Is there a Landau Pole Problem in QED?

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We investigate a lattice version of QED by numerical simulations. For the renormalized charge and mass we find results which are consistent with the renormalized charge vanishing in the continuum limit. A detailed study of the relation between bare and renormalized quantities reveals that the Landau pole lies in a region of parameter space which is made inaccessible by spontaneous chiral symmetry breaking.

I. INTRODUCTION

QED is the best tested of all quantum field theories. But all its success is in the context of perturbation theory. It has long been known that there are potential problems in the foundations of the theory due to the existence of the so-called Landau pole [1]. In the leading logarithmic calculation one finds

$$\frac{1}{e_R^2} - \frac{1}{e^2} = \beta_1 \ln \frac{\Lambda}{m_R}, \ \beta_1 = \frac{N_f}{6\pi^2}, \tag{1}$$

where $e(e_R)$ is the bare (renormalized) charge, m_R is the renormalized fermion mass, N_f is the number of flavors and Λ is the ultra-violet cutoff. When one attempts to send the cutoff to infinity while keeping e_R fixed, one finds that e diverges at

$$\Lambda = \Lambda_L \equiv m_R \mathrm{e}^{\frac{1}{\beta_1 e_R^2}},\tag{2}$$

the location of the Landau pole. The problem can also be seen by looking at the gauge invariant part of the photon propagator

$$\frac{D(k)}{k^2} = \frac{1}{k^2 \left[1 - (\beta_1/2) \ln(k^2/m_R^2)\right]},$$
(3)

which has a ghost pole at $k^2 = \Lambda_L^2$. This would mean that the entire theory is only applicable for momenta smaller than Λ_L . On the other hand, when one keeps e fixed and sends the cut-off to infinity, the renormalized charge goes to zero, meaning that the theory is trivial. The situation in two-loop perturbation theory is much the same. The (renormalized) β function

$$\beta_R \equiv m_R \frac{\partial e_R^2}{\partial m_R} \bigg|_{e^2} = \beta_1 e_R^4 + \beta_2 e_R^6 + \cdots$$
 (4)

remains positive for all e_R^2 , and the Landau pole is displaced to lower values

$$\Lambda_L = m_R \mathrm{e}^{\frac{1}{\beta_1 e_R^2}} \left(\frac{\beta_2 e_R^2}{\beta_1 + \beta_2 e_R^2} \right)^{\frac{\beta_2^2}{\beta_1^2}} \,. \tag{5}$$

QED is not the only theory with a Landau pole problem. Every theory which is not asymptotically free suffers from this problem. While $\Lambda_L \simeq 10^{227}$ GeV if only the electron is considered, $\Lambda_L \simeq 10^{34}$ GeV in the Standard Model. In the Minimal Supersymmetric Standard Model (MSSM) $\Lambda_L \simeq 10^{20}$ GeV, and in the MSSM with four Higgses, which offers a solution to the strong CP problem, the Landau pole moves down to $\Lambda_L \simeq 10^{17}$ GeV [2]. Thus the Landau pole is by no means academic.

To find a solution to this problem, one must consider a non-perturbative formulation of QED. Thus it is natural to investigate the problem on the lattice. On the lattice the inverse lattice spacing takes over the role of the ultraviolet cut-off, $a^{-1} \sim \Lambda$. Early calculations have shown that the non-compact formulation of the theory using staggered fermions undergoes a second order chiral phase transition at strong coupling [3,4]:



The solid line am = 0, m being the bare mass, $e^2 > e_c^2$ is a line of first order chiral phase transitions, where am_R , $a^3 \langle \bar{\psi}\psi \rangle \neq 0$, even though the bare mass is zero. The dashed line am = 0, $e_c^2 > e^2$ is a line of second order phase transitions on which am_R , $a^3 \langle \bar{\psi}\psi \rangle = 0$. A meaningful continuum limit can be taken at the tricritical point am = 0, $e^2 = e_c^2$, because here we can take a to zero while keeping m_R fixed.

To understand the continuum limit of the theory, we need to know the renormalized charge as a function of the cut-off in the critical region. We have recently computed the chiral condensate on large lattices [5]. In this letter we compute am_R and e_R in order to understand the fate of the Landau pole in QED.



FIG. 1. The renormalized mass against the bare coupling on $12^4~{\rm and}~16^4$ lattices.

II. FERMION MASS

We obtain the renormalized mass from the fermion propagator as outlined in Ref. [6]. We are using staggered fermions which in the continuum limit correspond to $N_f = 4$ flavors of dynamical Dirac fermions. The results are shown in Fig. 1.

Fitting and extrapolating this data it greatly helps that the chiral condensate $\sigma = a^3 \langle \bar{\psi}\psi \rangle$ is a function of am_R only [6]. In Fig. 2 we plot σ as a function of am_R . We observe that σ is well described by the polynomial

$$\sigma = 0.6197 a m_R - 0.321 (a m_R)^3 + 0.169 (a m_R)^5 -0.040 (a m_R)^7, \tag{6}$$

where the first coefficient is given by the one-loop result. This helps because we have already found an equation of state (EOS) that describes the σ data [5]. Combining the EOS with the polynomial (6) gives the curves shown in Fig. 1 and the extrapolation to am = 0. (Here we have used fit 1 of Ref. [5]. Our results do not change qualitatively if we use any of the other fits described there.) For $1/e^2 < 1/e_c^2$ chiral symmetry is broken, and even at am = 0 the renormalized mass is non-zero. This means there is an excluded region shown in white in Fig. 1 (the accessible region being shown in gray).

III. RENORMALIZED CHARGE

The renormalized charge is obtained from the residue of the photon propagator, $e_R^2 = Z_3 e^2$ and $Z_3 = \lim_{k\to 0} \lim_{V\to\infty} D(k)$. We can compute D(k) on the lattice, but not at k = 0. The smallest momentum that we can reach is $2\pi/aL$, where L is the lattice size (L = 16



FIG. 2. The chiral condensate against the renormalized mass on 12^4 and 16^4 lattices.

and 12 in our case). To extrapolate to k = 0 we need to make a fit to the photon propagator. The k dependence of the photon propagator is given by

$$\frac{1}{e^2 D(k)} - \frac{1}{e^2} = -\Pi(k, m_R, L), \qquad (7)$$

where Π is the polarization function. In the infinite volume limit we then have

$$\frac{1}{e_R^2} - \frac{1}{e^2} = -\Pi(0, m_R, \infty) \,. \tag{8}$$

We have already seen that the non-perturbative Π is actually very close to the result of one-loop renormalized perturbation theory [6]. So it is reasonable to make an ansatz which is inspired by renormalization group improved two-loop perturbation theory. In Ref. [7] it is shown that to next-to-leading logarithmic order the polarization function can be written

$$\Pi = U - \frac{V}{U} \ln(1 - e^2 U), \qquad (9)$$

where U is the one-loop perturbative result, and V the two-loop one. The lattice result for U is known [6]. For V we make the ansatz

$$V = v_0 + v_1 U \,. \tag{10}$$

This is motivated by the small k^2 and m_R^2 limits. For $a^2m_R^2 \ll a^2k^2 \ll 1$ we should have $V \simeq (\beta_2/2) \ln a^2k^2$, and for $a^2k^2 \ll a^2m_R^2 \ll 1$ we should have $V \simeq (\beta_2/2) \ln a^2m_R^2$. The one-loop result U has these properties. We fit this ansatz to a total of 52 photon propagators on 16⁴ and 12⁴ lattices for various values of am, e^2 in the range $0.005 \le am \le 0.16$, $0.16 \le 1/e^2 \le 0.22$ close to the critical point at $1/e_c^2 = 0.19040(9)$ [5]. A plot for



FIG. 3. The residue of the photon propagator against the momentum for $1/e^2 = 0.20$, am = 0.005 on the 16^4 lattice. The open symbols are the data, the solid symbols are the fit.

one particular parameter set is shown in Fig. 3. For the fit parameters we obtain $v_0 = -0.00207(2)$ and $v_1 = -0.0328(7)$, giving $\chi^2/\text{d.o.f.} = 1.7$. Two-loop continuum perturbation theory would give $v_1 \equiv \beta_2/\beta_1 = 3/16\pi^2 = 0.0190$. In Fig. 4 we show the resulting β function for $e^2 = e_c^2$. We compare this with the one-loop result. We see that the β function is a little smaller than the one-loop value and is positive. In particular this means that there is no ultra-violet stable zero in the β function out to $e_R^2 = e_c^2$, the maximal value e_R^2 can take because $Z_3 \leq 1$ [8,9]. As $am_R \to \infty$ fermion loops are suppressed and $e_R^2 \to e^2$, so that the β function vanishes. But this is of course not an interesting zero of the β function.



FIG. 4. The β function against the renormalized charge. The solid curve is our result, the dashed curve is the lattice one-loop result.



FIG. 5. A sketch of the mapping from the bare parameter plane (top) to the renormalized parameter plane (bottom).

IV. THE LANDAU POLE

Having calculated the renormalized mass and charge, we are now able to discuss the mapping from the bare parameters am, e to the renormalized parameters am_R , e_R . Qualitatively this is displayed in Fig. 5. One can choose any $e^2 \ge 0$, am shown in the top part of the figure by the gray region. This is then mapped onto the corresponding gray region in the bottom part of the figure. The line am = 0 from $e^2 = 0$ to $e^2 = e_c^2$ is mapped onto the point $am_R = e_R^2 = 0$. For $e^2 > e_c^2$ we have already seen in Fig. 1 that $am_R > 0$, even when am = 0, because of chiral symmetry breaking. Thus the line am = 0 from $e^2 = e_c^2$ to $e^2 = \infty$ is mapped onto the border line of the gray regions, and the white area is inaccessible for any combination of bare parameters. From this figure we evidently have triviality. Removing



FIG. 6. The (quantitative) mapping of the bare parameter plane (top) to the renormalized parameter plane (bottom).

the cut-off $(a \to 0)$ is only possible at $e_R^2 = 0$. At any finite value of e_R^2 there is a minimal possible value for $|am_R|$, namely the boundary of the gray region. The position of the Landau pole is sketched by the dotted line.

We now turn to the quantitative analysis of the problem. In Fig. 6 we plot again the bare and renormalized planes, this time using $1/e^2$ and $1/e_R^2$, respectively, as the horizontal variables because this displays the asymptotic behavior best. The curves are lines of constant $1/e_R^2$ (top part) and lines of constant am (bottom part), respectively. All lines of constant e_R^2 end on the first order phase transition line, and only the line $e_R^2 = 0$ goes into the critical point. This is another expression of triviality of the theory. In the bottom part of the figure the gray region is again the allowed region, and the white region is inaccessible. The border line is the line am = 0. From eq. (8) we find the Landau pole by setting the bare charge to infinity. This gives the dotted line. We see that it completely lies in the inaccessible region. It runs roughly parallel to the border line am = 0.

We also want to be sure that the photon propagator has no ghost pole for any k^2 . Beyond leading logarithmic order the extra pole in the photon propagator and the divergence in e^2 need not appear at the same place. The ghost pole position (in the infinite volume) is given by $1/k^2D(k) = 0$. It has a solution if

$$\frac{1}{e_R^2} < \max_k \Pi(k, m_R, \infty) - \Pi(0, m_R, \infty).$$
(11)

The solution is given by the dashed line in the bottom part of Fig. 6. This lies close to the Landau pole, even deeper in the inaccessible region.

V. CONCLUSION

From Figs. 5, 6 we see that the triviality of QED is intimately connected with chiral symmetry breaking. Any attempt to remove the cut-off is always thwarted by the dynamically generated fermion mass. In particular this means that spinor QED does not exist as an interacting theory, similar to what Coleman and Weinberg [10] found for scalar QED.

We have also seen that chiral symmetry breaking allows QED to escape the Landau pole problem. While the bare parameters of the theory can take any values, the renormalized parameters are restricted. The Landau pole and ghost problem only occur deep in the inaccessible e_R^2 , am_R region. Chiral symmetry breaking is always strong enough to push the Landau pole above the cut-off.

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