TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 336, Number 2, April 1993

UNLINKING VIA SIMULTANEOUS CROSSING CHANGES

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ABSTRACT. Given two distinct crossings of a knot or link projection, we consider the question: Under what conditions can we obtain the unlink by changing both crossings simultaneously? More generally, for which simultaneous twistings at the crossings is the genus reduced? Though several examples show that the answer must be complicated, they also suggest the correct necessary conditions on the twisting numbers.

Let L be an oriented link in S^3 with a generic projection onto the plane R^2. Let α be a short arc in R^2 transverse to both strands of L at a crossing, so that the strands pass through α in opposite directions. Then the inverse image of α contains a disk punctured twice, with opposite orientation, by L. Define a *crossing disk* D for a link L in S^3 to be a disk which intersects L in precisely two points, of opposite orientation. It is easy to see that any crossing disk arises in the manner described. Twisting the link q times as it passes through D is equivalent to doing $1/q$ surgery on ∂D in S^3 and adds $2q$ crossings to this projection of L. We say that this new link $L(q)$ is obtained by *adding q twists at D*. Call ∂D a *crossing circle* for L. A crossing disk D, and its boundary ∂D, are *essential* if ∂D bounds no disk disjoint from L. This is equivalent to the requirement that L cannot be isotoped off D in $S^3 - \partial D$.

Here we examine how a link can be turned into the unlink via simultaneous twists on disjoint crossing disks. In particular, we prove an analogue for pairs of crossing disks of the following theorem, a much more general version of which is proven in [ST, 1.4] (see also [Ga]):

0.1. **Theorem.** *If D is an essential crossing disk for the unlink L, then the link obtained by adding $q \neq 0$ twists to L at D is not the unlink.*

A moment's reflection suggests problems in finding an analogue for pairs of disks. Consider some examples:

0.2. **Example.** Suppose K_1 and K_2 are two essential crossing circles for any link L, and suppose that K_1 and K_2 are unlinked in S^3 and bound an annulus in $S^3 - L$. Then adding q twists at K_1 and $-q$ twists at K_2 leaves L unchanged. In particular, if L was the unlink before the twists were added, it will be the unlink afterwards.

Received by the editors January 21, 1991.
1991 *Mathematics Subject Classification*. Primary 57M25; Secondary 57M40.
The author was supported in part by a National Science Foundation grant.

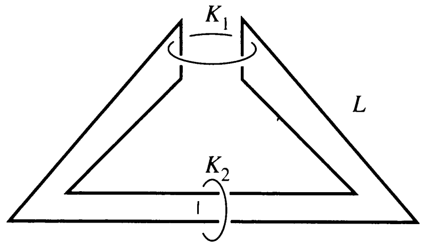


FIGURE 1

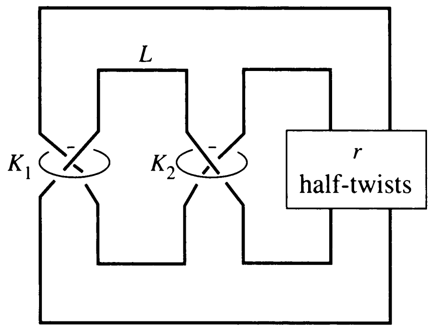


FIGURE 2

To see this, let D be a crossing disk for K_1. Using the annulus A we can find an imbedding $D \times I$ in S^3 so that $L \cap (D \times I) = (L \cap D) \times I$ and $D \times \partial I$ is the union of crossing disks at K_1 and K_2. Then the operation of adding q twists at one end and adding $-q$ twists at the other is clearly isotopic to the identity.

0.3. **Example.** Let K_1 and K_2 be crossing disks for the unknot, as shown in Figure 1. Then in general adding p twists at K_1 and q twists at K_2 gives a twist knot. But if *either* $p = 0$ or $q = 0$, then L is the unlink. Note here that K_1 (resp. K_2) is inessential if and only if $q = 0$ (resp. $p = 0$).

0.4. **Example.** Let K_1 and K_2 be crossing disks for the unknot L, as shown in Figure 2, with r odd. Then adding p twists at K_1 and q twists at K_2 gives the pretzel knot of type $(2p+1, 2q-1, r)$. According to [Bo; BZ, 12D], L is the unknot if and only if two of the three numbers $\{2p+1, 2q-1, r\}$ have opposite sign and absolute value 1. Thus, if $r \neq \pm 1$, L is the unknot if and only if $(p, q) = (-1, 1)$ or $(0, 0)$. If $r = 1$, then L is the unlink if and only if either $p = -1$ or $q = 0$. If $r = -1$, then L is the unlink if and only if either $p = 0$ or $q = 1$.

Example 0.4 suggests that the general solution could be quite complicated. In fact, 0.4 points to a general statement. We say that a finite set of disjoint crossing circles $\{K_i\}$ for L is an *essential set* of crossing circles if no K_i bounds a disk in $S^3 - (L \cup (\bigcup_j K_j))$. To deal with the problem raised by Example 0.2, define a pair ∂D_1, ∂D_2 of crossing circles for L to be *coannular* if there is an annulus A, disjoint from L, such that $\partial A \cap D_i = \partial D_i$. Given L and a pair K_1, K_2 of crossing circles for L, let $L(t_1, t_2)$ be the link obtained from L by adding t_1 twists at K_1 and t_2 twists at K_2.

0.5. **Theorem.** *Let K_1, K_2 be an essential pair of noncoannular crossing disks for a link $L \subset S^3$. Then there is a pair (s_1, s_2) of integers such that whenever $L(t_1, t_2)$ is the unlink, either $t_1 = s_1$ or $t_2 = s_2$.*

Clearly if L itself is the unlink then one of the s_i must be 0. In Example 0.3 when $r \neq \pm 1$, we can take for (s_1, s_2) either $(-1, 0)$ or $(0, 1)$. When $r = 1$, take $(s_1, s_2) = (-1, 0)$; when $r = -1$ take $(s_1, s_2) = (0, 1)$.

Theorem 0.5 is a consequence of a more general result, closer in spirit to [ST, 1.4]. Recall that a link in a 3-manifold is *split* if it is isotopic to the distant union of two sublinks. A link L has *splitting number* n if L is isotopic to the distant union of n nonsplittable sublinks. Suppose S is an orientable surface. Following Thurston [Th], define $\chi_-(S)$ to be $-\chi(C)$, where C is the union of nonsimply connected components of S. For L a link in S^3, define $\chi_-(L)$ to be the minimal value of $\chi_-(S)$ for all oriented incompressible surfaces S in S^3 with $\partial S = L$. Let K_1 and K_2 be boundaries of disjoint crossing circles for a link L in S^3. L bounds an orientable surface S disjoint from $K = K_1 \cup K_2$. Define $\chi_K(L)$ to be the minimal value of $\chi_-(S)$ for all oriented incompressible surfaces S in $S^3 - K$ with $\partial S = L$. Let $L(t_1, t_2)$ denote the link obtained from L by adding t_1 twists at K_1 and t_2 twists at K_2. A pair (t_1, t_2) of integers is *norm-reducing* for L at K if either the splitting number of $L(t_1, t_2)$ in S^3 is greater than that of L in $S^3 - (K_1 \cup K_2)$ or $\chi_-(L(t_1, t_2)) < \chi_K(L)$. We then have:

2.3. **Theorem.** *Let $K = K_1 \cup K_2$ be an essential pair of noncoannular crossing disks for a link $L \subset S^3$. Then there is a pair (s_1, s_2) of integers so that for every norm-reducing pair (t_1, t_2) for L at K, either $t_1 = s_1$ or $t_2 = s_2$.*

It is easy to see that Theorem 0.5 follows from Theorem 2.3. The only single-component link of trivial norm is the unknot. So if a link L in $S^3 - K$ splits in $S^3 - K$ into single component knots, each of trivial norm, then L bounds a collection of disks in $S^3 - K$ and K is not an essential pair for L. Thus either L does not completely split in $S^3 - K$ or one of its components is knotted and it has nontrivial norm. Hence, if $L(t_1, t_2)$ is the unlink, (t_1, t_2) is norm-reducing.

The outline is as follows: In §1 we treat what is apparently a very special case, that of links which lie on the boundary of a genus two handlebody. In §2 sutured manifold theory and [Sc2] are applied to the proof of 2.3, hence 0.5. It is shown that in fact §1 treated the critical case. There is also a Property P type theorem about surgery on strongly invertible pairs of knots. Section 3 contains a generalization to Dehn fillings on pairs of tori in arbitrary orientable compact 3-manifolds.

Much of §3 was prompted by a very helpful conversation with Abby Thompson.

1. A special case—links on a handlebody

Let H be a genus two handlebody in S^3 with specified meridian disks μ_1 and μ_2. Let $\Gamma \subset \partial H$ be an oriented collection of disjoint simple closed curves such that each curve of Γ intersects at least one meridian and each meridian is either disjoint from Γ or intersects Γ twice, with opposite orientation. Let $\Gamma(t_1, t_2)$ denote the curves obtained from Γ by twisting t_1 times around

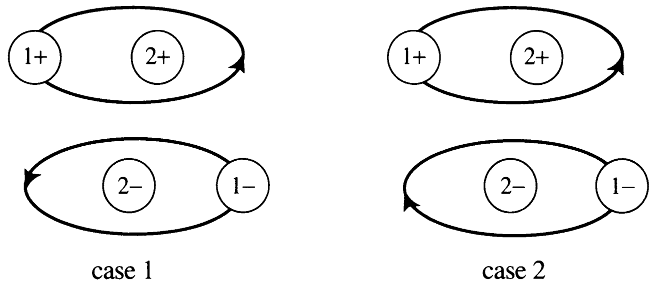


FIGURE 3

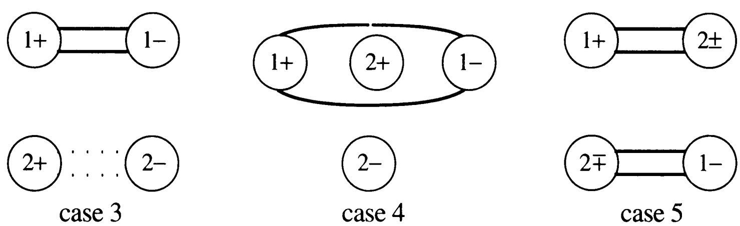


FIGURE 4

meridian μ_1 and t_2 times around meridian μ_2. Let X denote the closure of $S^3 - H$. Let Φ be the set of pairs (t_1, t_2) such that some component of $\Gamma(t_1, t_2)$ bounds a disk in X.

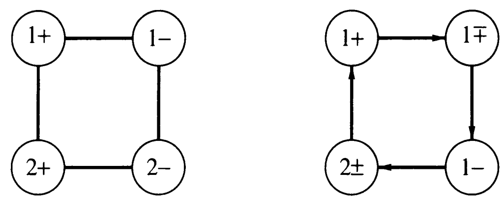
1.1. **Lemma.** *One of the following holds*:
(a) *There is a pair* (s_1, s_2) *in* Φ *such that* $\Phi = \{(t_1, t_2) | t_1 = s_1 \text{ or } t_2 = s_2\}$.
(b) *There is an* s_1 *such that* $\Phi = \{(t_1, t_2) | t_1 = s_1\}$.
(c) *There is an* s_2 *such that* $\Phi = \{(t_1, t_2) | t_2 = s_2\}$.
(d) Φ *has exactly two elements* (t_1, t_2), (t_1', t_2') *with* $|t_i - t_i'| = 1$, $i = 1, 2$.
(e) Φ *has at most one element.*

Proof. Let Q denote the 4-punctured sphere $\partial H - \{\mu_i\}$. We may assume that the 1-manifold $\gamma = \Gamma \cap Q$ consists of oriented essential arcs in Q. The four components of ∂Q may be labelled $\mu_i^{\pm}$, $i = 1, 2$, in the obvious manner. We may assume that $\Gamma \cap \mu_1$ contains two points. If there is a single arc of γ with ends at $\mu_1^{\pm}$, say, then, since any arc of γ is essential in Q, γ must be one of the configurations in Figure 3.

If the two arcs of γ having ends on μ_1^+ have their other end on the same component of ∂Q, then γ must be one of the three configurations of Figure 4.

If the two arcs of γ having ends on μ_1^+ have their other end on different components of ∂Q then γ must be one of the two configurations in Figure 5.

In cases 3 and 4 in Figure 4, there is a component k_1 of Γ which intersects μ_1 once and is disjoint from μ_2. Similarly, either Γ is disjoint from μ_2 or there is a component k_2 intersecting μ_2 once but not μ_1. At most one choice of t_1 allows k_1 to bound a disk in $S^3 - H$ since two different choices have nontrivial intersection number in ∂H. Similarly, at most one choice of t_2 allows k_2 to bound a disk in $S^3 - H$. If some such t_1 and t_2 exist we have case (a). If a choice of t_1 but not t_2 exists (or vice versa) we have case (b) (or (c)) of 1.1. If no choice of either exists, we have case (e).



case 6 case 7

FIGURE 5

Almost the same argument finishes case 6 of Figure 5; again we have the components k_1 and k_2, but also a component k_3 intersecting both meridia exactly once. Suppose there is a choice of t_1 after which k_1 bounds a disk in $S^3 - H$. Then just as before, k_3 cannot bound a disk after any twisting along μ_1, since its algebraic intersection with the original k_1 would be nontrivial. Similarly for t_2 and k_2. So the situation is the same as above. On the other hand, suppose there is no choice of t_1 or t_2 allowing k_1 or k_2 to bound disks. Then we may as well discard them and consider only k_3. This is equivalent to case 5 of Figure 4, treated below.

To treat the diagrams in cases 1, 2, 5, and 7, a bit more is needed. Suppose a component k of $\Gamma = \Gamma(0, 0)$ and a component k' of $\Gamma(t_1, t_2)$ bound disks D and D' respectively. Minimize $\partial D \cap \partial D'$ by isotoping across any bigons of $k \cup k'$ bounding disks in ∂H. A standard innermost disk argument allows us to assume that D and D' intersect in arcs. An outermost arc of $D' \cap D$ in D' cuts off a subdisk F along which D may be compressed to give two disks E and E', both disjoint from D. Both E and E' are compressing disks for ∂H in $S^3 - H$ which are disjoint from D. In sum, we have the following algorithm: Examine successive intersection points of k' with k. Let α' be the subarc of k' lying between them, so $\alpha' \cap k = \varnothing$. Let α be either arc of k lying between the intersection points. Then $\alpha \cup \alpha'$ is a simple closed curve in H which is disjoint from k. At least two curves constructed in this manner bound disks in X.

Apply this algorithm to the remaining diagrams: Applied to diagrams 1 or 2 it gives a simple closed curve in Q which is parallel to μ_2 and compresses in $S^3 - H$. Applied to diagram 5, the algorithm gives a simple closed curve in Q which separates μ_1^+ from μ_1^- and which compresses in X. Neither is possible. hence we have case (e).

When the algorithm is applied to diagram 7 we are only able to conclude that there is a compressing disk E in X so that E is disjoint from D and passes at most once over each meridian.

Case 1: *∂E intersects both meridians.* In diagram 7, ∂E appears as two arcs, parallel either to the vertical or horizontal arcs of ∂D in the diagram, say the former. Then a circle in S^2 separating the right two disks from the left two is disjoint from ∂E and intersects $k = \partial D$ in two points. Denote by μ_3 the disk this circle bounds in H. Then μ_3 is a meridian of the solid torus K obtained from H by attaching a 2-handle with core E.

Subcase (i): *K is unknotted*. This is essentially Example 0.4 above. Both k and k' are pretzel knots, with an odd number of half-twists in each band. H

can be viewed as a regular neighborhood of the natural Seifert surfaces of these pretzel knots. Such a Seifert surface consists of two disks connected by three bands, each dual to one of the μ_i and each having an odd number of half-twists. Let h_i [resp. h'_i] denote the number of half-twists in the band dual to μ_i, $i = 1, 2, 3$. Then k is unknotted if and only if $h_i = -h_j = \pm 1$ for some $i \neq j$ [Bo; BZ, 12D]. If $\pm 1 = -h_3 = h_1$ or h_2, then there is a compressing disk in X, disjoint from k, intersecting μ_1 or μ_2 exactly once. This case is essentially Case (2) below. The case in which $\pm 1 = -h_3 = h'_1$ or h'_2 is similar, with the roles of k and k' switched. Finally, if $h_1 = -h_2 = \pm 1$ and $-h'_1 = h'_2 = \pm 1$, then h_1 and h'_1, and h_2 and h'_2 each differ from each other by a single full twist. This is case (d) above.

Subcase (ii): *K is knotted.* Then all compressing disks for knots in K may be taken to lie in a neighborhood of K. Thus, at the risk of introducing possibly new elements of Φ, we may replace K by the unknot, hence k with a pretzel knot as above, with h_3 chosen to be large. This forces, at best, case (d). But we observe that in case (d) above the compressions of k and k' do indeed take place in a neighborhood of K, so we have introduced no new elements to Φ.

Case (2): *∂E intersects only one meridian.* Say ∂E is disjoint from μ_1, and intersects μ_2 once. We have $\Gamma(0, 0) = k(0, 0) = k$ and will show that either (c) or (a) holds with $s_2 = 0$. If E were disjoint from μ_2 and intersected μ_1, then we would have (b) or (a) with $s_1 = 0$.

First we show that any $(\cdot, 0)$ lies in Φ. Let K denote the solid torus with meridian μ_1 obtained from H by attaching a 2-handle along E. Then $k \subset \partial K$ bounds a disk in $S^3 - K$ and is null-homologous in K, so it is trivial in ∂K. Any $k(\cdot, 0)$ is disjoint from ∂E so it lies in ∂K, and differs from k by some twists around the meridian of K. Hence any $k(\cdot, 0)$ also bounds a disk in ∂K, so it bounds a disk in X.

On the other hand, let (t_1, t_2) be any nontrivial element of Φ, so $k' = k(t_1, t_2)$ bounds a disk D' in X. We wish to determine (t_1, t_2). The band sum of E to itself along μ_1 gives a separating disk F for X with ∂F disjoint from μ_1 and μ_2. F splits X into the boundary connect sum of two knot complements, one the unknot with longitude ∂E and the other $S^3 - K$. The curve ∂F intersects k' in four points, so an outermost arc of $F \cap D'$ in D' cuts off a disk from D' which is a longitude of one of the two knots, either ∂E or a longitude of K. In the former case, $t_2 = 0$. In the latter, K is the unknot and t_1 must be the specific slope s_1 for which some arc cut off from k' by ∂F is a longitude of K. Thus if K is knotted we have (c), and if K is unknotted we have (a). □

2. Sutures in genus two surfaces

2.1. **Definition.** Suppose M is a compact orientable 3-manifold M and $P \subset \partial M$ is a possibly disconnected closed surface. We say M is a *J-cobordism* on P if $H_2(M, \partial M - P) = 0$.

2.2. **Lemma.** *If M is a J-cobordism on P, then* $\operatorname{genus}(\partial M - P) \leq \operatorname{genus}(P)$.

Proof. Let $Q = \partial M - P$. Consider the following commutative diagram induced by inclusion:

$$\begin{tikzcd}
 & & 0 & \\
 & H_1(\partial M, Q) \arrow[r, "\cong"] & H_1(\partial M, Q) \arrow[u] \arrow[r] & 0 \\
H_2(M) \arrow[r, "j"] & H_2(M, \partial M) \arrow[u] \arrow[r, "\partial"] & H_1(\partial M) \arrow[u] \arrow[r] & H_1(M) \arrow[u] \\
H_2(M) \arrow[u, "\wr\|"'] \arrow[r] & H_2(M, Q) \arrow[u] \arrow[r] & H_1(Q) \arrow[u] \arrow[r] & H_1(M) \arrow[u, "\wr\|"'] \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

Using the fact that $H_2(M, Q) = 0$ and $\partial M - Q = P$ this simplifies to:

$$\begin{tikzcd}
 & & 0 & \\
 & H_1(P) \arrow[r, "\cong"] & H_1(P) \arrow[u] & \\
0 \arrow[r] & H_2(M, \partial M) \arrow[u] \arrow[r, "\partial"] & H_1(\partial M) \arrow[u] \arrow[r] & H_1(M) \\
 & 0 \arrow[u] \arrow[r] & H_1(Q) \arrow[u] \arrow[r] & H_1(M) \arrow[u, "\wr\|"'] \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

Let $p = \operatorname{rank} H_1(P)$, $q = \operatorname{rank} H_1(Q)$, and $s = \operatorname{rank} H_2(M, \partial M)$. A standard argument from Poincaré duality shows $2s = \operatorname{rank} H_1(\partial M) = p + q$. From the left-most vertical arrows we get $s \leq p$. Hence $q \leq p$. □

2.3. **Theorem.** *Let $K = K_1 \cup K_2$ be an essential pair of noncoannular crossing disks for a link $L \subset S^3$. Then there is a pair (s_1, s_2) of integers so that for every norm-reducing pair (t_1, t_2) for L at K, either $t_1 = s_1$ or $t_2 = s_2$.*

Proof. An innermost circle argument shows that any essential sphere in $S^3 - (K \cup L)$ can be isotoped off of a pair $D = D_1 \cup D_2$ of crossing disks bounded by K. If any essential sphere separates K_1 from K_2, then the theorem naturally follows from separate applications of [ST, 1.4] to K_1 and K_2. So suppose no essential sphere separates K_1 from K_2. Then any essential sphere in $S^3 - (L \cup D)$ splits off a sublink distant from D which is unaffected by twists at D, and so may be ignored. So we may henceforth assume that $S^3 - (K \cup L)$ is irreducible.

Let $\eta(K)$ and $\eta(L)$ denote tubular neighborhoods of K and L respectively, and let M be the irreducible manifold $S^3 - \eta(K \cup L)$. Let $T_i \subset \partial M$ be the torus $\partial\eta(K_i)$, and let λ_i and μ_i denote the longitude and meridian of $\partial\eta(K_i)$ respectively. Let $S \subset M$ be an incompressible surface so that $\partial S = L$ and $\chi_-(S)$ is as low as possible, so $\chi_-(S) = \chi_K(L)$.

Regard $(M, \partial M)$ as a sutured manifold (cf. [Sc1]). Construct a taut sutured manifold hierarchy

$$(M, \eta(L)) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_i} (M_i, \gamma_i) \xrightarrow{S_{i+1}} \cdots$$

so that $S_1 = S$, and each S_i is disjoint from $T = T_1 \cup T_2$. By [Sc$_1$, 2.6 and 4.17] such a sequence can be extended as long as $H_2(M_i, \partial M_i - T) \neq 0$, but eventually terminates in a sutured manifold (M_n, γ_n) with $H_2(M_n, \partial M_n - T) = 0$, i.e., a J-cobordism on T. Then by 2.2, genus$(\partial M_n - T) \leq 2$.

The crossing disks become a pair of 2-punctured disks in M. Regard $D \cap M$ as a parametrizing surface for the hierarchy [Sc1, §7]. Consider the effect of the decompositions in the hierarchy on a component of $D \cap M$. Take, for example, $Q = D_1 \cap M$. Q is a pair of pants, with two boundary components meridia of L and the other λ_1. Since $\chi(Q) = -1$, Q has index 2, and so its remnant Q_n in M_n is a planar surface of index no more than 2.

2.4. **Lemma.** *Some component of Q_n is an annulus A with one boundary component λ_1 and the other a simple closed curve on $\partial M_n - T$ which intersects the set of sutures at most twice.*

Proof. First we show that some component is an annulus from λ_1 to a curve in $\partial M_n - T$. This part is by induction in the sutured manifold hierarchy. We begin by observing that the initial Seifert surface S may be taken to intersect Q in a single arc connecting the two punctures (see [ST]), so Q_1 is an annulus from λ_1 to a circle crossing the suture $\partial\eta(L) - \partial S$ exactly twice. Suppose inductively that Q_i has an annular component A with one end λ_1 and the other end lying on $\partial M_i - T$. Examine the 1-manifold $\Gamma = S_{i+1} \cap A$. Since S_{i+1} is disjoint from T, no arc of Γ has an end on λ_1. Since S_{i+1} is incompressible, we may assume that any closed component is essential in A. If there is a closed component, then the one nearest to λ_1 in A cuts off the required annulus from A. If there are no closed components, then all components are arcs with neither end on λ_1, so the component of $A - S_{i+1}$ containing λ_1 remains an annulus.

It remains to check that the final annulus $A \subset Q_n$ intersects sutures in $\partial M_n - T$ no more than twice. Since Q_n is a parametrizing surface, no component of Q_n can have negative index, and each component must have even index, since each boundary component crosses the sutures an even number of times. The total index is no more than index $Q_1 = 2$. Hence Q_n can have no more than one component of positive index, and it must have index 2. In particular, A can have index at most 2, so ∂A intersects the set of sutures in $\partial M_n - T$ at most twice. □

To continue the proof of Theorem 2.3, assume that (M_n, γ_n) is taut, and that $H_2(M_n, \partial M_n - T) = 0$. Let $(\widetilde{M}, \gamma)$ denote the sutured manifold obtained from (M_n, γ_n) by filling back in $\eta(K_i)$. More generally, let $(\widetilde{M}, \gamma)(t_1, t_2)$ denote the sutured manifold obtained from (M_n, γ_n) by filling in a solid torus along $\partial\eta(K_i)$ with slope $1/t_i$, that is, by attaching solid tori so that the meridian of each torus is homologous to $t_i\lambda_i + \mu_i$.

From 2.4 (applied also to $Q = D_2 \cap M$), we know there are annuli A_i in M_n, each with one end on λ_i and with the other a curve c_i in $\partial M_n - T$ intersecting γ_n at most twice. We can assume that $|c_i \cap \gamma_n|$ cannot be reduced by an isotopy of c_i. Each c_i is essential in ∂M_n since T is incompressible, and the c_i are not parallel in ∂M_n since K_1 and K_2 are not coannular in $M_n \subset M$. We know that genus$(\partial M_n - T) \leq 2$; since $\partial M_n - T$ contains the two nonparallel inessential circles c_i we have that genus$(\partial M_n - T) = 2$. That is, the component(s) of $\partial M_n - T$ containing the c_i consist either of a single genus 2 surface, or two tori.

Since the meridian of the filling torus at $\eta(K_i)$ intersects λ_i once, there is a natural homeomorphism of $\widetilde{M}$ to $M_n - \eta(A_1 \cup A_2)$ under which c_i corresponds to a curve parallel to λ_i in M_n. In fact, then, the underlying manifolds of $(\widetilde{M}, \gamma)$ and $(\widetilde{M}, \gamma)(t_1, t_2)$ are the same; the only difference is that in $(\widetilde{M}, \gamma)(t_1, t_2)$ the sutures γ have been altered by Dehn twists on the curves c_i. Hence we will write $(\widetilde{M}, \gamma(t_1, t_2))$ instead of $(\widetilde{M}, \gamma)(t_1, t_2)$.

2.5. **Lemma.** *$\widetilde{M}$ is irreducible. If $(\widetilde{M}, \gamma(t_1, t_2))$ is not taut, then some suture in $\gamma(t_1, t_2)$ bounds a disk in $\widetilde{M}$.*

Proof. Without loss of generality, take $(t_1, t_2) = (0, 0)$ so $(\widetilde{M}, \gamma(t_1, t_2)) = (M, \gamma)$. Any sphere in $\widetilde{M}$ can be pushed along the annuli A_i and across the filling tori so that it lies in M_n. Since (M_n, γ_n) is taut, M_n is irreducible, so the sphere bounds a ball in $M_n \subset \widetilde{M}$. Hence $\widetilde{M}$ is irreducible.

$\chi_-(\partial \widetilde{M}) = 0$ or 2. Let $R_\pm$ denote the two submanifolds into which γ divides $\partial \widetilde{M}$. Then $\chi_-(R_+) = \chi_-(R_-) = \frac{1}{2}\chi_-(\partial \widetilde{M}) = 0$ or 1. Hence each of $R_\pm$ consists of a union of annuli and one other component, either a punctured torus or a 3-punctured sphere. There are two possible ways in which $(\widetilde{M}, \gamma)$ could fail to be taut:

Suppose γ bounds a surface R' in $\widetilde{M}$ with $\chi_-(R') < \chi_-(R_\pm)$. Then $\chi_-(R') = 0$ and $\chi_-(R_\pm) = 1$. Since $\chi_-(R_\pm) = 1$, γ has an odd number of components. The only way a surface with an odd number of boundary components can have trivial χ_- is if one component is a disk.

Suppose some component of $R_\pm$ is compressible. If the component is not an annulus, then the previous case applies. If it is an annulus, then after compression it becomes disks. □

Remark. Lemma 2.5 requires genus$(\partial \widetilde{M}) \leq 2$ and thereby limits the methods here to twists on at most two crossing circles.

2.6. **Lemma.** *There is a pair (s_1, s_2) of integers so that if $(\widetilde{M}, \gamma(t_1, t_2))$ is not taut, then either $t_1 = s_1$ or $t_2 = s_2$.*

Proof. There are two cases to consider:

Case (1): *$\partial \widetilde{M}$ contains two torus components, U_1 and U_2, with $c_i \subset U_i$.* If γ does not intersect c_i, then it is unaffected by Dehn twists at c_i. No component of $\gamma \cap U_i$ disjoint from c_i can bound a disk in $\widetilde{M}$, since (M_n, γ_n) is taut. All sutures in $\gamma \cap U_i$ which intersect c_i are parallel. If γ_0 is such a suture, and γ_1 is the image of γ_0 by a Dehn twist along c_i, then $\gamma_0 \cdot \gamma_1 \neq 0$, so at most one of them can bound a disk. Hence there is at most one twisting t_1 of c_1 and t_2 of c_2 after which $(\widetilde{M}, \gamma(t_1, t_2))$ is not taut.

Before considering the other case, consider this. Since we only care if a suture in γ bounds a disk in $\widetilde{M}$, we may reimbed $\widetilde{M}$ in S^3 with no effect on the problem. Excision of $\eta(A_1 \cup A_2)$ shows the first homomorphism in the composition $H_2(M_n, \partial M_n - T) \to h_2(\widetilde{M}, \partial \widetilde{M} - (c_1 \cup c_2)) \to H_2(\widetilde{M}, \partial \widetilde{M})$ is surjective. Since $H_2(M_n, \partial M_n - T) = 0$, the second homomorphism must be trivial. Then for any component W of $\widetilde{M}$, there is an imbedding of W in S^3 so that $S^3 - W$ is a union of handlebodies H, and $C = c_1 \cup c_2$ contains a complete system of meridia for H [Sc2].

Case (2): *There is a genus two component U of $\partial\widetilde{M}_n$ containing C.* Let W be the component of $\widetilde{M}_n$ such that $U \subset \partial W$. Any other component of ∂W must be a sphere, since $\text{genus}(\partial\widetilde{M}) = 2$. Since $\widetilde{M}$ is irreducible, we conclude that $\partial W = U$.

We have seen that we may take $S^3 - W$ to be a genus two handlebody H, on which C is a pair of meridia. The sutures γ intersect each c_i at most twice. No suture disjoint from the c_i compresses in W, for if it did, then it would compress in M_n, and (M_n, γ_n) is taut. The proof in this case now follows formally from 1.1. In cases (a) and (e) the choice of s_1 and s_2 is obvious. In case (b) (or (c)), the choice of s_1 (or s_2) is given, and the other is arbitrary. In case (d), take $s_1 = t_1$ and $s_2 = t'_2$. Lemma 2.6 is proved. □

The proof of Theorem 2.3 is now immediate from Lemma 2.6 and [Sc1, 3.9]. □

We give a modest "property P" type of application:

2.7. **Definition.** Let $L = L_1 \cup L_2 \subset S^3$ be a link of two components. L is *strongly invertible* if there is an involution of S^3 taking each L_i to itself, but reversing its orientation.

2.8. **Proposition.** *Suppose $L = L_1 \cup L_2$ is strongly invertible, L_1 and L_2 are not coannular in S^3, and neither component bounds a disk in the complement of the other. Let $M(p_1, p_2)$ be the manifold obtained from S^3 by surgery on L with slope $1/2p_i$ on each L_i. Then for one of the links L_i and some integer $q \neq 0$, if $M(p_1, p_2)$ is simply connected then either $p_1 = p_2 = 0$ or $p_i = q$.*

Proof. The proof is modelled on [BS, 1.7]. Since L is strongly invertible, the involution on S^3 must be rotation about an unknotted axis A which pierces each L_i twice. The axis A projects to a branch set B, which is the unknot in S^3, and each L_i projects to an arc α_i with ends on B.

The involution on S^3 induces an involution on $M(p_1, p_2)$. If $M(p_1, p_2)$ is simply connected, then the branch set B' for this action must be the unknot in S^3 [B]. There is a crossing circle K_i for B on the boundary of a regular neighborhood of α_i that lifts to a longitude of L_i, and B' can be viewed as having been obtained from B by adding p_i twists at K_i, $i = 1, 2$ (cf. [Mo, Li]). If $K = K_1 \cup K_2$ is not an essential pair, then one of the L_i bounds a disk in the complement of the other. If K_1 and K_2 were coannular in $S^3 - B$, then the lift of the annulus would make L_1 and L_2 coannular in S^3. Now apply Theorem 0.5. There is a pair of integers (q_1, q_2) such that B' is unknotted if and only if $p_1 = q_1$ or $p_2 = q_2$. Since B is unknotted, one of the q_i, say q_2, is trivial. Thus if B' is unknotted either $p_1 = q_1$ or $p_2 = 0$. But if $p_2 = 0$, then $p_1 = 0$ [BS]. □

Remark. Note that for $(1/2p_1, 1/2p_2)$ surgery to give a homology 3-sphere it is necessary that L_1 and L_2 have trivial algebraic linking. The above proposition can be generalized to surgeries with slopes of the form $(2r + 1/2p_1, 2s + 1/2p_2)$ by making a different choice of crossing circles K_i near the arcs α_i. This might be useful in that cases where L_1 and L_2 have nontrivial algebraic linking number.

3. A GENERALIZATION

3.1. **Definition.** Suppose M is a 3-manifold and T is a torus component of ∂M. Let σ be an essential simple closed curve in T. Then there is a homeomorphism $\varphi: \partial D^2 \times S^1 \to T$, well defined up to isotopy, such that $\varphi(\partial D^2) = \sigma$. The manifold $M(\sigma)$ obtained by attaching $D^2 \times S^1$ to M via φ is a *filling of* M *at* T *with slope* σ. $D^2 \times S^1 \subset M(\sigma)$ is called the *filling torus* and $\{0\} \times S^1$ is called the *core* of the filling torus. If σ and τ are two essential simple closed curves in T, isotoped to minimize $|\sigma \cap \tau|$, then $|\sigma \cap \tau|$ is denoted $\sigma \cdot \tau$ and is called the *difference in slope* between σ and τ.

Recall the following theorem, a reformulation of [Ga, 1.8].

3.2. **Theorem.** *Let (M, γ) be a connected taut sutured manifold with $\gamma \neq \varnothing$, and $T \subset \partial M$ a torus such that $\gamma \cap T = \varnothing$. Suppose the only J-cobordism on T contained in M is $T \times I$. Then there is at most one slope σ for which $(M(\sigma), \gamma)$ is not taut.*

Subsequent discoveries now allow a variant of this theorem, with the hypothesis on J-cobordisms removed, and the conclusion only slightly weakened:

3.3. **Theorem.** *Let (M, γ) be a connected taut sutured manifold with $\gamma \neq \varnothing$, and $T \subset \partial M$ a torus such that $\gamma \cap T = \varnothing$. Then there are at most three slopes σ for which $(M(\sigma), \gamma)$ is not taut. Indeed, if σ and τ are such exceptional slopes, then $\sigma \cdot \tau \leq 1$. At most one of the exceptional $M(\sigma)$ is irreducible.*

Proof. Special case: *M is a J-cobordism on T such that $\partial M - T$ is a torus* T'. Since (M, γ) is taut, all the sutures γ are essential in T', thus are parallel essential curves. Now $(M(\sigma), \gamma)$ can fail to be taut for two reasons: either $M(\sigma)$ is reducible, or the annuli $T' - \gamma$ compress in $M(\sigma)$. The latter can happen for at most one slope σ.

If $T' - \gamma$ compresses in $M(\sigma)$ for slope σ, and $M(\sigma')$ is reducible for slope σ', then it follows from [Sc3, 6.1] that the core of the filling torus in $M(\sigma)$ is a cabled knot k in $M(\sigma)$ and σ' is the slope of a cabling annulus A'. In particular, $\sigma \cdot \sigma' = 1$. Suppose there were another slope σ'' for which $M(\sigma'')$ is reducible. Then the slope σ'' is that of another cabling annulus A'' for k. A simple combinatorial argument on the intersection of A' and A'' shows this contradicts, for example, the irreducibility of M. Hence only the fillings σ and σ' can possibly produce nontaut sutured manifolds.

Suppose now that $T' - \gamma$ does not compress in any $M(\sigma)$, so the only way in which $(M(\sigma), \gamma)$ can fail to be taut is if $M(\sigma)$ is reducible. Gordon and Luecke [GL] have shown that any two fillings of T which produce reducible 3-manifolds differ in slope by at most one.

General case: The proof in general follows from the special case just as in [Ga, 1.8]: If M itself is not a J-cobordism on T, then one can construct a taut sutured manifold hierarchy of (M, γ), using always surfaces disjoint from T, until we reach a sutured manifold of the form (M_n, γ_n) with M_n a J-cobordism on T. Then, by 2.2, $\partial M_n - T$ has genus at most one. Since (M, γ) is taut, M is irreducible, so any sphere boundary component of M_n must bound a ball. Since M is connected, the component W of M_n containing T must have other boundary components, hence its boundary is the union of T and another torus T' containing sutures. As in [Ga, 1.8], $(M(\sigma), \gamma)$ is taut if the sutured manifold obtained from W by filling with slope σ at T is taut. The proof now follows from the special case applied to W. □

Below we shall present a theorem for simultaneous fillings on the union of two tori $T = T_1 \cup T_2$ which combines features of both 3.2 and 3.3. We begin with some remarks about J-cobordisms on tori.

3.4. **Definition.** Suppose W is a J-cobordisms on a torus T such that $\partial W - T$ is a torus T'. Then W is called a J-cobordism *between* T and T'.

3.5. **Lemma.** *Suppose W is a J-cobordism on a torus T such that $\partial W - T$ is a torus T'. Then W is also a J-cobordism on T'.*

There is an intersection-pairing preserving isomorphism

$$\theta\colon H_1(T, Q) \to H_1(T', Q)$$

defined by the requirement that $\theta(\alpha)$ is homologous to α in $H_1(W, Q)$.

Proof. This is an elementary consequence of Poincaré duality. See, for example, [Ga, 1.5]. □

3.6. **Definition.** A J-cobordism as in 3.5 is called a J-cobordism between T and T'.

3.7. **Lemma.** *Let W be a J-cobordism between tori T and T'. Suppose μ and λ are simple closed curves in T' so that $\mu \cdot \lambda = 1$. Suppose σ and τ are fillings at T so that μ compresses in $W(\sigma)$ and λ compresses in $W(\tau)$. Then $\theta(\mu) = \pm\sigma$, $\theta(\lambda) = \pm\tau$, and $\sigma \cdot \tau = 1$.*

Proof. Let D_μ be a compressing disk for μ in $W(\sigma)$. Then $D_\mu \cap W$ is a homology between μ and $\omega\sigma$, where ω is an integer. Hence $\theta(\mu) = \omega\sigma$. Similarly, $\theta(\lambda) = \omega'\tau$ for some integer ω'. Then $\mu\cdot\lambda = \theta(\mu)\cdot\theta(\lambda) = |\omega\omega'|\sigma\cdot\tau$. Since $\mu \cdot \lambda = 1$, $|\omega| = |\omega'| = \sigma \cdot \tau = 1$. □

As in 3.2, a simplifying assumption about J-cobordisms on T in M will be needed: A J-cobordism W on $T = T_1 \cup T_2$ is *split* if it is either the disjoint union or the boundary connected sum of J-cobordisms W_1 on T_1 and W_2 on T_2.

3.8. **Definition.** Let (M, γ) be a taut connected sutured manifold and $T = T_1 \cup T_2 \subset \partial M$ the union of tori T_1 and T_2 such that $\gamma \cap T = \varnothing$. Let $(M(\sigma_1, \sigma_2), \gamma)$ denote the sutured manifold obtained from (M, γ) by filling T_i with slope σ_i. Then we say the pair (σ_1, σ_2) is *norm-reducing* for M if $(M(\sigma_1, \sigma_2), \gamma)$ is not taut.

3.9. **Theorem.** *Let (M, γ) be a connected taut sutured manifold with $\gamma \neq \varnothing$, and $T = T_1 \cup T_2 \subset \partial M$ the union of tori T_1 and T_2 such that $\gamma \cap T = \varnothing$. Suppose any J-cobordism on T contained in M is split. Then there are slopes τ_1 and τ_2 such that for any norm-reducing slope pair (σ_1, σ_2) for M, either $\sigma_1 \cdot \tau_1 \leq 1$ or $\sigma_2 \cdot \tau_2 \leq 1$.*

If any $M(\sigma_1, \sigma_2)$ is reducible, so is $M(\tau_1, \tau_2)$.

Note that if no $M(\sigma_1, \sigma_2)$ is reducible, the theorem does not claim that (τ_1, τ_2) is necessarily norm-reducing for M.

Proof. Special case: M itself is a split J-cobordism on T. If M is the disjoint union of J-cobordisms on T_1 and T_2, just apply 3.3.

Suppose M is the boundary connected sum of J-cobordisms W_1 on T_1 and W_2 on T_2 along a disk E. Since M is irreducible, no boundary component is a sphere. Then $\partial M - T$ is a genus two surface F. Let F_1 and F_2 denote the punctured tori into which ∂E divides F. Then $\partial M = T_1 \cup T_2 \cup F$, and $F = F_1 \cup_{\partial E} F_2$.

Consider the J-cobordism W_1 on T_1. $\gamma \cap F_1$ is a set of disjoint essential arcs and circles in F_1. Let δ_1 be the set of simple closed curves in the torus $F_1 \cup E$ obtained by connecting the ends of each essential arc of $\gamma \cap F_1$ by an arc across E. Two curves in δ_1 intersect in at most one point.

Similarly define a set δ_2 of simple closed curves in the torus $F_2 \cup E \subset \partial W_2$.

Rule 1. If there is a filling σ of T_i which makes W_i reducible take $\tau_i = \sigma$.

Rule 2. If W_i remains irreducible after any filling of T_i, but there is a slope σ of T_i such that some curve in δ_i compresses in $W_i(\sigma)$, take τ_i to be one such slope σ.

Rule 3. If none of the δ_i compress and W_i remains irreducible after any filling of T, take τ_i to be any slope at all.

Suppose (σ_1, σ_2) is a norm-reducing slope pair.

If $M(\sigma_1, \sigma_2)$ is reducible then an innermost disk argument on the intersection of a reducing sphere with E shows that either $W_1(\sigma_1)$ or $W_2(\sigma_2)$ is reducible. Then by [GL] and Rule 1 either $\sigma_1 \cdot \tau_1 \leq 1$ or $\sigma_2 \cdot \tau_2 \leq 1$ and we are done.

Suppose that $M(\sigma_1, \sigma_2)$ is irreducible.

3.10. **Lemma.** *Either some component of δ_1 compresses in $W_1(\sigma_1)$ or some component of δ_2 compresses in $W_2(\sigma_2)$.*

Proof. By 2.5 some component c_σ of γ compresses in $M(\sigma_1, \sigma_2)$. Let $D \subset M(\sigma_1, \sigma_2)$ be a compressing disk for c_σ, chosen so as to minimize the number of components of intersection with the splitting disk E in $M \subset M(\sigma)$. Then a standard innermost disk argument shows that $D \cap E$ consists only of arcs. If $D \cap E = \varnothing$, then $c_\sigma \cap \partial E = \varnothing$, so c_σ is either in δ_1 or δ_2, proving the lemma. If $D \cap E \neq \varnothing$, an outermost arc α of $d_\sigma \cap E$ in D cuts off from $\partial D = c_\sigma$ an arc β so that β is essential in either $(F_1, \partial E)$ or $(F_2, \partial E)$, say the former. Then $\alpha \cup \beta$ is in δ_1 and the subdisk of D which α cuts off is a compressing disk for $\alpha \cup \beta$ in $W_1(\sigma_1)$. $\square$

To resume the proof of Theorem 3.9 we will assume without loss of generality that some component of δ_1 compresses in $W(\sigma_1)$.

If there is a slope τ_1 so that $W_1(\tau_1)$ is reducible (so Rule 1 applies), then $\sigma_1 \cdot \tau_1 \leq 1$ by [Sc3, 6.1]. If there is no such slope, so Rule 2 applies, then some (possibly different) component c_τ of δ_1 compresses in $W_1(\tau_1)$. Since c_σ and c_τ are both in δ_1, $c_\sigma \cdot c_\tau = 1$. Then, by 3.7, $\sigma_1 \cdot \tau_1 = 1$.

General case. If (M, γ) is not itself a J-cobordism, then construct a taut sutured manifold hierarchy

$$(M, \gamma) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_i} (M_i, \gamma_i) \xrightarrow{S_{i+1}} \cdots$$

so that each S_i is disjoint from $T = T_1 \cup T_2$. By [Sc1, 2.6 and 4.17] such a sequence eventually terminates in a taut sutured manifold (M_n, γ_n) with $H_2(M_n, \partial M_n - T) = 0$, i.e., M_n is a J-cobordism on T. Moreover, $(M(\sigma_1, \sigma_2), \gamma)$ is taut if $(M_n(\sigma_1, \sigma_2), \gamma_n)$ is taut. The proof now follows by applying the special case to $(M_n(\sigma_1, \sigma_2), \gamma_n)$. $\square$

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