Quantum Circuits Architecture

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We present a method for optimizing quantum circuits architecture. The method is based on the notion of *quantum comb*, which describes a circuit board in which one can insert variable subcircuits. The method allows one to efficiently address novel kinds of quantum information processing tasks, such as storing-retrieving, and cloning of channels.

Quantum Mechanics plays a crucial role in the technology of high precisions and high sensitivities, e. g. in frequency standards [1], quantum lithography [2], twophoton microscopy [3], clock synchronization [4], and reference-frame transfer [5]. In all these applications the essential problem is to achieve very high precision in: i) determining parameters; ii) executing a transformation that depends on unknown parameters. Since the parameters are generally encoded with a transformation [6], as in the whole class of quantum metrology problems [7], and since the estimation itself can be considered as a special case of transformation (with classical c-number output), both tasks i) and ii) can be reduced to the general problem of executing a desired transformation depending on an unknown transformation. Taking into account the possibility of exploiting N uses of the unknown transformation, the problem becomes to build a quantum circuit that has N circuits as input, and achieves the desired transformation as an output. This is what we call a quantum circuit board.

A quantum circuit board is a network of gates in which there are N slots with open ports for the insertion of N variable sub-circuits (see Fig. 1). Since generally it is im-

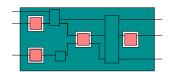


FIG. 1: A quantum circuit board.

possible even in principle to achieve the desired transformation exactly, the main task here is to optimize the circuit board according to a given figure of merit—e. g. the gate-fidelity with the target transformation. A typical example is the optimal cloning of an undisclosed transformation U, which will be operated by a board with N slotted uses of U, and achieving overall in-out unitary transformation which is the closest possible to $U^{\otimes M}$ with M>N. We emphasize that generally the overall in-out transformation of the board and the slotted ones can be of any kind, including measurements and state-preparations (i. e. transformations to and from c-numbers), and that in some specific situations the slotted transformations can be even different from each other.

In previous literature some special cases of circuit-

board optimization have been considered, e. g. in regards to the discrimination of two unitary transformations with N uses in parallel [8], or in sequence [9], and to the problem of phase estimation [7], where an optimal quantum circuit architecture has been determined [10]. However, the optimal architecture for estimating arbitrary unitaries is still unknown, and no systematic method for the general problem of circuit board optimization is available yet.

In this Letter we present a complete method for optimizing the architecture of quantum circuit boards. After providing a convenient description of circuit connectivity, we introduce the notion of quantum comb, which describes all possible transformations operated by a quantum circuits board, and generalizes the notion of quantum channel to the case where the inputs are quantum circuits, rather than quantum states. We then present the optimization method, based on the convex structure of the set of quantum combs. The method allows one to reduce the apparently untractable problem of optimal circuit architecture to the optimization of a single positive operator with linear constraints. Finally, we give two new applications in which the present approach dramatically simplifies the solution of the problem.

A quantum circuit operates a transformation from input to output, and is graphically represented by a box with input and output wires symbolizing the respective quantum systems. The quantum systems of different wires are generally different, and may also vary from input to output. Let us associate Hilbert spaces H_{in} (H_{out}) to all input (output) wires, and denote by ρ_{in} $(\rho_{\rm out})$ the corresponding states. The action of the circuit is generally probabilistic, i. e. different in-out transformations can randomly occur, as in a measurement process. Each transformation is described by a linear map $\rho_{\rm in} \to \mathscr{C}(\rho_{\rm in}) = k \rho_{\rm out}$, with the proportionality factor $0 \leqslant k = \text{Tr}[\mathscr{C}(\rho_{\text{in}})] \leqslant 1$ giving the probability that \mathscr{C} occurs on state $\rho_{\rm in}$. To describe a legitimate quantum transformation, the map $\mathscr{C}: Lin(H_{in}) \to Lin(H_{out})$ [11] has to be completely positive (CP) [12] and trace nonincreasing. Trace-preserving maps—i.e. deterministic transformations—are called quantum channels. Notice that a map \mathscr{C} , rather than representing a specific circuit, is univocally associated to the equivalence class of all circuits performing the same in-out transformation.

The linear map \mathscr{C} can be conveniently rewritten using

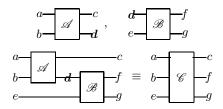
the so-called "Choi-Jamiołkowski" representation [13], corresponding to the following one-to-one correspondence between linear maps $\mathscr{C}: \mathsf{Lin}(\mathsf{H}_{\mathsf{in}}) \to \mathsf{Lin}(\mathsf{H}_{\mathsf{out}})$ and linear operators $C \in \mathsf{Lin}(\mathsf{H}_{\mathsf{out}} \otimes \mathsf{H}_{\mathsf{in}})$ given by

$$C = \operatorname{Choi}(\mathscr{C}) := \mathscr{C} \otimes \mathscr{I}(|\Omega\rangle\langle\Omega|). \tag{1}$$

$$\mathscr{C}(\rho) = \operatorname{Choi}^{-1}(C)(\rho) := \operatorname{Tr}_{\operatorname{in}}[(I_{\operatorname{out}} \otimes \rho^T)C], \quad (2)$$

where \mathscr{I} is the identity map, $|\Omega\rangle$ is the unnormalized maximally entangled state $|\Omega\rangle = \sum_n |n\rangle |n\rangle \in \mathsf{H}_{\mathrm{in}}^{\otimes 2}$, and T denotes transposition with respect to the orthonormal basis $\{|n\rangle\}$ for H_{in} . The map \mathscr{C} is CP if and only if the operator C—called *Choi operator*—is positive [14].

Two quantum circuits can be connected in all the ways allowed by the physical matchings between input and output wires (see e. g. Fig. 2, where the wires labelled d are connected): a connection will result in the composition of the corresponding CP maps, and hence of the corresponding Choi operators. Since building a quantum network means connecting many circuits, it is crucial to have a handy way to describe circuit connectivity with minimum overhead of notation. We provide here three simple rules that accomplish this goal:



$$A = \operatorname{Choi}(\mathscr{A}), B = \operatorname{Choi}(\mathscr{B}), C = \operatorname{Choi}(\mathscr{C}) = A * B$$

FIG. 2: Connection of two quantum circuits \mathscr{A} and \mathscr{B} . Wires are labelled according to Rule 1. The Choi operator the resulting circuit \mathscr{C} is given by the link product of Rule 3.

Rule 1 (Labelling) Each quantum wire is marked with a different label, except for wires that are connected, which are identified with the same label.

Rule 2 (Multiplication) The multiplication of two Choi operators $A \in \text{Lin}(\mathsf{H}_{a,b,c,\mathbf{d}})$ and $B \in \text{Lin}(\mathsf{H}_{\mathbf{d},e,f,g})$ is intended in the tensor fashion, i.e. $AB = (A \otimes I_{e,f,g})(I_{a,b,c} \otimes B)$.

Rule 3 (Composition) The connection of two circuits with Choi operators A and B—acting on Hilbert spaces labelled according to Rule 1—yields a new circuit with Choi operator C given by the link product

$$C = A * B = \text{Tr}_{\mathsf{J}}[A^{\theta_{\mathsf{J}}}B], \tag{3}$$

 θ_J denoting partial transposition over the Hilbert space J of the connected wires, and the multiplication in square brackets following Rule 2.

Rule 3 directly follows from Eqs. (1) and (2). Notice that due to invariance of trace under cyclic permutations, the link product is commutative: A * B = B * A. Using it, the action of a linear map \mathscr{C} on a state ρ in Eq. (2) can be rewritten as $\mathscr{C}(\rho) = C * \rho$. Assembling many circuits $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k$ yields a quantum network whose Choi operator is simply given by $C = C_1 * C_2 * \cdots * C_k$.

We are now ready to treat quantum circuit boards. To start with, we consider the case of a deterministic circuit board, i.e. a network of quantum channels with N open slots for the insertion of variable subcircuits. It is clear that by reshuffling and stretching the internal wires any circuit board can be reshaped in the form of a "comb", with an ordered sequence of slots, each between two successive teeth, as in Fig. 3. The order of the slots is the causal order induced by the flow of quantum information in the circuit board. We label the input systems (entering the board) with even numbers 2n, and the corresponding output systems (exiting the board) with odd numbers 2n + 1, with n ranging from 0 to N.

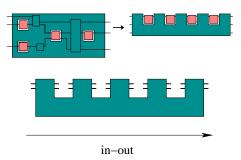


FIG. 3: Every circuit-board can be reshaped in form of a "comb", with an ordered sequence of slots, each between two successive teeth. The pins represent quantum systems, entering or exiting from the board (the horizontal arrow represents the quantum information flow).

A quantum comb with N slots is clearly equivalent to a concatenation of N+1 channels with memory, which is in turn equivalent to causal network, namely a network where the quantum state of the output systems up to time n does not depend on the state of the input systems at later times n' > n, with $n, n' \in \{0, 1, ..., N\}$ [15]. The causal network can be easily obtained by redrawing the comb as an equivalent circuit with all inputs on the left and all outputs on the right, as in Fig. 4. We define the

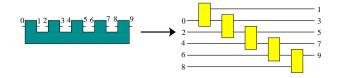


FIG. 4: Each quantum comb is equivalent to a causal network, with all inputs on the left and all outputs on the right. The Choi operator of a comb is the Choi operator of the corresponding causal network.

Choi operator of a quantum comb as the Choi operator R of the corresponding causal network. In terms of the Choi operator R, causality is equivalent to a set of linear constraints

$$\operatorname{Tr}_{2n+1}\left[R^{(n)}\right] = I_{2n} \otimes R^{(n-1)}, \quad n = 0, \dots N,$$

$$R^{(N)} \equiv R, \quad R^{(-1)} = 1.$$
(4)

where Tr_{2n+1} denotes the partial trace over the Hilbert space H_{2n+1} of the output wire labeled 2n+1, I_{2n} the identity operator over the Hilbert space H_{2n} of the input wire labeled 2n, $R^{(n)} = \operatorname{Choi}(\mathscr{C}^{(n)})$, and $\mathscr{C}^{(n)}$ is the map of the (n+1)-subnetwork from the first n+1 inputs to the first n+1 outputs. Precisely, we have the following

Theorem 1 Every positive operator $0 \leqslant R \in \text{Lin}(\bigotimes_{j=0}^{2N+1} \mathsf{H}_j)$ satisfying the linear constraints (4), is the Choi operator of a deterministic quantum comb.

Proof By definition, it is enough to show that any operator $R \ge 0$ normalized as in Eq. (4) is the Choi operator of a causal network. A causal network with N+1 input/output pairs is described by a family of channels $\mathscr{C}^{(n)}$, $n=0,1,\ldots,N$ with the property $\operatorname{Tr}_{2n+1}[\mathscr{C}^{(n)}(\rho^{(n)})] = \mathscr{C}^{(n-1)}(\operatorname{Tr}_{2n}[\rho^{(n)}])$, for any state $\rho^{(n)}$ of the first n+1 input systems. Using the correspondence of Eq. (2), one can easily see that this is equivalent to the normalization of Eq. (4).

A quantum comb transforms a series of N input circuits $\mathscr{C}_1, \ldots, \mathscr{C}_N$ into an output circuit \mathscr{C}' depending on them (Fig. 5a). This transformation of circuits corresponds to an N-linear CP-map that sends the input Choi operators into the ouput Choi operator according to $C' = C_1 * \cdots * C_N * R$, with R the Choi operator of the comb. We call the mapping between circuits $\{\mathscr{C}_1,\ldots,\mathscr{C}_N\} \mapsto \mathscr{C}'$ supermap as it sends channels into channels, rather than states into states. Notice that, depending on the number of slots that are saturated a quantum comb can transform a series of circuits into a comb (Fig. 5b), or, more generally, a comb into a comb (Fig. 5c). As a matter of fact, a quantum comb realizes many possible mappings, all obtained by the link product with its Choi operator R. Therefore, the quantum comb can be completely identified with its Choi operator. Remarkably, also the converse is true: any abstract supermap sending channels into channels in a CP fashion can be physically realized by a quantum comb [16].

The tools presented above provide a powerful method for optimizing quantum circuit architecture. Suppose we want to design a circuit board maximizing some convex figure of merit, e.g. the fidelity of the ouput circuit \mathscr{C}' with a desired unitary gate \mathscr{U} . In our framework the optimization of the board architecture is reduced to the search of the optimal operator $R \geq 0$ with the linear constraints (4). This is a standard problem of *convex optimization*, for which efficient algorithms are known. Ba-

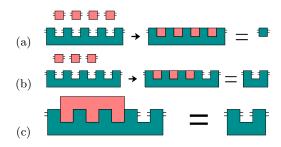


FIG. 5: A quantum comb realizes different transformations of quantum circuits, namely it can send (a) A series of channels into a channel. (b) A series of channels into a comb, or (c) An input comb into an output comb.

sically, we only need to implement the search on the extremal points of the convex set of Choi operators. Moreover, the complexity of the search can be dramatically reduced by exploiting additional constraints, e.g. symmetry properties of the circuit board. The optimal Choi operator will finally single out the optimal architecture, automatically deciding if the N slots of the circuit board have to be connected in a causal order or in parallel, or in any combination of the two.

We illustrate our method in two concrete applications. The first application is the optimal universal cloning of unitary transformations, i. e. the problem of designing a quantum board that optimally achieves the $N \to M$ cloning of an unknown unitary $U \in \mathbb{SU}(d)$ in dimension d. The board has N slots containing N identical uses of the unknown unitary U and performs a transformation which is the closest possible to $U^{\otimes M}$. Using channel fidelity as the figure of merit, the prob-

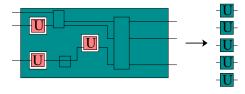


FIG. 6: Cloning of unitary transformations. A quantum board with N=3 input slots, designed to optimally emulate M=5 uses of the unknown unitary transformation U with maximum channel fidelity.

lem is to find the Choi operator R that maximizes the average over all unitaries of the overlap $F_U(N,M)=1/(d^{2M+N})\langle U|^{\otimes M}\langle U^*|^{\otimes N}R|U\rangle^{\otimes M}|U^*\rangle^{\otimes N}$, where $|U\rangle=(U\otimes I)|\Omega\rangle$, with $|\Omega\rangle=\sum_n|n\rangle|n\rangle$. The architecture optimization is then reduced to a standard convex analysis problem. For N=1 and M=2, we derived the optimal quantum board, achieving fidelity $F^{clon}(1,2)=(d+\sqrt{d^2-1})/d^3$, significantly higher than the classical threshold reached by the optimal estimation of a unitary $F^{est}(1,2)=6/d^4$ for d>2, $F^{est}(1,2)=5/16$ for qubits [17], this showing the advantage of coherent quantum in-

formation processing over any classical cloning strategy.

Another interesting application is the storage and retrieval of an undisclosed unitary transformation U from N uses, also called optimal quantum-algorithm learning. The problem arises from the need of running an undisclosed algorithm (available for N uses) on an input state ψ which will be available at later time. To this purpose one can slot the N uses of U in a quantum circuit board, put the output state of the board in a quantum memory, and, when the input state will be available, use the memory to recover the unitary. The series storing-retrieving is represented by the quantum comb in Fig. 7, which can be cut into two parts, a storing one including only the uses of U, and a retrieving one including ψ (the output state of the first part is stored in a quantum memory and is then fed in the second part). Also in this case

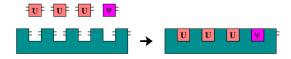


FIG. 7: Quantum-algorithm learning. One wants to run an undisclosed unitary U on a quantum state ψ which is available after the lapse of time in which the uses of U are available.

our method reduces the optimization to a convex analysis problem. For N=1 and N=2 we found average gate-fidelities $F=\frac{2}{d^2}$ and $F=\frac{3}{d^2}$, respectively, which coincide with the value attained by the optimal estimation of unitaries [18]. Remarkably, the optimal universal storing/retrieving of unitaries does not need a coherent interaction with the quantum memory at the retrieving stage—which is purely classical—, but only an entangled input state at the storing stage. This means that the quantum memory is not really needed, and that the learned algorithm can be executed an unlimited number of times with constant performance. The situation is radically different if no entanglement is allowed in the storing stage: in this case the optimal retrieving is purely quantum, yielding an optimal learning that is forgetful.

We conclude by mentioning the extension of our method to the optimization of probabilistic circuit boards, containing measuring devices that produce different transformations depending on random outcomes. The probabilistic comb corresponding to outcome i will have Choi operator R_i , with the sum over all outcomes $\sum_{i} R_{i} = R$ giving the Choi operator of a deterministic comb. Indeed, introducing a classical register with orthogonal states $|i\rangle \in \mathsf{H}_C$ we can define $\widetilde{R} = \sum_i R_i \otimes |i\rangle \langle i|$, which is the Choi operator of a deterministic comb with $\mathsf{H}_{2N+1} := \mathsf{H}_{2N+1} \otimes \mathsf{H}_C$. The comb corresponding to R_i is then obtained after applying the comb of R, by measuring the register on the basis $\{|i\rangle\}$ and postselecting outcome i. Probabilistic combs are a fundamental tool to address the optimized circuit architecture for estimating unknown transformations with multiple copies (see Fig. 8). Again, by optimizing the operators $\{R_i\}$ one will automatically determine the optimal disposition of the unitaries in the circuit, a problem whose solution is up to now known only in the very special case of phase estimation [10].



FIG. 8: Quantum comb for the estimation of unitary transformations with multiple uses.

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