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Continued fractions, by Andrew M. Rockett and Peter Szüsz. World Scientific, Singapore and Teaneck, NJ, 1992, ix+188 pp., $48.00. ISBN 981-02-1047-1

A continued fraction is an expression $a_0 + 1/c_1$ where

(A) $$c_1 = a_1 + 1/c_2, \qquad c_2 = a_2 + 1/c_3, \quad c_3 = a_3 + 1/c_4, \quad \ldots,$$

with zero denominators ruled out of consideration. If the process stops at, say, $c_n = a_n + 1/c_{n+1}$, the continued fraction is finite, denoted by $[a_0; a_1, a_2, a_3, \ldots, a_n, c_{n+1}]$. Otherwise, we have an infinite continued fraction, denoted by $[a_0; a_1, a_2, a_3, \ldots,]$. The initial question is whether the process converges in the infinite case. If a_0 is an integer and the rest of the a's are positive integers, the process converges to a real number; and all real numbers can be represented in this way. Simple (or "regular", as some authors write) continued fractions are those with these requirements on $a_0, a_1, a_2, \ldots$.

If this appears to be just another representation of the real numbers, and a slightly awkward one at that, it should be recalled that the first proof of the existence of transcendental numbers, by Joseph Liouville, used simple continued fractions. In effect, Liouville established that if the sequence $a_0, a_1, a_2, \ldots$ increases sufficiently sharply (spelled out precisely by Liouville), then the continued fraction represents a transcendental number. Liouville also did this without continued fractions by simply using such decimals as $0.1100010\ldots$ with 1's in the positions $1, 2, 6, 24, 120, 720, \ldots$ after the decimal place, those numbers being the factorials $1!, 2!, 3!, 4!, \ldots$ in order. Such a decimal can be shown to be transcendental by applying a basic result in the theory of Diophantine approximations establishing that there are limits on the closeness of approximation by rational numbers of a real algebraic number of degree n.

T. J. Stieltjes employed the continued fraction (A) with $a_0 = 0$, $a_1 = k_1 z$, $a_2 = k_2$, $a_3 = k_3 z$, $a_4 = k_4$, $a_5 = k_5 z, \ldots$ to solve a moment problem, z being a complex variable. Stieltjes used these so-called S-fractions to determine a distribution of mass at an increasing sequence of points on the real line so as to have preassigned moments at the given points.

C. F. Gauss obtained a continued fraction expansion for the quotient of two contiguous hypergeometric functions.

There are two standard books that treat the analytic theory of continued fractions, where the a's are functions of a complex variable: *Analytic theory of continued fractions*, by H. S. Wall, Chelsea, 1973; and *continued fraction*, by William B. Jones and Wolfgang Thron, Addison-Wesley, 1980.

The book under review, by Andrew M. Rockett and Peter Szüsz, restricts attention to simple continued fractions—a topic with an extensive literature wherein most of the work on continued fractions is done. Although simple continued fractions are discussed in many books on number theory, Rockett and Szüsz go considerably beyond the topics treated in such books. In addition, some forty percent of the book is devoted to applications in number theory, the geometry of numbers, and Diophantine approximations. One example is the use of continued fractions by Christiaan Huygens in his design of a gear-driven model of the solar system. Another is a proof of George Szekeres's empty parallelogram theorem giving the best possible lower bound for the area of a parallelogram centered about the origin in the Cartesian plane, containing no lattice points other than the origin. This is an improvement on an inequality of Mordell. The best possible upper bound for the area is given by Minkowski's classical theorem in the geometry of numbers.

The authors discuss approximation problems of the form $b\|bt\| < k$, where $\|bt\|$ denotes the distance from bt to the nearest integer, with k in a number of cases from $k = 1$ to $k = f(b)$ for certain special functions f. A well-known example of this is the result of Hurwitz that for every irrational number t there are infinitely many positive integers b such that $b\|bt\| < 1/\sqrt{5}$. The classical formulation of this is that there are infinitely many rational numbers a/b such that $|t-a/b| < 1/(\sqrt{5}b^2)$. But the use of the notation $\|bt\|$ is preferable, since a is somewhat superfluous. The best possible constant $1/\sqrt{5}$ is needed only for certain special values of t, and if these values are removed from consideration, there are infinitely many positive integers b such that $b\|bt\| < 1/\sqrt{8}$, a best possible result again. The constant $1/\sqrt{8}$ is needed only for certain special values of t, and if these values are removed from consideration, we reach the inequality $b\|bt\| < 5\sqrt{221}$. This process continues, and we have the Markhoff chain $1/\sqrt{5}$, $1/\sqrt{8}$, $5/\sqrt{221}$, $13/\sqrt{1517}, \dots$, a sequence with limit $\frac{1}{3}$.

The inhomogeneous case is somewhat different. Let s be any real number. A. Ya. Khintchine proved that for any irrational number t, $\liminf_{b\to\infty} b\|bt+s\| \leq 1/\sqrt{5}$ where b denotes a positive integer. If, on the other hand, b is allowed to be a positive real number, then for any irrational number t there exists an $s = s(t)$ such that

$$\liminf_{b\to\infty} b\|bt+s\| > 1/\sqrt{32}.$$

The authors establish a result of Marshall Hall that any real number can be written as a sum of an integer and two simple continued fractions, each of which has partial quotients from the set $\{1, 2, 3, 4\}$. (The partial quotients are the values $a_1, a_2, a_3, \dots$ from (A).) This result is surprising, since the set of numbers in the unit interval whose continued fraction expansions have bounded partial quotients has measure zero. This is a well-known example of what is called "metrical theory", namely, investigations of results that hold for almost all real numbers.

In a letter to Laplace in 1812 Gauss described a "curious problem" that had occupied him for twelve years and that he had not resolved to his satisfaction. For any n in the unit interval, let $m_n(x)$ be the probability that t in the interval $[0, 1)$ satisfies $1/\zeta_{n+1}(t) < x$. Here $\zeta_{n+1}(t)$ denotes the $(n+1)$st complete quotient $[a_{n+1}; a_{n+2}, a_{n+3}, \dots]$ of the continued fraction expansion of t. Gauss knew that $[\log(1+x)]/\log 2$ was a good approximation for $m_n(x)$, but he stated that he could not get a satisfactory estimate for the difference. The authors give a proof that

$$m_n(x) = \frac{\log(1+x)}{\log 2} + 0(q^n)$$

for some constant q satisfying $0 < q < 0.76$, which is a sharpening of a result by R. O. Kuzmin. (This has been improved again by Peter Szüsz, who has obtained $q \leq 0.485$.)

There are helpful notes at the ends of the chapters and a very complete bibliography at the end of the book. However, this reviewer noted one omission which may have been deliberate, since the book by Olds is quite elementary (*Continued fractions*, by C. D. Olds, no. 9 in the New Mathematical Library of the MAA). Elementary or not, this book contains some unusual items, such as the continued fraction expansion of π to 23 terms. This expansion does not seem to have any regularity, as contrasted with e, where we have $e - 1 = [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots]$. Rockett and Szüsz prove this nontrivial result in Chapter 1, whereas Olds merely states it.

The book presumes no knowledge of continued fractions, but a familiarity with the basic notions of number theory is taken for granted along with a "rudimentary acquaintance with Lebesgue theory". The probability theory needed is developed in the book. Although the authors include many remarks on the sources of their material, they refer the reader to other sources, such as the well-known book of André Weil, for the historical context of the classical results in the subject. There are no problems or exercises in the book.

In this review we have not attempted to give a catalog of all the topics covered in this book. For example, the authors present fairly recent or recent work by John Brillhart, Thomas Cusick, Harold Davenport, Mary E. Flahive, W. J. LeVeque, Alexander Ostrowski, and Wolfgang M. Schmidt, among others. Whether the book will take its place alongside the famous classics by A. Ya. Khintchine and Oscar Perron remains to be seen. Nevertheless, it provides a very substantial summary of the principal results in the theory of simple continued fractions. The authors have attempted to give the best possible results now known, with proofs that are the simplest and most direct. The book is an outstanding addition to the literature of mathematics.

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Transformations, transmutations, and kernel functions, Vol. 1, by Heinrich Begehr and Robert Gilbert. Longman Publishers, Harlow, 1992, 399 pp., \$79.00. ISBN 3-582-02695-4

There has been an enormous amount of activity in the last 45 years or so on topics involving elliptic systems of differential operators, integral operators in the theory of elliptic partial differential equations (PDE), pseudoanalytic and hyperanalytic function theory, Riemann-Hilbert problems, etc. This was partly connected with the explosion of mathematical and applied mathematical activity after the Second World War but has its roots still further back in classical problems of elasticity and fluid dynamics, for example. We cite here for background a few books [1, 3–8, 14–18, 20, 22–31] that have been particularly important in chronicling and delimiting the directions of concern in the book under review. No attempt at completeness is intended here; many important