Blocking and Persistence in the Zero-Temperature Dynamics of Homogeneous and Disordered Ising Models

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Abstract

A "persistence" exponent θ has been extensively used to describe the nonequilibrium dynamics of spin systems following a deep quench: for zero-temperature homogeneous Ising models on the d-dimensional cubic lattice Z^d, the fraction $p(t)$ of spins not flipped by time t decays to zero like $t^{-\theta(d)}$ for low d; for high d, $p(t)$ may decay to $p(\infty) > 0$, because of "blocking" (but perhaps still like a power). What are the effects of disorder or changes of lattice? We show that these can quite generally lead to blocking (and convergence to a metastable configuration) even for low d, and then present two examples — one disordered and one homogeneous — where $p(t)$ decays *exponentially* to $p(\infty)$.

In modelling the nonequilibrium dynamics of spin systems following a deep quench, the following question naturally arises [1, 2, 3, 4, 5, 6, 7, 8]: given a spin system at zero temperature with random starting configuration and evolving according to the usual Glauber dynamics, what is the probability $p(t)$ at time t that a spin has not yet flipped?

For the homogeneous ferromagnetic Ising model on Z^d, this probability has been found to decay at large times as a power law $p(t) \sim t^{-\theta(d)}$ [1, 2, 3] for $d < 4$. The "persistence" exponent $\theta(d)$ is considered to be a new universal exponent governing nonequilibrium dynamics following a deep quench [6]. The persistence problem can be extended to positive temperatures by considering the dynamics of the local order parameter rather than that of single spins [9, 10, 11].

In this paper we confine our attention to dynamics at zero temperature in infinite spin systems. In the usual case of asynchronous updating, a spin is chosen at random (this can be made precise for infinite systems, as in [12]) and then: always flips if the resulting

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configuration has lower energy, never flips if the energy is raised, and flips with probability 1/2 if the resulting energy change is zero. We will consider these dynamics for random initial configurations σ^0 (in which each spin is equally likely to be up or down, independent of the others) in both disordered ferromagnets and spin glasses with continuous coupling distributions, and also for uniform ferromagnets on lattices other than Z^d (e.g., hexagonal lattices in $2D$).

Our first result is that the persistence phenomenon as described above is unstable to the introduction of randomness into the spin couplings, or even to some changes in lattice structure. For the random ferromagnet, spin glass, $2D$ hexagonal ferromagnet, and others to be discussed below, we will see that a positive fraction of spins never flip and every spin flips only finitely many times.

The "frozenness" of a nonvanishing fraction of spins (sometimes referred to as "blocked" spins [5]) has been reported in numerical simulations of Ising ferromagnets on Z^d with $d > 4$ [1] and q-state Potts models on square lattices for $q > 4$ [5]. The problem can then be recast by restricting attention to only those spins that eventually do flip, and asking for the conditional probability that such spins haven't yet flipped by time t. Simulations of Potts models appeared to indicate that this probability (proportional to $p(t) - p(\infty)$) also decays as a power law at long times [5] (however, some curvature on their log-log plots was noted).

We will examine the same question for disordered Ising systems and also for homogeneous systems that show blocking. Although we cannot yet answer this question in general, we will present calculations on two systems, the homogeneous ferromagnet on a quasi-$1D$ "ladder" and the $1D$ disordered spin chain [13], showing that $p(t) - p(\infty)$ decays *exponentially* as $t \to \infty$. Exponential decay for $d \geq 2$ will also be discussed.

Persistence and local nonequilibration. The analysis of persistence exponents suggests that the fraction of sites that remain in the same phase (for $T > 0$) or spin value (at $T = 0$) from time t_1 to time t_2 tends to zero for $1 \ll t_1 \ll t_2$. It therefore implies the presence of *local nonequilibration* (LNE) [12]: that in any fixed, finite region, there exists no finite time after which the spins within remain in a single phase; that is, domain walls forever sweep across the region. At zero temperature, the presence of LNE means every spin flips infinitely often (in almost every sample).

Why does the decay to zero of $p(t)$ (coming from the analysis of persistence exponents at $T = 0$) suggest that every spin flips infinitely many times? Suppose instead that a positive fraction of spins flip only finitely many times. Then it is reasonable to expect that a (smaller but still positive) fraction of spins *never* flip, and $p(t)$ would not decay to zero. While not proved in general, this argument applies to all systems treated here.

It was proved in [14] (see also [12]) that, in the homogeneous Ising ferromagnet on the square lattice with a random initial spin configuration, every spin indeed flips infinitely often at zero temperature, consistent with persistence results in the literature. (Similar results apply to several other systems, and can be extended to positive temperature with the local order parameter in a region replacing individual spins [12].)

Blocking. What about the zero-temperature dynamics of systems with continuous disorder? In any dimension and on any lattice, it can be proved for these (and many other) systems that *every spin flips only finitely many times.* These systems exhibit "blocking" and for them $p(t)$ does not decay to zero.

These are examples of a general result [14] applying to the dynamical evolution (following a deep quench) of infinite-volume Ising spin systems with Hamiltonian

$$\mathcal{H} = -\sum_{<xy>} J_{xy}\sigma_x\sigma_y \ , \tag{1}$$

where the sum is over nearest neighbors. If the distribution of couplings is continuous with finite mean, then it can be proved that every spin flips only finitely many times (for almost every σ^0, realization ω of the dynamics and realization $\mathcal{J}$ of the couplings).

The proof of this theorem yields a more general result that shows that, even without the continuity assumption on the distribution of couplings, for almost every $\mathcal{J}$, σ^0, and ω, there can be only finitely many flips of any spin that cause a *nonzero* energy change. This is why the above result applies to ordinary spin glasses and random ferromagnets with a continuous distribution of couplings (e.g., Gaussian or uniform): the probability of a "tie" in any sum or difference of a given spin's nearest-neighbor coupling strengths (and therefore the probability of a spin flip costing zero energy) is zero, and the result follows.

We sketch the proof here; for further details, see [14]. Let σ_x^t be the value of σ_x at time t for fixed ω, σ^0 and $\mathcal{J}$. Let

$$E(t) = -(1/2)\overline{\sum_{y:|x-y|=1} J_{xy}\sigma_x^t\sigma_y^t} \tag{2}$$

where the bar indicates an average over $\mathcal{J}$, σ^0, and ω. By translation-ergodicity of the distributions from which $\mathcal{J}$, σ^0, and ω were chosen, and using the assumption that $\overline{|J_{xy}|} < \infty$, it follows that $E(t)$ exists, is independent of x, and equals the energy density (i.e., the average energy per site) at time t in almost every realization of $\mathcal{J}$, σ^0, and ω.

Because every spin flip lowers the energy, $E(t)$ monotonically decreases in time (note that $E(0) = 0$) and has a finite limit $E(\infty)$ ($\geq -d\overline{|J_{xy}|}$). Now choose any fixed number $\epsilon > 0$, and let N_x^ϵ be the number of spin flips (over all time) of the spin at x that lower the energy by an amount ϵ or greater. Then $-\infty < E(\infty) \leq -\epsilon\overline{N_x^\epsilon}$ so that for every x and $\epsilon > 0$, N_x^ϵ is finite. Let ϵ_x be the minimum energy (magnitude) change resulting from a flip of σ_x; then although ϵ_x varies (differently in each $\mathcal{J}$) with x, it is sufficient that it is strictly positive.

This result applies also to homogeneous systems on certain lattices, such as Ising ferromagnets on lattices with an *odd* number of nearest-neighbors so that ties in energy cannot occur. Such lattices include the hexagonal (or honeycomb) lattice in $2D$, and the double-layered cubic lattices $Z^d \times \{0, 1\}$ (i.e., a "ladder" when $d = 1$, two horizontal planes separated by unit vertical distance when $d = 2$, and so on) [15].

As for blocking in these systems, it is elementary to show that a positive fraction of spins will never flip. Consider first the hexagonal lattice. If the spins on any single hexagon are all up or all down, they will form a stable configuration that will never change. Such configurations (and of course similar larger-scale ones) occur with positive density in almost every σ^0. Similarly, in the ladder, any square with all spins up or down is stable. The extension to general $Z^d \times \{0,1\}$ is straightforward.

Turning to disordered systems, consider first the random ferromagnet on Z^2. For almost every $\mathcal{J}$, there will be a positive density of plaquettes whose couplings satisfy the following: on each of the four corners, the sum of the two couplings that connect to adjacent corners of the square is greater than the sum of the two couplings to sites outside the square. If the spins at the four corners initially are all up or all down, the spin configuration on the square will again be stable. A similar construction can be used for general d and for spin glasses.

To summarize, our first result has been to prove that many ordered and disordered spin systems display two important zero-temperature dynamical properties, which when taken together lead them to exhibit novel persistence behavior. The first concerns the presence of blocking, meaning a positive fraction of spins never flip. In the systems we treat, this is a zero time property in that some of the spins are blocked by the nature of σ^0 (and $\mathcal{J}$), regardless of the dynamics realization. The second property concerns infinite time: the existence of a limiting (metastable) spin configuration σ^∞, since every spin flips only finitely many times. Although the second of these properties probably implies the first, the first does not imply the second [16]. Our next result is to show that for at least some of these systems, these two properties lead to an exponential (as opposed to power law) decay of the quantity $p(t) - p(\infty)$ at large times.

Exponential decay. In this section we study the large-time behavior of $p(t) - p(\infty)$, the probability that a spin will flip at some time but has not yet flipped by time t. We will prove that this quantity decays exponentially by showing the same for the larger probability $\tilde{p}(t)$ that a spin will flip at some time after t (whether or not it has flipped before). We consider two systems, one homogeneous (the uniform Ising ferromagnet on the ladder) and one disordered (the $1D$ continuously disordered Ising chain).

Consider first a homogeneous system where every site has an odd number M of neighbors. (Systems such as $\pm J$ spin glasses where the *signs* are disordered but not the $|J_{xy}|$'s also fall into this category of examples.) Consider at time τ, all sites y such that the spin at y will flip after time τ, and denote by $\mathcal{C}_x(\tau)$ the cluster of such sites that contains x (an empty cluster if the spin at x will not flip after time τ). We will show below for the ladder model that with $\tau = 0$, the distribution of the number of sites $|\mathcal{C}_x(\tau)|$ in these clusters has an exponential tail; i.e., the probabilities for large cluster sizes are bounded by:

$$P(|\mathcal{C}_x(\tau)| \geq n) \;\leq\; A\, e^{-kn} \tag{3}$$

for some $A < \infty$ and $k > 0$. We next show that this implies exponential decay of $\tilde{p}(t)$.

Since each flip in $\mathcal{C}_x(\tau)$ lowers the energy of that cluster by at least 2 and since the total energy of the cluster lies somewhere between $-M|\mathcal{C}_x(\tau)|$ and $M|\mathcal{C}_x(\tau)|$ (we take $J = 1$ here), it follows that the entire cluster must reach its final configuration after no more than $M|\mathcal{C}_x(\tau)|$ flips. Let T_1 denote the (random) amount of time after τ until the first flip in $\mathcal{C}_x(\tau)$, T_2 the amount of time after $\tau + T_1$ until the second flip, etc. Clearly, as long as flips are possible, the T_i's are bounded above by independent exponential (mean one) random variables T'_i. Thus the time of the last flip of x is bounded above by $\tau + T'_1 + \cdots + T'_{M|\mathcal{C}_x(\tau)|}$ and so for $t > \tau$,

$$p(t) - p(\infty) \leq \tilde{p}(t) \leq \sum_{n=1}^{\infty} P(|\mathcal{C}_x(\tau)| = n) P(T'_1 + \cdots + T'_{Mn} \geq t - \tau) \,. \tag{4}$$

The probability density of $T'_1 + \cdots + T'_j$ is $f(s) = s^{j-1}e^{-s}/(j-1)!$ and so for $t > \tau$,

$$\begin{aligned} p(t) - p(\infty) &\leq \sum_{n=1}^{\infty} A e^{-kn} \int_{t-\tau}^{\infty} [s^{Mn-1}/(Mn-1)!] e^{-s} ds \\ &\leq \int_{t-\tau}^{\infty} \sum_{j=1}^{\infty} A (e^{-k/M})^j [s^{j-1}/(j-1)!] e^{-s} ds \\ &= A e^{-k/M} \int_{t-\tau}^{\infty} \exp(e^{-k/M} s - s) ds \\ &= A' e^{-k't} \end{aligned} \tag{5}$$

where the constants A' and k' depend on A, k, M and τ.

It remains to show that (3) is valid in the ladder ferromagnet with $\tau = 0$. (Similar arguments work for the ladder antiferromagnet or $\pm J$ spin glass.) To do this we note that a single plaquette has an initial probability $p_0 = 1/8$ of having its four corner spins all plus or all minus (we call such a blocked plaquette "frigid"). The lattice's sites take integer values (x_i, y_i), with $-\infty < x_i < \infty$ and $y_i = 0, 1$. A lower bound for the initial number of frigid plaquettes can be obtained by considering only those plaquettes whose left edges occur at even x_i (we define the location of a plaquette by the position of its left edge); such plaquettes do not overlap and so their probabilities of being frigid are independent. If the plaquette at the origin is frigid, and $P(0 \to 2n)$ is the probability that there is no frigid plaquette between 0 and x_{2n}, then

$$P(0 \to 2n) \leq (1 - p_0)^n = \exp(-kn) \ , \tag{6}$$

where $k = |\log(1 - p_0)|$. So the ladder is broken up into finite segments, bounded to either side by a frigid plaquette, whose length distribution has an exponential tail. This yields Eq. (3).

Our second example is a disordered $1D$ spin chain in zero field. The analysis is essentially the same for either the spin glass or the random ferromagnet, so for specificity we study a ferromagnet whose couplings $J_z \equiv J_{z,z+1}$ are independent random variables taken from the uniform distribution on $[0, 1]$. The key idea here is that the chain breaks up into finite, disjoint "influence segments" whose union is the infinite chain. An influence segment is a dynamical construct defined (for a given $\mathcal{J}$) as follows: two sites x and y belong to the same influence segment if and only if either the state of σ_x can dynamically induce a change in the state of σ_y or vice-versa (or both). To illustrate, suppose that the coupling J_x is larger than both J_{x-1} and J_{x+1}; i.e., J_x is a local maximum. Then it is clear that the state of σ_x can be dynamically influenced by σ_{x+1} but not by σ_{x-1} (and similarly, the state of σ_{x+1} can be dynamically influenced by σ_x but not by σ_{x+2}). That is, no state of the spin σ_{x-1} can alter the sign of the energy change $\Delta\mathcal{H}_x$ that would result from a flip of σ_x. To summarize, two sites x and y are defined to be in the same influence cluster if and only if either σ_x can influence σ_y, or σ_y can influence σ_x, or both [17].

Influence segments for the disordered $1D$ chain are then constructed as follows [14]. Consider the doubly infinite sequences $x = x_m$ of sites where J_x is a local maximum and $y = y_m \in (x_m, x_{m+1})$ where J_y is a local minimum: the couplings are strictly increasing from y_{m-1} to x_m and strictly decreasing from x_m to y_m. The set of spins at the sites $\{y_{m-1}+1, y_{m-1}+2, \ldots y_m\}$ determines a single influence segment.

To see this, note that the spin at y_m cannot influence the one at y_m+1 (or vice-versa); similarly, at the other end $y_{m-1}+1$ cannot influence y_{m-1}. Now consider the spins at x_m and x_m+1, which are within the interval $\{y_{m-1}+1, y_{m-1}+2, \ldots y_m\}$. Clearly, the spin at x_m-1 can never influence the spin at x_m, and the spin at x_m+2 can never influence the one at x_m+1. So once the spins at x_m and x_m+1 agree (either initially in σ^0 or after either spin flips) the final value of every spin in the interval $\{y_{m-1}+1, y_{m-1}+2, \ldots y_m\}$ is determined through a "cascade" of influence to either side of $\{x_m, x_m+1\}$ (which is put into effect as the Poisson clocks successively ring) until $y_{m-1}+1$ and y_m, respectively, are reached.

Given this, the analysis leading to Eqs. (4) and (5) applies as before. One needs only an estimate analogous to Eq. (3) for the probability distribution of influence segment sizes. In fact, the decay here for large size n is faster than in Eq. (3); the probability of n independent coupling random variables being ordered so as to have a single local maximum falls off as $1/n!$ (times an exponential factor).

In these two examples different factors determine the distribution of dynamical cluster sizes: for the homogeneous ferromagnet on the ladder, they're determined by the initial spin configuration σ^0; for the disordered $1D$ chain, they're determined by the coupling realization $\mathcal{J}$.

In this section we considered two examples, one ordered and one disordered, but both one-dimensional (or quasi-one-dimensional). There is another system that shows the same behavior in *any* dimension: the highly disordered spin glass (or ferromagnet) [18, 19, 20]. Using similar arguments, this system can also be shown to display an exponential decay to its final state [21]. We expect that a related model, in which coupling magnitudes are "stretched" in the manner of references [18, 19, 20] but only up to a finite length scale, would show similar behavior. This last model is of interest because its thermodynamic behavior is expected to be similar to that of the ordinary spin glass (or random ferromagnet).

Discussion. Most work on persistence at zero temperature has examined systems, such as the homogeneous Ising ferromagnet on Z^d in low d, where the quantity $p(t)$ decays to zero as a power law. We have shown here that there is a second class of models in which $p(\infty) > 0$: these include systems with continuous disorder and homogeneous systems on other lattices. In several of these the persistence decay is exponential rather than power law. It would be of interest to see whether this fast decay holds in other systems in this general class, such as the $2D$ homogeneous ferromagnet on a hexagonal lattice [22] or an ordinary spin glass with $d > 1$.

Although we don't know the answer in the general case, we can speculate using a rough argument that the answer may be yes. If every spin flips only finitely often, as time progresses an increasing number of spins will "freeze"; i.e., they cease to flip. It is reasonable to expect that after some finite time "unfrozen" spins no longer percolate, so that the dynamics is confined to noninteracting finite clusters, as in the examples treated here. Of course, there remain serious gaps: this is not independent percolation, and the dynamics in the localized clusters (should they exist) would need to be worked out, so the conclusion should be treated with caution.

There is a third class of systems not discussed here; in these, a positive fraction of spins flip infinitely often and a positive fraction flip only finitely many times. One such system is the two-dimensional $\pm J$ spin glass [16]. Although it appears that $p(\infty) > 0$, determining the large-time behavior of $p(t) - p(\infty)$ remains an open problem.

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