_{i}\right) \times n_{i}}$ be a parity-check matrix of the $\mathbb{F}_{q}$-linear vector space generated by the last $\rho_{i}$ rows in $B_{i} \in \mathbb{F}_{q}^{\left(2 t_{i}+\rho_{i}\right) \times n_{i}}$, for $i=1,2, \ldots, \ell$. Thus, it holds that

$$
\mathbf{c}^{(i)} A_{i}^{T}=\mathbf{x}_{i} B_{i} A_{i}^{T}=\left(\mathbf{y}_{i}, \mathbf{0}\right) B_{i} A_{i}^{T}
$$

where $\operatorname{Rk}\left(M_{\mathcal{A}}\left(\mathbf{y}_{i}, \mathbf{0}\right) B_{i} A_{i}^{T}\right) \leq \operatorname{Rk}\left(M_{\mathcal{A}}\left(\mathbf{y}_{i}\right)\right) \leq 2 t_{i}$, for $i=$ $1,2, \ldots, \ell$, and we conclude that $\mathrm{wt}_{S R}\left(\mathbf{c} A^{T}\right) \leq 2 t$, where $A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right)$. Hence there exists $\mathbf{e}_{1}, \mathbf{e}_{2} \in \mathbb{F}_{q^{m}}^{n-\rho}$ with $\mathrm{wt}_{S R}\left(\mathbf{e}_{1}\right), \mathrm{wt}_{S R}\left(\mathbf{e}_{1}\right) \leq t$, such that $\mathbf{c}_{1} A^{T}+\mathbf{e}_{1}=\mathbf{c}_{2} A^{T}+$ $\mathbf{e}_{2}$. Therefore the codewords $\mathbf{c}_{1}$ and $\mathbf{c}_{2}$ correspond to distinct secret messages, but cannot be distinguished by any decoder. Thus the scheme is not $t$-error and $\rho$-erasure-correcting.

We now prove Theorem 2 in Subsection III-B We will make repeated use of the following lemma, which is a particular case of [18, Lemma 15].
Lemma 6 ([|][). For matrices $X, Y \in \mathbb{F}_{q}^{m \times n}$, and integers $\rho \geq 0$ and $N \geq n-\rho>0$, it holds that

$$
\begin{aligned}
\min \{ & \operatorname{Rk}\left(X A^{T}-Y B^{T}\right) \mid \\
& \left.A, B \in \mathbb{F}_{q}^{N \times n}, \operatorname{Rk}(A), \operatorname{Rk}(B) \geq n-\rho\right\} \\
= & {[\max \{\operatorname{Rk}(X), \operatorname{Rk}(Y)\}} \\
& -\operatorname{dim}(\operatorname{Col}(X) \cap \operatorname{Col}(Y))-\rho]^{+}
\end{aligned}
$$

where $a^{+}=\max \{0, a\}$, for $a \in \mathbb{Z}$.
Proof of Theorem 2] First assume that the scheme is not $t$-error and $\rho$-erasure-correcting. Then there exist integers $0 \leq \rho_{i} \leq n_{i}$ and matrices $A_{i}, B_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$ with $\operatorname{Rk}\left(A_{i}\right)$, $\operatorname{Rk}\left(B_{i}\right) \geq n_{i}-\rho_{i}$, for $i=1,2, \ldots, \ell$, and there exist vectors $\mathbf{e}_{1}, \mathbf{e}_{2} \in \mathbb{F}_{q^{m}}^{N}, \mathbf{c}_{1} \in \mathcal{X}_{S}$ and $\mathbf{c}_{2} \in \mathcal{X}_{T}$, where $S \neq T$, such that

$$
\mathbf{c}_{1} A^{T}+\mathbf{e}_{1}=\mathbf{c}_{2} B^{T}+\mathbf{e}_{2}
$$

where $\rho=\rho_{1}+\rho_{2}+\cdots+\rho_{\ell}, A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right), B=$ $\operatorname{diag}\left(B_{1}, B_{2}, \ldots, B_{\ell}\right) \in \mathbb{F}_{q}^{N \times n}$, and $\mathrm{wt}_{S R}\left(\mathbf{e}_{1}\right), \mathrm{wt}_{S R}\left(\mathbf{e}_{2}\right) \leq$ $t$. We have that

$$
\mathrm{wt}_{S R}\left(\mathbf{c}_{1} A^{T}-\mathbf{c}_{2} B^{T}\right)=\mathrm{wt}_{S R}\left(\mathbf{e}_{2}-\mathbf{e}_{1}\right) \leq 2 t
$$

By Lemma 6, we also have that

$$
\begin{aligned}
\mathrm{wt}_{S R}\left(\mathbf{c}_{1} A^{T}-\mathbf{c}_{2} B^{T}\right) & \geq \sum_{i=1}^{\ell}\left[\max \left\{\mathrm{wt}_{R}\left(\mathbf{c}_{1}^{(i)}\right), \mathrm{wt}_{R}\left(\mathbf{c}_{2}^{(i)}\right)\right\}\right. \\
& \left.-\operatorname{dim}\left(\operatorname{Col}\left(\mathbf{c}_{1}^{(i)}\right) \cap \operatorname{Col}\left(\mathbf{c}_{2}^{(i)}\right)\right)-\rho_{i}\right]^{+} \\
& \geq \mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}\left(\mathbf{c}_{1}\right), \operatorname{Col}_{\Sigma}\left(\mathbf{c}_{2}\right)\right)-\rho,
\end{aligned}
$$

and we conclude that $2 t+\rho \geq \mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))\right)$.
Assume now that $2 t+\rho \geq \mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}(\operatorname{Supp}(F))\right)$. Take $\mathbf{c}_{1} \in$ $\mathcal{X}_{S}, \mathbf{c}_{2} \in \mathcal{X}_{T}, S \neq T$, such that $\mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}\left(\mathbf{c}_{1}\right), \operatorname{Col}_{\Sigma}\left(\mathbf{c}_{2}\right)\right) \leq$
$2 t+\rho$. Define now $m_{i}=\max \left\{\mathrm{wt}_{R}\left(\mathbf{c}_{1}^{(i)}\right), \mathrm{wt}_{R}\left(\mathbf{c}_{2}^{(i)}\right)\right\}-$ $\operatorname{dim}\left(\operatorname{Col}\left(\mathbf{c}_{1}^{(i)}\right) \cap \operatorname{Col}\left(\mathbf{c}_{2}^{(i)}\right)\right)$, for $i=1,2, \ldots, \ell$, and

$$
\delta=\left[\sum_{i=1}^{\ell} m_{i}-\rho\right]^{+}-\left(\sum_{i=1}^{\ell} m_{i}-\rho\right) \geq 0
$$

Next let $0 \leq \rho_{i} \leq m_{i}$, for $i=1,2, \ldots, \ell$, be such that $\rho=$ $\sum_{i=1}^{\ell} \rho_{i}+\delta$, which implies that

$$
\sum_{i=1}^{\ell}\left(m_{i}-\rho_{i}\right)=\left[\sum_{i=1}^{\ell} m_{i}-\rho\right]^{+}
$$

By Lemma 6, there exist matrices $A_{i}, B_{i} \in \mathbb{F}_{q}^{N_{i} \times n_{i}}$ such that $\operatorname{Rk}\left(A_{i}\right), \operatorname{Rk}\left(B_{i}\right) \geq n_{i}-\rho_{i}$, for $i=1,2, \ldots, \ell$, and

$$
\begin{gathered}
\mathrm{wt}_{S R}\left(\mathbf{c}_{1} A^{T}-\mathbf{c}_{2} B^{T}\right)=\sum_{i=1}^{\ell}\left(m_{i}-\rho_{i}\right)=\left[\sum_{i=1}^{\ell} m_{i}-\rho\right]^{+} \\
=\left[\mathrm{d}_{S I}\left(\operatorname{Col}_{\Sigma}\left(\mathbf{c}_{1}\right), \operatorname{Col}_{\Sigma}\left(\mathbf{c}_{2}\right)\right)-\rho\right]^{+} \leq 2 t
\end{gathered}
$$

where $A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{\ell}\right)$ and $B=\operatorname{diag}\left(B_{1}, B_{2}\right.$, $\left.\ldots, B_{\ell}\right)$. As in the previous proof, the codewords $\mathbf{c}_{1}$ and $\mathbf{c}_{2}$ correspond to distinct secret messages, but cannot be distinguished by any decoder. Thus the scheme is not $t$-error and $\rho$-erasure-correcting.

We conclude this appendix with the proof of Lemma 1.
Proof of Lemma $\mathbb{7}$ Define the map $f: \mathbb{F}_{q^{m}}^{n} \longrightarrow \mathbb{F}_{q^{m}}^{\mu}$ given by

$$
f(\mathbf{c})=\mathbf{c} B^{T}
$$

for $\mathbf{c} \in \mathbb{F}_{q^{m}}^{n}$. Observe that $f$ is a linear map over $\mathbb{F}_{q^{m}}$. For the random variable $X=F(S)$, it follows that

$$
\begin{gathered}
H\left(X B^{T}\right)=H(f(X)) \leq \log _{q^{m}}\left|f\left(\mathcal{C}_{1}\right)\right| \\
=\operatorname{dim}\left(f\left(\mathcal{C}_{1}\right)\right)=\operatorname{dim}\left(\mathcal{C}_{1}\right)-\operatorname{dim}\left(\operatorname{ker}(f) \cap \mathcal{C}_{1}\right),
\end{gathered}
$$

where the last equality is the well-known first isomorphism theorem. On the other hand, we may similarly compute the conditional entropy as follows. Recall that $X$ is uniform given $S$. We have that

$$
\begin{aligned}
& H\left(X B^{T} \mid S\right)=H(f(X) \mid S)=\log _{q^{m}}\left|f\left(\mathcal{C}_{2}\right)\right| \\
& =\operatorname{dim}\left(f\left(\mathcal{C}_{2}\right)\right)=\operatorname{dim}\left(\mathcal{C}_{2}\right)-\operatorname{dim}\left(\operatorname{ker}(f) \cap \mathcal{C}_{2}\right)
\end{aligned}
$$

We leave for the reader to prove as an exercise that $\operatorname{ker}(f)=$ $\mathcal{V}_{\left(\mathcal{L}^{\perp}\right)}=\mathcal{V}_{\mathcal{L}}^{\perp}$, where $\mathcal{L}^{\perp}=\left(\mathcal{L}_{1}^{\perp}, \mathcal{L}_{2}^{\perp}, \ldots, \mathcal{L}_{\ell}^{\perp}\right)$ (use that $B_{i}$ is a parity-check matrix of $\mathcal{L}_{i}^{\perp}$, for $\left.i=1,2, \ldots, \ell\right)$. Therefore

$$
\begin{gathered}
I\left(S ; X B^{T}\right)=H\left(X B^{T}\right)-H\left(X B^{T} \mid S\right) \\
\leq\left(\operatorname{dim}\left(\mathcal{C}_{1}\right)-\operatorname{dim}\left(\mathcal{V}_{\mathcal{L}}^{\perp} \cap \mathcal{C}_{1}\right)\right)-\left(\operatorname{dim}\left(\mathcal{C}_{2}\right)-\operatorname{dim}\left(\mathcal{V}_{\mathcal{L}}^{\perp} \cap \mathcal{C}_{2}\right)\right) \\
=\left(\operatorname{dim}\left(\mathcal{V}_{\mathcal{L}}\right)-\operatorname{dim}\left(\mathcal{C}_{1}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right)\right)-\left(\operatorname{dim}\left(\mathcal{V}_{\mathcal{L}}\right)-\operatorname{dim}\left(\mathcal{C}_{2}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right)\right) \\
=\operatorname{dim}\left(\mathcal{C}_{2}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right)-\operatorname{dim}\left(\mathcal{C}_{1}^{\perp} \cap \mathcal{V}_{\mathcal{L}}\right)
\end{gathered}
$$

where the first equality follows from the dimensions formulas for duals, sums and intersections of vector spaces.

Finally, if $S$ is uniform in $\mathcal{S}$, then all inequalities in this proof are equalities.

## Appendix B <br> Newton Interpolation for Skew Polynomials

In this appendix, we show how to find the skew polynomials $F_{k}$ and $G$ necessary to initialize the algorithm in Subsection V-C. We use the idea behind Newton's interpolation algorithm for conventional polynomials. Newton interpolation for skew polynomials was investigated in [51]. An algorithm with quadratic complexity based on Kötter interpolation was given in [59, Sec. 4]. The algorithm in this appendix is analogous, but with notation in accordance with the rest of this paper.

Let $\mathcal{B}_{k}=\left\{b_{1}, b_{2}, \ldots, b_{k}\right\} \subseteq \mathbb{F}_{q^{m}}$ be a P-independent set and let $a_{1}, a_{2}, \ldots, a_{k} \in \mathbb{F}_{q^{m}}$. By definition of P-independence, the sets $\mathcal{B}_{i}=\left\{b_{1}, b_{2}, \ldots, b_{i}\right\}$ are also P-independent, for $i=$ $1,2, \ldots, k$. It also holds that $F_{\mathcal{B}_{i}}\left(b_{i+1}\right) \neq 0$, for $i=1,2$, $\ldots, k-1$, by definition.

We have that $F_{\mathcal{B}_{1}}=x-b_{1}$, and $G_{1}=a_{1}$ satisfies that $\operatorname{deg}\left(G_{1}\right)<1$ and $G_{1}\left(b_{1}\right)=a_{1}$. Now, let $1 \leq i \leq k-1$ and assume that we have computed $F_{\mathcal{B}_{i}}$ and $G_{i}$ such that $\operatorname{deg}\left(G_{i}\right)<i$ and $G_{i}\left(b_{j}\right)=a_{j}$, for $j=1,2, \ldots, i$. Then by Lemma 5 it holds that

$$
\begin{aligned}
F_{\mathcal{B}_{i+1}} & =\left(x-b_{i+1}^{F_{\mathcal{B}_{i}}\left(b_{i+1}\right)}\right) F_{\mathcal{B}_{i}} \\
G_{i+1} & =G_{i}+\left(a_{i+1}-G_{i}\left(b_{i+1}\right)\right) F_{\mathcal{B}_{i}}\left(b_{i+1}\right)^{-1} F_{\mathcal{B}_{i}}
\end{aligned}
$$

where $G_{i+1}$ is the unique skew polynomial with $\operatorname{deg}\left(G_{i+1}\right)<$ $i+1$ and $G_{i+1}\left(b_{j}\right)=a_{j}$, for $j=1,2, \ldots, i+1$. Thus the required skew polynomials are $F_{k}=F_{\mathcal{B}_{k}}$ and $G=G_{k}$.

This algorithm requires computing the evaluations $F_{\mathcal{B}_{i}}\left(b_{i+1}\right)$ and $G_{i}\left(b_{i+1}\right)$, where $\operatorname{deg}\left(F_{\mathcal{B}_{i}}\right), \operatorname{deg}\left(G_{i}\right) \leq i<k$, which requires $\mathcal{O}(k)$ multiplications and $\mathcal{O}(k)$ additions. It requires another $\mathcal{O}(k)$ multiplications and $\mathcal{O}(k)$ additions in each of the $k-1$ steps. Thus its complexity is of $\mathcal{O}\left(k^{2}\right)$ multiplications and $\mathcal{O}\left(k^{2}\right)$ additions in $\mathbb{F}_{q^{m}}$.

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