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ON TOPOLOGICALLY INVARIANT MEANS ON A LOCALLY COMPACT GROUP

BY
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Abstract. Let $\mathscr{M}$ be the set of all probability measures on βN. Let G be a locally compact, noncompact, amenable group. Then there is a one-one affine mapping of $\mathscr{M}$ into the set of all left invariant means on $L^\infty(G)$. Note that $\mathscr{M}$ is a very big set. If we further assume G to be σ-compact, then we have a better result: The set $\mathscr{M}$ can be embedded affinely into the set of two-sided topologically invariant means on $L^\infty(G)$. We also give a structure theorem for the set of all topologically left invariant means when G is σ-compact.

1. **Introduction.** Let G be a σ-compact, locally compact, amenable group. Then there exists a sequence of compact neighborhoods U_n of the identity e which satisfies the following two conditions (cf. [10]):

$$\text{(F1)} \qquad U_n \subset U_{n+1}, \quad n = 1, 2, \ldots; \quad \bigcup_{n=1}^{\infty} U_n = G$$

and

$$\text{(F2)} \qquad \lim_n \left(|xU_n \,\Delta\, U_n|/|U_n|\right) = 0$$

uniformly on compact subsets of G. Here $A \,\Delta\, B=(A\backslash B) \cup (B\backslash A)$, the symmetric difference of A and B; for a Borel set B, $|B|$ is the measure of B with respect to a fixed left Haar measure on G. We shall call such a sequence an F-sequence. (F stands for Følner.)

Let $\varphi_n=\chi_{U_n}/|U_n|$, where χ_{U_n} is the characteristic function of U_n, $n=1, 2, \ldots$. Let (x_n) be a sequence in G. Then a sequence of linear functionals μ_n on $L^\infty(G)$ can be defined as follows: $\mu_n(f)=(\varphi_n * f)(x_n)$. Denote the set of all w^*-limit points of the sequence μ_n by $I'(x_n)$ and then set

$$I'(G) = \bigcup \{I'(x_n) : (x_n) \text{ ranges over all sequences in } G\}.$$

Let $MTl(G)$ be the set of all left topologically invariant means on $L^\infty(G)$. Then we have the following.

THEOREM. *Let G be a σ-compact, locally compact, amenable group with a fixed F-sequence. Construct $I'(G)$ as in the previous paragraph. Then the w^*-closed convex hull of $I'(G)$ is $MTl(G)$.*

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When $G=R$, the additive group of reals, the above theorem, in a different form, was proved by Raimi [15].

Let G be a locally compact amenable group and $Ml(C(G))$ be the set of all left invariant means on $C(G)$. It is natural to ask how big is the set $Ml(C(G))$. If G is an infinite discrete amenable group, Granirer [7] proved that the space of left invariant functionals on $C(G)$ is infinite dimensional. When G is a locally compact group the following two results are known:

(1) *If G is a locally compact abelian group then $Ml(C(G))$ is a singleton if and only if G is compact* (cf. [13]);

(2) *If G is a separable, locally compact, noncompact group which is amenable as a discrete group then the space of left invariant functionals on $C(G)$ is infinite dimensional,* (cf. [8]).

In this paper we shall prove that the above two results can be generalized to every locally compact amenable group.

THEOREM. *Let G be a locally compact amenable group. Then* card $Ml(C(G))=1$ *or* $\geqq 2^c$. *It is one if and only if G is compact.*

Here c is the cardinality of the Continuum and, for any arbitrary set A, card A denotes its cardinality.

Under the same assumption as in (2) above, Granirer actually proved that the space of left invariant functionals in LUC (G), the space of bounded left uniformly continuous functions, is infinite dimensional. We are able to improve his result to show the following.

THEOREM. *Let G be a σ-compact, locally compact, noncompact, amenable group. Then G has at least 2^c two-sided topologically invariant means.*

Note that every separable group is σ-compact. We believe that the above theorem is new even for the discrete case. Another result along this line is the following.

THEOREM. *Let G be a locally compact, noncompact, amenable group such that G has equivalent right and left uniform structures. Then* card $MTl(G)\geqq 2^c$.

In particular, the above theorem is true for every locally compact abelian group.

For an arbitrary locally compact group G, LUC $(G)^*$ is a Banach algebra with convolution as multiplication, cf. [8]. A consequence of the above two theorems is the following.

COROLLARY. *Let G be a locally compact amenable group which satisfies one of the following two conditions*: (i) *G is σ-compact*; (ii) *the right and left uniform structures on G are equivalent. Then* dim $R(G)=0$ *or* $\geqq 2^c$, *where $R(G)$ is the radical of* LUC $(G)^*$. *It is* 0 *if and only if G is compact.*

P. Civin and B. Yood in *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. **11** (1961), 847–870 (especially p. 853), proved this corollary for the case that G is the group of additive integers. They conjectured it to hold for any infinite abelian discrete group. Much more than this conjecture has subsequently been proved by Granirer in [7, pp. 48–58], and [8, pp. 131–132]. The above corollary improves Theorem 4 and part of Theorem 6 in [8].

Let $\mathcal{M}$ be the set of all probability Borel measures on the compact set βN, the Stone-Čech compactification of N, the discrete set of positive integers. Then the above theorems are actually consequences of the following embedding theorem.

THEOREM. (1) *Let G be a locally compact, noncompact, amenable group. Then there is a one-one affine mapping of $\mathcal{M}$ into $Ml(C(G))$.*

(2) *Let G be a σ-compact, locally compact, noncompact, amenable group. Then there exists an affine* (w^*-w^*) *homeomorphism of $\mathcal{M}$ into $MTl(G)$.*

In (2), if G is further assumed to be unimodular, then we can actually embed $\mathcal{M}$ into the set of two-sided topologically invariant means on $L^\infty(G)$.

A portion of this paper is contained in the last chapter of the author's Ph.D. thesis. He wishes to thank Professor Raimi for his advice and encouragement.

2. **Preliminaries and notation.** Let G be a locally compact group with a fixed left Haar measure. If f is a Borel measurable function on G and B is a Borel subset of G, the integral of f on B with respect to the left Haar measure is denoted by $\int_B f(x)\,dx$. The Banach space of all essentially bounded real-valued Borel functions, with ess. sup-norm $\|\cdot\|$, is denoted by $L^\infty(G)$. The space of integrable real functions with respect to the fixed Haar measure is denoted by $L^1(G)$.

Let $\varphi \in L^1(G)$, $\varphi^\sim$ denotes the function: $\varphi^\sim(x)=\varphi(x^{-1})$. For $\varphi \in L^1(G)$ and $f \in L^\infty(G)$, the convolutions $\varphi * f$ and $f * \varphi^\sim$ are defined by

$$(\varphi * f)(x) = \int_G f(t^{-1}x)\varphi(t)\,dt, \qquad (f * \varphi^\sim)(x) = \int_G f(t)\varphi(x^{-1}t)\,dt.$$

For $f \in L^\infty(G)$ and $x \in G$, $l_x f$ $\{r_x f\}$, the left {right} translation of f by x is defined by $(l_x f)(y)=f(xy)$ $\{(r_x f)(y)=f(yx)\}$. A function $f \in L^\infty(G)$ is called left {right} uniformly continuous if, given $\varepsilon>0$, there is a neighborhood U of the identity e in G such that $\|f-l_x f\|<\varepsilon$ $\{\|f-r_x f\|<\varepsilon\}$ for all $y \in U$. The space of left {right} uniformly continuous functions on G will be denoted by LUC (G) {RUC (G)}. Let UC $(G)=$ LUC $(G) \cap$ RUC (G) and let $C(G)$ be the space of all bounded real continuous functions on G. All the above spaces are closed subspaces of $L^\infty(G)$.

We shall need the following well-known elementary fact.

LEMMA 2.1. *Let G be a locally compact group, $\varphi \in L^1(G)$ and $f \in L^\infty(G)$. Then $\varphi * f \in$ LUC (G) and $f * \varphi^\sim \in$ RUC (G).*

Let E be a subspace of $L^\infty(G)$ which contains the constant function 1. $\mu \in E^*$ is called a mean if $\mu(1)=\|\mu\|=1$. A subspace E of $L^\infty(G)$ is said to be left {right} invariant, if $1 \in E$, $f \in E$ and $x \in G$ imply $l_x f \in E$ $\{r_x f \in E\}$. If E is a left {right} invariant subspace, a mean $\mu \in E^*$ is called a left {right} invariant mean if $\mu(l_x f)$ $=\mu(f)$ $\{\mu(r_x f)=\mu(f)\}$ for $x \in G$ and $f \in E$.

A subspace E of $L^\infty(G)$ is said to be l-admissible $\{r$-admissible$\}$ if it is left $\{$right$\}$ invariant and $\mathrm{LUC}(G)\subset E$ $\{\mathrm{RUC}(G)\subset E\}$. Denote the set $\{\varphi\in L^1(G) : \varphi\geqq 0, \|\varphi\|_1=1\}$ by $P(G)$. A mean $\mu\in E^*$, E l-admissible $\{r$-admissible$\}$, is said to be topologically left $\{$right$\}$ invariant if $\mu(\varphi * f)=\mu(f)$ $\{\mu(f*\varphi^\sim)=\mu(f)\}$ for $f\in E$ and $\varphi\in P(G)$.

The set of all left $\{$right$\}$ invariant means on a left $\{$right$\}$ invariant subspace E will be denoted by $Ml(E)$ $\{Mr(E)\}$ and the set of topologically left $\{$right$\}$ invariant means for an l-admissible $\{r$-admissible$\}$ space E will be denoted by $MTl(E)$ $\{MTr(E)\}$. For convenience, we shall denote $Ml(L^\infty(G))$ and $MTl(L^\infty(G))$ by $Ml(G)$ and $MTl(G)$ respectively. We shall denote the set of all two-sided topologically invariant means on $L^\infty(G)$ by $MT(G)$ and the set of two-sided invariant means on $L^\infty(G)$ by $M(G)$. It is obvious that the sets $Ml(E)$, $MTl(E)$, ... etc. are w^*-compact and convex in E^*.

When $Ml(G)$ is not empty we say G is amenable. It is well known that $MTl(G)\subset Ml(G)$ and when G is amenable, $MT(G)\neq\varnothing$ (cf. [10]). Abelian groups and compact groups are amenable.

The following lemma is implicitly contained in [10]. We state it here for later quotations.

LEMMA 2.2. *Let G be a locally compact amenable group and let E be an l-admissible subspace of $L^\infty(G)$. Then*

(1) *if $\mu\in Ml(E)$ and $\varphi_1, \varphi_2\in P(G)$ then $\mu(\varphi_1 * f)=\mu(\varphi_2 * f)$;*

(2) *$MTl(\mathrm{LUC}(G))=Ml(\mathrm{LUC}(G))$ and $MT(\mathrm{UC}(G))=M(\mathrm{UC}(G))$;*

(3) *$MTl(G)|E=MTl(E)$ and the restriction mapping is one-one; $M(\mathrm{UC}(G))=MT(G)|\mathrm{UC}(G)$ and the restriction mapping is one-one.*

(1) tells us that if $\mu\in Ml(E)$ and for each $f\in E$ there exists $\varphi_f\in P(G)$ such that $\mu(\varphi_f * f)=\mu(f)$ then μ is actually topologically left invariant. (On the other hand, it is not true, in general, that $Ml(E)=MTl(E)$, cf. §5.)

(2) tells us that for each l-admissible space E, $MTl(E)$ and $MTl(G)$ can be considered as the same set.

Let (y_n) be a sequence in a topological space. The set of limit points of (y_n) will be denoted by $lp(y_n)$, in other words, $y\in lp(y_n)$ if and only if $y=\lim_\alpha y_{n_\alpha}$ for some subset n_α of n.

Let X be a subset of a topological vector space. Then the closed convex hull of X will be denoted by cl co X.

3. **Structure of $MTl(G)$ for a σ-compact group G.**

In this section we assume that G is a σ-compact, locally compact, amenable group with a fixed F-sequence (U_n). Note that (F2) can be replaced by a stronger condition:

$$\lim_n (|U_n \,\Delta\, KU_n|/|U_n|) = 0$$

for each compact subset K of G [5]. But we do not need this.

For each n, we define a linear operator T_n from $L^\infty(G)$ into LUC (G) as follows:

$$(T_n f)(x) = \frac{1}{|U_n|}\int_G f(t^{-1}x)\,dt, \qquad f \in L^\infty(G),\ x \in G.$$

That $T_n f \in \mathrm{LUC}\,(G)$ follows from Lemma 2.1. Note that $\|T_n\| = 1$ and $T_n f \geqq 0$ if $f \geqq 0$. For $x \in G$, x' will denote the linear functional on LUC (G) defined by $x'(f) = f(x)$. Since x' is a mean, $T_n^* x'$ is also a mean (on $L^\infty(G)$).

Let $V(G) = \{f \in L^\infty(G) : \lim_n T_n f = c, \text{ a constant function, in norm}\}$. $V(G)$ is a closed subspace of $L^\infty(G)$. Define a linear functional m_0 on $V(G)$ as follows: $m_0(f) = c$, if $\lim_n f = c$. Note that m_0 is a mean on $V(G)$. Let

$$L'(G) = \{\mu \in L^\infty(G)^* : \mu | V(G) = m_0,\ \|\mu\| = 1\},$$

i.e., the set of all linear norm-preserving extensions of m_0 to $L^\infty(G)$.

If E is an admissible subspace of $L^\infty(G)$, we shall use the following notation: $L'(E) = L'(G)|E$, $V(E) = V(G) \cap E$. By the Hahn-Banach theorem

$$L'(E) = \{\mu \in E^* : \mu | V(E) = m_0 | V(E),\ \|\mu\| = 1\}.$$

Let

$$I'(E) = \bigcup \{w^*\text{-}lp((T_n|E)^* x_n') : x_n \text{ ranges over all sequences in } G\}$$

and let the set $I'(L^\infty(G))$ be denoted by $I'(G)$.

It appears that $V(E)$, $L'(E)$ depend on the choice of the F-sequence (U_n). But actually they are independent of it. We first prove the following.

LEMMA 3.1. *Let $\varphi \in P(G)$ and $f \in L^\infty(G)$. Then* $\lim_n \|T_n(\varphi * f) - T_n f\| = 0$.

Proof. Let $\varepsilon > 0$ be given. Choose a compact set $K \subset G$ such that $\int_K \varphi(t)\,dt > 1 - \varepsilon$. For an arbitrary $y \in G$,

$$\begin{aligned}
|(T_n(\varphi * f) - T_n f)(y)| &= \frac{1}{|U_n|}\left|\int_{U_n}\int_G (f(t^{-1}x^{-1}y) - f(x^{-1}y))\varphi(t)\,dt\,dx\right| \\
&\leqq \frac{1}{|U_n|}\left|\int_{U_n}\int_K (f(t^{-1}x^{-1}y) - f(x^{-1}y))\varphi(t)\,dt\,dx\right| + 2\|f\|\varepsilon \\
&\leqq \frac{1}{|U_n|}\left|\int_K \varphi(t)\left[\int_{t^{-1}U_n} f(x^{-1}y) - \int_{U_n} f(x^{-1}y)\right]dx\,dt\right| \\
&\leqq \sup_{t \in K}\,(|t^{-1}U_n \,\Delta\, U_n|/|U_n|)\|f\| + 2\|f\|\varepsilon.
\end{aligned}$$

Since $|t^{-1}U_n \,\Delta\, U_n|/|U_n|$ converges to zero uniformly on K, there is a positive integer n_0 such that $n \geqq n_0$ implies $|t^{-1}U_n \,\Delta\, U_n|/|U_n| < \varepsilon$ $(t \in K)$. Thus

$$\|T_n(\varphi * f) - T_n f\| < 3\|f\|\varepsilon$$

if $n \geqq n_0$ and the proof is completed.

Now we are in a position to state and prove the main theorem of this section.

THEOREM 3.2. *Let G be a σ-compact, locally compact, amenable group with a fixed F-sequence U_n and let E be an l-admissible subspace of $L^\infty(G)$. Define T_n, $V(E)$, $L'(E)$, and $I'(E)$ as above. Then*

(a) $MTl(E)=L'(E)=w^*\text{-cl co}(I'(E))$;

(b) $V(E)=$ *closed linear span of* $\{T_nf-f: f\in E, n \text{ positive integer}\}\cup\{1\}$
$=\{f\in E: \mu(f)\equiv a \text{ constant as } \mu \text{ runs through } MTl(E)\}$.

Proof. When $G=R$, this theorem, in a different form, is contained in Raimi [10]. Part of the proof here is similar to his.

(1) $w^*\text{-cl co}(I'(E))=L'(E)$: It is directly checked that $I'(E)\subset L'(E)$. Since $L'(E)$ is w^*-compact and convex, to show $w^*\text{-cl co}(I'(E))=L'(E)$, by one form of the Krein-Milman theorem [4, p. 80], it suffices to show that for each $f\in E$

$$\sup\{\mu(f): \mu\in I'(E)\} = \sup\{\mu(f): \mu\in L'(E)\}.$$

Let $f\in E$ be given and denote the left-hand side of the above formula by λ_1 and the right-hand side by λ_2. Clearly $\lambda_2\geqq\lambda_1$.

Choose a sequence x_n such that $\sup\{(T_nf)(x): x\in G\}-T_nf(x_n)<1/n$. Let μ be a w^*-limit point of the sequence $(T_n|E)^*x_n'$. Then $\mu\in I'(E)$ and

$$\lambda_1 \geqq \mu(f) \geqq \liminf_n T_nf(x_n) = \liminf_n \sup\{(T_nf)(x): x\in G\} \equiv \lambda_3.$$

Note that for each n, $\lim_k T_k(T_nf-f)=0$ (Lemma 3.1). Thus if $\nu\in L'(E)$, $\nu(T_nf)=\nu(f)$. But ν is a mean, we have $\sup\{T_nf(x): x\in G\}\geqq\nu(T_nf)=\nu(f)$. Therefore for each n,

$$\sup\{T_nf(x): x\in G\} \geqq \sup\{\nu(f): \nu\in L'(E)\} = \lambda_2.$$

Thus $\lambda_3\geqq\lambda_2$ and the proof is completed.

(2) $w^*\text{-cl co}(I'(E))\subset MTl(E)$: Since $MTl(E)$ is w^*-closed and convex, we only need to show that $I'(E)\subset MTl(E)$. Let $\mu=w^*\text{-}\lim_\alpha (T_{n_\alpha}|E)^*x_{n_\alpha}'\in I'(E)$. Let $\varphi\in P(G)$ and $f\in E$. Then

$$\begin{aligned}\mu(\varphi*f-f) &= \lim_\alpha (T_{n_\alpha}|E)^*x_{n_\alpha}'(\varphi*f-f) = \lim_\alpha T_{n_\alpha}(\varphi*f-f)(x_{n_\alpha})\\ &= 0, \quad \text{by Lemma 3.1,}\end{aligned}$$

and hence $\mu\in MTl(E)$.

(3) $MTl(E)\subset L'(E)$: Let $\mu\in MTl(E)$. Then $\mu(T_nf)=\mu(f)$ for each $f\in E$. If $f\in V(E)$, say, $\lim_n T_nf=c$, then $c=\mu(c)=\lim_n\mu(T_nf)=\mu(f)$. Thus $\mu\in L'(E)$.

(4) $V(E)=$ closed linear span of $\{T_nf-f: n\in N, f\in E\}\cup\{1\}\equiv V_1(E)$: By Lemma 3.1, for $n\in N$ and $f\in E$, $T_nf-f\in V(E)$. Since $V(E)$ is closed in E, $V_1\subset V(E)$. Conversely, if $g\in V(E)$, then $\lim_n T_ng=c$ exists. Thus $g=\lim_n(g-T_ng)+c\in V_1(E)$.

(5) $V(E)=\{f\in E: \mu(f)=\text{a constant as } \mu \text{ runs through } MTl(E)\}\equiv V_2(E)$: If $f\notin V(E)$ then there exist two subsequences $k(n)$, $j(n)$ of n and two sequences x_n, y_n in G such that

$$\lim_n (T_{k(n)}f)(x_n) = c_1 \neq c_2 = \lim_n (T_{j(n)}f)(y_n).$$

Let μ_1 be a w^*-limit point of the sequence $(T_{k(n)}|E)^*x'_n$ and μ_2 be a w^*-limit point of $(T_{j(n)}|E)^*y'_n$. By (2), $\mu_i \in MTl(E)$, $i=1,2$, and $\mu_1(f)=c_1 \neq c_2=\mu_2(f)$. Thus $f \notin V_2(E)$.

Conversely, if $f \in V(E)$, say $\lim_n T_n f=c$, and $\mu \in MTl(E)$ then $\mu(f)=\mu(T_n f) =\lim_k \mu(T_k f)=\mu(\lim_k T_k f)=\mu(c)=c$. Thus $f \in V_2(E)$.

4. **Embeddings of $\mathscr{M}$ into** $MT(G)$ **and** $MTl(G)$. Let N be the additive semigroup of positive integers and $m(N)$ the Banach space of bounded real functions on N with sup norm. Let $C=\{f \in m(N): \lim_n f(n) \text{ exists}\}$. C is a closed linear subspace of $m(N)$. There is a bounded linear functional ν_0 on C defined by $\nu_0(f)=\lim_n f(n)$. Note that ν_0 is a mean on C. Let $\mathscr{F}=\{\nu \in m(N)^*: \|\nu\|=1, \nu|C=\nu_0\}$. Then $\mathscr{F}$ is w^*-compact and convex.

For each $w \in \beta N$, the Stone-Čech compactification of the discrete space N, cf. [6], there corresponds a linear functional w' on $m(N)$, defined by $w'(f)=f^-(w)$, where f^- denotes its continuous extension to βN. Then $(\beta N)'=\{w': w \in \beta N\}$, with the w^*-topology, is homeomorphic to βN and w^*-cl co $(\beta N)'=\mathscr{M}$, the set of all means on $m(N)$.

The set $\mathscr{F}$ is a big subset of $\mathscr{M}$ as the following lemma shows.

Lemma 4.1. (1) $\mathscr{F}=w^*$-cl co $(\beta N\backslash N)'$.

(2) *There exists a one-one affine* (w^*-w^*) *homeomorphism of* $\mathscr{M}$ *into* $\mathscr{F}$.

Proof. (1) Let $w \in \beta N\backslash N$. Then $w=\lim_\alpha n_\alpha$, n_α a net in N, with $\lim_\alpha n_\alpha=\infty$. Therefore, if $f \in C$, $w'(f)=f^-(w)=\lim_\alpha f(n_\alpha)=\lim_n f(n)$. Thus, $w' \in \mathscr{F}$ and hence $(\beta N\backslash N)' \subset \mathscr{F}$. Since $\mathscr{F}$ is w^*-compact and convex, we have w^*-cl co $(\beta N\backslash N)' \subset \mathscr{F}$. Conversely, if $\mu \in \mathscr{F}$ then, considering μ as a measure on βN, the support of μ is contained in $\beta N\backslash N$. By the Hahn-Banach theorem, $\mu \in w^*$-cl co $(\beta N\backslash N)'$.

(2) It is well known that $\beta N\backslash N$ contains a topological copy K of βN, cf. [6]. Then the set of Borel probability measures on K is affinely homeomorphic to $\mathscr{M}$. Thus the proof is completed.

Theorem 4.2. *Let G be a σ-compact, locally compact, noncompact, amenable group. Then the set $\mathscr{M}$ can be embedded into $MTl(G)$ affinely and* (w^*-w^*) *topologically.*

Proof. By Lemma 4.1, it suffices to prove that the set $\mathscr{F}$ can be embedded into $MTl(G)$ affinely and (w^*-w^*) topologically. Let (U_n) be an F-sequence for G. Since G is noncompact, $\lim_n |U_n|=\infty$. So, by choosing a subsequence if necessary, we may assume that (U_n) also satisfies

$$\text{(F3)} \qquad |U_{n+1}| \geqq (n+1)|U_n|, \qquad n = 1,2,\ldots.$$

We define a mapping π of $L^\infty(G)$ into $m(N)$ as follows:
For $f \in L^\infty(G)$ and $n \in N$

$$(\pi f)(n) = \frac{1}{|U_{n+1}\backslash U_n|}\int_{U_{n+1}\backslash U_n} f(t)\, dt.$$

Note that π is linear, $\|\pi\|=1$ and $\pi(L^\infty(G))=m(N)$. Therefore π^*, the conjugate of π, is a linear, one-one and (w^*-w^*) continuous mapping from $m(N)^*$ into $L^\infty(G)^*$. Since $\mathscr{F}$ is w^*-compact, if we can prove that $\pi^*(\mathscr{F})\subset MTl(G)$ then π^* is the embedding mapping we are looking for. To this end, we have to prove that (1) if $\nu\in\mathscr{F}$ then $\pi^*\nu$ is a mean, and (2) if $f\in L^\infty(G)$, $\varphi\in P(G)$ and $\nu\in\mathscr{F}$ then $\pi^*\nu(\varphi * f)=\pi^*\nu(f)$. (1) is obvious. For (2), it suffices to show that for $f\in L^\infty(G)$ and $\varphi\in P(G)$ then $\lim_n \pi(\varphi * f-f)(n)=0$. (It implies that if $\nu\in\mathscr{F}$ then $\pi^*\nu(\varphi * f)$ $=\pi^*\nu(f)$ by the definition of $\mathscr{F}$.)

Let $f\in L^\infty(G)$ and $\varphi\in P(G)$. We may assume that $\varphi=\chi_U/|U|$, the normalized characteristic function on U where U is open and relatively compact. Then

$$\begin{aligned}|\pi f(n)-\pi((\chi_U/|U|) * f)(n)| &= \left|\frac{1}{|U_{n+1}\backslash U_n|}\int_{U_{n+1}\backslash U_n}\left(\frac{1}{|U|}\int_U (f(t)-f(x^{-1}t))\,dx\right)dt\right| \\ &= \frac{1}{|U|}\left|\int_U\left[\frac{1}{|U_{n+1}\backslash U_n|}\int_{U_{n+1}\backslash U_n}(f(t)-f(x^{-1}t))\,dt\right]dx\right| \\ &\leqq \|f\|\sup_{x\in U}\frac{|(U_{n+1}\backslash U_n)\,\Delta\, x^{-1}(U_{n+1}\backslash U_n)|}{|U_{n+1}\backslash U_n|} \\ &\leqq \left[\sup_{x\in U}\frac{|U_{n+1}\,\Delta\, x^{-1}U_{n+1}|}{|U_{n+1}|}\frac{n+1}{n}+\frac{2}{n}\right]\|f\|.\end{aligned}$$

To get the last inequality, we used the following easily verified inclusion relation:

$$(U_{n+1}\backslash U_n)\,\Delta\, x^{-1}(U_{n+1}\backslash U_n) \subset (U_{n+1}\,\Delta\, x^{-1}U_{n+1})\cup U_n\cup xU_n,$$

and condition (F3). By (F2)

$$\lim_n\,(|U_n\,\Delta\, x^{-1}U_n|/|U_n|) = 0$$

uniformly on U. Thus we have $(\pi f)(n)-\pi((\chi_U/|U|) * f)(n)\to 0$ as $n\to\infty$, as required.

REMARKS. (1) Let G be as in the above theorem and let E be an l-admissible subspace of $L^\infty(G)$. Then $MTl(E)$ and $MTl(G)$ are affinely homeomorphic, cf. Lemma 2.2. So the set $\mathscr{M}$ can be embedded into $MTl(E)$, and hence $Ml(E)$, affinely and homeomorphically.

(2) Let $M(N)$ be the set of invariant means on $m(N)$. Note that $M(N)$ is a very small portion of $\mathscr{M}$. (It is known that there exists a nowhere dense compact subset K of βN such that each $\mu\in M(N)$ is supported on K, cf. [1].) On the other hand there is an affine homeomorphism of $\mathscr{M}$ into $M(N)$: Let k_n be an increasing sequence of positive integers satisfying the following condition: $k_{n+1}\geqq(n+1)k_n$. We define a mapping θ: $m(N)\to m(N)$ as follows: For $f\in m(N)$,

$$(\theta f)(n) = \frac{1}{k_{n+1}-k_n}\sum_{l=k_n+1}^{k_{n+1}} f(l).$$

Then using the same proof as the above theorem, we see that the set $\mathscr{F}$, and hence $\mathscr{M}$, can be embedded into $M(N)$ topologically and affinely. This also gives us a new proof that card $M(N)=2^c$, cf. [2].

In the previous theorem if we further assume that G is unimodular then we have a better result:

THEOREM 4.3. *Let G be a unimodular, σ-compact, noncompact, amenable group. Then the set $\mathscr{M}$ can be embedded into $MT(G)$ affinely and $(w^*\text{-}w^*)$ topologically.*

Recall that $MT(G)$ is the set of all two-sided topologically invariant means on $L^\infty(G)$. Indeed, we have

THEOREM 4.4. *Let G be a unimodular, σ-compact, amenable group. Then G has an F-sequence (U_n) which also satisfies*

$$\text{(S)} \qquad U_n = U_n^{-1}, \quad \textit{for } n = 1, 2, 3, \ldots.$$

Let G be a group satisfying the hypothesis in Theorem 4.3. Then by applying Theorem 4.4, there is an F-sequence which also satisfies (F3) and (S). Note that

$$\frac{|U_n \,\Delta\, U_n x|}{|U_n|} = \frac{|(U_n \,\Delta\, U_n x)^{-1}|}{|U_n|} = \frac{|U_n^{-1} \,\Delta\, x^{-1} U_n^{-1}|}{|U_n|} = \frac{|U_n \,\Delta\, x^{-1} U_n|}{|U_n|},$$

since G is unimodular and (S) holds. Thus

$$\text{(F2R)} \qquad \lim_n (|U_n \,\Delta\, U_n x|/|U_n|) = 0$$

uniformly on compact subsets of G. Define π as in the proof of Theorem 4.2. By (F2R) we see that for $\nu \in \mathscr{F}$, $\pi^*\nu$ is also topologically right invariant. Thus $\pi^*(\mathscr{F}) \subset MT(G)$, and Theorem 4.3 is proved.

When G is discrete, Theorem 4.4 is proved by Namioka in the last section of [14]. For the general case, we have to modify his proof and combine with results of Hulanicki [11] and Nyll-Nardzewski (cf. [5]). We sketch the proof here.

For convenience, $\varphi \in L^1(G)$ is called symmetric if $\varphi(x) = \varphi(x^{-1})$ almost everywhere and a Borel set B with finite Haar measure is called symmetric if $|B \,\Delta\, B^{-1}| = 0$.

LEMMA 4.5. *Let G be a unimodular amenable group. Let a compact subset K and $\varepsilon > 0$ be given. Then there exists a symmetric $\varphi \in P(G)$ such that $\|l_x\varphi - \varphi\|_1 < \varepsilon$ for all $x \in K$.*

Proof. Since G is amenable, there exists $\psi \in P(G)$ such that $\|l_x\psi - \psi\|_1 < \varepsilon$ for all $x \in K$, cf. [11, Theorem 3.21]. Set $\varphi = \psi * \psi^\sim$, where $\psi^\sim(x) = \psi(x^{-1})$. Then φ is symmetric and $\varphi \in P(G)$. Moreover, since G is unimodular

$$\begin{aligned} \|l_x\varphi - \varphi\|_1 &= \|l_x(\psi * \psi^\sim) - \psi * \psi^\sim\|_1 = \|(l_x\psi) * \psi^\sim - \psi * \psi^\sim\|_1 \\ &\leqq \|l_x\psi - \psi\|_1 \|\psi^\sim\|_1 = \|l_x\psi - \psi\|_1. \end{aligned}$$

Thus, φ is the function we are looking for.

LEMMA 4.6. *Let G be a unimodular amenable group. Let K be a given compact subset of G and let $\varepsilon > 0$. Then there exists a symmetric compact set A, $0 < |A|$, such that*

$$|xA \,\Delta\, A|/|A| < \varepsilon \quad \textit{for all } x \in K.$$

Proof. First of all, we have to prove the following weaker result: Let $\varepsilon>0$, $\delta>0$ and a compact set $K\subset G$, $|K|>0$, be given. Then there exists a symmetric compact set A with $0<|A|$, and a Borel set $B\subset K$ with $|B|<\delta$, such that $|xA \,\Delta\, A|/|A|<\varepsilon$ for all $x\in K\backslash B$.

By Lemma 4.5, there is a symmetric function $\varphi\in P(G)$ such that $\|l_x\varphi-\varphi\|_1 <\delta\varepsilon|K|$. Then we use the same proof as Proposition 5.1 of Hulanicki [11] to get the symmetric set A and the Borel set B we are looking for.

To complete the proof of our lemma, we only need to apply the above weaker result and use the same proof as Lemma 1.4.3 of [5].

PROPOSITION 4.7. *Let G be a unimodular amenable group. Let compact sets $K\subset G$, $F\subset G$ and $\varepsilon>0$ be given. Then there exists a symmetric compact set A, $|A|>0$, $A\supset F$ and $|xA \,\Delta\, A|/|A|<\varepsilon$ for all $x\in K$.*

Proof. Clearly, it is equivalent to prove that for a given $0<k<1$, and given compact sets F and K, there is a compact symmetric set A such that $A\supset F$ and $|xA\cap A|\geqq k|A|$ for all $x\in K$. If G is compact this proposition is trivial. Therefore, we assume that G is noncompact. We may also assume that F is symmetric. Choose a number $c>0$ such that $\frac{1}{2}<k(1+|F|c^{-1})<1$. Suppose that we can find a symmetric compact set $B\subset G$ such that (1) $|B|\geqq c$ and (2) $|xB\cap B|\geqq k(1+|F|c^{-1})|B|$ for each $x\in K$, then $A=B\cup F$ is the symmetric set we are looking for. Cf. Namioka [14] for the details. Therefore, it remains to produce a compact symmetric set B such that (1) and (2) are satisfied. Choose a symmetric compact set K_1 such that $K_1\supset K$ and $|K_1|\geqq 2c^2$. Then by Lemma 4.6, there is a symmetric compact set B such that

$$|xB\cap B| \;\geqq\; k(1+|F|c^{-1})|B| \quad \text{for each } x\in K_1,$$

and hence for each $x\in K$. Thus B satisfies (2). To see the set B satisfies (1), we consider the function $\chi_B*\chi_B$. Note that since B is symmetric $(\chi_B*\chi_B)(x)=|B\cap xB|$. Also, note that $\|\chi_B*\chi_B\|_1=|B|^2$. Let $D=\{x\colon \chi_B*\chi_B(x)\geqq\frac{1}{2}\}$. Since $k(1+|F|c^{-1}) >\frac{1}{2}$, the set $K_1\subset D$. Thus, $|B|^2\geqq\int_D\chi_B*\chi_B(x)\,dx\geqq\frac{1}{2}|D|\geqq\frac{1}{2}|K_1|\geqq c^2$. So $|B|\geqq c$, as we wanted.

Finally, note that Theorem 4.4 is an easy consequence of Proposition 4.7 (cf. [14]).

Now we want to generalize our result to general locally compact amenable groups.

LEMMA 4.8. *Let G be a locally compact, noncompact group. Then G contains a σ-compact, noncompact, open subgroup.*

Proof. Let V_1 be an arbitrary relatively compact symmetric neighborhood of e. Set $W_1=\bigcup_{n=1}^{\infty}V_1^n$. If W_1 is not compact, then it is the subgroup we are looking for. If W_1 is compact, choose a relatively compact symmetric neighborhood V_2 of e such that $V_2\supsetneqq W_1$. Set $W_2=\bigcup_{n=1}^{\infty}V_2^n$. Continue this process. If W_k is not compact we stop there. Otherwise, we choose V_{k+1} such that V_{k+1} is a relatively compact symmetric neighborhood of e and $V_{k+1} \supsetneqq W_k$. If all the W_k's are compact we set $V = V_1 \cup V_2 \cup \cdots$. V is clearly a noncompact, σ-compact, open subgroup of G.

LEMMA 4.9. *Let G be a locally compact amenable group and H an open subgroup of G. Then there is a one-one affine mapping of $Ml(C(H))$ into $Ml(C(G))$.*

Proof. The proof is similar to the discrete case in [3]. Fix $\mu_0 \in Ml(G)$. We define a mapping θ: $Ml(C(H)) \to MlC(G))$ as follows: For $\nu \in Ml(C(H))$, $\theta\nu(f) = \mu_0(\nu^{\sim}(f))$ where $f \in C(G)$ and $\nu^{\sim}(f)(x) = \nu((l_x f)|H)$ $(x \in G)$. Note that $\nu^{\sim}f$ is constant on each right coset and is, therefore, continuous. It is easy to check that $\theta\nu \in Ml(C(G))$.

Let T be a transversal for the right cosets of H and let $x = \tau(x)\eta(x) \in T \cdot H$ be the unique factorization of $x \in G$ with respect to T. Let $g \in C(H)$. Extend g to $g_1 \in C(G)$: $g_1(x) = g(\eta(x))$ $(x \in G)$. Then $(\theta\nu)(g_1) = \nu(g)$. So the mapping θ is one-one. θ is clearly an affine mapping.

REMARKS. (1) The above lemma has two shortcomings: (i) It is a one-sided theorem. We do not know how to embed the set $M(C(H))$ into $M(C(G))$. (ii) When $\nu \in Ml(C(H))$ is actually left topologically invariant, we do not know whether $\theta\nu$ is left topologically invariant.

(2) Let $\nu \in Ml(\mathrm{LUC}(H))$. Define $\theta\nu \in Ml(\mathrm{LUC}(G))$ by $(\theta\nu)(f) = \mu_0(\nu^{\sim}f)$ as in the proof of the above lemma. In general, we do not know whether this θ is one-one. But note that if g is a right uniformly continuous function on H then its extension g_1 to G, as in the proof, is also right uniformly continuous. Thus if G has equivalent left and right uniform structures, e.g. G is abelian or discrete, then the mapping θ is one-one. In this case, we know that the set $Ml(\mathrm{LUC}(H))$ can be embedded into $Ml(\mathrm{LUC}(G))$ affinely. But $MTl(G)|\mathrm{LUC}(G) = Ml(\mathrm{LUC}(G))$ and the restriction mapping is one-one (Lemma 2.2); so we know that the set $MTl(H)$ can be embedded into $MTl(G)$ affinely.

A combination of Lemmas 4.8, 4.9 and Theorem 4.2, gives us the following

THEOREM 4.10. *Let G be a locally compact, noncompact, amenable group. Then the set $\mathcal{M}$ can be embedded into $Ml(C(G))$ affinely.*

By the above remark, we also have the following

THEOREM 4.11. *Let G be a locally compact, noncompact, amenable group such that* (1) *G is σ-compact or* (2) *G has equivalent right and left uniform structures. Then the set $\mathcal{M}$ can be embedded into $MTl(G)$ affinely.*

5. **Remarks and consequences.**

(I) *The cardinalities of $Ml(E)$ and $MTl(E)$.* It is clear that card $\mathcal{M} = 2^c$. It is also well known that if G is a compact group then G has a unique invariant mean: the normalized Haar measure on G. Thus by Theorem 4.10, we have

THEOREM 5.1. *Let G be a locally compact amenable group. Then* card $Ml(C(G)) = 1$ *or $\geqq 2^c$. It is one if and only if G is compact.*

When G is discrete, a stronger result is contained in [2]. By Theorem 4.11, we get

THEOREM 5.2. *Let G be a locally compact amenable group which satisfies one of the following two conditions* (1) *G is σ-compact*, (2) *right and left uniform structure on G are equivalent. Then* card $MTl(G)=1$ *or* $\geqq 2^c$. *It is one if and only if G is compact.*

Granirer [8] proved that if G is a locally compact, noncompact, separable group, amenable as a discrete group (which implies that G is amenable) then the dimensionality of the space of left invariant functionals on LUC (G) is infinite. Since separability implies σ-compactness, our result is an improvement of his (cf. Lemma 2.2). Unfortunately, we are unable to prove Theorem 5.2 for general locally compact amenable groups. We would like to give the following

CONJECTURE. *Let G be a locally compact amenable group. Then* card $MTl(G)=1$ *or* $\geqq 2^c$.

THEOREM 5.3. *Let G be a locally compact amenable group which satisfies one of the following conditions* (1) *G is σ-compact*, (2) *G is nonunimodular. Then* card $MT(G)=1$ *or* $\geqq 2^c$. *It is one if and only if G is compact.*

Proof. If G is σ-compact and unimodular then this theorem is a consequence of Theorem 4.3. So, we assume that G is not unimodular. Let $H=\{x\in G| : \Delta(x)=1\}$. Here Δ is the modular function for G. Note that H is a proper closed normal subgroup of G and G/H can be identified with a nontrivial subgroup of R, the group of real numbers. Thus G/H is not compact and hence, card $MT(G/H)\geqq 2^c$ by Theorem 5.2. It follows that card $MT(G)\geqq 2^c$ as the following lemma shows.

LEMMA 5.4. *Let G be a locally compact amenable group. Let H be a closed normal subgroup of G. Then there exists an affine mapping of $MT(G)$ onto $MT(G/H)$.*

Proof. Note that the restriction mapping $\mu\to\mu|\mathrm{UC}\,(G)$ of $MT(G)$ into $M(\mathrm{UC}\,(G))$ is one-one and onto (Lemma 2.2). Therefore we only need to show that there exists an affine mapping of $M(\mathrm{UC}\,(G))$ onto $M(\mathrm{UC}\,(G/H))$. The proof is similar to the discrete case in [3]. We shall only give the outline of the proof here.

Consider the natural mapping $\theta\colon \mathrm{UC}\,(G/H)\to\mathrm{UC}\,(G)$ defined by $(\theta f)(x)=f(x+H)$, $x\in G$. Let θ^* be the conjugate of θ. Then clearly

$$\theta^*(M(\mathrm{UC}\,(G)))\subset M(\mathrm{UC}\,(G/H)).$$

To see

$$\theta^*(M(\mathrm{UC}\,(G))) = M(\mathrm{UC}\,(G/H)),$$

let $\nu\in M(\mathrm{UC}\,(G/H))$. Set $E=\theta(\mathrm{UC}\,(G/H))$. E is a two-sided invariant subspace of UC (G/H) and $1\in E$. Define a functional μ on E as follows: $\mu(\theta f)=\nu(f)$. Then μ is a two-sided invariant mean on E. Now set

$$K = \{\lambda\colon \lambda \text{ is a mean on UC}(G) \text{ and } \lambda|E = \mu\}.$$

Clearly, K is w^*-compact and convex and $l_x^*K\subset K$, $r_x^*K\subset K$ for each $x\in G$. By Rickert's fixed point theorem [16], there exist λ_1, $\lambda_2\in K$ such that $l_x^*\lambda_1=\lambda_1$ and $r_x^*\lambda_2=\lambda_2$ for all $x\in G$, i.e., $\lambda_1\in Ml(\mathrm{UC}(G))$, $\lambda_2\in Mr(\mathrm{UC}(G))$ and $\lambda_i|E=\mu$. Finally, define λ as follows: $\lambda(f)=\lambda_1(f^{\sim})$ where $f\in \mathrm{UC}(G)$ and $f^{\sim}(x)=\lambda_2(l_xf)$ $(x\in G)$. Then $\lambda|E=\mu$, i.e., $\theta^*\lambda=\nu$, and $\lambda\in M(\mathrm{UC}(G))$.

(II) *The algebra* $\mathrm{LUC}(G)^*$. Let G be a locally compact group. Then it is well known that $\mathrm{LUC}(G)^*$ is a Banach algebra with convolution, cf. [8]. Denote its radical by $R(G)$. If G is amenable, then as in [8], for a fixed $\mu_0\in Ml(\mathrm{LUC}(G))$, we have $Ml(\mathrm{LUC}(G))-\mu_0\subset R(G)$. Thus, as a consequence of Theorem 5.2 and the fact that if G is compact then $\mathrm{LUC}(G)^*$ is semisimple, we have

THEOREM 5.5. *Assume G is a locally compact amenable group such that* (1) *G is σ-compact or* (2) *the uniform structures on G are equivalent. Then* $\dim R(G)=0$ *or* $\geqq 2^c$. *It is zero if and only if G is compact.*

We would like to single out an important special case here. Note that every abelian group is amenable.

COROLLARY 5.6. *Let G be a locally compact abelian group. Then the algebra* $\mathrm{UC}(G)^*$ *is semisimple if and only if G is compact. When G is noncompact, dimensionality of the radical of* $\mathrm{UC}(G)^*$ *is* $\geqq 2^c$.

It is conceivable that the above theorem should be true for every locally compact group. But we cannot even prove it for amenable groups.

(III) *Relations between invariant means and topological invariant means.* It is well known that for an l-admissible subspace E of $L^\infty(G)$, G a locally compact amenable group, $MTl(E)\subset Ml(E)$ and when $E=\mathrm{LUC}(G)$, $MTl(E)=Ml(E)$. In [9] Granirer asked whether $MTl(C(R))=Ml(C(R))$, where R is the additive group of reals with the usual topology. (Greenleaf [10], also asked the same question.) Granirer's problem had actually been solved by Raimi [15]. He proved that there is a $\mu\in Ml(C(R))\backslash L'(C(R))$. (The above notation is defined in §3.) But, by Theorem 3.2, $L'(C(R))=MTl(C(R))$, therefore $MTl(C(R))\neq Ml(C(R))$.

In general, we have the following

LEMMA 5.7. *Let G be a locally compact amenable group and E be an l-admissible subspace of $L^\infty(G)$. Then the following are equivalent*:

(1) $Ml(E)=MTl(E)$;

(2) $E\subset \mathrm{LUC}(G)+F(G)$;

(3) *the restriction mapping of* $Ml(E)\to Ml(\mathrm{LUC}(G))$ *is one-one.*

Here, $F(G)=\{f\in L^\infty(G)\colon \mu(f)=a$ *constant as* μ *runs through* $Ml(G)\}$, *the space of almost convergent functions in* $L^\infty(G)$.

Proof of Lemma 5.7. (1) $\Leftrightarrow$ (3) is a consequence of Lemma 2.2.

(1) $\Rightarrow$ (2). Let $f\in E$, $f\notin \mathrm{LUC}(G)+F(G)$. Choose a $\varphi\in P(G)$. $\varphi*f\in \mathrm{LUC}(G)$, so $f-\varphi*f\notin F(G)$. Therefore there exists $\mu\in Ml(G)$ such that $\mu(f)\neq\mu(\varphi*f)$ and hence, $\mu|E\notin MTl(E)$.

(2) $\Rightarrow$ (3). Assume (2) holds. Let $\mu_i \in Ml(E)$, $i=1,2$, and $\mu_1 \neq \mu_2$. Then there exists $f=g+h \in E$, where $g \in \mathrm{LUC}(G)$ and $h \in F(G)$ such that $\mu_1(f) \neq \mu_2(f)$. Since $\mu_1(h)=\mu_2(h)$, we have $\mu_1(g) \neq \mu_2(g)$ and hence, (3) is true.

It is known that $\mathrm{LUC}(G) \neq C(G)$ if G is a noncompact, nondiscrete, locally compact group, cf. [12]. By Theorem 5.1 if G is noncompact amenable then $Ml(C(G))$ is not a singleton and hence $F(C(G)) \neq C(G)$. So the following conjecture seems reasonable.

CONJECTURE. *Let G be a locally compact group. If G is nondiscrete and noncompact then* $MTl(C(G)) \neq Ml(C(G))$.

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