

Nonexistence of conformally flat slices of the Kerr spacetime

Alcides Garat¹, Richard H. Price²

1. *Department of Physics, University of Utah, Salt Lake City, Utah 84112. On leave from Universidad de la República, Montevideo, Uruguay.*

2. *Department of Physics, University of Utah, Salt Lake City, Utah 84112.*

Initial data for black hole collisions are commonly generated using the Bowen-York approach based on conformally flat 3-geometries. The standard (constant Boyer-Lindquist time) spatial slices of the Kerr spacetime are not conformally flat, so that use of the Bowen-York approach is limited in dealing with rotating holes. We investigate here whether there exist foliations of the Kerr spacetime that are conformally flat. We limit our considerations to foliations that are axisymmetric and that smoothly reduce in the Schwarzschild limit to slices of constant Schwarzschild time. With these restrictions, we show that no conformally flat slices can exist.

I. INTRODUCTION

Perhaps the most exciting source that might be detected by gravitational wave detectors now in development is radiation from a merger of black holes. This has been one of the motivations for the effort being put into the application of numerical relativity to black hole collisions. In this work supercomputers are used to evolve initial value solutions of Einstein's field equations. The computation of the initial value solutions is itself a difficult task, and much of the work has taken advantage of the Bowen-York [1] program for initial value solutions, in which the restriction is made that the initial 3-geometry is conformally flat.

Despite the elegance and convenience of the Bowen-York approach, a conformally flat initial solution has a serious shortcoming for work with black holes. Astrophysically realistic black holes will be rapidly rotating. The spatial geometry of the Kerr spacetime of rotating holes is not conformally flat. More specifically, for Kerr spacetime described in standard Boyer-Lindquist [2] coordinates t, r, θ, ϕ , a slice at constant t is not conformally flat. Because of this, the use of Bowen-York initial data to study colliding holes entails two difficulties. First, the Bowen-York representation of a rotating hole will be that of a distorted Kerr hole. Numerical evolution of this solution will produce a burst of radiation as each of the colliding holes “relaxes” to an approximately Kerr form [3]. If it were possible to start the collision of the holes at large separation, this initial burst would be easily distinguishable from the radiation arising from the merger itself. But for the present, numerical evolutions must start quite close to the final state of the merger. The second difficulty is that the Bowen-York program cannot give an initial value solution that is a perturbation of the final single stationary hole, since that final solution does not have conformally flat spatial slices. This precludes the application of “close limit” perturbation theory that has proven to be very successful as an approximation scheme for black hole mergers [4].

It would be extremely useful if the need to deal with Kerr black holes could be reconciled with the convenience of the Bowen-York conformally flat scheme. For that reason, we inquire here whether there might exist conformally flat slices of the Kerr spacetime. We know that the $t = \text{constant}$ slices are not flat, but it may be that the geometry of a different kind of slicing, of the form $t = f(r, \theta, \phi)$ is conformally flat. The question of conformal flatness of any 3-geometry can most conveniently be answered with the use of the the Bach (Cotton-York) tensor [5],

$$B^{ij} = 2\epsilon^{ikp}[R_k^j - \frac{1}{4}\delta_k^j R]_{;p}, \quad (1)$$

where R_k^j is the Ricci tensor for the 3-geometry, $R \equiv R_k^k$ is its Ricci scalar, and the semicolon refers to covariant differentiation with respect to the metric of the 3-geometry. The vanishing of the Bach tensor is a necessary and sufficient condition for a 3-geometry to be conformally flat.

Using the Bach tensor, we investigate here whether a conformally flat slice of the Kerr spacetime might exist, but we make certain restrictions on the search that limit the generality of our conclusion. One restriction is that we consider only axisymmetric slices of the axisymmetric Kerr geometry. One reason for this is practical: Gaining the advantages of conformal flatness while losing axisymmetry would be a Pyrrhic victory. A second reason is that the extension of our conclusion to a nonaxisymmetric slicing turns out to be quite difficult. We restrict ourselves, therefore, to slices of the form $t = f(a, r, \theta)$, where a is the Kerr spin parameter.

A second restriction we make is to consider only families of slicings that have the property that in the Schwarzschild ($a \rightarrow 0$) limit, the slicings smoothly go to slices of constant Schwarzschild time. This means that the slicing function $f(a, r, \theta)$, would have the limit zero as $a \rightarrow 0$, since the Boyer-Lindquist coordinates become the Schwarzschild coordinates as ($a \rightarrow 0$). We assume, furthermore, that f can be expanded in a so that

$$t = aF(r, \theta) + \mathcal{O}(a^2) . \quad (2)$$

Our approach is to compute the Bach tensor for the 3-geometry induced by the slicing in eq. (2) and to expand the Bach tensor in a . We will show that no slicing can be found that makes the tensor vanish to lowest nontrivial order in a , and we conclude that no family of slices of this type can be conformally flat.

The assumption in eq. (2) means that our conclusion does not rule out a conformally flat slicing for an isolated value of a , or a family of conformally flat slicings for a range of a that does not include $a = 0$. These limitations are inherent to our method of expanding about $a = 0$. A more subtle shortcoming of our method is that it does not rule out a family of slicings of the form

$$t = G(r, \theta) + aF(r, \theta) + \mathcal{O}(a^2) . \quad (3)$$

where G is not a constant. That is, it does not rule out a family of slicings that, in the Schwarzschild ($a \rightarrow 0$) limit takes the form $t = G(r, \theta)$. But there *are* slicings of Schwarzschild other than $t = \text{constant}$ that are conformally flat. In fact any spherically symmetric 3 geometry is conformally flat, so any slicing of the form $t = G(r)$ is conformally flat. Such slicings of Schwarzschild, of course, are not orthogonal to the timelike Killing vector for the spacetime geometry. The use of such a slicing would have disadvantages for initial data similar to the disadvantage of a nonaxisymmetric slicing. For the Kerr spacetime of course, the Killing vector is not hypersurface orthogonal, so there is no foliation that is singled out by the symmetry. Still, intuition suggests that a useful slicing for Kerr should have extrinsic curvature that only describes the rotation, and that vanishes as $a \rightarrow 0$. This is equivalent to a slicing that reduces to one of constant Schwarzschild time.

II. KERR METRIC TENSOR

In Boyer-Lindquist [2] coordinates, the components of the Kerr metric have the explicit form

$${}^{(4)}g_{tt} = -1 + 2Mr/(r^2 + a^2 \cos^2 \theta) \quad (4)$$

$${}^{(4)}g_{t\phi} = -2Mra \sin^2 \theta / (r^2 + a^2 \cos^2 \theta) \quad (5)$$

$${}^{(4)}g_{rr} = (r^2 + a^2 \cos^2 \theta) / (r^2 - 2Mr + a^2) \quad (6)$$

$${}^{(4)}g_{\theta\theta} = r^2 + a^2 \cos^2 \theta \quad (7)$$

$${}^{(4)}g_{\phi\phi} = [(r^2 + a^2)^2 - (r^2 - 2Mr + a^2)a^2 \sin^2 \theta] \sin^2 \theta / (r^2 + a^2 \cos^2 \theta) . \quad (8)$$

The 3-geometry ${}^{(3)}g_{ij}$ induced on a spatial slice given by eq. (2) has components

$${}^{(3)}g_{rr} = {}^{(4)}g_{rr} + {}^{(4)}g_{tt}a^2 (\partial_r F)^2 + \mathcal{O}(a^3) \quad (9)$$

$${}^{(3)}g_{\theta\theta} = {}^{(4)}g_{\theta\theta} + {}^{(4)}g_{tt}a^2 (\partial_\theta F)^2 + \mathcal{O}(a^3) \quad (10)$$

$${}^{(3)}g_{r\phi} = 2 {}^{(4)}g_{t\phi} a (\partial_r F) + \mathcal{O}(a^3) \quad (11)$$

$${}^{(3)}g_{\theta\phi} = 2 {}^{(4)}g_{t\phi} a (\partial_\theta F) + \mathcal{O}(a^3) \quad (12)$$

$${}^{(3)}g_{\phi\phi} = {}^{(4)}g_{\phi\phi} . \quad (13)$$

From these expressions, and from the fact that $g_{t\phi}$ is proportional to a , it follows that the metric functions ${}^{(3)}g_{ij}$ have no terms first order in a . The deviations of ${}^{(3)}g_{ij}$ from a $t = \text{constant}$ slice of the Schwarzschild spacetime are therefore of order a^2 and higher. Since the $t = \text{constant}$ Schwarzschild slice is conformally flat, and hence has a vanishing Bach tensor, the components of the Bach tensor for eqs. (9) – (13) vanish to first order in a .

III. DIAGONAL COMPONENTS OF THE BACH TENSOR

To second order in a , the diagonal components of the Bach tensor turn out to be given by

$$B^{rr} = 6Ma^2(r-2M) \left(r^5 \sqrt{r^5 \sin^2 \theta / (r-2M)} \right)^{-1} \times [3(\sin^2 \theta - \cos^2 \theta) \partial_\theta F(r, \theta) - 5 \cos \theta \sin \theta \partial_\theta^2 F(r, \theta) - \sin^2 \theta \partial_\theta^3 F(r, \theta)] \quad (14)$$

$$B^{\theta\theta} = 6Ma^2 \sqrt{r^5 / (r-2M)} (r^{12} \sin \theta)^{-1} \times [(-r^4 + 4M^2 r^2 \cos^2 \theta + r^4 \cos^2 \theta - 4Mr^3 \cos^2 \theta - 4M^2 r^2 + 4Mr^3) \partial_\theta \partial_r^2 F(r, \theta) + (-8r^2 - 56M^2 + 11r^2 \cos^2 \theta - 50Mr \cos^2 \theta + 56M^2 \cos^2 \theta + 44Mr) \partial_\theta F(r, \theta) + (5r^3 - 26M^2 r \cos^2 \theta + 23Mr^2 \cos^2 \theta + 26M^2 r - 5r^3 \cos^2 \theta - 23Mr^2) \partial_\theta \partial_r F(r, \theta) + r(r-2M) \cos \theta \sin \theta \partial_\theta^2 F(r, \theta)] \quad (15)$$

$$B^{\phi\phi} = 6Ma^2 \left(r^7 \sin^2 \theta \sqrt{r^5 / (r-2M)} \right)^{-1} \times [(5r - 28M) \sin \theta \partial_\theta F(r, \theta) + 4r \cos \theta \partial_\theta^2 F(r, \theta) + (13M - 5r) r \sin \theta \partial_\theta \partial_r F(r, \theta) + r^2 (r-2M) \sin \theta \partial_\theta \partial_r^2 F(r, \theta) + r \sin \theta \partial_\theta^3 F(r, \theta)] \quad (16)$$

These three diagonal components are not independent, but are related by the fact that the trace B_i^i vanishes, as can easily be checked (to second order in a) for eqs. (14)–(16).

If B^{rr} is to vanish, we have from eq. (14) that

$$0 = \partial_\theta [3 \cos \theta \sin \theta \partial_\theta F(r, \theta) + \sin^2 \theta \partial_\theta^2 F(r, \theta)] . \quad (17)$$

This equation can be solved by three integrations with respect to θ to give the general solution

$$F(r, \theta) = \frac{h(r)}{2 \sin^2 \theta} + u(r) \left[\frac{-\cos \theta}{2 \sin^2 \theta} + \frac{1}{2} \ln(\tan[\theta/2]) \right] + v(r) , \quad (18)$$

in which $h(r)$, $u(r)$ and $v(r)$ are the “constants” introduced in the three integration steps. When this form of $F(r, \theta)$ is put into the right hand side of eq. (15), the equation $B^{\theta\theta} = 0$ takes the form

$$0 = \hat{O}_u u(r) + \cos \theta \hat{O}_h h(r) , \quad (19)$$

and the vanishing of $\hat{O}_h h(r)$ and of $\hat{O}_u u(r)$ yield the two differential equations

$$0 = (9r^2 + 56M^2 - 46Mr) h(r) + (-26M^2 r + 23Mr^2 - 5r^3) \partial_r h(r) + (-4Mr^3 + r^4 + 4M^2 r^2) \partial_r^2 h(r) \quad (20)$$

$$0 = (-8r^2 + 44Mr - 56M^2) u(r) + (26M^2 r + 5r^3 - 23Mr^2) \partial_r u(r) + (-4M^2 r^2 - r^4 + 4Mr^3) \partial_r^2 u(r) . \quad (21)$$

Finally, the solutions to these equations are

$$h(r) = A \frac{r^{7/2}}{\sqrt{r-2M}} + B \frac{r^{7/2} \cosh^{-1}(\sqrt{r/2M})}{\sqrt{r-2M}} \quad (22)$$

$$u(r) = C \frac{(r-M)r^{7/2}}{M\sqrt{r-2M}} + D \frac{r^4}{M} . \quad (23)$$

At this point we have reduced the freedom in $F(r, \theta)$ to the constants A, B, C, D and the function $v(r)$, if the Bach tensor is to vanish.

IV. OFFDIAGONAL COMPONENTS OF THE BACH TENSOR

When the form of $F(r, \theta)$ required by eqs. (18), (22), and (23) is used, we find that the component $B^{r\theta}$ is

$$B^{r\theta} = 3a^2 B M r^{-9/2} \sqrt{r-2M} / \sin \theta . \quad (24)$$

It follows that a conformally flat slicing requires that $B = 0$.

With B set to zero in the form of $F(r, \theta)$ required by eqs. (18), (22), and (23), the rather lengthy expression for $B^{\theta\phi}$ can be written as

$$B^{\theta\phi} = a^2 \Xi (\sin^{15} \theta r^{12} (r - 2M)^3 M^2)^{-1}, \quad (25)$$

where Ξ is

$$\begin{aligned} \Xi = & c_1(r) \cos \theta + c_2(r) \cos^2 \theta + \dots + c_{16}(r) \cos^{16} \theta \\ & + [b_1(r) \cos \theta + b_2(r) \cos^2 \theta + \dots + b_{12}(r) \cos^{12} \theta] \ln(\sin \theta / [1 + \cos \theta]) \end{aligned} \quad (26)$$

For $B^{\theta\phi}$ to vanish, each of the terms $c_1(r) \dots c_{16}(r), b_1(r) \dots b_{12}(r)$ must vanish. The sum of b_k terms is explicitly

$$\sum_k b_k(r) \cos^k \theta = (-2) (1 - \cos^2 \theta)^5 P_b(r, \theta),$$

with

$$\begin{aligned} P_b = & [(30Mr^{15} - 246M^2r^{14} + 777M^3r^{13} - 1158M^4r^{12} + 780M^5r^{11} - 168M^6r^{10}) \cos^2 \theta \\ & + 10Mr^{15} - 82M^2r^{14} + 259M^3r^{13} - 386M^4r^{12} + 260M^5r^{11} - 56M^6r^{10}] DA \\ & - \left[(-30Mr^{29/2} + 216M^2r^{27/2} - 576M^3r^{25/2} + 672M^4r^{23/2} - 288M^5r^{21/2}) \cos^2 \theta \right. \\ & \left. - 10Mr^{29/2} + 72M^2r^{27/2} - 192M^3r^{25/2} + 224M^4r^{23/2} - 96M^5r^{21/2} \right] \sqrt{r - 2M} CA \\ & + [-48r^{31/2} + 388Mr^{29/2} - 1208M^2r^{27/2} + 1776M^3r^{25/2} - 1184M^4r^{23/2} + 256M^5r^{21/2}] \cos \theta \sqrt{r - 2M} D^2 \\ & + [-48r^{31/2} + 388Mr^{29/2} - 1224M^2r^{27/2} + 1872M^3r^{25/2} - 1376M^4r^{23/2} + 384M^5r^{21/2}] \cos \theta \sqrt{r - 2M} C^2 \\ & - [96r^{16} - 872Mr^{15} + 3160M^2r^{14} - 5740M^3r^{13} + 5320M^4r^{12} - 2192M^5r^{11} + 224M^6r^{10}] \cos \theta DC. \end{aligned} \quad (27)$$

From the terms in P_b that are proportional to $\cos \theta$ it follows that C and D must vanish if $B^{\theta\phi}$ vanishes.

When the simplification $C = D = 0$ is made in eq.(26) the sum of the c_k terms takes the form

$$\begin{aligned} \sum_k c_k(r) \cos^k \theta = & 84 M^4 (r - 2M)^{7/2} r^{5/2} (1 - \cos^2 \theta)^8 \\ & - r^7 M^2 (r - 2M)^3 (3 \cos^2 \theta + 1) (1 - \cos^2 \theta)^5 \\ & \times ([r^4 - 4r^3M + 4r^2M^2] \partial_r^3 v(r) + [2r^3 - 2r^2M - 4rM^2] \partial_r^2 v(r) + [-2r^2 + 8rM - 5M^2] \partial_r v(r)) A \\ & + 2 M^2 r^{19/2} (r - 2M)^{7/2} (1 - \cos^2 \theta)^3 \\ & \times [(21M - 5r) \cos^4 \theta - 21M \cos^2 \theta - 3r] A^2. \end{aligned} \quad (28)$$

In this expression the terms proportional to $\cos^{11} \theta \dots \cos^{16} \theta$ do not vanish for any choice of A or $v(r)$. It follows that there is no function $F(r, \theta)$ for which the slicing in eq. (2) gives a 3-geometry that is conformally flat to second order in a . Under the assumptions stated at the outset this implies that there is no slicing of the Kerr spacetime that is conformally flat.

V. DISCUSSION

We have shown that there can be no spatial slicing of the Kerr spacetime with the following properties: (i) The slicing is conformally flat. (ii) The slicing is axisymmetric. (iii) The slicing, as a function of the Kerr parameter a , goes smoothly to a slice of constant Schwarzschild time as $a \rightarrow 0$ and the spacetime approaches the Schwarzschild spacetime. It follows that the Bowen-York method of generating initial value solutions, when applied to configurations with initial Kerr holes, or for the close-limit approximation method, entails the difficulties outlined in Sec. I.

It is natural to ask, aside from the ‘‘practical’’ questions related to black hole collisions, whether any of the restrictions in our conclusion can be modified or removed, so that a more general conclusion can be stated about conformally flat slicings of the Kerr spacetime. The axisymmetric restriction might appear to be particularly simple to remove, since we use what amounts to a perturbation (in a) expansion in which the ‘‘background’’ (the Kerr spacetime) is axisymmetric. But some of the terms in the Bach tensor are second order in the slicing function F , so a Fourier decomposition of $F(r, \theta, \phi)$ will result in a mixing of Fourier modes. A solution without a Fourier decomposition would appear to be quite difficult. The key to the relatively simple result we have presented is that the diagonal components of the Bach tensor, to second order in a , are linear in F for axisymmetric slicings. That

simplicity disappears in we allow for ϕ dependence in F , and greatly complicates an approach of the type we have used. Since this is a difficult task and has little connection with questions of initial data sets, we have not pursued it.

An attempt to look at slicings with the form in eq. (3) runs into different problems. In this case, for nontrivial G , the Bach tensor will have terms to first order in a . These terms will be linear in both G and in F . Choices of G and F for which the slicing is conformally flat to first order in a cannot be ruled out. (If, for example, one chooses both F and G to be functions only of r , the slicing to first order in a is spherically symmetric and hence conformally flat). It follows that from the first order Bach terms one can only infer restrictions on G and on F . These restrictions must be applied to the equations that arise from the terms to second order in a , equations that include terms quadratic in F . Again, we have not pursued a generalization along these lines.

ACKNOWLEDGMENTS

We thank Walter Landry, William Krivan and Karel Kuchař for useful discussions. We thank James Bardeen for bringing to our attention the possibility that the Schwarzschild geometry may have slices that are conformally flat other than those of constant Schwarzschild time. Special thanks go to John Whelan for helpful suggestions at the early stages of this work. We gratefully acknowledge the support of the National Science Foundation under grant PHY9734871.

-
- [1] J. Bowen and J. W. York, Jr., Phys. Rev. **D21**, 2047 (1980).
 - [2] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967).
 - [3] O. Nicasio, R. Gleiser, R. Price, and J. Pullin, Phys. Rev. D **57**, 3401-3407 (1998).
 - [4] R. H. Price and J. Pullin, Phys. Rev. Lett. **72**, 3297 (1994); A. Abrahams and R. H. Price, Phys. Rev. D **53**, 1963 (1996); J. Pullin, Fields Inst. Commun., **15** 117 (1997); P. Anninos, R. H. Price, J. Pullin, E. Seidel and W.-M. Suen, Phys. Rev. D **52**, 4462 (1995); J. Baker, A. Abrahams, P. Anninos, S. Brandt, R. Price, J. Pullin, E. Seidel, Phys. Rev. D **55**, 829 (1997); O. Nicasio, R. Gleiser, R. Price, and J. Pullin, Phys. Rev. D **59**, 044024 (1999).
 - [5] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, and E. Schmutzner in Exact Solutions of Einstein's Field Equations, Cambridge Univ. Press, (1980)