The star-structure connectivity and star-substructure connectivity of hypercubes and folded hypercubes

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Abstract

As a generalization of vertex connectivity, for connected graphs G and T, the T-structure connectivity $\kappa(G,T)$ (resp. T-substructure connectivity $\kappa^s(G,T)$) of G is the minimum cardinality of a set of subgraphs F of G that each is isomorphic to T (resp. to a connected subgraph of T) so that G - F is disconnected. For n-dimensional hypercube Q_n , Lin et al. [6] showed $\kappa(Q_n, K_{1,1}) = \kappa^s(Q_n, K_{1,1}) = n - 1$ and $\kappa(Q_n, K_{1,r}) = \kappa^s(Q_n, K_{1,r}) = \lceil \frac{n}{2} \rceil$ for $2 \leq r \leq 3$ and $n \geq 3$. Sabir et al. [11] obtained that $\kappa(Q_n, K_{1,4}) = \kappa^s(Q_n, K_{1,4}) = \lceil \frac{n}{2} \rceil$ for $n \geq 6$, and for n-dimensional folded hypercube FQ_n , $\kappa(FQ_n, K_{1,1}) = \kappa^s(FQ_n, K_{1,1}) = n$, $\kappa(FQ_n, K_{1,r}) = \kappa^s(FQ_n, K_{1,r}) = \lceil \frac{n+1}{2} \rceil$ with $2 \leq r \leq 3$ and $n \geq 7$. They proposed an open problem of determining $K_{1,r}$ -structure connectivity of Q_n and FQ_n for general r. In this paper, we obtain that for each integer $r \geq 2$, $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$ and $\kappa(FQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$ for all integers n larger than r in quare scale. For $4 \leq r \leq 6$, we separately confirm the above result holds for Q_n in the remaining cases.

Keywords: Structure connectivity; Substructure connectivity; Star graph; Hypercube; Folded hypercube.

1 Introduction

It is well known that the topology of an interconnection network is often modeled by a connected graph. Let G be a graph with vertex set V(G) and edge set E(G), where each vertex represents a processor or node and every edge a communication link. For a subgraph H of G, we use G - H to denote the subgraph G - V(H). For a set $F = \{T_1, T_2, \ldots, T_m\}$ of subgraphs of G, let $G - F = G - V(T_1) - V(T_2) - \ldots - V(T_m)$. A

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good interconnection network should have some good performances, such as uniformity, symmetry, high fault tolerance, expansibility and small fixed vertex degree. One of the important parameters of high fault tolerance is connectivity. A vertex-cut of a graph G is a set $S \subseteq V(G)$ such that G-S has more than one component. The connectivity $\kappa(G)$ of G is defined as the minimum cardinality of a vertex-cut S such that G-S is disconnected or has only one vertex. In 1994, Fabrega et al. [3] proposed g-extra connectivity, providing more accurate measures for fault tolerance of large-scale parallel processing systems. For a connected non-complete graph G and a non-negative integer g, a vertex cut S of G is an g-extra cut if G - S is disconnected and every component of G - S has more than gvertices. The g-extra connectivity $\kappa_g(G)$ of G is defined as the minimum cardinality of g-extra cut of G.

In reality of network reliability and fault-tolerance, the neighbors of a faulty node might be more vulnerable. For networks and subnetworks made into chips, when any node on the chip becomes faulty, the whole chip can be considered faulty. To study the fault-tolerance of some structures of the network, Lin et al. [6] introduced the concepts of structure connectivity and substructure connectivity of networks. Let T be a connected subgraph of graph G. Let F be a set of subgraphs of G such that every member in F is isomorphic to T. Then F is called a T-structure-cut of G if the deletion of all members of F disconnects G, i.e. G - F is disconnected. The T-structure connectivity $\kappa(G,T)$ of G is defined as the minimum cardinality of a T-structure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T)$ of G is defined as the minimum cardinality of a T-substructure connectivity $\kappa^s(G,T) \in \kappa(G,T)$. Note that K_1 -structure connectivity reduces to the classical vertex connectivity.



Figure 1. C_3 -structure cut and C_3 -substructure cut.

In the study of *T*-structure connectivity, much of the work has been focused on certain special structures of some given networks. Let P_k denote a path with k vertices, C_k a cycle with k vertices, and $K_{1,r}$ a star with $r \ge 1$ leaves. For the bubble-sort star graph BS_n ,

Zhang et al. [15] obtained $\kappa(BS_n, T)$ and $\kappa^s(BS_n, T)$ for $T \in \{P_k, C_{2k}\}$. For k-ary n-cube network Q_n^k , Lv et al. [7] showed $\kappa(Q_n^k, K_{1,r})$ and $\kappa^s(Q_n^k, K_{1,r})$ with $1 \leq r \leq 3$; further, Lu et al. [9] showed $\kappa(Q_n^k, T)$ and $\kappa^s(Q_n^k, T)$ for $T \in \{P_k, C_k\}$ where $3 \leq k \leq 2n$; For n-dimensional twisted hypercube TQ_n , Li et al. [5] obtained $\kappa(TQ_n, T)$ and $\kappa^s(TQ_n, T)$ for $T \in \{K_{1,3}, K_{1,4}, P_k\}$ where $1 \leq k \leq n$.

For *n*-dimensional hypercube Q_n , Lin et al. [6] showed

$$\kappa(Q_n, K_{1,r}) = \kappa^s(Q_n, K_{1,r}) = \begin{cases} n-1, & \text{if } r=1, \ n \ge 3, \\ \lceil \frac{n}{2} \rceil, & \text{if } 2 \le r \le 3, \ n \ge 3. \end{cases}$$
(1.1)

Moreover, Sabir et al. [11] established

$$\kappa(Q_n; K_{1,4}) = \kappa^s(Q_n; K_{1,4}) = \lceil \frac{n}{2} \rceil, \text{ for } n \ge 6;$$
(1.2)

and for *n*-dimensional folded hypercubes FQ_n , they also determined for $n \ge 7$,

$$\kappa(FQ_n, K_{1,r}) = \kappa^s(FQ_n, K_{1,r}) = \begin{cases} n, & \text{if } r = 1, \\ \lceil \frac{n+1}{2} \rceil, & \text{if } r = 2, 3. \end{cases}$$

From the above results we can see that for Q_n and FQ_n the structure connectivity of only small stars $K_{1,r}$ $(1 \leq r \leq 4)$ have been already determined. So Sabir et al. [11] pointed out that determining the $K_{1,r}$ -structure connectivity and $K_{1,r}$ -substructure connectivity of Q_n and FQ_n with general r remain open. In this paper, we treat general star-structure connectivity for n-dimensional hypercube Q_n and folded hypercubes FQ_n and obtain the following results: for each integer $r \geq 2$, $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$ and $\kappa(FQ_n; K_{1,r}) = \kappa^s(FQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$ for all integers n larger than r in quare scale. To describe clearly the extent of n exceeding r we introduce two functions f(r) and g(r). For details, see Theorems 3.13 and 5.12. Such results partly solve the open problem. For Q_n , from the above-mentioned results (1.1), (1.2) and Theorems 3.13 we find that the $K_{1,r}$ -structure and substructure connectivity of Q_n for $4 \leq r \leq 6$ and n = r and r + 1 have not been determined yet. So in section 4, we separately discuss such low dimensional cases and get the same result. That is, for $4 \leq r \leq 6$ and $n \geq r$ we have that, $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$.

2 Preliminaries

We only consider finite and simple graphs G. Two vertices u and v of G are adjacent if they are the end-vertices of an edge. A neighbor of a vertex x of G means a vertex of G adjacent to x. The neighborhood of a vertex x in G is the set of neighbors of x, denoted by $N_G(x) = \{y | xy \in E(G)\}$. The neighborhood of a vertex set A in G is denoted by $N_G(A) = \bigcup_{x \in A} N_G(x) - A$. A path $P_k = v_1 v_2 \dots v_k$ of length k - 1 is a sequence of k distinct vertices such that $v_{i-1}v_i \in E(G)$ for every $2 \le i \le k$. If the end-vertices of a path P of length $k \ge 3$ are identified, then it becomes a cycle of length k, denoted by C_k .

An *n*-dimensional hypercube Q_n is a simple graph on the all binary strings of length n, such two strings $u_1u_2\cdots u_n$ and $u'_1u'_2\cdots u'_n$, $u_i, u'_i \in \{0,1\}$ for $1 \leq i \leq n$, are adjacent if and only if they differ in exactly one position [8], that is, $\sum_{i=1}^n |u_i - u'_i| = 1$. For any vertex $u = u_1u_2u_3\ldots u_n$ in Q_n , we set $u^i = u_1^iu_2^iu_3^i\ldots u_n^i$ is the neighbor of u in dimension i of Q_n where $u_j^i = u_j$ for $j \neq i$ and $u_i^i = 1 - u_i$. In general, for $A \subseteq \{1, 2, \ldots, n\}$, let u^A be the vertex of Q_n so that $(u^A)_i = \overline{u_i} = 1 - u_i$ if and only if $i \in A$. Obviously, for $A, B \subseteq \{1, 2, \ldots, n\}, u^A = u^B$ if and only if A = B. So u^{i_1,i_2} is the neighbor of u^{i_1} in dimension i_2 and u^{i_1,i_2,i_3} is the neighbor of u^{i_1,i_2} in dimension i_3 . We make a convention: the elements in $\{1, 2, \ldots, n\}$ are taken arithmetic operations on module n. It is known that Q_n is a bipartite and n-regular graph.



Figure 2. FQ_3 and Q_3 .

As one of the popular variants of the hypercube, the *n*-dimensional folded hypercube FQ_n proposed by El-Amawy and Latifi [1] is a graph obtained from hypercube Q_n by adding 2^{n-1} edges, each of them being between vertices $u = u_1u_2u_3...u_n$ and $\overline{u} = \overline{u_1u_2u_3...u_n}$, where $\overline{u_i} = 1 - u_i$. FQ_n is a highly symmetric graph as a underlying topology of several parallel systems, such as ATM Switches [10], PM21 networks [4] and 3D-FolH-NOC network [2]. For example, the FQ_3 and Q_3 are illustrated in Figure 2.

3 The star-structure connectivity of hypercubes

To determine the star-structure connectivity and star-substructure connectivity of n-hypercubes, we first list some preliminary results.

Lemma 3.1. [13] Any two vertices in $Q_n (n \ge 3)$ have exactly two common neighbors, if they have any.

The following two lemmas in the case $3 \le r \le n$ are taken from Lemmas 2.4 and 2.5 in reference [11] respectively. We find that they also hold for r = 2 by Lemma 3.1, since Q_n is triangle-free.

Lemma 3.2. Let $K_{1,r}$ be a star in Q_n with $2 \le r \le n$. If u is a vertex in $Q_n - K_{1,r}$, then $|N_{Q_n}(u) \cap V(K_{1,r})| \le 2$, and equality holds if and only if u is adjacent two leaves of $K_{1,r}$.

Lemma 3.3. Let $K_{1,r}$ be a star in Q_n with $2 \leq r \leq n$. If u and v are two adjacent vertices of $Q_n - K_{1,r}$, then $|N_{Q_n}(u,v) \cap V(K_{1,r})| \leq 2$.

Now we extend two adjacent vertices u and v in Lemma 3.3 to a connected subgraph C in $Q_n - K_{1,r}$ with $|V(C)| \ge 2$ as follows.

Lemma 3.4. Let $K_{1,r}$ be a star in Q_n with $2 \le r \le n$. If C is a connected subgraph in $Q_n - K_{1,r}$ with $k = |V(C)| \ge 2$, then $|N_{Q_n}(C) \cap V(K_{1,r})| \le 2(k-1)$, and equality holds only if C is a star in Q_n .

Proof. Let $V(K_{1,r}) = \{x, x_1, x_2, \dots, x_r\}$ and $E(K_{1,r}) = \{xx_i | 1 \le i \le r\}$. Then x is the center of $K_{1,r}$. Let $V(C) = \{u_1, u_2, \dots, u_k\}$.

First, we prove that $|N_{Q_n}(C) \cap V(K_{1,r})| \leq 2(k-1)$. Suppose to the contrary that $|N_{Q_n}(C) \cap V(K_{1,r})| \geq 2(k-1) + 1 = 2k - 1$. By Lemma 3.2, each vertex u_i in C has at most 2 neighbors in $K_{1,r}$, and if $|N_{Q_n}(u_i) \cap V(K_{1,r})| = 2$, then u_i is adjacent to two leaves in $K_{1,r}$, so $2k \geq |N_{Q_n}(C) \cap V(K_{1,r})| \geq 2k - 1$. It means that there exists at least k-1 vertices in C which each has two neighbors in $K_{1,r}$, and such neighbors are pairwise distinct. Without loss of generality, we assume $\{u_i x_{2i-3}, u_i x_{2i-2}\} \subset E(Q_n)$ for $2 \leq i \leq k$. Since C is connected, u_1 is adjacent to u_i for some $2 \leq i \leq k$. If $u_1 x_j \in E(Q_n)$ with $2k-1 \leq j \leq r$, then there exists an odd cycle $u_1 x_j x x_{2i-3} u_i u_1$, a contradiction. Otherwise, $u_1 x \in E(Q_n)$. Then $N_{Q_n}(u_i) \cap N_{Q_n}(x) = \{x_{2i-3}, x_{2i-2}, u_1\}$, contradicting Lemma 3.1. Hence $|N_{Q_n}(C) \cap V(K_{1,r})| \leq 2(k-1)$.

Next we show that if $|N_{Q_n}(C) \cap V(K_{1,r})| = 2(k-1)$, then C is a star in Q_n . Suppose to the contrary that C is not a star in Q_n . Then we have that there exists a 4-vertex path P_4 in C by taking a longest path of C, so $4 \leq k$ and $6 \leq |N_{Q_n}(P_4) \cap V(K_{1,r})| \leq 8$ by Lemma 3.2. However, by Lemma 3.3, any two consecutive vertices in P_4 together have at most two neighbors in $V(K_{1,r})$, which implies that P_4 has at most four neighbors in $V(K_{1,r})$, a contradiction.

Yang et al. came to the following two results in the g-extra connectivity of Q_n .

Lemma 3.5. [14] Let C be a subgraph of Q_n with |V(C)| = g + 1 for $n \ge 4$. Then $|N_{Q_n}(C)| \ge (g+1)n - 2g - \binom{g}{2}$.

Lemma 3.6. [14] For $n \ge 4$,

$$\kappa_g(Q_n) = \begin{cases} (g+1)n - 2g - \binom{g}{2}, & \text{if } 0 \le g \le n-4, \\ \frac{n(n-1)}{2}, & \text{if } n-3 \le g \le n. \end{cases}$$

Lemma 3.7. For $n \ge r \ge 2$ and $n \ge 3$, $\kappa(Q_n; K_{1,r}) \le \lceil \frac{n}{2} \rceil$ and $\kappa^s(Q_n; K_{1,r}) \le \lceil \frac{n}{2} \rceil$.

Proof. Since $\kappa^s(Q_n; K_{1,r}) \leq \kappa(Q_n; K_{1,r})$, we only prove $\kappa(Q_n; K_{1,r}) \leq \lceil \frac{n}{2} \rceil$. Let $u = 000 \cdots 0$ be a vertex in Q_n . Then $N_{Q_n}(u) = \{u^i | 1 \leq i \leq n\}$.

If $n \geq 3$ is odd, let $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} | 1 \leq j \leq r-2\}$ for $1 \leq i \leq \frac{n-1}{2}$, and let $S_{\frac{n+1}{2}} = \{u^n, u^{n,1}\} \cup \{u^{n,1,j} | 2 \leq j \leq r\}$ for n > r and $S_{\frac{n+1}{2}} = \{u^n, u^{n,1}, u^1\} \cup \{u^{n,1,j} | 2 \leq j \leq r-1\}$ for n = r. Then S_i induces a star $K_{1,r}$ with the center $u^{2i-1,2i}$ for $1 \leq i \leq \frac{n-1}{2}$ and with the center $u^{n,1}$ for $i = \frac{n+1}{2}$ respectively (see Fig. 3(right)).

Let $S = \bigcup_{i=1}^{\frac{n+1}{2}} S_i$. Then $N_{Q_n}(u) \subseteq S$, and u is an isolated vertex of $Q_n - S$. If $n \ge 4$, then the vertex \overline{u} belongs to $Q_n - S$, so the S_i 's for $1 \le i \le \frac{n+1}{2}$ form a $K_{1,r}$ -structure cut of Q_n . If n = 3 and r = 2, then $S = \{100, 010, 110\} \cup \{001, 101, 111\}$, so S forms a $K_{1,2}$ -structure cut of Q_3 since $u^{2,3} = 011$ belongs to $Q_3 - S$. If n = r = 3, then $S = \{100, 010, 110, 111\} \cup \{001, 101, 100, 111\}$, so S forms a $K_{1,3}$ -structure cut of Q_3 since $u^{2,3} = 011$ belongs to $Q_3 - S$.

If $n \geq 4$ is even, let $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} | 1 \leq j \leq r-2\}$ for $1 \leq i \leq \frac{n}{2}$. Then S_i induces a star $K_{1,r}$ with the center $u^{2i-1,2i}$ for $1 \leq i \leq \frac{n}{2}$. Then u is an isolated vertex in $Q_n - S$ and \overline{u} belongs to $Q_n - S$, where $S = \bigcup_{i=1}^{\frac{n}{2}} S_i$. Also S forms a $K_{1,r}$ -structure cut of Q_n .

Remark 3.8. Obviously Q_2 has no $K_{1,2}$ -structure cut.

Remark 3.9. For the $K_{1,r}$ -structure cut S_i 's, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, in the proof of Lemma 3.7, $S_m \cap S_k = \emptyset$ for each pair $1 \leq m < k \leq \lfloor \frac{n}{2} \rfloor$, and $S_m \cap S_{\lceil \frac{n}{2} \rceil} \neq \emptyset$ for $1 \leq m < \lceil \frac{n}{2} \rceil$ if and only if m = 1 and $n = r \geq 3$ is odd (in this case, $S_1 \cap S_{\frac{n+1}{2}} = \{u^1, u^{1,2,n}\}$). We now give a proof as follows. Recall that $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} \mid 1 \leq j \leq r-2\}$ for $1 \leq i \leq \frac{n-1}{2}$. For $1 \leq m < k \leq \lfloor \frac{n}{2} \rfloor$, $\{2m - 1, 2m\} \cap \{2k - 1, 2k\} = \emptyset$, and thus $\{2m - 1, 2m, 2m + j_1\} \neq \{2k - 1, 2k, 2k + j_2\}$ for $1 \leq j_1, j_2 \leq r-2$, which implies that $S_m \cap S_k = \emptyset$. Next suppose that $S_m \cap S_{\lceil \frac{n}{2} \rceil} \neq \emptyset$ for $1 \leq m < \lceil \frac{n}{2} \rceil$. Then $n \geq 3$ is odd. If n > r, then $S_{\frac{n+1}{2}} = \{u^n, u^{n,1}\} \cup \{u^{n,1,j} \mid 2 \leq j \leq r\}$. Since 1 < 2m < n, there are $1 \leq j_1 \leq r-2$ and $2 \leq j_2 \leq r-1$ such that $\{2m - 1, 2m, 2m + j_1\} = \{n, 1, j_2\}$, which implies that $2m + j_1 = n$ and 2m - 1 = 1. So m = 1, and $n = 2 + j_1 \leq r$, a contradiction. So we may assume that $n = r \geq 3$. Then $S_{\frac{n+1}{2}} = \{u^n, u^{n,1}, u^1\} \cup \{u^{n,1,j} \mid 2 \leq j \leq r-1\}$. Similarly we have that m = 1. Conversely, if m = 1 and $n = r \geq 3$ is odd, then we can find that $S_1 \cap S_{\frac{n+1}{2}} = \{u^1, u^{1,2,n}\}$. The proof is complete.

To describe our main result about *n*-hypercube Q_n , we define the function f(r) for all integers $r \ge 2$ as follows.

$$\int_{0}^{\infty} = \begin{cases} f_1(r) = \max\{\frac{r+7}{2}, \frac{r^2+4r+3}{8}\}, & \text{if } r \ge 3 \text{ is odd}, \end{cases}$$
(3.1)

$$f(r) = \begin{cases} 2 & 8\\ f_2(r) = \max\{\frac{r^2 + 2r}{8}, \frac{r+8}{2}, \frac{r^2 + 6r + 12}{12}\}, & \text{if } r \ge 2 \text{ is even.} \end{cases}$$
(3.2)

It is not difficult to find that $f_1(r)$ and $f_2(r)$ are both strictly increasing functions for $r \ge 2$ by considering the property of a quadratic function. As Table 1 shows some initial



Figure 3. $K_{1,6}$ -structure cut of Q_{12} and $K_{1,7}$ -structure cut of Q_{11} .

values of f(r) for $2 \le r \le 20$, in general we can prove that f(r) is an increasing function and integral except at r = 8 in the following lemma.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
f(r)	5	5	6	6	7	10	$\frac{31}{3}$	15	15	21	21	28	28	36	36	45	45	55	55

Table 1. The values of f(r) for $2 \le r \le 20$.

Lemma 3.10. f(r) is an increasing function for $r \ge 2$ and integral except at r = 8, and for odd $r \ge 9$,

$$f(r) = f(r+1) = \frac{r^2 + 4r + 3}{8}.$$
(3.3)

Proof. By Eq. (3.1), we know that

$$f_1(r) = \begin{cases} \frac{r+7}{2}, & \text{if } r = 3, \\ \frac{r^2+4r+3}{8}, & \text{if } r \ge 5 \text{ is odd} \end{cases}$$

and by Eq. (3.2),

$$f_2(r) = \begin{cases} \frac{r+8}{2}, & \text{if } 2 \le r \le 6 \text{ is even}, \\ \frac{r^2+6r+12}{12}, & \text{if } r = 8, \\ \frac{r^2+2r}{8}, & \text{if } r \ge 10 \text{ is even}. \end{cases}$$

So we have

$$f_1(r) = \frac{r^2 + 4r + 3}{8}$$
, for odd $r \ge 5$, and
 $f_2(r) = \frac{r^2 + 2r}{8}$, for even $r \ge 10$.

Further, if $r_2 = r_1 + 1$, then

$$\frac{r_1^2 + 4r_1 + 3}{8} = \frac{r_2^2 + 2r_2}{8}$$

The above three equalities imply that for odd $r \ge 9$, $f(r) = f(r+1) = \frac{r^2+4r+3}{8}$, so Eq. (3.3) holds. Together with Table 1 we know that f(r) is an increasing function for $r \ge 2$.

It remains to prove that f(r) is integral for $r \ge 9$. Let $r + 1 = 2k \ge 10$. Then $f(r) = f(r+1) = f_2(2k) = \frac{(2k)^2 + 2 \times 2k}{8} = \frac{k(k+1)}{2}$, which is an integer.

Lemma 3.11. For all integers $r \ge 2$, r < f(r).

Proof. Obviously, we have

$$f_1(3) - 3 = \frac{3+7}{2} - 3 = 2 > 0$$
, and
 $f_1(r) - r = \frac{r^2 + 4r + 3}{8} - r = \frac{1}{8}(r-1)(r-3) > 0$, for $r \ge 5$

which implies that for odd $r \geq 3$, the result holds.

For even $r \geq 2$, we have

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$$f_2(r) - r = \frac{r+8}{2} - r = \frac{8-r}{2} > 0$$
, for $2 \le r \le 6$, $f_2(8) - 8 = \frac{7}{3} > 0$, and
 $f_2(r) - r = \frac{r^2 + 2r}{8} - r = \frac{1}{8}r(r-6) > 0$, for $r \ge 10$,

so the result holds.

Lemma 3.12. If integers $r \geq 2$ and n > f(r), then we have $\kappa(Q_n; K_{1,r}) \geq \lceil \frac{n}{2} \rceil$ and $\kappa^s(Q_n; K_{1,r}) \geq \lceil \frac{n}{2} \rceil$.

Proof. Since $\kappa^s(Q_n; K_{1,r}) \leq \kappa(Q_n; K_{1,r})$, we only show $\kappa^s(Q_n; K_{1,r}) \geq \lceil \frac{n}{2} \rceil$. Suppose to the contrary that $\kappa^s(Q_n; K_{1,r}) < \lceil \frac{n}{2} \rceil$. Then Q_n has a set F of subgraphs that each is a star of at most r leaves so that $|F| \leq \lceil \frac{n}{2} \rceil - 1$ and $Q_n - F$ is disconnected. So

$$|V(F)| \le (1+r)|F| \le (1+r)(\lceil \frac{n}{2} \rceil - 1) \le \frac{1}{2}(r+1)(n-1).$$
(3.4)

Let C be a smallest component of $Q_n - F$ and k = |V(C)|. We distinguish the following three cases by considering the neighborhood of C and g-extra connectivity in Q_n .

Case 1. k = 1.

Let $C = \{u\}$. By Lemma 3.2, $|N_{Q_n}(u) \cap V(K_{1,r'})| \leq 2$ for each member $K_{1,r'}$ in F, $0 \leq r' \leq r$. Thus

$$n = |N_{Q_n}(u)| \leq \sum_{K \in F} |N_{Q_n}(u) \cap V(K)| \leq 2|F|$$

$$\leq 2(\lceil \frac{n}{2} \rceil - 1) \leq 2(\frac{n+1}{2} - 1) = n - 1,$$

a contradiction.

Case 2. $2 \le k \le \frac{r}{2} + 1$.

From the given conditions, we know that $n \ge 6$. By Lemma 3.5, we have $|N_{Q_n}(C)| \ge nk - 2(k-1) - \binom{k-1}{2}$. By Lemma 3.4, $|N_{Q_n}(C) \cap V(K_{1,r'})| \le 2(k-1)$ for each member $K_{1,r'}$ in $F, 0 \le r' \le r$. We have

$$nk - 2(k-1) - \binom{k-1}{2} \le |N_{Q_n}(C)| \le 2(k-1)|F| \le 2(k-1)(\lceil \frac{n}{2} \rceil - 1)$$
$$\le (k-1)(n-1),$$

which implies that

$$n \leq \frac{k(k-1)}{2}$$

If r is even, then $n \leq \frac{r^2+2r}{8}$, contradicting $n > \max\{\frac{r^2+2r}{8}, \frac{r+8}{2}, \frac{r^2+6r+12}{12}\} = f(r)$. If r is odd, then $n \leq \frac{r^2-1}{8}$, contradicting $n > \max\{\frac{r+7}{2}, \frac{r^2+4r+3}{8}\} = f(r)$.

Case 3. $k \ge \frac{r+1}{2} + 1$.

If r is even, then $k \ge \frac{r}{2} + 2$. Since $n > f(r) \ge \frac{r+8}{2}$, $0 < \frac{r}{2} + 1 \le n - 4$, by Lemma 3.6 we have

$$\kappa_{\frac{r}{2}+1}(Q_n) = (2+\frac{r}{2})n - 2(\frac{r}{2}+1) - (\frac{r}{2}+1) = \frac{-r^2}{8} + \frac{rn}{2} + 2n - \frac{5r}{4} - 2.$$
(3.5)

Since $Q_n - F$ is disconnected and C is a smallest component of $Q_n - F$, $|V(F)| \ge \kappa_{\frac{r}{2}+1}(Q_n)$, so by Ineq. (3.4) and Eq. (3.5) we have

$$\frac{1}{2}(r+1)(n-1) \ge \frac{-r^2}{8} + \frac{rn}{2} + 2n - \frac{5r}{4} - 2,$$

which implies that $n \leq \frac{r^2 + 6r + 12}{12}$, contradicting $n > \max\{\frac{r^2 + 2r}{8}, \frac{r + 8}{2}, \frac{r^2 + 6r + 12}{12}\} = f(r)$. If r is odd, then $k \geq \frac{r-1}{2} + 2$. Since $n > f(r) \geq \frac{r+7}{2}$, $0 < \frac{r-1}{2} + 1 \leq n - 4$, by Lemma

If r is odd, then $k \ge \frac{r}{2} + 2$. Since $n > f(r) \ge \frac{r}{2}, 0 < \frac{r}{2} + 1 \le n - 4$, by Lemma 3.6 we have

$$\kappa_{\frac{r-1}{2}+1}(Q_n) = \left(2 + \frac{r-1}{2}\right)n - 2\left(\frac{r-1}{2} + 1\right) - \left(\frac{r-1}{2} + 1\right) \\ = \frac{-r^2}{8} + \frac{(r+3)n}{2} - r - \frac{7}{8}.$$
(3.6)

Since $Q_n - F$ is disconnected and C is a smallest component of $Q_n - F$, $|V(F)| \ge \kappa_{\frac{r-1}{2}+1}(Q_n)$. So by Ineq. (3.4) and Eq. (3.6) we have

$$\frac{1}{2}(r+1)(n-1) \ge \frac{-r^2}{8} + \frac{(r+3)n}{2} - r - \frac{7}{8},$$

which implies $n \le \frac{r^2 + 4r + 3}{8}$, a contradiction to $n > \max\{\frac{r+7}{2}, \frac{r^2 + 4r + 3}{8}\} = f(r).$

From Lemma 3.11 we know that the condition of Lemma 3.12 implies that of Lemma 3.7. Hence we obtain the following main result in this section.

Theorem 3.13. If $r \ge 2$ and n > f(r), then $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$.

4 Some low dimensional cases of Q_n

For $r \leq 6$ and Q_n , the remaining cases not solved are to determine the values of $\kappa(Q_n; K_{1,r})$ and $\kappa^s(Q_n; K_{1,r})$ for $4 \leq r \leq 6$ and n = r and r + 1.

In this section, we will solve separately the low dimensional cases, which cannot be treated in the previous unified way. Latifi [4] express $Q_n = Q_n^0 \bigotimes Q_n^1$, where $Q_n^0 \cong Q_{n-1}$ and $Q_n^1 \cong Q_{n-1}$. Q_n^0 and Q_n^1 induced by the vertices with the *i*th coordinates 0 and 1 respectively, where $1 \le i \le n$. In following, $N_{G-A}(A) = \{x | xy \in E(G), x \in G-A, y \in A\}$.

Lemma 4.1. $\kappa^{s}(Q_4, K_{1,4}) \geq 2$.

Proof. We set F being a star of at most 4 leaves, $F_i = F \cap Q_4^i (i = 0, 1)$. It is sufficient to prove that $Q_4 - F$ is connected. Without loss of generality, assume the center of F belongs to Q_4^0 . It is noticed that $Q_4^0 - F_0$ and $Q_4^1 - F_1$ are both connected since $\kappa^s(Q_3, K_{1,3}) = 2$. Since there exists at least $V(Q_4^0 - F_0) = 2^3 - 4 = 4$ edges between $Q_4^0 - F_0$ and $Q_4^1 - F_1$ but $|V(F_1)| \leq 1$, there is an edge between $Q_4^0 - F_0$ and $Q_4^1 - F_1$. Thus $Q_4 - F$ is connected.

Lemma 4.2. $\kappa^{s}(Q_5, K_{1,4}) \geq 3.$

Proof. We set F_i be a star of at most 4 leaves. It is sufficient to prove that $Q_5 - F_1 - F_2$ is connected. If $F_1 \not\cong K_{1,4}$ and $F_2 \not\cong K_{1,4}$, then the result holds since $\kappa^s(Q_5, K_{1,3}) = 3$. Thus we assume that $F_i \cong K_{1,4}$ and $Q_5^i \cap F_2 = F_2^i(i = 0, 1)$. Without loss of generality, assume $F_1 \subseteq Q_5^0$. Since $\kappa^s(Q_4, K_{1,4}) \ge 2$ by Lemma 4.1, $Q_5^1 - F_2^1$ is connected. If $Q_5^0 - F_1 - F_2^0$ is connected, then $Q_5 - F_1 - F_2$ is connected since $|V(Q_5^0 - F_1 - F_2^0)| \ge 2^4 - 10 = 6$ and $|V(F_2^1)| \le 5$, there is a vertex in $Q_5^0 - F_1 - F_2^0$ which has a neighbor in $Q_5^1 - F_2^1$. If $Q_5^0 - F_1 - F_2^0$ is disconnected and each component of $Q_5^0 - F_1 - F_2^0$ connects to $Q_5^1 - F_2^1$, then $Q_5 - F_1 - F_2$ is connected. Hence, we consider there is a component C of $Q_5^0 - F_1 - F_2^0$ which is not connecting to $Q_5^1 - F_2^1$. Then $N_{Q_5}(C) \subseteq (F_1 \cup F_2)$ and $N_{Q_5^1}(C) \subseteq F_2^1$, which implies that $|V(C)| = |N_{Q_5^1}(C)| \le |V(F_2^1)| \le 5$. If $1 \le |V(C)| \le 2$, then $5 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |N_{Q_5}(C) \cap F_i| \le 4$ by Lemma 3.2 and 3.3, a contradiction. If $4 \le |V(C)| \le 5$, then $|N_{Q_5}(C)| \ge 11$ by Lemma 3.5, so we have $11 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |V(F_i)| \le 10$, a contradiction. Thus $Q_5 - F_1 - F_2$ is connected. □

By the above two lemmas and Lemma 3.7, we obtain the following theorem.

Theorem 4.3. For $4 \le n \le 5$, $\kappa(Q_n, K_{1,4}) = \kappa^s(Q_n, K_{1,4}) = \lceil \frac{n}{2} \rceil$.

Lemma 4.4. $\kappa^{s}(Q_5, K_{1,5}) \geq 3.$

Proof. Suppose to the contrary that $\kappa^s(Q_5; K_{1,5}) \leq 2$. Let F_i be a star of at most 5 leaves such that $Q_5 - F_1 - F_2$ is disconnected and C be a smallest component of $Q_5 - F_1 - F_2$. We assume that for $i = 0, 1, F_1^i = F_1 \cap Q_5^i, F_2^i = F_2 \cap Q_5^i$ and $C_i = C \cap Q_5^i$. If $F_1 \ncong K_{1,5}$ and $F_2 \ncong K_{1,5}$, then $Q_5 - F_1 - F_2$ is connected since $\kappa^s(Q_5, K_{1,4}) = 3$ by Theorem 4.3. Thus we assume that $F_i \cong K_{1,5}$.

Case 1. $F_1 \cong K_{1,5}$ and $F_2 \ncong K_{1,5}$. We set x is the center of F_1 . Without loss of generality, assume that $F_2 \subseteq Q_5^0$. Since $Q_5^1 \cong Q_4$ and $\kappa^s(Q_4, K_{1,4}) \ge 2$ by Lemma 4.1, $Q_5^1 - F_1^1$ is connected. If $Q_5^0 - F_1^0 - F_2$ is connected, then $C = Q_5 - F_1 - F_2$ since $|V(Q_5^0 - F_1^0 - F_2)| \ge 2^4 - 10 = 6$ and $|V(F_1^1)| \le 5$, there is a vertex in $Q_5^0 - F_1^0 - F_2$ which has a neighbor in $Q_5^1 - F_1^1$. If $Q_5^0 - F_1^0 - F_2$ is disconnected and each component of $Q_5^0 - F_1 - F_2^0$ connects to $Q_5^1 - F_2^1$, then $C = Q_5 - F_1 - F_2$ is connected. Hence, there exists a smallest component C' of $Q_5^0 - F_1 - F_2^0$ which is not connecting to $Q_5^1 - F_1^1$. Then $N_{Q_5}(C') \subseteq (F_1 \cup F_2)$ and $N_{Q_{5}^{1}}(C') \subseteq F_{1}^{1}$, which implies that $|V(C')| = |N_{Q_{5}^{1}}(C')| \leq |V(F_{1}^{1})| \leq 5$. As we know that C' is also a component of $Q_5 - F_1 - F_2$. Since $Q_5^1 - F_1^1$ is connected, the components of $Q_5 - F_1 - F_2$ either contains $Q_5^1 - F_1^1$ or not. If $C \neq C'$, then $(Q_5^1 - F_1^1) \subseteq C$ and $|V(C)| \geq C$ $|V(Q_5^1 - F_1^1)| \ge 2^4 - 5 = 11 > |V(C')|, \text{ it is a contradiction since } C \text{ is a smallest component.}$ Thus C = C'. If $1 \le |V(C)| \le 2$, then $5 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |N_{Q_5}(C) \cap F_i| \le 4$ by Lemmas 3.2 and 3.3, a contradiction. If |V(C)| = 3, then $10 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |N_{Q_5}(C) \cap F_i| \le 8$ by Lemma 3.4, a contradiction. If $4 \leq |V(C)| \leq 5$, then $|N_{Q_5}(C)| \geq 11$ by Lemma 3.5, so we have $11 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |V(F_i)| \le 10$, a contradiction. Thus $Q_5 - F_1 - F_2$ is connected.

Case 2. $F_i \cong K_{1,5}(i = 1, 2)$. We set $F_1 = \{x, x_1, x_2, x_3, x_4, x_5\}$, $F_2 = \{y, y_1, y_2, y_3, y_4, y_5\}$ where $\{xx_i, yy_i\} \subset E(Q_5)(i \in \{1, 2, ..., 5\})$. We have the following cases by the positions of x and y.

Case 2.1. Both x and y belong to $V(Q_5^i)$. Without loss of generality, assume that x and y belong to $V(Q_5^0)$ and $\{x_5, y_5\} \subset V(Q_5^1)$, Then $Q_5^1 - x_5 - y_5$ is connected since $\kappa(Q_5^1) =$ 4. Thus $C_1 = Q_5^1 - x_5 - y_5$ or $C_1 = \emptyset$. If $C_1 = Q_5^1 - x_5 - y_5$, then $|V(C)| \ge |V(C_1)| =$ $2^4 - 2 = 14$. We have $|V(Q_5 - F_1 - F_2)| - |V(C)| \le 2^5 - 12 - 14 = 6$, it is a contradiction since C is a smallest component. Therefore $C = C_0$. Then $N_{Q_5^1}(C) \subseteq \{x_5, y_5\}$ and $|V(C)| = |N_{Q_5^1}(C)| \le 2$. If $1 \le |V(C)| \le 2$, then $5 \le |N_{Q_5}(C)| \le \sum_{i=1}^2 |N_{Q_5}(C) \cap F_i| \le 4$ by Lemmas 3.2 and 3.3, a contradiction. Thus $Q_5 - F_1 - F_2$ is connected.

Case 2.2. Either x or y belongs to $V(Q_5^i)$. Without loss of generality, assume that $x \in V(Q_5^0)$, $y \in V(Q_5^1)$ and $x_5 \in V(Q_5^1)$, $y_5 \in V(Q_5^0)$. Then $N_{Q_5^0}(C_0) \subseteq (F_1^0 \cup \{y_5\})$. If $1 \leq |V(C_0)| \leq 2$, then $4 \leq |N_{Q_5^0}(C_0)| \leq |N_{Q_5^0}(C_0) \cap F_1^0| + 1 \leq 3$ by Lemmas 3.2 and 3.3, a contradiction. If $3 \leq |V(C_0)| \leq 5$, then $6 \leq |N_{Q_5^0}(C_0)| \leq |\{x_1, x_2, x_3, x_4, y_5\}| = 5$ by Lemmas 3.5, a contradiction. Thus $|V(C_0)| \geq 6$ and $|V(C_1)| \geq 6$ by a similar argument. So $|V(C)| \geq 12$ and $|V(Q_5 - F_1 - F_2)| - |V(C)| \leq 2^5 - 12 - 12 = 8$. It contradicts to that C is a smallest component. Thus $Q_5 - F_1 - F_2$ is connected.

Lemma 4.5. For $5 \le r \le 6$, $\kappa^s(Q_6, K_{1,r}) \ge 3$.

Proof. Suppose to the contrary that $\kappa^s(Q_6; K_{1,r}) \leq 2$. Let $F_i(i = 1, 2)$ be a star of at most r leaves and $Q_6 - F_1 - F_2$ is disconnected. Let C be a smallest components of $Q_6 - F_1 - F_2$.

If $1 \leq |V(C)| \leq 2$, by Lemmas 3.2 and 3.3, $|N_{Q_6}(C) \cap V(F_i)| \leq 2$. Thus $6 \leq |N_{Q_6}(C)| \leq \sum_{i=1}^2 |N_{Q_6}(C) \cap V(F_i)| \leq 4$ a contradiction.

If |V(C)| = 3, then $|N_{Q_6}(C) \cap V(F_i)| \le 4$ by Lemma 3.4 and $|N_{Q_6}(C)| \ge 13$ by Lemma 3.5. Thus $13 \le |N_{Q_6}(C)| \le \sum_{i=1}^2 |N_{Q_6}(C) \cap V(F_i)| \le 8$, a contradiction.

If $|V(C)| \ge 4$, by Lemma 3.6 we have $\kappa_3(Q_6) = 15$. Since $Q_6 - F_1 - F_2$ is disconnected and C is a smallest component of $Q_6 - F_1 - F_2$, $14 \ge |V(F_1)| + |V(F_2)| \ge \kappa_3(Q_6) = 15$. A contradiction.

By Lemmas 4.4 and 4.5, we have $\kappa^{s}(Q_{n}, K_{1,5}) \geq 3$ with $5 \leq n \leq 6$. Thus we get the following theorem by Lemma 3.7.

Theorem 4.6. For $5 \le n \le 6$, $\kappa(Q_n, K_{1,5}) = \kappa^s(Q_5, K_{1,5}) = 3$.

Lemma 4.7. $\kappa^{s}(Q_7, K_{1,6}) \geq 4.$

Proof. Suppose to the contrary that $\kappa^{s}(Q_{7}; K_{1,6}) \leq 3$. Let F_{i} be star of at most 6 leaves such that $Q_{7} - \bigcup_{i=1}^{3} F_{i}$ is disconnected and C be a smallest components of $Q_{7} - \bigcup_{i=1}^{3} F_{i}$ with |V(C)| = g + 1. Then $N_{Q_{7}}(C) \subseteq \bigcup_{i=1}^{3} F_{i}$ and $N_{Q_{7}}(C) \subseteq \bigcup_{i=1}^{3} (N_{Q_{7}}(C) \cap F_{i})$. By Lemma 3.5, we have

$$7(g+1) - 2g - \frac{1}{2}g(g-1) \le |N_{Q_7}(C)| \le \sum_{i=1}^3 |F_i| \le 21,$$

it implies that $g \leq 4$ or $g \geq 7$; for $g \leq 4$, it contradicts to $|N_{Q_7}(C)| \leq \sum_{i=1}^3 |N_{Q_7}(C) \cap F_i|$ by Lemmas 3.2 and 3.4. Thus we have $|V(C)| \ge 8$. We assume that $C_i = C \cap Q_7^i (i = 0, 1)$. If $F_i \ncong K_{1,6}$ (i = 1, 2, 3), then $Q_7 - \bigcup_{i=1}^3 F_i$ is connected since $\kappa^s(Q_7, K_{1,5}) = 4$ by Theorem 3.13. So we consider $F_i \cong K_{1,6}$. Without loss of generality, we assume that $F_1 \cong K_{1,6}$, $F_1 \subseteq Q_7^0$ and $Q_7^i \cap F_2 = F_2^i, Q_7^i \cap F_3 = F_3^i (i = 0, 1)$. Since $Q_7^i \cong Q_6$ and $\kappa^s(Q_6, K_{1,6}) \ge 3$ by Lemma 4.5, $Q_7^1 - F_2^1 - F_3^1$ is connected. Thus we know the components of $Q_7 - \bigcup_{i=1}^3 F_i$ either contains $Q_7^1 - F_2^1 - F_3^1$ or not. We have $C_1 = Q_7^1 - F_2^1 - F_3^1$ or $C_1 = \emptyset$. If $C_1 = Q_7^1 - F_2^1 - F_3^1$, then $|V(C_1)| \geq 2^6 - 14 = 50$. Since each vertex in C_1 has exactly one neighbor in $N_{Q_7^0}(C_1), |N_{Q_7^0}(C_1)| = |V(C_1)| \ge 50 \text{ and } N_{Q_7^0}(C_1) \subseteq (F_1 \cup F_2^0 \cup F_3^0 \cup C_0).$ We know $|N_{Q_2^0}(C_1)| \le |V(F_1 \cup F_2^0 \cup F_3^0)| + |V(C_0)| \le 21 + |V(C_0)|$, which implies that $|V(C_0)| \ge 29$. We get $|V(C)| = |V(C_1)| + |V(C_0)| \ge 79$ and $|V(Q_7 - \bigcup_{i=1}^3 F_i)| - |V(C)| < 2^7 - 79 = 49$, it is a contradiction since C is a smallest component. Therefore $C \subseteq Q_7^0 - F_1 - F_2^0 - F_3^0$ with $|V(C)| \ge 8$. We have $8 \le |V(C)| = |N_{Q_7^1}(C)| \le |V(F_2^1 \cup F_3^1)|$, then $|V(F_2^1 \cup F_3^1)| \ge 8$. By the above analysis, we know the component of $Q_7 - \bigcup_{i=1}^3 F_i$ which contains $Q_7^1 - F_2^1 - F_3^1$ have at least 29 vertices in $Q_7^0 - F_1 - F_2^0 - F_3^0$, so the component of $Q_7^0 - F_1 - F_2^0 - F_3^0$ has at least 8 vertices. Then $\kappa_6(Q_7^0) \le |V(F_1 \cup F_2^0 \cup F_3^0)| \le |V(\cup_{i=1}^3 F_i)| - |V(F_2^1 \cup F_3^1)| \le 13.$ As we know $\kappa_6(Q_6) = 15$ by Lemma 3.6. It is a contradiction. Thus $Q_7 - \bigcup_{i=1}^3 F_i$ is connected.

By Lemma 3.7 and Lemmas 4.5, 4.7, we have the following theorem.

Theorem 4.8. $\kappa(Q_6, K_{1,6}) = \kappa^s(Q_6, K_{1,6}) = 3; \ \kappa(Q_7, K_{1,6}) = \kappa^s(Q_7, K_{1,6}) = 4.$

5 The star-Structure connectivity of Fold Hypercube

In this section, we study the $\kappa(FQ_n; K_{1,r})$ and $\kappa^s(FQ_n; K_{1,r})$ for $r \ge 2$. It is known that FQ_n is triangle-free for $n \ge 3$.

Lemma 5.1. [18] Any two vertices in FQ_n exactly have two common neighbors for $n \ge 4$ if they have any.

It is easy to find the above lemma is true when n = 2. For n = 3, we find $N_{FQ_n}(011) \cap N_{FQ_n}(110) = \{010, 001, 100, 111\}$, so Lemma 5.1 is fault when n = 3.

Lemma 5.2. [12] Let FQ_n be a folded hypercube. Then

- i) $\kappa(FQ_n) = n+1;$
- ii) FQ_n is a bipartite graph if and only if n is odd;

iii) If FQ_n contains an odd cycle, then a shortest odd cycle has the length n + 1.

Lemma 5.3. Let $K_{1,r}$ be a star in FQ_n with $n \ge 4$ and $n+1 \ge r \ge 2$. If u is a vertex in $FQ_n - K_{1,r}$, then $|N_{FQ_n}(u) \cap V(K_{1,r})| \le 2$, and equality holds if and only if u is adjacent to exactly two leaves of $K_{1,r}$.

Proof. Since FQ_n is triangle-free for $n \ge 4$, u cannot be adjacent to both a leaf and the center of $K_{1,r}$. If u is just adjacent to a leaf of $K_{1,r}$, then by Lemma 5.1 u has at most two neighbors in the leaves of $K_{1,r}$, which are adjacent to the center of $K_{1,r}$. \Box

For $n \ge 5$, FQ_n has no 5-cycle or 3-cycle. So we can derive the following result by analogous arguments as Lemma 3.4.

Lemma 5.4. Let $K_{1,r}$ be a star in FQ_n with $n \ge 5$ and $n+1 \ge r \ge 2$. If C is a connected subgraph in $FQ_n - K_{1,r}$ with $|V(C)| = k \ge 2$, then $|N_{FQ_n}(C) \cap V(K_{1,r})| \le 2(k-1)$, and equality holds only if C is a star in FQ_n .

Proof. Let $V(K_{1,r}) = \{x, x_1, x_2, \dots, x_r\}$ and $E(K_{1,r}) = \{xx_i | 1 \le i \le r\}$. Then x is the center of $K_{1,r}$. Let $V(C) = \{u_1, u_2, \dots, u_k\}$.

First, we prove that $|N_{FQ_n}(C) \cap V(K_{1,r})| \leq 2(k-1)$. Suppose to the contrary that $|N_{FQ_n}(C) \cap V(K_{1,r})| \geq 2(k-1) + 1 = 2k-1$. By Lemma 5.3, each vertex u_i in C has at most 2 neighbors in $K_{1,r}$, and if $|N_{FQ_n}(u_i) \cap V(K_{1,r})| = 2$, then u_i is adjacent to two leaves in $K_{1,r}$, so $2k \geq |N_{FQ_n}(C) \cap V(K_{1,r})| \geq 2k-1$. It means that there exists at least k-1 vertices in C which each has two neighbors in $K_{1,r}$, and such neighbors are pairwise distinct. Without loss of generality, we assume $\{u_i x_{2i-3}, u_i x_{2i-2}\} \subset E(FQ_n)$ for $2 \leq i \leq k$. Since C is connected, u_1 is adjacent to u_i for some $2 \leq i \leq k$. If $u_1 x_j \in E(FQ_n)$ with $2k-1 \leq j \leq r$, then there exists a 5-cycle $u_1 x_j x x_{2i-3} u_i u_1$, a contradiction. Otherwise, $u_1 x \in E(FQ_n)$. Then $N_{FQ_n}(u_i) \cap N_{FQ_n}(x) = \{x_{2i-3}, x_{2i-2}, u_1\}$, contradicting Lemma 5.1. Hence $|N_{FQ_n}(C) \cap V(K_{1,r})| \leq 2(k-1)$.

Next we show that if $|N_{FQ_n}(C) \cap V(K_{1,r})| = 2(k-1)$, then C is a star in FQ_n . Suppose to the contrary that C is not a star in FQ_n . Then there exists a 4-vertex path P_4 in C, so $4 \leq k$ and $6 \leq |N_{FQ_n}(P_4) \cap V(K_{1,r})| \leq 8$ by Lemma 5.3. However, any two consecutive vertices in P_4 have at most two neighbors in $V(K_{1,r})$, since $FQ_n(n \geq 5)$ has no 5-cycle and $|N_{FQ_n}(u) \cap N_{FQ_n}(x)| \leq 2$ for $u \in V(P_4)$. This implies that P_4 has at most four neighbors in $V(K_{1,r})$, a contradiction.

The following lemma can be obtained from Theorem 2.11 of [17].

Lemma 5.5. [17] Let C be a subgraph of FQ_n with |V(C)| = g + 1, for $n \ge 5$, $1 \le g \le n+2$. Then $|N_{FQ_n}(C)| \ge (n+1)(g+1) - 2g - \binom{g}{2}$.

Lemma 5.6. [16] For $n \ge 7$,

$$\kappa_g(FQ_n) = \begin{cases} (g+1)(n+1) - 2g - \binom{g}{2}, & \text{if } 0 \le g \le n-3, \\ \frac{n(n+1)}{2}, & \text{if } n-2 \le g \le n+1 \end{cases}$$

Lemma 5.7. For $n + 1 \ge r \ge 2$, $n \ge 3$, $\kappa(FQ_n; K_{1,r}) \le \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(FQ_n; K_{1,r}) \le \lceil \frac{n+1}{2} \rceil$.

Proof. Since $\kappa^s(FQ_n; K_{1,r}) \leq \kappa(FQ_n; K_{1,r})$, we only prove $\kappa(FQ_n; K_{1,r}) \leq \lceil \frac{n+1}{2} \rceil$. Let $u = 000 \cdots 0$ be a vertex in Q_n . Then $N_{FQ_n}(u) = \{\overline{u}\} \cup \{u^i | 1 \leq i \leq n\}$.

Case 1. $n \geq 3$ is odd. For $1 \leq i \leq \frac{n-1}{2}$, let $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} | 1 \leq j \leq n-2\} \cup \{\overline{u}^{2i-1,2i}\}$ with $r \leq n$ and $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} | 1 \leq j \leq n-2\} \cup \{\overline{u}^{2i-1,2i}\}$ with r = n + 1. Let $S_{\frac{n+1}{2}} = \{u^n, \overline{u}, \overline{u^n}\} \cup \{\overline{u}^{n,j} | 1 \leq j \leq r-2\}$. Noting that $\overline{u^n} = \overline{u^n}$, we also know that S_i induces a star $K_{1,r}$ with the center $u^{2i-1,2i}$ for $1 \leq i \leq \frac{n-1}{2}$ and with the center $\overline{u^n}$ for $i = \frac{n+1}{2}$ respectively. Let $S = \bigcup_{i=1}^{\frac{n+1}{2}} S_i$. Then $N_{FQ_n}(u) \subseteq S$, and u is an isolated vertex of $FQ_n - S$. We can see that vertex $u^{1,n} \notin S_i$ for each $1 \leq i \leq \frac{n+1}{2}$. So the $S'_i s$ for $1 \leq i \leq \frac{n+1}{2}$ form a $K_{1,r}$ -structure cut of FQ_n .

Case 2. $n \geq 4$ is even. For $r \leq n$, let $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j}|1 \leq j \leq r-2\}$ when $1 \leq i \leq \frac{n}{2}$ and $S_{\frac{n+2}{2}} = \{\overline{u}\} \cup \{\overline{u}^j|1 \leq j \leq r\}$. For r = n + 1, let $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j}|1 \leq j \leq n-2\} \cup \{\overline{u}^{2i-1,2i}\}$ when $1 \leq i \leq \frac{n}{2}$ and $S_{\frac{n+2}{2}} = \{\overline{u}, u^1, \overline{u}^1\} \cup \{\overline{u}^{1,j}|2 \leq j \leq n\}$. Then S_i induces a star $K_{1,r}$ with the center $u^{2i-1,2i}$ for $1 \leq i \leq \frac{n}{2}$, and $S_{\frac{n+2}{2}}$ also induces a star $K_{1,r}$ with the center \overline{u} for $r \leq n$ and with the center \overline{u}^1 for r = n + 1 respectively. Let $S = \bigcup_{i=1}^{\frac{n+2}{2}} S_i$. We can see that u is an isolated vertex in $FQ_n - S$ and $u^{1,n}$ belongs to $FQ_n - S$. So S forms a $K_{1,r}$ -structure cut of FQ_n .

Remark 5.8. For the $K_{1,r}$ -structure cut S_i 's, $1 \le i \le \lceil \frac{n+1}{2} \rceil$, in the proof of Lemma 5.7, any pair of distinct S_i and S_j are disjoint for $n \ge 6$ and $r \le n$. A proof is presented here. Recall that $S_i = \{u^{2i-1}, u^{2i}, u^{2i-1,2i}\} \cup \{u^{2i-1,2i,2i+j} | 1 \le j \le r-2\}$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$. For $1 \le m < k \le \lfloor \frac{n}{2} \rfloor$, $\{2m - 1, 2m\} \cap \{2k - 1, 2k\} = \emptyset$, and thus $\{2m - 1, 2m, 2m + j_1\} \ne \{2k-1, 2k, 2k+j_2\}$ for $1 \le j_1, j_2 \le r-2$, which implies that $S_m \cap S_k = \emptyset$. We now consider S_m and $S_{\lceil \frac{n+1}{2} \rceil}$ for $1 \le m \le \lfloor \frac{n}{2} \rfloor$. If n is odd, $S_{\frac{n+1}{2}} = \{u^n, \overline{u}, \overline{u}^n, \overline{u}^{n,j} | 1 \le j \le r-2\}$, and 1 < 2m < n. Since $\overline{u}^{n,j}$ agrees with u in exactly 2 positions, and $u^{2m-1,2m,2m+j_1}$ agrees with u in exactly n-3 positions, $\overline{u}^{n,j} \ne u^{2m-1,2m,2m+j_1}$ for n > 5. So $S_m \cap S_{\lceil \frac{n+1}{2} \rceil} = \emptyset$. If n is even, $S_{\frac{n}{2}+1} = \{\overline{u}, \overline{u}^j | 1 \le j \le r\}$, and $S_m \cap S_{\frac{n}{2}+1} = \emptyset$ for $n \ge 6$.

For r = n + 1 and $n \ge 6$, we have a unique pair of intersecting $K_{1,r}$ -stars in the S_i 's, that is, $S_1 \cap S_{\frac{n+2}{2}} = \{u^1, \overline{u}^{1,2}\}$. For $3 \le n \le 5$, however, there are many pairs of intersecting $K_{1,r}$ -stars in the S_i 's.

In order to describe our main result about *n*-dimensional folded hypercube FQ_n , we define the function g(r) as follows.

$$g(r) = \begin{cases} g_1(r) = \max\{6, \frac{r+5}{2}, \frac{r^2+4r-5}{8}\}, & \text{if } r \ge 3 \text{ is odd}, \\ m^2+2m-8 - m+6 - m^2+6m \end{cases}$$
 (5.1)

$$g_2(r) = \max\{6, \frac{r^2 + 2r - 8}{8}, \frac{r + 6}{2}, \frac{r^2 + 6r}{12}\}, \quad \text{if } r \ge 2 \text{ is even.} \quad (5.2)$$

We find that $g_1(r)$ and $g_2(r)$ are both increasing functions. Table 2 lists the values of g(r) for $2 \leq r \leq 20$. We also have the following monotonicity and integrality of function g(r).

Table 2. The values of g(r) for $2 \le r \le 20$.

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
g(r)	6	6	6	6	6	9	$\frac{28}{3}$	14	14	20	20	27	27	35	35	44	44	54	54

Lemma 5.9. g(r) is an increasing function for $r \ge 2$ and integral except at r = 8, and for odd $r \ge 9$,

$$g(r) = g(r+1) = \frac{r^2 + 4r - 5}{8}.$$
(5.3)

Proof. By Eq. (5.1), we find that

$$g_1(r) = \begin{cases} 6, & \text{if } 3 \le r \le 5 \text{ is odd} \\ \frac{r^2 + 4r - 5}{8}, & \text{if } r \ge 7 \text{ is odd}; \end{cases}$$

and by Eq. (5.2),

$$g_2(r) = \begin{cases} 6, & \text{if } 2 \le r \le 6 \text{ is even} \\ \frac{r^2 + 6r}{12}, & \text{if } r = 8 \\ \frac{r^2 + 2r - 8}{8}, & \text{if } r \ge 10 \text{ is even.} \end{cases}$$

So, we have

$$g_1(r) = \frac{r^2 + 4r - 5}{8}, \text{ for odd } r \ge 7,$$

 $g_2(r) = \frac{r^2 + 2r - 8}{8}, \text{ for even } r \ge 10.$

Moreover, if $r_1 \ge 9$ and $r_2 = r_1 + 1$, then

$$\frac{r_1^2 + 4r_1 - 5}{8} = \frac{r_2^2 + 2r_2 - 8}{8},$$

which means that for odd $r \ge 9$, Eq. (5.3) holds. Together with Table 2, we know that g(r) is a monotonically increasing function for $r \ge 2$.

We now only prove that g(r) is integral for $r \ge 9$. Let $r + 1 = 2k \ge 10$. Then $g(r) = g(r+1) = g_2(2k) = \frac{(2k)^2 + 2(2k) - 8}{8} = \frac{(k+2)(k-1)}{2}$, which is an integer.

Lemma 5.10. For all integers $r \ge 2$, $g(r) \ge r$.

Proof. We have

$$g_1(r) - r = 6 - r > 0, \text{ for } 3 \le r \le 5;$$

$$g_1(r) - r = \frac{r^2 + 4r - 5}{8} - r = \frac{1}{8}(r+1)(r-5) > 0, \text{ for } r \ge 7$$

Therefore, if $r \ge 3$ is odd, then $g_1(r) - r > 0$.

On the other hand,

$$g_2(r) - r = 6 - r \ge 0, \text{ for } 2 \le r \le 6;$$

$$g_2(8) - 8 = \frac{8^2 + 48}{12} - 8 = \frac{4}{3} > 0;$$

$$g_2(r) - r = \frac{r^2 + 2r - 8}{8} - r = \frac{1}{8}(r^2 - 6r - 8) > 0, \text{ for } r \ge 10.$$

So, if $r \ge 2$ is even, then $g_2(r) - r \ge 0$.

Lemma 5.11. If integers $r \ge 2$ and n > g(r), then we have $\kappa(FQ_n; K_{1,r}) \ge \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(FQ_n; K_{1,r}) \ge \lceil \frac{n+1}{2} \rceil$.

Proof. Since $\kappa^s(FQ_n; K_{1,r}) \leq \kappa(FQ_n; K_{1,r})$, it suffices to show that $\kappa^s(FQ_n; K_{1,r}) \geq \lceil \frac{n+1}{2} \rceil$. Suppose to the contrary that $\kappa^s(FQ_n; K_{1,r}) < \lceil \frac{n+1}{2} \rceil$. Then there are a set F of subgraphs of FQ_n that each is a star of at most r leaves so that $|F| \leq \lceil \frac{n+1}{2} \rceil - 1$ and $FQ_n - F$ is disconnected. Let C be a smallest component of $FQ_n - F$ and k := |V(C)|. We consider following three cases.

Case 1. k = 1.

Let $C = \{u\}$. By Lemma 5.3, $|N_{FQ_n}(u) \cap V(K_{1,r'})| \leq 2$ for each member $K_{1,r'}$ in F, $0 \leq r' \leq r$. Thus

$$n+1 = |N_{FQ_n}(u)| \leq \sum_{K \in F} |N_{FQ_n}(u) \cap V(K)| \leq 2|F|$$
$$\leq 2(\lceil \frac{n+1}{2} \rceil - 1) \leq 2(\frac{n+2}{2} - 1) = n$$

a contradiction.

Case 2. $2 \le k \le \frac{r}{2} + 1$.

Since $n > g(r) \ge \frac{r+5}{2}$, $2 \le k \le \frac{r}{2} + 1 \le n-2$. For $n \ge 7$, by Lemma 5.5, we have $|N_{FQ_n}(C)| \ge (n+1)k - 2(k-1) - \binom{k-1}{2}$. By Lemma 5.4, $|N_{FQ_n}(C) \cap V(K_{1,r'})| \le 2(k-1)$ for each member $K_{1,r'}$ in $F, 0 \le r' \le r$. We have

$$(n+1)k - 2(k-1) - {\binom{k-1}{2}} \le |N_{FQ_n}(C)| \le 2(k-1)|F| \le 2(k-1)(\lceil \frac{n+1}{2} \rceil - 1) \le (k-1)n,$$

which implies that $n \leq \frac{(k-2)(k+1)}{2}$. If *r* is even, then $n \leq \frac{r^2+2r-8}{8}$, contradicting $n > \max\{6, \frac{r^2+2r-8}{8}, \frac{r+6}{2}, \frac{r^2+6r}{12}\} = g(r)$. If *r* is odd, then $n \leq \frac{r^2-9}{8} < \frac{r^2+4r-5}{8} \leq g(r)$, a contradiction.

Case 3. $k \ge \frac{r+1}{2} + 1$.

If r is even, then $k \ge \frac{r}{2} + 2$. Since $n > g(r) \ge \max\{6, \frac{r+6}{2}\}, 2 \le \frac{r}{2} + 1 \le n-3$, by Lemma 5.6 we have

$$\kappa_{\frac{r}{2}+1}(FQ_n) = \left(\frac{r}{2}+2\right)(n+1) - 2\left(\frac{r}{2}+1\right) - \left(\frac{r}{2}+1\right)$$
$$= \frac{-r^2}{8} + \frac{rn}{2} + 2n - \frac{3r}{4}.$$

We also have that

$$|V(F)| \le (1+r)|F| \le (1+r)(\lceil \frac{n+1}{2} \rceil - 1) \le \frac{1}{2}(r+1)n.$$
(5.4)

Since $FQ_n - F$ is disconnected, and C is a smallest component of $FQ_n - F$ and $|C| = k \ge \frac{r}{2} + 2$, we have $|V(F)| \ge \kappa_{\frac{r}{2}+1}(FQ_n)$, so

$$\frac{1}{2}(r+1)n \ge \frac{-r^2}{8} + \frac{rn}{2} + 2n - \frac{3r}{4},$$

which implies that $n \leq \frac{r^2+6r}{12}$, contradicting $n > \max\{6, \frac{r^2+2r-8}{8}, \frac{r+6}{2}, \frac{r^2+6r}{12}\} = g(r)$. If r is odd and $n > g(r) \geq \max\{6, \frac{r+5}{2}\}$, then $2 \leq \frac{r+1}{2} \leq n-3$. By Lemma 5.6 we

If r is odd and $n > g(r) \ge \max\{6, \frac{r+3}{2}\}$, then $2 \le \frac{r+1}{2} \le n-3$. By Lemma 5.6 we have that

$$\kappa_{\frac{r+1}{2}}(FQ_n) = \left(\frac{r+1}{2} + 1\right)(n+1) - 2\left(\frac{r+1}{2}\right) - \left(\frac{\frac{r+1}{2}}{2}\right)$$
$$= \frac{-r^2}{8} + \frac{(r+3)n}{2} - \frac{r}{2} + \frac{5}{8}.$$
(5.5)

Since $FQ_n - F$ is disconnected, and C is a smallest component of $FQ_n - F$ and $|C| = k \ge \frac{r+1}{2} + 1$, we have that $|V(F)| \ge \kappa_{\frac{r+1}{2}}(FQ_n)$. From Ineq. (5.4) and Eq. (5.5) we also have

$$\frac{1}{2}(r+1)n \ge |V(F)| \ge \frac{-r^2}{8} + \frac{(r+3)n}{2} - \frac{r}{2} + \frac{5}{8},$$

which implies that $n \leq \frac{r^2 + 4r - 5}{8}$, contradicting $n > \max\{6, \frac{r+5}{2}, \frac{r^2 + 4r - 5}{8}\} = g(r)$.

From Lemma 5.10 we know that the condition of Lemma 5.11 implies that of Lemma 5.7, and thus have the following main result of this section.

Theorem 5.12. If $r \geq 2$ and n > g(r), then $\kappa(FQ_n; K_{1,r}) = \kappa^s(FQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$

6 Conclusion

For n-dimensional hypercubes Q_n and folded hypercubes FQ_n , in this paper we have showed that for all integers $r \geq 2$ and n > f(r), $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$, and for all integers $r \geq 2$ and n > g(r), $\kappa(FQ_n; K_{1,r}) = \kappa^s(FQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$; see Theorems 3.13 and 5.12. In particular, both functions f(r) and g(r) have simple expressions: $f(r) = f(r+1) = \frac{r^2+4r+3}{8}$ and $g(r) = g(r+1) = \frac{r^2+4r-5}{8}$ for odd $r \geq 9$. But for $2 \leq r \leq 8$, f(r) and g(r) are piecewise functions with $5 \leq f(r) \leq \frac{31}{3}$ and $6 \leq g(r) \leq \frac{28}{3}$. Especially, for low dimensional hypercubes Q_n , we also obtain for all integers $4 \leq r \leq 6$, $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$ where $n \geq r$. For $2 \leq r \leq 3$, Lin et al. [6] has determined $\kappa(Q_n, K_{1,r})$ and $\kappa^s(Q_n, K_{1,r})$. Our results solved partly the open problem of determining $K_{1,r}$ -structure connectivity of Q_n and FQ_n for general r. Setting r = 2, 3 in Theorem 5.12, we obtain the results given by Sabir et al. in [11]. But for the cases that $7 \leq r \leq n \leq f(r)$ and $1 \leq r - 1 \leq n \leq g(r)(r \geq 2; n \geq 3)$, the open problem has not been solved yet.

From the above facts obtained already we can propose the following general conjectures:

Conjecture 6.1. For any integers $n \ge r \ge 2$ and $n \ge 3$, $\kappa(Q_n; K_{1,r}) = \kappa^s(Q_n; K_{1,r}) = \lceil \frac{n}{2} \rceil$.

Conjecture 6.2. For any integers $n+1 \ge r \ge 2$ and $n \ge 3$, $\kappa(FQ_n; K_{1,r}) = \kappa^s(FQ_n; K_{1,r}) = \lfloor \frac{n+1}{2} \rfloor$.

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