Nonbinary Polar Codes Constructions Based on k-means Clustering

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***Abstract*—This study investigates an efficient code construction method based on the Density evolution scheme, which has a component of channel merge routine. In this paper, k-means clustering algorithm is proposed for channel merging. By inspecting the merged channels, bit channels are ordered in estimated reliability. Simulation results show that the proposed channel merging algorithm achieves marginal capacity loss of nonbinary polar coded modulation.**

***Index Terms*—Polar codes, bit error rates, coded modulations, successive cancellation decoding, density evolution**

I. Introduction

Compared to the 5G ultra-reliable low-latency communications (URLLC), 6G URLLC will emerge as three different services: broadband, scalable and extreme URLLC [1]. Each URLLC service requires different values in key performance indicators (KPI) such as connection density, reliability, latency and data rates. Especially, the broadband URLLC should apply nonbinary modulation to deliver large packets in a short duration.

In coded modulation systems, bit-interleaved coded modulation (BICM) scheme is widely adopted since binary channel codes can be reused. However, it is known that BICM scheme requires iterative receiver for optimality, and excessive processing delays are induced [2]. In practice, capacity approaching channel decoders fairly correct randomized errors without feedback to demodulator. Instead, the sub-optimality may not be excused by 6G URLLC applications. By matching the dimensions of encoder output symbols to modulation order, nonbinary codes are beneficial to the lossless reception of broadband URLLC with lower latency in despite of increased decoding complexity.

Polar codes were invented assuming binary memoryless channels [3], and extended to arbitrary channels [4]. Binary polar codes have been extensively studied and embedded in 5G New Radio access networks. Unlike efforts to develop short block length coding in 5G URLLC, the broadband URLLC will require both low latency and higher data rates up to 1Tbps. Nonbinary Polar codes will support the requirements of the broadband URLLC with high-order modulation. Also, [5] confirmed that the optimal performance can be obtained under successive cancellation (SC) decoding at the expense of complexity due to the channel dependent code design.

As one of code construction methods, the density evolution (DE) scheme was originally proposed for low-density parity check (LDPC) codes [6] and later applied to binary polar codes [7]. The DE scheme traces the transition probabilities of polarized channels, but its hurdle is the exponential growth of channel output combinations according to higher order modulation and longer code block length. The channel merge scheme based on channel upgrading and degrading was proposed for binary channels [8] and extended to arbitrary channels [9].

In this paper, we apply the DE scheme to polar code construction in nonbinary additive white Gaussian noise (AWGN) channels, and investigate the reliability measure per bit channel. Section II describes the system model of polar coded modulation, section III describes polar code construction algorithms, and section IV shows the simulation results of the proposed channel merge algorithm. Finally, section V concludes this paper.

II. System Model

In this section, system model of polar coded modulation is described. A binary $q \times 2^n$ matrix $\mathbf{D}$ is transformed into $\mathbf{C}$ of same size, in such a way that

$$\mathbf{C}^T\mathbf{b} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^{\otimes n} \mathbf{D}^T\mathbf{b} \tag{1}$$

where $\otimes n$ denotes n times repeated Kronecker product and a basis vector with a primitive element $\alpha \in \mathrm{GF}(2^q)$ is defined as

$$\mathbf{b} = \begin{pmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{q-1} \end{pmatrix}^T \tag{2}$$

The elements of $\mathbf{D}$ are served as synthesized bit channels and ordered according to reliability. Information bits are filled in the most reliable channels and frozen bits are filled in the remaining channels.

Assuming even numbered q, the i-th element of $\mathbf{C}^T\mathbf{b}$ is mapped to a quadrature amplitude modulation (QAM) symbol with Gray bit labeling [10] and is given by

$$x_i = -\sqrt{\frac{3}{2^q - 1}} \sum_{m=0}^{\frac{q}{2}-1} 2^{\frac{q}{2}-1-m} \frac{1}{\sqrt{2}} \left(\prod_{k=0}^{m} (2c_{2k,i} - 1) + j \prod_{k=0}^{m} (2c_{2k+1,i} - 1) \right) \tag{3}$$

Transmit symbols undergo memoryless circularly symmetric complex-valued AWGN channel, so that the likelihood is given by

$$\Pr(y_i|x_i) = \frac{1}{\pi\sigma^2}\exp\left(-\frac{|y_i - x_i|^2}{\sigma^2}\right) \tag{4}$$

where σ^2 is a noise variance. The likelihood is denoted as

$$W_i^{(0)}(y_i|\mathbf{c}_i^T\mathbf{b}) = \Pr(y_i|x_i) \tag{5}$$

The likelihoods are fed to the receiver for $u \in \mathrm{GF}(2^q)$. The SC decoding is performed resorting to the recursions [9] as following.

$$W_{i_\mathrm{e}}^{(\lambda+1)}(z_0, z_1|u) = \frac{1}{2^q}\sum_{v\in\mathrm{GF}(2^q)} W_{i_\mathrm{e}}^{(\lambda)}(z_0|u+v\alpha)W_{i_\mathrm{o}}^{(\lambda)}(z_1|v) \tag{6}$$

$$W_{i_\mathrm{o}}^{(\lambda+1)}(z_0, z_1, u|v) = \frac{1}{2^q}W_{i_\mathrm{e}}^{(\lambda)}(z_0|u+v\alpha)W_{i_\mathrm{o}}^{(\lambda)}(z_1|v), \tag{7}$$

where phase $0 \le i_\mathrm{o} = i_\mathrm{e} + 2^{n-1-\lambda} < 2^n$, layer $0 \le \lambda < n$ and channel input symbols $u, v \in \mathrm{GF}(2^q)$. The polarization kernel can be depicted in terms of input symbols as shown in Fig. 1.

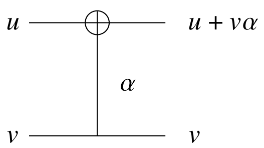

Fig. 1. Polarization kernel over $\mathrm{GF}(2^q)$

At layer n, a data symbol is sliced as

$$\widehat{\mathbf{d}}_i(y) = \arg\max_{\{\mathbf{a}\in\mathbb{Z}_2^q | a_j=0\ \forall j\in\mathcal{F}_i\}} W_i^{(n)}(y|\mathbf{a}^T\mathbf{b}) \tag{8}$$

where $\mathcal{F}_i$ denotes the set of bit indices filled with frozen bits.

III. Code Constructions

For layer $\lambda = 0$, continuous output alphabet set $\mathbb{C}$ is quantized into finite areas

$$\mathcal{A}_{m_0+2Lm_1} = \left\{a_0 + ja_1 \in \mathbb{C} \;\middle|\; \left\lfloor\frac{a_k}{\Delta}\right\rfloor = m_k - L, k \in \{0, 1\}\right\} \tag{9}$$

where resolution Δ and the number $M = (2L)^2$ of area depends on noise variance σ^2. Also, indices m_0 and m_1 of area are between 0 and $2L$. The outputs of $W_i^{(0)}$ in an area are merged to obtain discrete channels as following.

$$\widetilde{W}_i^{(0)}(m|u) = \int_{\mathcal{A}_m} W_i^{(0)}(y_i|u)dy_i \tag{10}$$

where $0 \le m < M$.

For layers $0 < \lambda \le n$, the recursions (6) and (7) are approximated as

Input: $Q \times \mu$ TPM $\mathbf{R}$, number N_iter of iterations for k-means clustering, number M of output symbols in merged channel
Output: merged $Q \times M$ TPM $\mathbf{R}$
while $\mu > M$ **do**
  /* Normalizations */
  for $k = 0, 1, \cdots, \mu - 1$ **do**
    $v_k \leftarrow \mathbf{1}^T\mathbf{r}_k/Q$
    $\mathbf{r}_k \leftarrow \mathbf{r}_k/v_k$
  end
  /* k-means clustering */
  $\mathbf{h}_0 \leftarrow \mathbf{1}$
  while $m = 1, 2, \cdots, M - 1$ **do**
    for $k = 0, 1, \cdots, \mu - 1$ **do**
      $\phi(k) \leftarrow \arg\min_{0\le j<m} \|\mathbf{r}_k - \mathbf{h}_j\|_1$
    end
    $i \leftarrow \arg\max_k \|\mathbf{r}_k - \mathbf{h}_{\phi(k)}\|_1$
    $\mathbf{h}_m \leftarrow \mathbf{r}_i$
  end
  for $t = 0, 1, \cdots, N_{iter} - 1$ **do**
    for $j = 0, 1, \cdots, M - 1$ **do**
      $\mathbf{h}_j \leftarrow \mathrm{median}\{\mathbf{r}_k|\phi(k) = j\}$
    end
    for $k = 0, 1, \cdots, \mu - 1$ **do**
      $\phi(k) \leftarrow \arg\min_{0\le j<M} \|\mathbf{r}_k - \mathbf{h}_j\|_1$
    end
  end
  /* Proximity ordering */
  for $k = 0, 1, \cdots, \mu - 1$ **do**
    $g_k \leftarrow v_k \cdot \|\mathbf{r}_k - \mathbf{h}_{\phi(k)}\|_1$
  end
  Find a permutation θ on $\{0, 1, \cdots, \mu - 1\}$ such that $g_{\theta(k)} \le g_{\theta(k+1)}$ for $0 \le k < \mu - 1$
  /* Merging selected outputs */
  for $j = 0, 1, \cdots, M - 1$ **do**
    $\mathbf{s}_j \leftarrow \sum_{\{0\le k<\mu/2+M | \phi(k)=j\}} \mathbf{r}_{\theta(k)} v_{\phi(k)}$
  end
  for $j = M, M + 1, \cdots, \mu/2 - 1$ **do**
    $\mathbf{s}_j \leftarrow \mathbf{r}_{\theta(j+\mu/2)}$
  end
  $\mathbf{R} \leftarrow \mathbf{S}$
  $\mu \leftarrow \mu/2$
end

Algorithm 1: Channel merge algorithm

$$\widetilde{W}_{i_\mathrm{e}}^{(\lambda+1)}(m|u) = \sum_{(m_0,m_1)\in\mathcal{S}_m^{(i_\mathrm{e})}} \frac{1}{2^q} \sum_{v\in\mathrm{GF}(2^q)} \widetilde{W}_{i_\mathrm{e}}^{(\lambda)}(m_0|u+v\alpha)\widetilde{W}_{i_\mathrm{o}}^{(\lambda)}(m_1|v) \tag{11}$$

$$\widetilde{W}_{i_\mathrm{o}}^{(\lambda+1)}(m|v) = \sum_{(m_0,m_1,u)\in\mathcal{S}_m^{(i_\mathrm{o})}} \frac{1}{2^q} \widetilde{W}_{i_\mathrm{e}}^{(\lambda)}(m_0|u+v\alpha)\widetilde{W}_{i_\mathrm{o}}^{(\lambda)}(m_1|v) \tag{12}$$

where sets $S_m^{(i_e)}$ and $S_m^{(i_o)}$ should be determined to maximize channel capacities of $\widetilde{W}_{i_e}^{(\lambda+1)}$ and $\widetilde{W}_{i_o}^{(\lambda+1)}$ respectively [8] [9]. The summations over the sets correspond to the channel merge and can be performed by Algorithm. 1, where the original channel is formatted to a transition probability matrix (TPM) $\mathbf{R}$ as in [11], i.e., the element $r_{i,j}$ corresponds to the likelihood of the i-th input symbol observing the j-th output symbols.

The normalized column vectors of TPM are clustered in L_1 distance measure so that $\mathbf{r}_k$ is associated with a centroid $\mathbf{h}_{\phi(k)}$. The distance between $\mathbf{r}_k$ and $\mathbf{h}_{\phi(k)}$ is scaled with the normalization factor ν_k and served as cost. The vectors with $(\mu/2+M)$ smallest costs are merged to M clusters and collected with remaining $(\mu/2-M)$ vectors, so the number of columns in updated TPM amounts to $\mu/2$.

Finally, given the approximate channel $\widetilde{W}_i^{(n)}$ of the i-th symbol, the expected error rates for the k-th bit are given by

$$P_{\text{err}}(k,i) = \frac{1}{2^q}\sum_{y} \sum_{\left\{\mathbf{a}\in\mathbb{Z}_2^q \;\mid\; a_k\neq \widehat{d}_{k,i}(y)\right\}} \widetilde{W}_i^{(n)}(y|\mathbf{a}^T\mathbf{b}) \qquad (13)$$

where $\widehat{d}_{k,i}(y)$ denotes the k-th element of the estimator $\widehat{\mathbf{d}}_i(y)$. The bit channels of $\mathbf{D}$ with the smallest P_{err} are utilized for information transmissions.

IV. Simulations

The proposed code construction methods were evaluated with parameters $q = 4, n = 5, \sigma = 0.1$, and $M = 256$. Approximate channel capacities at layer 0 and 5 are compared in Fig. IV, where symbol indices are sorted in ascending capacity. Channel capacities at layer 5 are obtained from the proposed channel merge scheme and approximate the original channel capacities at layer 0.

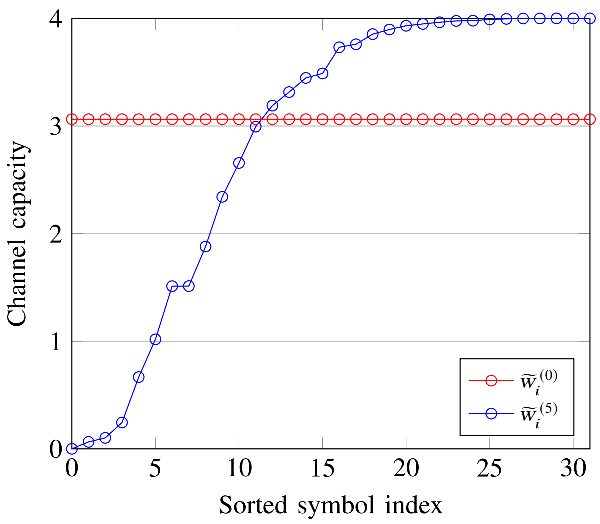

Fig. 2. Approximate channel capacities

Also, bit error probabilities P_{err} are illustrated in Fig. IV, where the symbol indices are same as ones in Fig. IV. Error rates mainly depend on symbol index, but some deviations are observed across bit channels. Linear ordering on the pairs of symbol and bit indices will be necessary for more reliable communications.

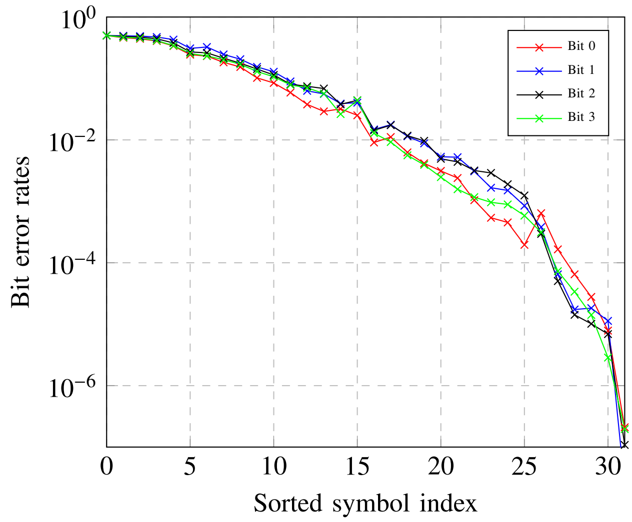

Fig. 3. Error rates per bit channel

V. Conclusions

Assuming the system model of polar coded modulation, nonbinary polar codes can be constructed using density evolution schemes. Channel merging operations are essential to resolve the growth of channel output alphabets, and k-means clustering algorithms are proposed to find suitable candidates for merging. Also, each input symbol can be decomposed into bit channels, and bit error rates were derived from the synthesized channels. Finding reliability order on the pairs of symbol and bit indices completes code construction.

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