Comment on "Solitary waves in optical fibers governed by higher-order dispersion"

Hans Werner Schürmann*
Department of Physics, University of Osnabrück, Osnabrück, Germany

Valery Serov†
Department of Mathematical Sciences, University of Oulu, Oulu, Finland

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Mainly with respect to the mathematical part of the article by Kruglov and Harvey [Phys. Rev. A **98**, 063811 (2018)] some (supplementary) remarks on the solution method, on the conditions of existence, and on the parameter dependence are presented. For elucidation, numerical examples are included.

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In a recent article by Kruglov and Harvey [1], an exact stationary solitonlike solution of the generalized nonlinear Schrödinger equation (NLSE) with second-, third-, and fourth-order dispersion terms was presented (without deriving it). We note in advance that solution (2) in Ref. [1] is correct.

Some years ago, we proposed the famous Weierstrass solution [2] of the ordinary nonlinear differential equation ("basic equation")

$$[f'(z)]^2 = \alpha f^4(z) + 4\beta f^3(z) + 6\gamma f^2(z) + 4\delta f(z) + \epsilon \equiv R(f) \tag{1}$$

($\alpha, \beta, \gamma, \delta, \epsilon$ real) to construct solutions of various nonlinear partial differential equations [3–5]. As a background to the following remarks, it seems suitable to shortly outline this approach and its connection with Ref. [1].

The general solution of (1) is given by [2,6] (the prime denotes the derivative with respect to f)

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)\right]} \tag{2}$$

if a simple (real) root f_0 of $R(f)$ exists (it turns out that this is sufficient for the present comment). The invariants g_2, g_3 of the Weierstrass elliptic function $\wp(z; g_2, g_3)$ are related to the coefficients of $R(f)$ by [6]

$$g_2 = \alpha\epsilon - 4\beta\delta + 3\gamma^2,$$
$$g_3 = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \gamma^3 - \epsilon\beta^2. \tag{3}$$

The discriminant [of $\wp$ and $R(f)$ [6]] is given by

$$\Delta = g_2^3 - 27g_3^2. \tag{4}$$

The behavior of $f(z)$ can be classified by (3) and (4): If $\Delta \neq 0$ or $\Delta = 0$ and $g_2 > 0, g_3 > 0$, $f(z)$ is periodic (oscillatory), and if

$$\Delta = 0, \quad g_2 \geqslant 0, \quad g_3 \leqslant 0, \tag{5}$$

$f(z)$ is solitonlike. Subject to (5), due to the limit of $\wp$ for $\Delta = 0$ [7], Eq. (2) reads

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[e_1 + 3e_1 \operatorname{csch}^2(z\sqrt{3e_1}) - \frac{1}{24}R''(f_0)\right]}, \tag{6}$$

with $e_1 = \frac{1}{2}\sqrt[3]{-g_3}$. Obviously, $f(z)$ according to (6) is not real and bounded in general. To specify the real and bounded ("physical") solutions it is suitable to consider the graph $\{[f'(z)]^2, f(z)\}$, denoted as the phase diagram. Physical solutions are related to the existence of a finite interval $[f_1, f_2]$, where $R(f)$ is non-negative and bounded with real roots f_1, f_2 of $R(f)$. This condition has been denoted as the phase diagram condition (PDC) [6].

Returning now to Eq. (1) in Ref. [1], a traveling-wave ansatz (k, c, r, λ real)

$$\Phi(x, t) = f(kx - ct)e^{i(rx - \lambda t)} \tag{7}$$

inserted in Eq. (1) in Ref. [1], and changing the notation of coefficients ($\alpha \to a, \epsilon \to m, \gamma \to g$, assuming f is real and $\mu = 0$), the real and imaginary parts lead to a system of two equations,

$$m^4c^4 f^{(4)}(z) = \left(m^3ac^2 - \frac{3}{8}m^2c^2\sigma^2\right)f''(z) + \left(m^3r - \frac{ma\sigma^2}{16} + \frac{3\sigma^4}{256}\right)f(z) - m^3gf^3(z),$$
$$(\sigma c^3 + 4m\lambda c^3)f^{(3)}(z) + (k - 2\lambda ca - 4mc\lambda^3 - 3\sigma\lambda^2)f'(z) = 0, \tag{8}$$

with $z = kx - ct$ and with

$$\lambda = -\frac{\sigma}{4m}, \quad k = \frac{c\sigma^3 - 4cam\sigma}{8m^2}, \quad m \neq 0, \quad c \neq 0. \tag{9}$$

Taking now into account the basic equation (1) and inserting the derivatives

$$f''(z) = 2\alpha f^3(z) + 6\beta f^2(z) + 6\gamma f(z) + 2\delta,$$
$$f^{(4)}(z) = (12\alpha + 12\beta)[f'(z)]^2 + [6\alpha f^2(z) + 12\beta f(z) + 6\gamma]f''(z)$$

*hwschuer@uos.de
†vserov@cc.oulu.fi

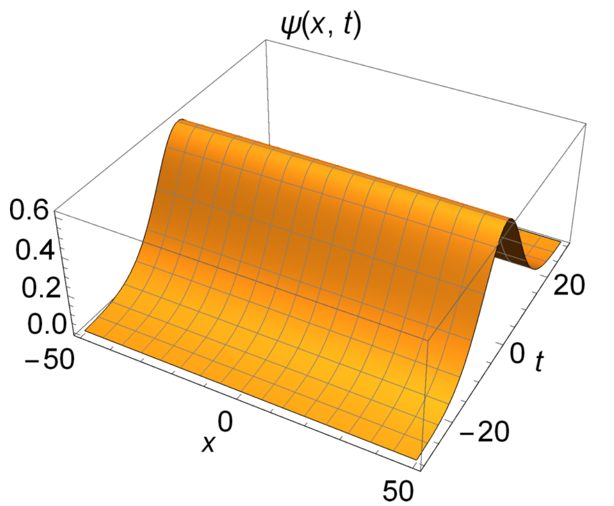

FIG. 1. Field pattern $\psi(x,t)$ according to Eqs. (11) and (19). Parameters $a=-0.1$, $g=0.01$, $\sigma=0.14$, $m=-0.6$, $c=-\frac{1}{4}\sqrt{\frac{8am-3\sigma^2}{10m^2}}$, and $r=0.0017$ (corresponding to w and κ in Ref. [1], respectively).

into the first Eq. (8), we obtain

$$\alpha=0,\quad \beta^2=-\frac{g}{120mc^4},\quad \gamma=\frac{8am-3\sigma^2}{240m^2c^2},$$
$$\delta=\frac{B}{72\beta mc^4}+\frac{A^2}{450\beta m^2c^8},\quad \epsilon=\frac{2A\delta}{15\beta mc^4},$$
$$A=ac^2-\frac{3c^2\sigma^2}{8m},\quad B=r-\frac{a\sigma^2}{16m^2}+\frac{3\sigma^4}{256m^3}. \tag{10}$$

The condition $\frac{g}{m}<0$ must hold otherwise no solution exists.

Thus, the basic equation (1), with coefficients according to (10), defines a class of solutions $f(z)$ (having been denoted as Weierstrass solutions [8]) and hence by (7) a class of solutions $\Phi(x,t)$. The solution $\psi(z,\tau)$ presented in Ref. [1] is a particular element of this class: Since r corresponds to κ [given by Eq. (5) in Ref. [1]], setting $r=\kappa$, it turns out that $\delta=0$, $\epsilon=0$. Hence $\Delta=0$, $g_3=-\gamma^3$, and an evaluation of Eq. (6) yields

$$f(z)=-\frac{3\gamma}{2\beta\cosh^2\left(z\sqrt{\frac{3}{2}\gamma}\right)},$$

and thus

$$f(z)=\mp\frac{8am-3\sigma^2}{8\sqrt{-\frac{10}{3}gm^3}}\frac{1}{\cosh^2\left(\frac{z}{4}\sqrt{\frac{8am-3\sigma^2}{10m^2c^2}}\right)}, \tag{11}$$

where $\mp$ corresponds to different roots for $\beta=\pm\sqrt{\frac{-g}{120mc^4}}$ [see (10)]. The PDC ($\gamma=\frac{8am-3\sigma^2}{240m^2c^2}>0$) in this case is consistent with $g_3<0$ so that solution (11) is physical (see Fig. 1, with all quantities dimensionless). Obviously, comparing the amplitudes of $\psi(z,\tau)$ and $\Phi(x,t)$, using (10), there is agreement: The arguments of ψ and Φ are $w\tau-\frac{w}{v}z$ (disregarding the constants η and ϕ) and $kx-ct$, respectively. If we specify c according to Eq. (3) in Ref. [1], we obtain agreement between k and $\frac{w}{v}$. Hence, ψ and Φ are identical for the particular values of r and c given by w and κ in Eqs. (3) and (5) in Ref. [1].

For coefficients (10) the discriminant Δ and the invariant g_3 are [see (3)]

$$\Delta=(-4\beta\delta+3\gamma^2)^3-27(2\beta\gamma\delta-\gamma^3-\epsilon\beta^2)^2,$$
$$g_3=2\beta\gamma\delta-\gamma^3-\epsilon\beta^2. \tag{12}$$

Obviously, there are various possibilities to satisfy $\Delta=0$, $g_3<0$ in order to find solitonlike solutions. Solving $\Delta=0$ with respect to ϵ we get

$$\epsilon_\pm=\frac{-9\gamma^3+18\beta\gamma\delta\pm\sqrt{3(3\gamma^2-4\beta\delta)^3}}{9\beta^2}. \tag{13}$$

In this case, choosing ϵ_+, the solution (6) reads

$$f(z)=\frac{\sqrt{\gamma^2-\frac{4}{3}\beta\delta}-\gamma}{2\beta}-\frac{\sqrt{9\gamma^2-12\beta\delta}}{2\beta\cosh^2\left(z\sqrt[4]{\frac{9}{4}\gamma^2-3\beta\delta}\right)}. \tag{14}$$

Using $\beta,\gamma,\delta,\epsilon_+$ according to (10), $f(z)$ can be written as

$$f(z)=\frac{1}{\sqrt{-gm^3}}\left[\frac{\sqrt{d}}{48}-\frac{8am-3\sigma^2}{8\sqrt{30}}-\frac{\sqrt{d}}{16}\mathrm{sech}^2\left(\frac{z}{4}\sqrt[4]{\frac{d}{120m^4c^4}}\right)\right], \tag{15}$$

with $d=-128m^2(a^2+10mr)+176am\sigma^2-33\sigma^4$. The PDC

$$\beta\neq 0,\quad 4\beta\delta<3\gamma^2$$

reads

$$128m^2(a^2+10mr)+33\sigma^4<176am\sigma^2,\quad g\neq 0, \tag{16}$$

equivalent to $d>0$, $g\neq 0$.

Equation (15) represents a generalization of Eq. (11) [if c and r are specified according to w and κ in Ref. [1], respectively, Eqs. (15) and (11) are identical]. The asymptotic behavior of solutions (11) and (15) is different (see Figs. 1 and 2). The limit of (11) is zero, when $|z|\to\infty$, and for (15) it is $\omega=\frac{1}{48}\sqrt{\frac{d}{-gm^3}}-\frac{8am-3\sigma^2}{8\sqrt{-30gm^3}}$. Remarkably, ω is zero if r is chosen equal to κ in Ref. [1]. As mentioned above, it

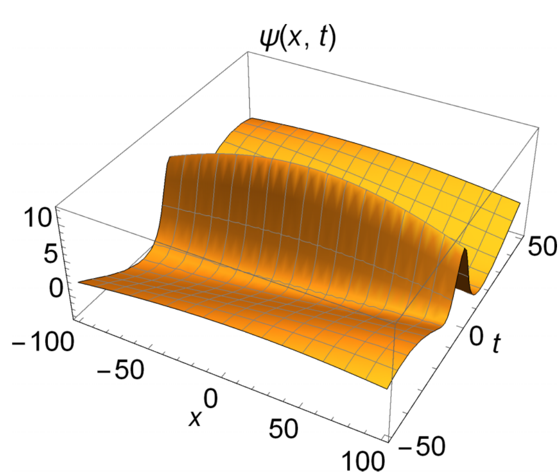

FIG. 2. Field pattern $\psi(x,t)$ according to Eqs. (15) and (19). Parameters as for Fig. 1, but $r=0.0086$ and $c\neq 0$ arbitrary.

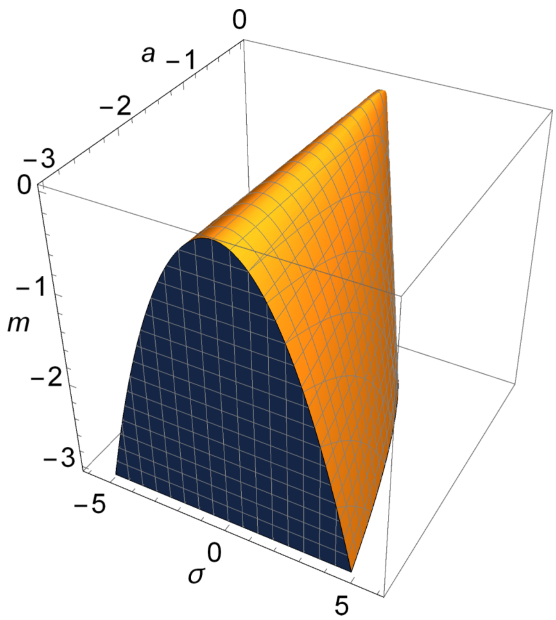

FIG. 3. Parameter region for allowed $\{\sigma, a, m\}$ according to PDC: $\gamma = \frac{8am - 3\sigma^2}{240m^2c^2} > 0$. Parameters c, r, g as for Fig. 1.

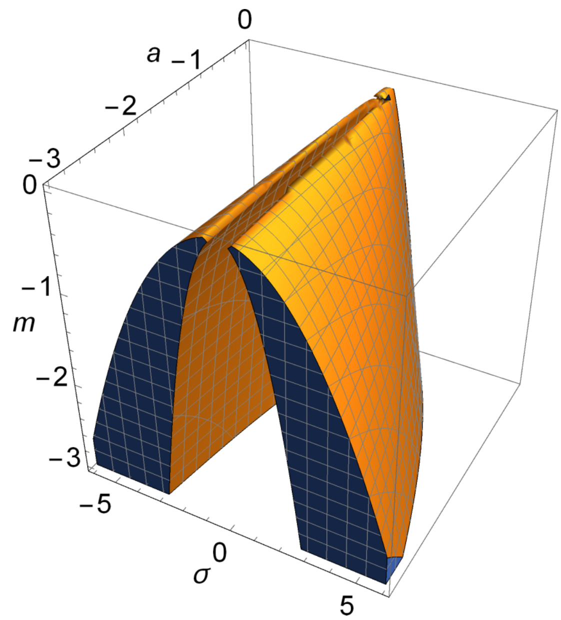

FIG. 4. Parameter regions for allowed $\{\sigma, a, m\}$ according to PDC: $128m^2(a^2 + 10mr) + 176am\sigma^2 - 33\sigma^4 > 0$. Parameters $g = 0.01$, $r = 0.0086$ [consistent with the PDC for $\sigma \in (-6, 6)$, $a \in (-3, 0)$, $m \in (-3, 0)$], $c \neq 0$ arbitrary.

follows $\delta = \epsilon = 0$ in this case, and hence solutions (11) and (15) are identical. If r is selected arbitrarily [but subject to the PDC (16)], the condition $\omega = 0$ can be satisfied, e.g., by selecting parameter a appropriately related to m, σ, r. In this case, solutions (11) and (15) are different. Another difference should be noted. In PDC $\gamma > 0$, the ansatz parameter c (w in Ref. [1]) is irrelevant. Due to Eq. (3) in Ref. [1], c is related to the problem's parameters and thus enters the solution (2) in Ref. [1]. Though PDC $\beta \neq 0$, $4\beta\delta < 3\gamma^2$ formally depends on it, parameter c drops out [see (16)]. The same holds, with k according to (9), for solutions (11) and (15). Thus, parameter c is irrelevant for solutions and associated PDCs.

In general, the PDC is suitable to study the dependence of f on the parameters. In particular, for solution (15), the PDC (16) implies restrictions of the problem's parameters a, σ, m, g and the ansatz parameter r [k and λ are related to the problem's parameters by Eq. (9)]. For practical applications it is important to know the parameter range where solution (15) is real and bounded. Region plots of the PDC, depicted in Figs. 3 and 4, represent the corresponding ranges in parameter space.

The foregoing statements can be summarized as follows. The generalized NLSE

$$\begin{aligned} i\Phi'_x(x,t) &= a\Phi''_t(x,t) - i\sigma\Phi^{(3)}_t(x,t) \\ &\quad - m\Phi^{(4)}_t(x,t) - g|\Phi(x,t)|^2\Phi(x,t) \end{aligned} \tag{17}$$

has (Weierstrass) solutions

$$\Phi(x,t) = f(kx - ct)e^{i(rx-\lambda t)}, \tag{18}$$

where f is a solution of the basic equation (1) with coefficients $\alpha = 0, \beta, \gamma, \delta, \epsilon$ defined by Eqs. (10), subject to the necessary condition $\frac{g}{m} < 0$. The amplitude f is real and bounded (physical) if the coefficients of the basic equation satisfy the phase diagram condition (PDC). If the PDC is valid,

$$\psi(x,t) = \mathrm{Re}[\Phi(x,t)] = f(kx - ct)\cos(rx - \lambda t) \tag{19}$$

is a solution of Eq. (17). Solution (2) in Ref. [1] is a Weierstrass solution. The solution (11) of the associated basic equation

$$[f'(z)]^2 = 4\beta f^3(z) + 6\gamma f^2(z)$$

is equal to the corresponding factor u of solution (2) in Ref. [1] if c and r are chosen as in (3) and (5) in Ref. [1], respectively, leading to $\delta = \epsilon = 0$. The PDC reads $8am - 3\sigma^2 > 0$, and parameter r is a free parameter restricted by (16). Solution $f(kx - ct)$ according to Eq. (15) is another Weierstrass solution (not in the literature, to the best of our knowledge). The associated basic equation is Eq. (1), with $\alpha = 0, \epsilon = \epsilon_+$ [see Eq. (13)].

Solution (11) is not physical if $8am - 3\sigma^2 < 0$. If $d > 0$, solution (15) is physical in this case. If $d < 0$, irrespective of the sign of $8am - 3\sigma^2$, solution (15) is not physical.

To sum up: The class of traveling-wave solitonlike (Weierstrass) solutions of the NLSE (17) [(1) in Ref. [1]] is defined by (7), where f is a solution of the basic equation (1) with coefficients $\alpha, \beta, \gamma, \delta, \epsilon$ given by (10) and subject to (5). The ansatz parameters λ and k are related to the problem's parameters by Eqs. (9). Ansatz parameter c is irrelevant. Restricted by (16), ansatz parameter r is free. In general, the subclass of real and bounded ("physical") solutions is specified by the phase diagram condition (PDC). The solution $\psi(z, \tau)$ in Ref. [1] is a particular element of this subclass.

Our criticism refers to the following points:

(1) The authors do not justify (neither mathematically nor physically) the unnecessarily restrictive choice of w and κ in Ref. [1]. As mentioned above, only if c and r are specified as in Ref. [1], the solution is given by (2) in Ref. [1].

(2) In addition, the conditions $\alpha < 0$, $\epsilon < 0$, $8\alpha\epsilon > 3\sigma^2$, $\gamma > 0$ presented in Ref. [1] are consistent with (2) in Ref. [1], but are not necessary in general for physical Weierstrass solutions ($gm < 0$ is necessary in general).

(3) Since the authors do not derive solution (2), not only w and κ [see point (1) above] but also u, v, δ must appear to the readers as being selected *ad hoc*. It would be interesting to know "the regular method" [leading to (2)], in order to compare with the approach above.

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