# THE ASYMPTOTIC SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER, WITH SPECIAL REFERENCE TO A TURNING POINT

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1. **Introduction.** The subject with which this discussion is to be concerned is the differential equation of the type

$$(1.1)\qquad \frac{d^2w}{dx^2} + \lambda P_1(x, \lambda)\frac{dw}{dx} + \lambda^2 P_2(x, \lambda)w = 0,$$

with coefficient functions of the form

$$P_j(x, \lambda) = \sum_{\mu=0}^{\infty} \frac{p_{j,\mu}(x)}{\lambda^\mu}, \qquad j = 1, 2.$$

The variable $x$ is to be real and on an interval $a \leqq x \leqq b$. The parameter $\lambda$ is to be large in absolute value, but otherwise unrestricted—real or complex. The matter of primary concern is to be the derivation of analytic forms which represent the solutions of the equation asymptotically as to $\lambda$. By way of hypotheses it is to be assumed that the functions $p_{j,\mu}(x)$ are indefinitely differentiable on $(a, b)$, and that the series $P_j(x, \lambda)$ are convergent and differentiable term by term when $|\lambda|$ is sufficiently large[2].

It is familiar that the functional forms of the solutions of an equation (1.1) depend in large measure upon the nature of the discriminant of the auxiliary algebraic equation

$$\gamma^2 + p_{1,0}(x)\gamma + p_{2,0}(x) = 0,$$

namely of the function

$$(1.2)\qquad [p_{2,0}(x) - p_{1,0}^2(x)/4].$$

When $\lambda$ is real this is very evident, for the solutions are then of an oscillatory or an exponential type according as the function (1.2) is positive or negative.

Whether $\lambda$ is real or complex, the simplest case—we shall refer to it as the *classical case*—is that in which the function (1.2) is bounded from zero, and therefore maintains its sign over the interval $(a, b)$. For that case algorithms

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(2) These conditions could be materially relaxed in ways that will be evident. They are adopted here to prevent the discussion from losing itself in matters other than those which are of primary interest.

are known by which series in negative powers of $\lambda$ may be determined to satisfy the given differential equation at least formally. Though these series are in general divergent, it can be shown that their segments serve to represent true solutions of the equation in specifiable regions of the $\lambda$ plane to degrees of approximation that are characterized by appropriate powers of $1/\lambda$.

The facts are different when the interval $(a, b)$ includes a so-called *turning point*, namely, a point at which the discriminant (1.2) changes its sign. In the neighborhood of such a point the largeness of $|\lambda|$ is in a certain sense offset. In the case of a real $\lambda$ the solutions change there from the oscillatory to the exponential type. If a restriction to the use of elementary functions is made, no single analytic form suffices for the representation of a solution over the whole interval, but, on the contrary, different forms are found to be requisite on opposite sides of the turning point. This is known as the *Stokes' phenomenon*. When it is present the forms representing one and the same solution for different values of the variable must be associated with each other, and this requires the deduction of so-called *connection formulas*. Heretofore, such formulas have been given only to the extent that the leading terms of the asymptotic representations have been involved. They have not been given for the representations as a whole.

If the restriction to elementary functions is dropped and the use of Bessel functions is admitted, the theory of asymptotic representation is greatly simplified. The Stokes' phenomenon is obviated, and the representation of a solution by a single form over the entire interval is made possible. While such representations have been given, and have even been expressed as series in powers of $\lambda$, it is nevertheless true that only the leading terms of these series have heretofore been given in any simple way. For the terms after the leading ones only very complicated expressions in repeated integrals over products of Bessel functions, and so on, have been known. In this respect the theory of asymptotic representation for the case of a turning point has remained conspicuously behind that for the classical case.

In the present paper this gap is in large part to be filled. An algorithm of an elementary type is to be given for the deduction of formal solutions for the differential equation with a simple turning point, that is, when the zero of the discriminant (1.2) is simple. It is to be shown that the segments of these formal solutions represent true solutions to within terms which are of an arbitrary order as powers of $1/\lambda$.

It is convenient for the discussion to normalize the equation (1.1) by subjecting it to the transformation

$$w = u \exp\left[-\frac{\lambda}{2}\int P_1(x, \lambda)dx\right].$$

In its resulting form, the equation is then

$$\frac{d^2u}{dx^2} + [\lambda^2 q_0(x) + \lambda q_1(x) + R(x, \lambda)]u = 0, \tag{1.3}$$

with

$$R(x, \lambda) = \sum_{\nu=0}^{\infty} \frac{r_\nu(x)}{\lambda^\nu}. \tag{1.4}$$

The functions $q_0(x)$, $q_1(x)$, and $r_\nu(x)$ are again indefinitely differentiable, and the series (1.4) again converges when $|\lambda| > N$(3).

For the differential equation in the form (1.3) the discriminant analogous to (1.2) is simply the coefficient $q_0(x)$. It is therefore this function, the coefficient of $\lambda^2$, which vanishes at a turning point. We shall consider the equation without imposing any restriction upon the coefficient $q_1(x)$. Heretofore it has invariably been found necessary to suppose that $q_1(x)$ is identically zero, or at least that it vanishes where $q_0(x)$ does so.

The discussion below consists of two parts. Part one, which is brief and entirely formal, applies to the classical case. For that it gives a new algorithm by which formal solutions are obtainable, one which seems simpler than that heretofore known, especially when $q_1(x) \not\equiv 0$. Part two, which constitutes the main and essential part of the paper, deals with the case of a turning point. In this are given both the formal deductions by which the asymptotic forms are to be derived, and also the rigorous analysis by which the properties of the forms are established. The method of the latter seems simpler and more direct than those which have heretofore been used.

## Part I. The classical case

2. **The formal solutions.** When the coefficient $q_0(x)$ of the equation (1.3) is nonvanishing, and hence is bounded from zero on the (closed) interval $(a, b)$, we may choose either determination of $q_0^{1/2}(x)$ as a root of $q_0(x)$, and with $x_0$ as any point of $(a, b)$ that is independent of $x$, define $\phi$, $\xi$, and $\Theta$ as functions of $x$ and $\lambda$ by the formulas

$$\begin{aligned} \phi(x, \lambda) &= q_0^{1/2}(x) + \frac{q_1(x)}{2\lambda q_0^{1/2}(x)}, \qquad \xi = \lambda \int_{x_0}^{x} \phi(x, \lambda)dx, \\ \Theta(x, \lambda) &= R(x, \lambda) - \frac{q_1^2(x)}{4q_0(x)}. \end{aligned} \tag{2.1}$$

The equation (1.3) is then expressible in the form

$$\frac{d^2u}{dx^2} + [\lambda^2\phi^2(x, \lambda) + \Theta(x, \lambda)]u = 0, \tag{2.2}$$

(3) The symbolism "$|\lambda| > N$" is to be read throughout the paper as standing for the phrase "when the absolute value of $\lambda$ is sufficiently large." The letter $N$ is not to be regarded as denoting any specific constant in this connection.

with

$$\Theta(x, \lambda) = \sum_{\mu=0}^{\infty} \frac{\theta_\mu(x)}{\lambda^\mu}, \tag{2.3}$$

this series converging when $|\lambda| > N$, and being indefinitely differentiable term by term.

Let the functions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be given by the formulas

$$y_j(x, \lambda) = e^{\pm i\xi} A_j(x, \lambda), \qquad j = 1, 2(^4), \tag{2.4}$$

in which the factors $A_j(x, \lambda)$ remain for the moment unspecified. It is then found by direct substitution that

$$\frac{d^2 y_j}{dx^2} + [\lambda^2\phi^2 + \Theta] y_j = e^{\pm i\xi} C_j(x, \lambda), \qquad j = 1, 2, \tag{2.5}$$

in which

$$C_j(x, \lambda) = \pm\, i\lambda[2\phi A_j' + \phi' A_j] + [A_j'' + \Theta A_j], \qquad j = 1, 2(^5). \tag{2.6}$$

With an arbitrarily chosen positive integer $m$, let the functions $A_j(x, \lambda)$ now be taken to be of the form

$$A_j(x, \lambda) = \sum_{\mu=0}^{m} \frac{\alpha_{j,\mu}}{\lambda^\mu}, \qquad j = 1, 2. \tag{2.7}$$

Under the substitutions (2.3) and (2.7) the formula (2.6) may be made to appear more explicitly as

$$\begin{aligned} C_j(x, \lambda) = {} & \pm\, i\lambda[2\phi\alpha_{j,0}' + \phi'\alpha_{j,0}] \\ & + \sum_{\mu=0}^{\infty} \lambda^{-\mu}\left[\pm i(2\phi\alpha_{j,\mu+1}' + \phi'\alpha_{j,\mu+1}) + \alpha_{j,\mu}'' + \sum_{\nu=0}^{\mu} \theta_{\mu-\nu}\alpha_{j,\nu}\right], \end{aligned} \tag{2.8}$$

it being understood in this that every symbol $\alpha$ with a subscript greater than $m$ is to be interpreted as having the value zero. Now in the formula (2.8) the terms in $\lambda^{1-k}$, for $k = 0, 1, \cdots, m$, will all vanish if the relations

$$2\phi\alpha_{j,0}' + \phi'\alpha_{j,0} = 0,$$

$$\pm i\,(2\phi\alpha_{j,k}' + \phi'\alpha_{j,k}) = -\alpha_{j,k-1}'' - \sum_{\nu=0}^{k-1} \theta_{k-1-\nu}\alpha_{j,\nu}, \qquad k = 1, 2, \cdots, m,$$

are fulfilled. These will be fulfilled if suitable values are assigned successively

(4) It will often be convenient to combine the expression of two relations in this way. It is then always to be understood that the upper signs are to be associated with $j=1$, and the lower ones with $j=2$.

(5) The accents indicate differentiations with respect to $x$.

to $\alpha_{j,0}, \alpha_{j,1}, \cdots, \alpha_{j,m}$, in particular if

$$\alpha_{j,0} = \phi^{-1/2}(x, \lambda),$$

$$(2.9) \qquad \alpha_{j,k} = \frac{\pm i}{2\phi^{1/2}(x, \lambda)} \int_{x_0}^{x} \frac{\alpha''_{j,k-1} + \sum_{\nu=0}^{k-1} \theta_{k-1-\nu}\alpha_{j,\nu}}{\phi^{1/2}(x, \lambda)} dx.$$

Since $q_0(x)$ is bounded from zero, it follows from the first of the relations (2.1) that $\phi(x, \lambda)$ is similarly bounded when $|\lambda| > N$. No singularities are therefore involved in the relations (2.9). It results from these relations that the functions $C_j(x, \lambda)$ reduce to the order of $\lambda^{-m}$, namely that the products $\lambda^m C_j(x, \lambda)$, $j = 1, 2$, are bounded when $|\lambda| > N$.

It will be observed at once that if in the discussion above the integer $m$ were to be replaced by $\infty$, the scheme outlined would serve to remove all terms from the right-hand members of the relations (2.8). Formally, therefore, the functions $C_j(x, \lambda)$—and with them the right-hand members of the equations (2.5)—would then reduce to zero, and the functions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ would formally solve the given differential equation (2.2). This equation thus admits of the pair of formal solutions

$$(2.10) \qquad e^{\pm i\xi} \sum_{\mu=0}^{\infty} \frac{\alpha_{j,\mu}}{\lambda^{\mu}}, \qquad j = 1, 2,$$

with coefficients that are given by the formulas (2.9).

3. **The approximating differential equation.** The functions $y_1(x, \lambda)$, $y_2(x, \lambda)$, as defined under (2.4) are the initial segments of $(m+1)$ terms of the respective formal solutions (2.10). Like any pair of linearly independent and suitably differentiable functions, they determine a linear differential equation of the form

$$(3.1) \qquad \frac{d^2y}{dx^2} + P_1(x, \lambda)\frac{dy}{dx} + P_2(x, \lambda)y = 0,$$

of which they are solutions. For the coefficients of this equation we have the familiar formulas

$$P_1(x, \lambda) = -\begin{vmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{vmatrix} \div \begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix},$$

$$P_2(x, \lambda) = \begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} \div \begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix}.$$

If in these we substitute for $y_1''$ and $y_2''$ their respective values as given by the equations (2.5), and for $y_1$, $y_2$ and $y_1'$, $y_2'$ their values as obtainable from the relations (2.4), the formulas are found to reduce to

$$P_1(x, \lambda) = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \div \begin{vmatrix} A_1' + i\lambda\phi A_1 & A_1 \\ A_2' - i\lambda\phi A_2 & A_2 \end{vmatrix},$$

$$P_2(x, \lambda) = \lambda^2\phi^2 + \Theta + \begin{vmatrix} C_1 & A_1' + i\lambda\phi A_1 \\ C_2 & A_2' - i\lambda\phi A_2 \end{vmatrix} \div \begin{vmatrix} A_1' + i\lambda\phi A_1 & A_1 \\ A_2' - i\lambda\phi A_2 & A_2 \end{vmatrix}.$$

Since the functions $A_j(x, \lambda)$ are bounded when $|\lambda| > N$, while the functions $C_j(x, \lambda)$ are of the order of $\lambda^{-m}$, it is evident that

$$P_1(x, \lambda) = \frac{\Omega_1(x, \lambda)}{\lambda^{m+1}}, \qquad P_2(x, \lambda) = \lambda^2\phi^2 + \Theta + \frac{\Omega_2(x, \lambda)}{\lambda^m},$$

with $\Omega_1$ and $\Omega_2$ standing for functions that are bounded when $|\lambda| > N$. The equation which is solved by $y_1$ and $y_2$ is thus

$$(3.2) \qquad \frac{d^2y}{dx^2} + \frac{\Omega_1(x, \lambda)}{\lambda^{m+1}} \frac{dy}{dx} + \left[\lambda^2\phi^2(x, \lambda) + \Theta(x, \lambda) + \frac{\Omega_2(x, \lambda)}{\lambda^m}\right] y = 0.$$

This differential equation, of which the solutions are known, approximates the given equation (2.2) in an obvious sense when $|\lambda| > N$, the degree of the approximation being the better the larger the value of $m$. From the similarity between the two equations it is possible to prove that the solutions of the one are asymptotically representable by those of the other. We shall not give such a proof here. The reasoning upon which it may be based is, however, precisely such as will be used below in §§10, 11, and 12.

## Part II. The case of a turning point

4. **The first approximating equation.** When the coefficient $q_0(x)$ of the equation (1.3) has a zero at a point $x_1$ in the interior of the interval $(a, b)$, the deductions of the preceding sections fail. Either the function $\phi(x, \lambda)$ or its reciprocal—they both play a crucial rôle—is then unbounded, according as $q_1(x)$ is different from or equal to zero at $x_1$. In either case the formulas (2.9) give no usable evaluations. It is the purpose of the following discussion to present an alternative algorithm, one which retains its validity even when a zero of $q_0(x)$ is present.

The hypotheses to be made are the following:

(i) *The interval* $(a, b)$ *contains a point* (*just one*) *at which the function* $q_0(x)$ *vanishes.*

(ii) *This zero of* $q_0(x)$ *is simple, that is, of the first order.*

(iii) *The function* $q_0(x)$ *is real, except possibly for a constant complex factor, when* $x$ *is real and is expressible by a power series in* $(x-x_1)$ *near* $x_1$.

It is convenient to make certain adjustments which involve no loss of generality. To begin with, the zero of $q_0(x)$ may be taken to be the origin of the variable $x$. Further, the function $q_0(x)$ may be taken to be real on $(a, b)$, and to have everywhere on this interval the same sign as $x$. Under the hypothesis (iii) this adjustment may be attained by at most the transfer of a suitable constant factor from $q_0(x)$ to the parameter $\lambda^2$. We shall assume these normalizations to have been made, and observe therefore that henceforth $a<0$ and $0<b$.

With $q_0^{1/2}(x)$ determined to be that root of $q_0(x)$ which is real and positive for positive values of $x$, let the functions $\phi(x)$, $\xi$, and $\Psi(x)$ be defined by the formulas

$$(4.1)\qquad \begin{aligned} &\text{(a)} \quad \phi(x) = q_0^{1/2}(x), \\ &\text{(b)} \quad \xi = \lambda \int_0^x \phi(x)dx, \\ &\text{(c)} \quad \Psi(x) = \frac{\left[\int_0^x \phi(x)dx\right]^{1/6}}{[\phi(x)]^{1/2}}. \end{aligned}$$

It will be observed from these that $\phi(x)$ and its integral both vanish at $x=0$, and that near this point they are respectively of the orders of $x^{1/2}$ and $x^{3/2}$. For positive $x$ they are both real and positive, so that then $\arg \xi = \arg \lambda$. For negative $x$, on the other hand, $\phi(x)$ and its integral are both pure imaginary, with arguments that are respectively $\pi/2$ and $3\pi/2$. In this case $\arg \xi = \arg \lambda + 3\pi/2$. Finally, it is clear that the formula (4.1c) is indeterminate at the origin. It is, however, easily verified that $\lim_{x\to 0} \Psi(x)$ exists and is different from zero([6]). With proper definition at $x=0$ the function $\Psi(x)$ is, therefore, continuous, and in fact indefinitely differentiable, over the interval $(a, b)$. Both $\Psi(x)$ itself and its reciprocal are bounded.

Consider the function

$$(4.2)\qquad v(x, \lambda) = \Psi(x)V(\xi),$$

in which the factor $V(\xi)$ is any solution of the differential equation

$$(4.3)\qquad \frac{d^2V}{d\xi^2} + \frac{1}{3\xi}\frac{dV}{d\xi} + V = 0.$$

The formula (4.1b) yields the relation

$$(4.4)\qquad \frac{d}{dx} = \lambda\phi(x)\frac{d}{d\xi},$$

and by virtue of this it is found that

$$v'' = \Psi''V(\xi) + [2\lambda\phi\Psi' + \lambda\phi'\Psi]\frac{dV}{d\xi} + \lambda^2\phi^2\Psi\frac{d^2V}{d\xi^2}.$$

([6]) Its value is $[3q_0'(0)/2]^{-1/6}$.

But from the relation (4.1c) it is readily deduced that

$$[2\lambda\phi\Psi' + \lambda\phi'\Psi] = \frac{\lambda^2\phi^2\Psi}{3\xi}.$$

The evaluation of $v''$ above therefore reduces to the form

$$v'' = \Psi''V(\xi) + \lambda^2\phi^2\Psi\left[\frac{d^2V}{d\xi^2} + \frac{1}{3\xi}\frac{dV}{d\xi}\right]$$

or, because of the equation (4.3), to

$$v'' = \Psi''V(\xi) - \lambda^2\phi^2\Psi V(\xi).$$

From this and the formula (4.2) it follows that $v(x, \lambda)$ solves the differential equation

$$\frac{d^2 v}{dx^2} + [\lambda^2 q_0(x) + \theta(x)]\, v = 0, \tag{4.5}$$

in which

$$\theta(x) = -\frac{\Psi''(x)}{\Psi(x)}. \tag{4.6}$$

The differential equation (4.3) is a familiar one. It admits as explicit solutions the product $\xi^{1/3}H_{1/3}(\xi)$, in which $H_{1/3}$ stands for any Bessel function of the order 1/3. The equation (4.5), which to the extent of the term in $\lambda^2$ resembles the given equation (1.3), is thus also explicitly solvable, its solutions being the functions

$$v(x, \lambda) = \Psi(x)\xi^{1/3}H_{1/3}(\xi). \tag{4.7}$$

We shall refer to (4.5) as the *first solvable approximating equation.*

5. **The second approximating equation.** Whenever the coefficient $q_1(x)$ in the given differential equation (1.3) is not identically zero, the resemblance between the equations (1.3) and (4.5) is still remote when $|\lambda|$ is large. It is to be shown that an equation yielding a closer approximation, but which is still explicitly solvable, is then derivable.

With $v_1(x,\lambda)$, $v_2(x,\lambda)$ as any pair of linearly independent solutions of the equation (4.5), and with tentatively undetermined coefficients $\mu_0(x)$, $\mu_1(x)$, let the functions $\zeta_1(x, \lambda)$, $\zeta_2(x, \lambda)$ be defined by the formulas

$$\zeta_j(x, \lambda) = \mu_0(x)\, v_j(x, \lambda) + \frac{\mu_1(x)}{\lambda}\, v_j'(x, \lambda), \qquad j = 1, 2. \tag{5.1}$$

The differentiation of these, followed by an elimination of the $v_j''$ through use of the equation (4.5), leads to the companion formulas

$$\zeta_j'(x,\lambda) = \left[\mu_0' - \lambda q_0\mu_1 - \frac{\theta\mu_1}{\lambda}\right] v_j + \left[\mu_0 + \frac{\mu_1'}{\lambda}\right] v_j', \qquad j = 1, 2, \tag{5.2}$$

and a repetition of the process shows further that

$$\zeta_j''(x,\lambda) = \left[\mu_0'' - 2\lambda q_0\mu_1' - \lambda q_0'\mu_1 - \frac{2\theta\mu_1' - \theta'\mu_1}{\lambda} - \lambda^2 q_0\mu_0 - \theta\mu_0\right] v_j$$
$$+ \left[2\mu_0' - \lambda q_0\mu_1 - \frac{\theta\mu_1 - \mu_1''}{\lambda}\right] v_j', \qquad j = 1, 2.$$

From these results it follows that

$$\begin{aligned} \zeta_j'' + [\lambda^2 q_0 + \lambda q_1]\zeta_j &= \{-\lambda(2q_0\mu_1' + q_0'\mu_1 - q_1\mu_0) + g_1(x,\lambda)\} v_j \\ &\quad + \{\lambda(2\mu_0' + q_1\mu_1) + g_2(x)\}\frac{v_j'}{\lambda}, \qquad j = 1, 2, \end{aligned} \tag{5.3}$$

with

$$g_1(x,\lambda) = \mu_0'' - \theta\mu_0 - \frac{2\theta\mu_1' + \theta'\mu_1}{\lambda}, \qquad g_2(x) = \mu_1'' - \theta\mu_1. \tag{5.4}$$

Let the functions $\mu_0(x)$ and $\mu_1(x)$ now be chosen thus,

$$\mu_0(x) = \cos\left\{\int_0^x \frac{q_1(x)}{2\phi(x)}\,dx\right\}, \qquad \mu_1(x) = \frac{\sin\left\{\int_0^x \frac{q_1(x)}{2\phi(x)}\,dx\right\}}{\phi(x)}. \tag{5.5}$$

It then follows that

$$2q_0\mu_1' + q_0'\mu_1 - q_1\mu_0 = 0, \qquad 2\mu_0' + q_1\mu_1 = 0,$$

and because of this, that the relations (5.3) reduce to the forms

$$\zeta_j'' + [\lambda^2 q_0 + \lambda q_1]\zeta_j = g_1(x,\lambda)\, v_j + g_2(x)\frac{v_j'}{\lambda}, \qquad j = 1, 2. \tag{5.6}$$

When $q_1(x)$ differs from zero at $x=0$, the integral which appears in the formulas (5.5) is improper. Since its integrand becomes infinite only like $x^{-1/2}$, however, the integral is convergent and is itself of the order of $x^{1/2}$. The formula for $\mu_0(x)$ is thus definitive, and although that for $\mu_1(x)$ is indeterminate at $x=0$, it is easily found to yield the result $\lim_{x\to 0}\mu_1(x) = q_1(0)/q_0'(0)$. With proper definition at $x=0$—and we shall assume that—the functions (5.5) are accordingly continuous, and in fact indefinitely differentiable over the interval $(a, b)$. It is to be observed for future use that

$$\mu_0^2(x) + q_0(x)\mu_1^2(x) \equiv 1. \tag{5.7}$$

When the index $j$ is understood to stand for either one of the values 1, 2, but to be fixed, the equations (5.1) and (5.2) together constitute a linear algebraic system in which the values $v_j$ and $v_j'$ take the rôle of the unknowns. The determinant of this system is

$$D_0(x,\lambda) = \begin{vmatrix} \mu_0 & \dfrac{\mu_1}{\lambda} \\ \mu_0' - \lambda q_0\mu_1 - \dfrac{\theta\mu_1}{\lambda} & \mu_0 + \dfrac{\mu_1'}{\lambda} \end{vmatrix}, \tag{5.8}$$

and this, because of the relation (5.7), has the value

$$D_0(x,\lambda) = 1 + \frac{\mu_0\mu_1' - \mu_0'\mu_1}{\lambda} + \frac{\theta\mu_1^2}{\lambda^2}. \tag{5.9}$$

It is clear that $D_0$ is thus both bounded and bounded from zero when $|\lambda| > N$. The solution of the system (5.1), (5.2) is now found to be

$$\begin{aligned} v_j &= \frac{1}{D_0}\left\{\left(\mu_0 + \frac{\mu_1'}{\lambda}\right)\zeta_j - \frac{\mu_1}{\lambda}\zeta_j'\right\}, \\ v_j' &= \frac{1}{D_0}\left\{-\left(\mu_0' - \lambda q_0\mu_1 - \frac{\theta\mu_1}{\lambda}\right)\zeta_j + \mu_0\zeta_j'\right\}. \end{aligned} \tag{5.10}$$

With the substitution of these values, the relation (5.6) becomes an equation in $\zeta_j$ and its derivatives. Since the coefficients in this equation are independent of $j$ it is thus found that $\zeta_1(x,\lambda)$ and $\zeta_2(x,\lambda)$ are solutions of the differential equation

$$\frac{d^2\zeta}{dx^2} + H_0(x,\lambda)\frac{d\zeta}{dx} + [\lambda^2 q_0(x) + \lambda q_1(x) + K_0(x,\lambda)]\zeta = 0, \tag{5.11}$$

in which

$$\begin{aligned} H_0(x,\lambda) &= \frac{1}{\lambda D_0}[g_1(x,\lambda)\mu_1 - g_2(x)\mu_0], \\ K_0(x,\lambda) &= \frac{-1}{D_0}\left[g_1(x,\lambda)\left(\mu_0 + \frac{\mu_1'}{\lambda}\right) + g_2(x)\left(q_0\mu_1 - \frac{\mu_0'}{\lambda} + \frac{\theta\mu_1}{\lambda^2}\right)\right]. \end{aligned} \tag{5.12}$$

Now by virtue of the values (5.4) it is easily verified that the first one of the formulas (5.12) is in fact expressible as

$$H_0(x,\lambda) = -\frac{D_0'(x,\lambda)}{D_0(x,\lambda)}.$$

The change of variable from $\zeta$ to $z$ under the relation

$$z = D_0^{-1/2}\zeta \tag{5.13}$$

therefore reduces the equation (5.11) to the form

$$\frac{d^2z}{dx^2} + [\lambda^2 q_0(x) + \lambda q_1(x) + K(x, \lambda)]z = 0, \tag{5.14}$$

in which the term in the first derivative is lacking, and in which

$$K(x, \lambda) = K_0(x, \lambda) + \frac{D_0''(x, \lambda)}{2D_0(x, \lambda)} - \frac{3}{4}\left[\frac{D_0'(x, \lambda)}{D_0(x, \lambda)}\right]^2. \tag{5.15}$$

The function $K(x, \lambda)$ is evidently bounded when $|\lambda| > N$. It is in fact expansible in powers of $\lambda^{-1}$, thus

$$K(x, \lambda) = \sum_{\nu=0}^{\infty} \frac{k_\nu(x)}{\lambda^\nu}, \tag{5.16}$$

with coefficients $k_\nu(x)$ that are differentiable. The equation (5.14) is thus of precisely the type of the given equation (1.3), and resembles it to the extent of all terms in positive powers of $\lambda$. It is, moreover, explicitly solvable, its solutions being the functions

$$z_j(x, \lambda) = D_0^{-1/2}(x, \lambda)\left\{\mu_0(x)\, v_j(x, \lambda) + \frac{\mu_1(x)}{\lambda}\, v_j'(x, \lambda)\right\}, \quad j = 1, 2. \tag{5.17}$$

We shall refer to (5.14) as the *second solvable approximating equation.*

The deductions of this section are significant and lead to an equation (5.14) which is an improvement over (4.5), whenever the coefficient $q_1(x)$ in the given differential equation is not identically zero. If $q_1(x)$ is identically zero the step represented by these deductions may be completely omitted. Formally the evaluations (5.5) then reduce to $\mu_0 = 1$, $\mu_1 = 0$, so that $D_0 \equiv 1$ and the functions $z_j(x, \lambda)$ are identical with the $v_j(x, \lambda)$. The equations (4.5) and (5.14) are then the same.

6. **The related differential equation.** By a method which is essentially that of §5 it is now possible to derive a solvable differential equation which approximates the given one, not only to the extent of the terms in positive powers of $\lambda$, but to the terms in arbitrarily prescribed powers of $1/\lambda$ as well.

Let $m$ be chosen as any positive integer. With tentatively undetermined coefficients $\alpha_\nu(x)$, $\beta_\nu(x)$, let the functions $A(x, \lambda)$ and $B(x, \lambda)$ be given by the relations

$$A(x, \lambda) = \sum_{\nu=0}^{m-1} \frac{\alpha_\nu(x)}{\lambda^\nu}, \qquad B(x, \lambda) = \sum_{\nu=0}^{m-1} \frac{\beta_\nu(x)}{\lambda^\nu}. \tag{6.1}$$

Finally with $z_1(x, \lambda)$, $z_2(x, \lambda)$ as any linearly independent solutions of the differential equation (5.14), let the functions $\eta_j(x, \lambda)$ be defined by the formulas

$$(6.2) \qquad \eta_j(x,\lambda) = A(x,\lambda)z_j(x,\lambda) + \frac{B(x,\lambda)}{\lambda^2} z_j'(x,\lambda), \qquad j = 1, 2.$$

A differentiation of these, followed by an elimination of the $z_j''$ through the use of the equation (5.14), leads to the companion formulas

$$(6.3) \qquad \eta_j'(x,\lambda) = \left[A' - q_0 B - \frac{q_1 B}{\lambda} - \frac{KB}{\lambda^2}\right] z_j + \left[A + \frac{B'}{\lambda^2}\right] z_j', \qquad j = 1, 2,$$

while a repetition of this process gives a corresponding evaluation of the $\eta_j''$ in terms of $z_j$ and $z_j'$. It follows from these results that

$$(6.4) \qquad \eta_j'' + [\lambda^2 q_0 + \lambda q_1 + R]\eta_j = S_0(x,\lambda)z_j + S_1(x,\lambda)z_j', \qquad j = 1, 2,$$

with

$$(6.5) \qquad \begin{aligned} S_0(x,\lambda) &= \Bigg[- 2q_0B' - q_0' B + A'' + (R - K)A \\ &\qquad - \frac{2q_1B' + q_1' B}{\lambda} - \frac{2KB' + K'B}{\lambda^2}\Bigg], \\ S_1(x,\lambda) &= \left[2A' + \frac{B'' + (R-K)B}{\lambda^2}\right]. \end{aligned}$$

Through the use of the relations (1.4), (5.16), and (6.1), the functions (6.5) may be expressed as power series in $1/\lambda$. If it is agreed to interpret all symbols with negative subscripts, and all symbols $\alpha$ and $\beta$ with subscripts greater than $(m-1)$, as having the value zero, the formulas (6.5) thus become more explicitly:

$$(6.6) \qquad \begin{aligned} S_1 &= \sum_{\nu=0}^{\infty} \lambda^{-\nu} \left\{2\alpha_\nu' + \beta_{\nu-2}'' + \sum_{\mu=0}^{m-1} (r_{\nu-\mu-2} - k_{\nu-\mu-2})\beta_\mu\right\}, \\ S_0 &= \sum_{\nu=0}^{\infty} \lambda^{-\nu} \Bigg\{- 2q_0\beta_\nu' - q_0'\beta_\nu + \alpha_\nu'' - 2q_1\beta_{\nu-1}' - q_1'\beta_{\nu-1} \\ &\qquad + \sum_{\mu=0}^{m-1} [(r_{\nu-\mu} - k_{\nu-\mu})\alpha_\mu - 2k_{\nu-\mu-2}\beta_\mu' - k_{\nu-\mu-2}'\beta_\mu]\Bigg\}. \end{aligned}$$

Now in these series the leading terms are in general those which are free from $\lambda$. They vanish, however, if

$$2\alpha_0' = 0,$$

$$- 2q_0\beta_0' - q_0'\beta_0 + \alpha_0'' + (r_0 - k_0)\alpha_0 = 0.$$

These equations may be fulfilled by the choices of $\alpha_0(x)$ and $\beta_0(x)$, and they will be fulfilled if in particular

$$\alpha_0(x) \equiv 1, \qquad \beta_0(x) = \frac{1}{\phi(x)} \int_0^x \frac{r_0(x) - k_0(x)}{2\phi(x)} dx. \tag{6.7a}$$

We shall assume that $\alpha_0(x)$ and $\beta_0(x)$ have been given these values. The series (6.6) therefore lack the terms which are constant as to $\lambda$.

In general, now, the leading terms in the series (6.6) are those in $\lambda^{-1}$. These, however, vanish if

$$2\alpha_1' = 0,$$

$$-2q_0\beta_1' - q_0'\beta_1 + \alpha_1'' - 2q_1\beta_0' - q_1'\beta_0 + \sum_{\mu=0}^{1} (r_{1-\mu} - k_{1-\mu})\alpha_\mu = 0.$$

To achieve this we shall choose

$$\alpha_1(x) \equiv 0, \qquad \beta_1(x) = \frac{1}{\phi(x)} \int_0^x \frac{-2q_1\beta_0' - q_1'\beta_0 + (r_1 - k_1)}{2\phi(x)} dx. \tag{6.7b}$$

The expressions (6.6) are thus reduced to be formally of at most the order of $\lambda^{-2}$. It will be clear, now, that the procedure admits of repetition to make the terms in $\lambda^{-\nu}$ vanish successively for $\nu = 2, 3, \cdots, (m-1)$, by the successive choices of $\alpha_\nu(x)$ and $\beta_\nu(x)$. The formulas expressing these choices in terms of those previously made may, at the $\nu$th stage, be taken to be

$$\begin{aligned} \alpha_\nu(x) &= \frac{-1}{2} \int_0^x \left[ \beta_{\nu-2}'' + \sum_{\mu=0}^{\nu-2} (r_{\nu-\mu-2} - k_{\nu-\mu-2})\beta_\mu \right] dx, \\ &\qquad\qquad \nu = 2, 3, \cdots, (m-1), \\ \beta_\nu(x) &= \frac{1}{\phi(x)} \int_0^x \left( \frac{\alpha_\nu'' - 2q_1\beta_{\nu-1}' - q_1'\beta_{\nu-1} + \sum_{\mu=0}^{\nu} (r_{\nu-\mu} - k_{\nu-\mu})\alpha_\mu}{2\phi(x)} \right. \\ &\qquad \left. - \frac{\sum_{\mu=0}^{\nu-2} (2k_{\nu-\mu-2}\beta_\mu' - k_{\nu-\mu-2}'\beta_\mu)}{2\phi(x)} \right) dx. \end{aligned} \tag{6.7c}$$

It will be observed that the formula for the function $\beta_0(x)$ involves an integral which is improper, and also involves $\phi(x)$ to a negative power. However it is clear, that since $\phi(x)$ vanishes only like $x^{1/2}$, the integral is convergent, and the entire formula only indeterminate at $x=0$. With $\beta_0(x)$ defined to be continuous at the origin it is a function which is continuous and indefinitely differentiable over the interval $(a, b)$. The like observations may now be made upon the definition of the function $\beta_1(x)$, and then in turn upon those of $\beta_\nu(x)$ for $\nu = 2, 3, \cdots, (m-1)$. The determinations (6.7), and therewith the specifications of the functions (6.1), are, therefore, significant. As a consequence of them the functions $S_0(x, \lambda)$ and $S_1(x, \lambda)$ in the relations

(6.4) are of the order of $\lambda^{-m}$.

When $j$ is fixed at either one of its possible values, (6.2) and (6.3) constitute an algebraic system in $z_j$ and $z_j'$. The determinant of this system is

$$(6.8)\qquad D_1(x, \lambda) = \begin{vmatrix} A & B/\lambda^2 \\ A' - q_0B - q_1B/\lambda - KB/\lambda^2 & A + B'/\lambda^2 \end{vmatrix},$$

and since this differs from $A^2$ by an expression of the order of $\lambda^{-2}$, whereas, by (6.7a) and (6.7b), $A^2$ differs in turn from unity by a similar quantity, it is clear that $D_1$ and its reciprocal are both bounded when $|\lambda| > N$. The system at hand is therefore solvable, and yields the evaluations

$$(6.9)\qquad \begin{aligned} z_j &= \frac{1}{D_1}\left\{\left[A + \frac{B'}{\lambda^2}\right]\eta_j - \frac{B}{\lambda^2}\eta_j',\right. \\ z_j' &= \frac{-1}{D_1}\left\{\left[A' - q_0B - \frac{q_1B}{\lambda} - \frac{KB}{\lambda^2}\right]\eta_j - A\eta_j'\right\}. \end{aligned}$$

By virtue of these the relation (6.4) is expressible as an equation in $\eta_j$ and its derivatives. It is found thus that $\eta_1(x, \mu)$ and $\eta_2(x, \mu)$ are solutions of the differential equation

$$(6.10)\qquad \frac{d^2\eta}{dx^2} + H_1(x, \lambda)\frac{d\eta}{dx} + [\lambda^2q_0 + \lambda q_1 + R - K_1]\eta = 0,$$

with

$$(6.11)\qquad \begin{aligned} H_1(x, \lambda) &= \frac{-1}{D_1}\begin{vmatrix} A & B/\lambda^2 \\ S_0 & S_1 \end{vmatrix}, \\ K_1(x, \lambda) &= \frac{1}{D_1}\begin{vmatrix} A' - q_0B - q_1B/\lambda - KB/\lambda^2 & A + B'/\lambda^2 \\ S_0 & S_1 \end{vmatrix}. \end{aligned}$$

The differentiation of (6.8), and a comparison of the result with the first one of the equations (6.11), shows that $H_1(x, \lambda) = -D_1'(x, \lambda)/D_1(x, \lambda)$. The change of variable from $\eta$ to $y$, where

$$(6.12)\qquad y = D_1^{-1/2}\eta,$$

accordingly reduces the equation (6.10) to the form

$$(6.13)\qquad \frac{d^2y}{dx^2} + \left[\lambda^2q_0(x) + \lambda q_1(x) + R(x, \lambda) + \frac{\Omega(x, \lambda)}{\lambda^m}\right]y = 0,$$

in which

$$(6.14)\qquad \Omega(x, \lambda) = \lambda^m\{K_1(x, \lambda) - H_1^2(x, \lambda)/4 - H_1'(x, \lambda)/2\}.$$

Since the functions (6.5) are both of the order of $\lambda^{-m}$, the same is true of the functions (6.11). Hence $\Omega(x, \lambda)$ is bounded when $|\lambda| > N$, and the differential equation (6.13) accordingly resembles the given equation (1.3) to the extent of all terms in powers of $\lambda$ to that in $\lambda^{-m}$. We shall refer to (6.13) as the differential equation *related* to the given equation, or simply as the *related equation.* Its explicit solutions are

$$(6.15)\qquad y_j(x, \lambda) = \frac{1}{D_1^{1/2}(x, \lambda)}\left\{A(x, \lambda)z_j(x, \lambda) + \frac{B(x, \lambda)}{\lambda^2} z_j'(x, \lambda)\right\}, \qquad j = 1, 2.$$

7. **The solutions of the related equation.** Inasmuch as the solutions $v_j(x, \lambda)$ of the first approximating equation are of the wholly familiar forms (4.7), it is desirable to refer the expressions for the functions $y_j(x, \lambda)$ directly into terms of them. That is easily done. By virtue of the relation (5.13), the formulas (6.15) assume the form

$$y_j = \{D_0 D_1\}^{-1/2}\left\{\left[A - \frac{D_0' B}{2\lambda^2 D_0}\right]\zeta_j + \frac{B}{\lambda^2}\zeta_j', \qquad j = 1, 2,\right.$$

and from this the evaluations (5.1) and (5.2) lead at once to the forms

$$(7.1)\qquad y_j(x, \lambda) = E_0(x, \lambda)\, v_j(x, \lambda) + E_1(x, \lambda)\frac{v_j'(x, \lambda)}{\lambda}, \qquad j = 1, 2,$$

with

$$(7.2)\qquad \begin{aligned} E_0(x, \lambda) &= \mu_0 A - \frac{B}{\lambda}\left[q_0\mu_1 - \frac{\mu_0'}{\lambda} + \frac{\mu_0 D_0'}{2\lambda D_0} + \frac{\mu_1\theta}{\lambda^2}\right], \\ E_1(x, \lambda) &= \mu_1 A + \frac{B}{\lambda}\left[\mu_0 + \frac{\mu_1'}{\lambda} - \frac{\mu_1 D_0'}{2\lambda D_0}\right]. \end{aligned}$$

Let the Wronskian of two functions $y_1$, $y_2$ be designated by $W(y_1, y_2)$, thus

$$(7.3)\qquad W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

The relations (6.9), (6.8), and (6.12) show then that $W(z_1, z_2) = D_1^{-1} W(\eta_1, \eta_2) = W(y_1, y_2)$, whereas the relations (5.10), (5.8), and (5.13) show that

$$W(v_1, v_2) = D_0^{-1} W(\zeta_1, \zeta_2) = W(z_1, z_2).$$

Thus it follows that

$$(7.4)\qquad W(y_1, y_2) = W(v_1, v_2).$$

These Wronskians are evidently independent of $x$, since the equations (6.13) and (4.5) lack the terms in the first derivatives.

In the formula (4.7)$H_{1/3}$ is any Bessel function of the order 1/3. Its rôle may therefore be filled by any constant linear combination of the standard functions $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$, the so-called Bessel functions of the third kind([7]) and for the deductions which follow it is convenient to do that. We may thus write

$$(7.5) \qquad v_j(x, \lambda) = \Psi(x)\xi^{1/3}\{c_{j,1}H_{1/3}^{(1)}(\xi) + c_{j,2}H_{1/3}^{(2)}(\xi)\}, \qquad j = 1, 2,$$

and these two functions will be linearly independent if the determinant of their coefficients $c_{j,i}$ is different from zero; in particular if

$$(7.6) \qquad \begin{vmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{vmatrix} = 1.$$

To adopt the relation (7.6) is a mere normalization which it is convenient in the following to adhere to.

To obtain the derivatives $v_j'$ we may draw upon the familiar relations

$$\frac{d}{d\xi}\{\xi^{1/3}H_{1/3}^{(j)}(\xi)\} = \xi^{1/3}H_{-2/3}^{(j)}(\xi), \qquad j = 1, 2([8]),$$

together with (4.4) which may be alternatively written in the manner

$$(7.7) \qquad \frac{d}{dx} = \frac{\lambda}{\Psi^2(x)}\left(\frac{\xi}{\lambda}\right)^{1/3}\frac{d}{d\xi}.$$

It is thus found that

$$(7.8) \qquad \begin{aligned} v_j'(x, \lambda) = \lambda\Bigg\{c_{j,1}\left[\frac{\Psi'(x)}{\lambda}\xi^{1/3}H_{1/3}^{(1)}(\xi) + \frac{1}{\lambda^{1/3}\Psi(x)}\xi^{2/3}H_{-2/3}^{(1)}(\xi)\right] \\ + c_{j,2}\left[\frac{\Psi'(x)}{\lambda}\xi^{1/3}H_{1/3}^{(2)}(\xi) + \frac{1}{\lambda^{1/3}\Psi(x)}\xi^{2/3}H_{-2/3}^{(2)}(\xi)\right]\Bigg\}, \\ j = 1, 2. \end{aligned}$$

From the formulas (7.1) we find accordingly that

$$(7.9) \qquad \begin{aligned} y_j(x, \lambda) = \Bigg\{c_{j,1}\left[\left(\Psi E_0 + \frac{\Psi' E_1}{\lambda}\right)\xi^{1/3}H_{1/3}^{(1)}(\xi) + \frac{E_1}{\lambda^{1/3}\Psi}\xi^{2/3}H_{-2/3}^{(1)}(\xi)\right] \\ + c_{j,2}\left[\left(\Psi E_0 + \frac{\Psi' E_1}{\lambda}\right)\xi^{1/3}H_{1/3}^{(2)}(\xi) + \frac{E_1}{\lambda^{1/3}\Psi}\xi^{2/3}H_{-2/3}^{(2)}(\xi)\right]\Bigg\}, \\ j = 1, 2. \end{aligned}$$

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([7]) Cf. Watson, G. N. *A treatise on the theory of Bessel functions*, Cambridge, 2d ed., 1944, p. 73.

([8]) Cf. Watson, p. 74.

It is familiar that, with any positive index $\gamma$, the functions $\xi^\gamma H^{(j)}_{\pm\gamma}(\xi)$ are bounded in any bounded domain of the variable $\xi$. The relations (7.9) therefore show at once that with some constant $M$

$$(7.10) \qquad | y_j(x, \lambda) | < M, \qquad \text{when } | \xi | \leqq N.$$

The Bessel functions, as is well known, become infinite, or approach zero, or remain of an oscillatory character, when the variable itself becomes infinite. Which of these alternatives maintains depends in general upon the argument of the variable as a complex quantity. The configurations which determine the behavior of the standard functions $H^{(j)}_{1/3}$ are the following ones:

$$(7.11) \qquad \begin{aligned} H^{(1)}_{1/3}(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} e^{iz-5\pi i/12}, &\quad &\text{when } -\pi+\delta \leqq \arg z \leqq 2\pi-\delta, \\ H^{(2)}_{1/3}(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz+5\pi i/12}, &\quad &\text{when } -2\pi+\delta \leqq \arg z \leqq \pi-\delta, \end{aligned}$$

with $\delta>0$(9). The sense in which these relationships are to be understood is in each case the following one: that the quotient of the member on the left of the symbol $\sim$ by the member on the right differs from unity by an amount which is arbitrarily small when $|z|>N$. For values of arg $z$ which do not fall within the ranges for which the relations (7.11) are indicated to apply, these latter must be used in conjunction with the classical equations

$$(7.12) \qquad \begin{aligned} H^{(1)}_{1/3}(ze^{k\pi i}) &= \frac{\sin(1-k)\pi/3}{\sin\pi/3} H^{(1)}_{1/3}(z) - e^{-\pi i/3}\frac{\sin k\pi/3}{\sin \pi/3} H^{(2)}_{1/3}(z), \\ H^{(2)}_{1/3}(ze^{k\pi i}) &= e^{\pi i/3}\frac{\sin k\pi/3}{\sin\pi/3} H^{(1)}_{1/3}(z) + \frac{\sin(1+k)\pi/3}{\sin \pi/3} H^{(2)}_{1/3}(z). \end{aligned}$$

In these $k$ may be any integer, positive, negative, or zero(10).

It will be clear from this that the forms of the functions (7.9), when $|\xi|$ is large, depend both upon the constants $c_{j,i}$, and also upon arg $\xi$, which is to say upon $x$ and arg $\lambda$. The further discussion of this matter is therefore to be deferred to the appropriate sections below.

8. **The representation formulas.** If $u(x)$ is a function which fulfills an equation

$$(8.1) \qquad y(x) = u(x) + \int_{x_0}^{x}\left[\frac{y_1(x)y_2(t) - y_2(x)y_1(t)}{W(y_1, y_2)}\right]\frac{\Omega(t,\lambda)}{\lambda^m} u(t)dt,$$

in which $x_0$ may be any fixed point of the interval $(a, b)$, and in which $y_1(x)$, $y_2(x)$, and $y(x)$ are any solutions of the equation (6.13), with $y_1(x)$ and $y_2(x)$

(9) Cf. Watson, pp. 197, 198.

(10) Cf. Watson, p. 75.

linearly independent, then $u(x)$ is a solution of the given differential equation (1.3). This is easily verified. The second derivative of the equation (8.1) is

$$y''(x) = u''(x) + \int_{x_0}^{x}\left[\frac{y_1''(x)y_2(t) - y_2''(x)y_1(t)}{W(y_1, y_2)}\right]\frac{\Omega(t, \lambda)}{\lambda^m}u(t)dt$$
$$- \frac{\Omega(x, \lambda)}{\lambda^m}u(x),$$

and from this and the fact that $y_1(x)$ and $y_2(x)$ satisfy the equation (6.13), it is to be seen that

$$y'' + \left[\lambda^2 q_0 + \lambda q_1 + R + \frac{\Omega}{\lambda^m}\right]y = u'' + [\lambda^2 q_0 + \lambda q_1 + R]u.$$

Since the left member of this is zero, the assertion as to $u(x)$ is clearly established. Equations of the form (8.1) evidently associate solutions of the given differential equation with those of the related equation. It is to the analysis of such associations that the following discussion applies.

From the relations (7.5) and (7.6) it is clear that

$$\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = \xi^{2/3}\Psi(x)\begin{vmatrix} H_{1/3}^{(1)}(\xi) & H_{1/3}^{(2)}(\xi) \\ \dfrac{d}{dx}H_{1/3}^{(1)}(\xi) & \dfrac{d}{dx}H_{1/3}^{(2)}(\xi)\end{vmatrix},$$

and this equation, by the use of the formula (7.7), may be written thus:

$$W(v_1, v_2) = \lambda^{2/3}\xi\begin{vmatrix} H_{1/3}^{(1)}(\xi) & H_{1/3}^{(2)}(\xi) \\ \dfrac{d}{d\xi}H_{1/3}^{(1)}(\xi) & \dfrac{d}{d\xi}H_{1/3}^{(2)}(\xi)\end{vmatrix}.$$

Under (7.4) the left-hand member of this may be replaced by $W(y_1, y_2)$, and since the determinant on the right is known(11) to have the value $-4i/\pi\xi$, it follows that

$$W(y_1, y_2) = \frac{-4i}{\pi}\lambda^{2/3}. \tag{8.2}$$

Let $u_1(x)$ be defined now as the solution of the equation (8.1) when $x_0 = a$ and $y(x) \equiv y_1(x)$, and let $u_2(x)$ be correspondingly defined as the solution of that equation when $x_0 = b$ and $y(x) \equiv y_2(x)$. Then because of the evaluation (8.2) we have

$$u_2(x) = y_1(x) + \frac{1}{\lambda^{m+1}}\int_a^x \frac{\pi}{4i}\{y_1(x)y_2(t) - y_2(x)y_1(t)\}\lambda^{1/3}\Omega(t, \lambda)u_1(t)dt \tag{8.3}$$

(11) Cf. Watson, p. 76.

and

$$(8.4)\quad u_2(x) = y_2(x) + \frac{1}{\lambda^{m+1}}\int_b^x \frac{\pi}{4i}\{y_1(x)y_2(t) - y_2(x)y_1(t)\}\lambda^{1/3}\Omega(t, \lambda)u_2(t)dt.$$

With appropriate choices of the functions $y_1(x)$ and $y_2(x)$—and these choices will depend upon the value of $\arg \lambda$—we shall deduce from these equations the forms of the solutions $u_1(x)$, $u_2(x)$.

The equations (8.3) and (8.4) are integral equations of the Volterra type

$$(8.5)\qquad w(x, \lambda) = f(x, \lambda) + \frac{1}{\lambda^{m+1}}\int_{x_0}^x K(x, t, \lambda)w(t, \lambda)dt,$$

with coefficients $f(x, \lambda)$ and kernels $K(x, t, \lambda)$ that are continuous over the ranges of the variables that come into question. Their solutions are therefore likewise continuous. Beyond that, the fact which is asserted by the following lemma will be of use in the deductions that follow.

LEMMA. *If for $x_0$ and $x$ on a given interval and $|\lambda| > N$, the coefficient and kernel of the equation* (8.5) *are such that*

$$(8.6)\quad \begin{array}{ll} \text{(a)} & |f(x, \lambda)| \leqq M, \\ \text{(b)} & \left|\int_{x_0}^x |K(x, t, \lambda)|\, dt\right| \leqq M, \end{array}$$

*with $M$ standing for some constant (independent of $x$ and $\lambda$), then*

$$(8.7)\qquad w(x, \lambda) = f(x, \lambda) + \frac{O(1)}{\lambda^{m+1}},$$

*with $O(1)$ standing for a function of $x$ and $\lambda$ that is bounded when $x$ is on the given interval and $|\lambda| > N$.*

To show this, let $w_m(\lambda)$ be used to denote the maximum of $|w(x, \lambda)|$ on the interval in question, and let $x_m$ be a point at which this maximum is taken on. Then the equation (8.5) at $x = x_m$ shows readily that

$$w_m(\lambda) \leqq M + \frac{Mw_m(\lambda)}{|\lambda|^{m+1}},$$

namely, when $|\lambda| > N$, that

$$(8.8)\qquad w_m(\lambda) \leqq \frac{M}{1 - M/|\lambda|^{m+1}}.$$

But by virtue of this, the equation (8.5) with the coefficient $f(x, \lambda)$ transposed shows that

$$| w(x, \lambda) - f(x, \lambda) | \leqq \frac{1}{|\lambda|^{m+1}} \left( \frac{M}{1 - M/|\lambda|^{m+1}} \right) M.$$

With this the assertion of the lemma is evidently established.

9. **Some appraisals which apply when**

$$(9.1) \qquad 0 \leqq \arg \lambda < \pi/2.$$

According as $x<0$ or $0<x$, the value of $\arg \phi$ is $\pi/2$ or 0, and the integral of $\phi(x)$ therefore has an argument that is respectively $3\pi/2$ or 0. Accordingly, by (4.1), $\arg \xi = \arg \lambda + 3\pi/2$ when $x<0$, and $\arg \xi = \arg \lambda$ when $0<x$, and the range (9.1) thus corresponds to

$$(9.2) \qquad 3\pi/2 \leqq \arg \xi < 2\pi, \qquad \text{when } x < 0,$$

and to

$$(9.3) \qquad 0 \leqq \arg \xi < \pi/2, \qquad \text{when } 0 < x.$$

Let us choose as the functions $v_j(x)$, in this case, those which are given by the formulas (7.5) with the coefficients

$$(9.4) \qquad (c_{1,1}, c_{1,2}) = (e^{\pi i/3}, 1), \qquad (c_{2,1}, c_{2,2}) = (-1, 0),$$

namely

$$(9.5) \qquad \begin{aligned} v_1(x) &= \Psi(x)\xi^{1/3}[e^{\pi i/3} H_{1/3}^{(1)}(\xi) + H_{1/3}^{(2)}(\xi)], \\ v_2(x) &= -\Psi(x)\xi^{1/3} H_{1/3}^{(1)}(\xi). \end{aligned}$$

The reasons for these particular choices will soon appear. It is to be observed that they conform to the relation (7.6).

When $x<0$ the variable $\xi$, lying in the region (9.2), does not fulfill the conditions upon $z$ for which the relations (7.11) are valid. The variable $\xi e^{-2\pi i}$ on the other hand does fulfill those conditions. For that reason we shall rewrite the formulas (9.5), as we may by the use of the equations (7.12), to appear thus

$$(9.6) \qquad \begin{aligned} v_1(x) &= -\Psi(x)\xi^{1/3} H_{1/3}^{(2)}(\xi e^{-2\pi i}), \\ v_2(x) &= \Psi(x)\xi^{1/3}[H_{1/3}^{(1)}(\xi e^{-2\pi i}) + e^{-\pi i/3} H_{1/3}^{(2)}(\xi e^{-2\pi i})]. \end{aligned}$$

The relations (7.11) with $z = \xi e^{-2\pi i}$ may now be applied to show through the formulas (9.6) that

$$(9.7) \qquad \begin{aligned} v_1(x) &\sim \left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\Psi(x)\xi^{-1/6}e^{-i\xi}, \\ v_2(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\Psi(x)\xi^{-1/6}[e^{i\xi} + ie^{-i\xi}], \qquad \text{when } x < 0. \end{aligned}$$

Now for $\xi$ in the region (9.2) the exponential $e^{-i\xi}$ is bounded, while $e^{i\xi}$ in general is unbounded. Of all the linear combinations of $v_1(x)$ and $v_2(x)$, it is therefore clear from the relations (9.7) that $v_1(x)$ alone is invariably bounded for negative $x$ and $\lambda$ in the region (9.1). The reason for the particular choice of $v_1(x)$ is thereby manifested.

To derive relations analogous to (9.7) but applying to the derivatives $v_1'(x)$, $v_2'(x)$, the deductions above could be applied to the formulas (7.8) with the coefficients (9.4). The result of that, however, would be merely to show that the relations (9.7) are differentiable by differentiating only the exponentials that are involved. Thus it is found that

$$
(9.8)\qquad
\begin{aligned}
v_1'(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\lambda\phi(x)\Psi(x)\xi^{-1/6}e^{-i\xi},\\
v_2'(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\lambda\phi(x)\Psi(x)\xi^{-1/6}\left[e^{i\xi} - ie^{-i\xi}\right],
\end{aligned}
$$

when $x < 0$.

Because of the relations (9.7) and (9.8), and by virtue of the equations (7.1), the following observation may now be made. If the functions $Y_j(x)$ and $U_j(x)$ are defined by the relations

$$
(9.9)\qquad
\begin{aligned}
Y_1(x) &= \xi^{1/6}e^{i\xi}y_1(x), & Y_2(x) &= \xi^{1/6}e^{-i\xi}y_2(x),\\
U_1(x) &= \xi^{1/6}e^{i\xi}u_1(x), & U_2(x) &= \xi^{1/6}e^{-i\xi}u_2(x),
\end{aligned}
$$

then the functions $Y_1(x)$ and $Y_2(x)$ are bounded when $x<0$.

For positive $x$, the variable $\xi$, lying in the region (9.3), fulfills the conditions upon $z$ which pertain to the relations (7.11). These relations are therefore directly applicable to the formulas (9.5), and show that

$$
(9.10)\qquad
\begin{aligned}
v_1(x) &\sim \left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\Psi(x)\xi^{-1/6}\left[e^{-i\xi} - ie^{i\xi}\right],\\
v_2(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\Psi(x)\xi^{-1/6}e^{i\xi}, \qquad \text{when } 0 < x,
\end{aligned}
$$

and hence also that

$$
(9.11)\qquad
\begin{aligned}
v_1'(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\lambda\phi(x)\Psi(x)\xi^{-1/6}\left[e^{-i\xi} + ie^{i\xi}\right],\\
v_2'(x) &\sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\lambda\phi(x)\Psi(x)\xi^{-1/6}e^{i\xi}, \qquad \text{when } 0 < x.
\end{aligned}
$$

In the region (9.3) it is the exponential $e^{i\xi}$ which is bounded, while $e^{-i\xi}$ is in general not so. Since $v_2(x)$ thus appears as the only one of the linear combinations of $v_1$ and $v_2$ which is bounded for positive $x$ and $\lambda$ in the region (9.1), the reason for its particular choice will be clear. Through the equations (7.1) and the relations (9.10) and (9.11) it is now to be observed that the functions $Y_1(x)$ and $Y_2(x)$, as they have been defined under (9.9), are bounded also when $0<x$.

With any specific value of $N$ the relation $|\xi|>N$ is fulfilled for the values of $x$ on a negative and on a positive subinterval, say for $a\leqq x<x_{-N}$, and for $x_N<x\leqq b$. In these the points $x_{-N}$ and $x_N$ lie respectively on the negative and the positive sides of the origin, and both depend upon $\lambda$. Since, when $|x|$ is small, $|\xi|$ is in the order of $|\lambda x^{3/2}|$, it is to be concluded that $|x_{-N}|$, $x_N$, and the length of the subinterval $(x_{-N}, x_N)$ are all of the order of $|\lambda|^{-2/3}$.

10. **The forms of a pair of solutions when**

$$0 \leqq \arg \lambda < \pi/2.$$

For use in the discussion of the equations (8.3) and (8.4) it is convenient to define a value $\tau$ in terms of the variable of integration $t$, as $\xi$ is defined in terms of $x$, namely

$$\tau = \lambda \int_0^t \phi(t)dt,$$

and also to list certain functional combinations which may then subsequently be abbreviated. These combinations, with the designations to be assigned them, are the following ones:

$$K_0(x, t, \lambda) = \frac{\pi}{4i}\{y_1(x)y_2(t) - y_2(x)y_1(t)\}\lambda^{1/3}\Omega(t, \lambda),$$

$$K_1(x, t, \lambda) = \frac{\pi}{4i}\{Y_1(x)Y_2(t) - Y_2(x)Y_1(t)e^{2i(\xi-\tau)}\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),$$

$$K_2(x, t, \lambda) = \frac{\pi}{4i}\{y_1(x)Y_2(t) - y_2(x)Y_1(t)e^{-2i\tau}\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),$$

$$(10.1)\quad K_3(x, t, \lambda) = \frac{\pi}{4i}\{Y_1(x)y_2(t) - Y_2(x)y_1(t)e^{2i\xi}\}\lambda^{1/3}\Omega(t, \lambda),$$

$$K_4(x, t, \lambda) = \frac{\pi}{4i}\{Y_1(x)Y_2(t)e^{-2i(\xi-\tau)} - Y_2(x)Y_1(t)\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),$$

$$K_5(x, t, \lambda) = \frac{\pi}{4i}\{y_1(x)Y_2(t)e^{2i\tau} - y_2(x)Y_1(t)\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),$$

$$K_6(x, t, \lambda) = \frac{\pi}{4i}\{Y_1(x)y_2(t)e^{-2i\xi} - Y_2(x)y_1(t)\}\lambda^{1/3}\Omega(t, \lambda).$$

Let us, to begin with, consider the equation (8.3) when $x$ is on the range $a \leqq x < x_{-N}$. Upon multiplying the equation by $\xi^{1/6}e^{i\xi}$ and suitably grouping the factors of the terms in the integrand, the equation may be made to appear in the form

$$(10.2) \qquad U_1(x) = Y_1(x) + \frac{1}{\lambda^{m+1}} \int_a^x K_1(x, t, \lambda) U_1(t) dt,$$

the function $U_1(x)$ having been defined by the formulas (9.9). This is manifestly an equation of the type (8.7), of which the coefficient $Y_1(x)$ was observed in §9 to be bounded. In the course of the indicated integration $\arg \xi$ and $\arg \tau$ both have the value $\arg \lambda + 3\pi/2$, and $|\tau| \geqq |\xi|$. Thus $\arg \{i(\xi - \tau)\} = \arg \lambda + \pi$, and for the values of $\arg \lambda$ presently in question this means that the exponential $e^{2i(\xi-\tau)}$ is bounded. In the formula (10.1) for $K_1(x, t, \lambda)$, only the factor $(\lambda/\tau)^{1/3}$ is therefore unbounded. However near $\tau = 0$ this is of the order of $t^{-1/2}$. It is therefore seen that $K_1(x, t, \lambda)$ fulfills the condition which is imposed upon the kernel $K(x, t, \lambda)$ by the relations (8.6). The lemma of §8 is thus applicable to the equation (10.2), and justifies the conclusion that

$$(10.3) \qquad U_1(x) = Y_1(x) + \frac{O(1)}{\lambda^{m+1}}.$$

Let us now consider values of $x$ on the interval $(x_{-N}, x_N)$. If $f_1(x, \lambda)$ is defined by the relation

$$(10.4) \qquad f_1(x, \lambda) = y_1(x) + \frac{1}{\lambda^{m+1}} \int_a^{x_{-N}} K_2(x, t, \lambda) U_1(t) dt,$$

the equation (8.3) may be written thus,

$$(10.5) \qquad u_1(x) = f_1(x, \lambda) + \frac{1}{\lambda^{m+1}} \int_{x_{-N}}^x K_0(x, t, \lambda) u_1(t) dt.$$

This is again an equation of the type (8.7). We wish to show that its coefficient and kernel again fulfill the conditions (8.6). Since $\xi$ is now upon the bounded range $|\xi| \leqq N$, the functions $y_1(x)$, $y_2(x)$ are bounded, as was shown in (7.10). For the integration in the formula (10.4) the relation (10.3), which has already been deduced, maintains, and since the functions $Y_j(t)$ and the exponential $e^{-2i\tau}$ are all bounded, the integrand of (10.4) is bounded except for the factor $(\lambda/\tau)^{1/3}$. The integral is therefore bounded, and since $f_1(x, \lambda)$ thus differs from $y_1(x)$ by a term of the order of $\lambda^{-m-1}$, it fulfills the condition (8.6). For the integration in the equation (10.5), the functions $y_j(t)$ are bounded by (7.10). The kernel $K_0(x, t, \lambda)$ is thus of the order of $\lambda^{1/3}$, and since its integration is extended only over an interval whose length is of the order of $\lambda^{-2/3}$, it likewise fulfills the conditions (8.6). The lemma of §8 is thus applicable to the equation (10.5), and by consequence

$$u_1(x) = y_1(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \text{when } |\xi| \leqq N. \tag{10.6}$$

Finally let us consider values of $x$ on the interval $x_N < x \leqq b$. With $f_2(x, \lambda)$ defined by the formula

$$\begin{aligned} f_2(x, \lambda) = Y_1(x) &+ \frac{1}{\lambda^{m+1}} \int_a^{x_{-N}} K_1(x, t, \lambda) U_1(t) dt \\ &+ \frac{1}{\lambda^{m+1}} \int_{x_{-N}}^{x_N} K_3(x, t, \lambda) u_1(t) dt, \end{aligned} \tag{10.7}$$

the equation (8.3) may be written in the form

$$U_1(x) = f_2(x, \lambda) + \frac{1}{\lambda^{m+1}} \int_{x_N}^{x} K_1(x, t, \lambda) U_1(t) dt. \tag{10.8}$$

For the values of $x$ now in question the functions $Y_j(x)$ are again bounded. The functions $U_1(t)$ and $u_1(t)$ on the respective ranges of integration in the relation (10.7) have already been shown to be bounded, so that it is clear that $f_2(x, \lambda)$ differs from $Y_1(x)$ by an amount which is of the order of $\lambda^{-m-1}$. In the course of the integration in the equation (10.8) arg $\xi$, arg $\tau$, and arg $(\xi - \tau)$ all have the value arg $\lambda$, and thus the exponential in the formula for $K_1(x, t, \lambda)$ is bounded. The conditions of the lemma are accordingly fulfilled, and the equation (10.8) again yields an evaluation (10.3).

In a quite similar way we may now consider the equation (8.4). When $x$ is on the range $x_N < x \leqq b$, the equation may be given the form

$$U_2(x) = Y_2(x) + \frac{1}{\lambda^{m+1}} \int_b^x K_4(x, t, \lambda) U_2(t) dt.$$

In the course of this integration arg $\{-i(\xi - \tau)\} = \arg \lambda + \pi/2$, and hence the exponential in the formula for $K_4(x, t, \lambda)$ is bounded. An application of the lemma is permissible, and yields the result

$$U_2(x) = Y_2(x) + \frac{O(1)}{\lambda^{m+1}} \cdot \tag{10.9}$$

When $x_{-N} \leqq x \leqq x_N$, we may write the equation

$$u_2(x) = f_3(x, \lambda) + \frac{1}{\lambda^{m+1}} \int_{x_N}^{x} K_0(x, t, \lambda) u_2(t) dt,$$

with

$$f_3(x, \lambda) = y_2(x) + \frac{1}{\lambda^{m+1}} \int_b^{x_N} K_5(x, t, \lambda) U_2(t) dt,$$

and by reasoning that is now familiar, this may be made to show that

$$u_2(x) = y_2(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \text{when } |\xi| \leqq N. \tag{10.10}$$

Finally when $a \leqq x < x_{-N}$, the equation is

$$U_2(x) = f_4(x, \lambda) + \frac{1}{\lambda^{m+1}} \int_{x_{-N}}^{x} K_4(x, t, \lambda) U_2(t) dt,$$

with

$$f_4(x, \lambda) = Y_2(x) + \frac{1}{\lambda^{m+1}} \int_b^{x_N} K_4(x, t, \lambda) U_2(t) dt$$

$$+ \frac{1}{\lambda^{m+1}} \int_{x_N}^{x_{-N}} K_6(x, t, \lambda) u_2(t) dt.$$

From this a relation (10.9) is again derivable.

The results (10.3), (10.6), (10.9), and (10.10) effectively express the forms of the solutions $u_1(x)$, $u_2(x)$ over the entire interval $(a, b)$. By way of summary, they may be set forth thus:

THEOREM 1. *The given differential equation* (1.3) *admits of a pair of solutions* $u_1(x)$, $u_2(x)$, *which, when* $0 \leqq \arg \lambda < \pi/2$, *have the forms*

$$u_j(x) = y_j(x) + \xi^{-1/6} e^{\mp i\xi} \frac{O(1)}{\lambda^{m+1}}, \qquad j = 1, 2, \ \textit{when } |\xi| > N,$$

$$u_j(x) = y_j(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } |\xi| \leqq N.$$

*The functions* $y_1(x)$, $y_2(x)$ *are in this instance those which are given by the formulas* (7.9) *with the coefficients* $(c_{1,1}, c_{1,2}) = (e^{\pi i/3}, 1)$, *and* $(c_{2,1}, c_{2,2}) = (-1, 0)$.

## 11. The forms of a pair of solutions when

$$\pi/2 \leqq \arg \lambda < \pi. \tag{11.1}$$

When $\lambda$ lies in the region (11.1) we may choose as the functions $v_1(x)$, $v_2(x)$, those which are given by the formulas (7.5) with the coefficients $(c_{1,1}, c_{1,2}) = (0, e^{-\pi i/3})$ and $(c_{2,1}, c_{2,2}) = (-e^{\pi i/3}, 0)$(12). These coefficients fulfill the relation (7.6), and give explicitly the formulas

$$v_1(x) = e^{-\pi i/3} \Psi(x) \xi^{1/3} H_{1/3}^{(2)}(\xi), \qquad v_2(x) = -e^{\pi i/3} \Psi(x) \xi^{1/3} H_{1/3}^{(1)}(\xi). \tag{11.2}$$

(12) It will, of course, be clear that the functions thus to be designated by $U_1$, $U_2$ in this section are not the same as those so designated in §10. The same observation will apply also to the functions which are to be denoted by $y_1$, $y_2$, and by $u_1$, $u_2$.

When $x<0$ we have

$$(11.3) \qquad 2\pi \leqq \arg \xi < 5\pi/2,$$

and thus the variable $\xi$ is such that the relations (7.11) are applicable when $z=\xi e^{-2\pi i}$. By means of the equations (7.12), the formulas (11.2) may be given the alternative forms

$$v_1(x) = \Psi(x)\xi^{1/3}H_{1/3}^{(1)}(\xi e^{-2\pi i}),$$
$$v_2(x) = \Psi(x)\xi^{1/3}[e^{\pi i/3}H_{1/3}^{(1)}(\xi e^{-2\pi i}) + H_{1/3}^{(2)}(\xi e^{-2\pi i})],$$

and as applied to these, the relations (7.11) show that

$$v_1(x) \sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\Psi(x)\xi^{-1/6}e^{i\xi},$$
$$v_2(x) \sim -\left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\Psi(x)\xi^{-1/6}[e^{-i\xi} - ie^{i\xi}], \qquad \text{when } x < 0.$$

On the range (11.3) the exponential $e^{i\xi}$ is bounded. Hence, as will readily be seen, the functions $Y^{(1)}(x)$, $Y^{(2)}(x)$ that are defined in the relations

$$(11.4) \qquad \begin{aligned} Y^{(1)}(x) &= \xi^{1/6}e^{-i\xi}y_1(x), & Y^{(2)}(x) &= \xi^{1/6}e^{i\xi}y_2(x), \\ U^{(1)}(x) &= \xi^{1/6}e^{-i\xi}u_1(x), & U^{(2)}(x) &= \xi^{1/6}e^{i\xi}u_2(x), \end{aligned}$$

are bounded when $x<0$.

When $0<x$, on the other hand,

$$(11.5) \qquad \pi/2 \leqq \arg \xi < \pi,$$

and $\xi$ is accordingly such that the relations (7.11) are applicable when $z=\xi e^{-\pi i}$. Under the equations (7.12). the formulas (11.2) are alternatively expressible in the manner

$$v_1(x) = \Psi(x)\xi^{1/3}[H_{1/3}^{(1)}(\xi e^{-\pi i}) + e^{-\pi i/3}H_{1/3}^{(2)}(\xi e^{-\pi i})],$$
$$v_2(x) = \Psi(x)\xi^{1/3}H_{1/3}^{(2)}(\xi e^{-\pi i}),$$

and on the basis of these the relations (7.11) show that

$$v_1(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{\pi i/12}\Psi(x)\xi^{-1/6}[e^{-i\xi} + ie^{i\xi}],$$
$$v_2(x) \sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-\pi i/12}\Psi(x)\xi^{-1/6}e^{i\xi}, \qquad \text{when } 0 < x.$$

The region (11.5) is one in which the exponential $e^{i\xi}$ is bounded, and it is accordingly seen that $Y_1(x)$, $Y_2(x)$, as they were defined in (9.9), are bounded.

Let us now agree to the following designations:

$$
(11.6)\quad
\begin{aligned}
K^{(1)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{Y^{(1)}(x)Y^{(2)}(t) - Y^{(2)}(x)Y^{(1)}(t)e^{-2i(\xi-\tau)}\right\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),\\
K^{(2)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{y_1(x)Y^{(2)}(t) - y_2(x)Y^{(1)}(t)e^{2i\tau}\right\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),\\
K^{(3)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{Y_1(x)Y^{(2)}(t) - Y_2(x)Y^{(1)}(t)e^{2i(\xi+\tau)}\right\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),\\
K^{(4)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{Y^{(1)}(x)Y_2(t)e^{2i(\xi+\tau)} - Y^{(2)}(x)Y_1(t)\right\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda),\\
K^{(5)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{Y^{(1)}(x)y_2(t)e^{2i\xi} - Y^{(2)}(x)y_1(t)\right\}\lambda^{1/3}\Omega(t, \lambda),\\
K^{(6)}(x, t, \lambda) &= \frac{\pi}{4i}\left\{Y^{(1)}(x)Y^{(2)}(t)e^{2i(\xi+\tau)} - Y^{(2)}(x)Y^{(1)}(t)\right\}\left(\frac{\lambda}{\tau}\right)^{1/3}\Omega(t, \lambda).
\end{aligned}
$$

The equations (8.3) and (8.4) may then be written in the following ways:

$$
\begin{aligned}
U^{(1)}(x) &= Y^{(1)}(x) + \frac{1}{\lambda^{m+1}}\int_a^x K^{(1)}(x, t, \lambda)U^{(1)}(t)dt, && \text{when } a \leqq x < x_{-N},\\
u_1(x) &= \left\{y_1(x) + \frac{1}{\lambda^{m+1}}\int_a^{x_{-N}} K^{(2)}(x, t, \lambda)U^{(1)}(t)dt\right\}\\
&\quad + \frac{1}{\lambda^{m+1}}\int_{x_{-N}}^x K_0(x, t, \lambda)u_1(t)dt, && \text{when } x_{-N} \leqq x \leqq x_N,\\
U_1(x) &= \left\{Y_1(x) + \frac{1}{\lambda^{m+1}}\int_a^{x_{-N}} K^{(3)}(x, t, \lambda)U^{(1)}(t)dt\right.\\
&\quad \left. + \frac{1}{\lambda^{m+1}}\int_{x_{-N}}^{x_N} K_3(x, t, \lambda)u_1(t)dt\right\}\\
&\quad + \frac{1}{\lambda^{m+1}}\int_{x_N}^x K_1(x, t, \lambda)U_1(t)dt, && \text{when } x_N < x \leqq b,
\end{aligned}
$$

and

$$
\begin{aligned}
U_2(x) &= Y_2(x) + \frac{1}{\lambda^{m+1}}\int_b^x K_4(x, t, \lambda)U_2(t)dt, && \text{when } x_N < x \leqq b,\\
u_2(x) &= \left\{y_2(x) + \frac{1}{\lambda^{m+1}}\int_b^{x_N} K_5(x, t, \lambda)U_2(t)dt\right\}\\
&\quad + \frac{1}{\lambda^{m+1}}\int_{x_N}^x K_0(x, t, \lambda)u_2(t)dt, && \text{when } x_{-N} \leqq x \leqq x_N,
\end{aligned}
$$

$$U^{(2)}(x) = \left\{ Y^{(2)}(x) + \frac{1}{\lambda^{m+1}} \int_b^{x_N} K^{(4)}(x, t, \lambda) U^{(2)}(t) dt \right.$$
$$\left. + \frac{1}{\lambda^{m+1}} \int_{x_N}^{x_{-N}} K^{(5)}(x, t, \lambda) u_2(t) dt \right\}$$
$$+ \frac{1}{\lambda^{m+1}} \int_{x_{-N}}^{x} K^{(6)}(x, t, \lambda) U^{(2)}(t) dt, \qquad \text{when } a \leqq x < x_{-N}.$$

The reasoning which was used in §10 is successively applicable to the equation in each of these forms, and shows that in each instance the equation's left-hand member differs from its first term on the right by an amount which is of the order of $\lambda^{-m-1}$. These facts may be summarized as follows.

THEOREM 2. *The given differential equation* (1.3) *admits of a pair of solutions* $u_1(x)$, $u_2(x)$ *which, when* $\pi/2 \leqq \arg \lambda < \pi$, *have the forms*

$$u_j(x) = y_j(x) + \xi^{-1/6 \pm i\xi} \frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } x < 0 \textit{ and } |\xi| > N,$$

$$u_j(x) = y_j(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } |\xi| \leqq N,$$

$$u_j(x) = y_j(x) + \xi^{-1/6} e^{\mp i\xi} \frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } 0 < x \textit{ and } |\xi| > N.$$

*The functions* $y_1(x)$, $y_2(x)$ *are in this instance those which are given by the formula* (7.9) *with the coefficients* $(c_{1,1}, c_{1,2}) = (0, e^{-\pi i/3})$ *and* $(c_{2,1}, c_{2,2}) = (-e^{\pi i/3}, 0)$.

12. **The forms of pairs of solutions for other values of $\lambda$.** When $\lambda$ lies in

$$\pi \leqq \arg \lambda < 3\pi/2, \tag{12.1}$$

the choice of solutions $v(x)$ to be used as $v_1(x)$ and $v_2(x)$ is

$$\begin{aligned} v_1(x) &= -\Psi(x)\xi^{1/3} H_{1/3}^{(2)}(\xi), \\ v_2(x) &= \Psi(x)\xi^{1/3}[H_{1/3}^{(1)}(\xi) + e^{-\pi i/3} H_{1/3}^{(2)}(\xi)]. \end{aligned} \tag{12.2}$$

For negative values of $x$ these may be rewritten by use of the equations (7.12) into the forms

$$\begin{aligned} v_1(x) &= \Psi(x)\xi^{1/3} H_{1/3}^{(2)}(\xi e^{-3\pi i}), \\ v_2(x) &= -\Psi(x)\xi^{1/3}[H_{1/3}^{(1)}(\xi e^{-3\pi i}) + e^{-\pi i/3} H_{1/3}^{(2)}(\xi e^{-3\pi i})], \end{aligned}$$

whereupon the relations (7.11) with $z = \xi e^{-3\pi i}$ are applicable, and show that

$$v_1(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-\pi i/12}\Psi(x)\xi^{-1/6}e^{i\xi},$$

$$v_2(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{\pi i/12}\Psi(x)\xi^{-1/6}[e^{-i\xi} + ie^{i\xi}], \qquad \text{when } x < 0.$$

The exponential $e^{i\xi}$ and therefore the functions $Y^{(1)}(x)$, $Y^{(2)}(x)$ of (11.4) are bounded.

For positive $x$ the formulas (12.2), rewritten into the forms

$$v_1(x) = -\Psi(x)\xi^{1/3}[e^{\pi i/3}H_{1/3}^{(1)}(\xi e^{-\pi i}) + H_{1/3}^{(2)}(\xi e^{-\pi i})],$$

$$v_2(x) = \Psi(x)\xi^{1/3}H_{1/3}^{(1)}(\xi e^{-\pi i}),$$

show that

$$v_1(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-\pi i/12}\Psi(x)\xi^{-1/6}[e^{i\xi} - ie^{-i\xi}],$$

$$v_2(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{\pi i/12}\Psi(x)\xi^{-1/6}e^{-i\xi}, \qquad \text{when } 0 < x.$$

The functions $Y^{(j)}(x)$ are thus again found to be bounded.

The equations (8.3) and (8.4) yield in this instance to an analysis which is analogous to that of §10, and differs from this latter only to the extent that the functions $Y^{(j)}(x)$ replace the $Y_j(x)$. The conclusions to be found are the following ones.

THEOREM 3. *The given differential equation* (1.3) *admits of a pair of solutions* $u_1(x)$, $u_2(x)$, *which, when* $\pi \leqq \arg\lambda < 3\pi/2$, *have the forms*

$$u_j(x) = y_j(x) + \xi^{-1/6}e^{\pm i\xi}\frac{O(1)}{\lambda^{m+1}}, \qquad \text{when } |\xi| > N,$$

$$u_j(x) = y_j(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \text{when } |\xi| \leqq N.$$

*The functions* $y_j(x)$ *in this instance are those which are given by the formulas* (7.9) *with the coefficients* $(c_{1,1}, c_{1,2}) = (0, -1)$ *and* $(c_{2,1}, c_{2,2}) = (1, e^{-\pi i/3})$.

For values of $\lambda$ in the region $3\pi/2 \leqq \arg\lambda < 2\pi$, the choices

$$v_1(x) = \Psi(x)\xi^{1/3}H_{1/3}^{(1)}(\xi),$$

$$v_2(x) = \Psi(x)\xi^{1/3}[e^{\pi i/3}H_{1/3}^{(1)}(\xi) + H_{1/3}^{(2)}(\xi)]$$

are to be made. For negative values of $x$ the alternative forms

$$v_1(x) = -\Psi(x)\xi^{1/3}H_{1/3}^{(1)}(\xi e^{-3\pi i}),$$
$$v_2(x) = -\Psi(x)\xi^{1/3}[e^{\pi i/3}H_{1/3}^{(1)}(\xi e^{-3\pi i}) + H_{1/3}^{(2)}(\xi e^{-3\pi i})]$$

yield the relations

$$v_1(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{\pi i/12}\Psi(x)\xi^{-1/6}e^{-i\xi},$$
$$v_2(x) \sim -\left(\frac{2}{\pi}\right)^{1/2} e^{-\pi i/12}\Psi(x)\xi^{-1/6}[e^{i\xi} - ie^{-i\xi}], \qquad \text{when } x < 0,$$

and these show that the functions $Y_j(x)$ are bounded.

For positive $x$ the formulas

$$v_1(x) = -\Psi(x)\xi^{1/3}[H_{1/3}^{(1)}(\xi e^{-2\pi i}) + e^{-\pi i/3}H_{1/3}^{(2)}(\xi e^{-2\pi i})],$$
$$v_2(x) = -\Psi(x)\xi^{1/3}H_{1/3}^{(2)}(\xi e^{-2\pi i})$$

yield the relations

$$v_1(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-5\pi i/12}\Psi(x)\xi^{-1/6}[e^{i\xi} + ie^{-i\xi}],$$
$$v_2(x) \sim \left(\frac{2}{\pi}\right)^{1/2} e^{5\pi i/12}\Psi(x)\xi^{-1/6}e^{-i\xi}, \qquad \text{when } 0 < x,$$

and thus the functions $Y^{(j)}(x)$ are bounded. An analysis of the equations (8.3) and (8.4), which is analogous to that of §11, leads in this case to the following conclusion.

THEOREM 4. *The given differential equation* (1.3) *admits of a pair of solutions* $u_1(x)$, $u_2(x)$, *which, when* $3\pi/2 \leqq \arg \lambda < 2\pi$, *have the forms*

$$u_j(x) = y_j(x) + \xi^{-1/6}e^{\mp i\xi}\frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } x < 0 \textit{ and } |\xi| > N,$$
$$u_j(x) = y_j(x) + \frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } |\xi| \leqq N,$$
$$u_j(x) = y_j(x) + \xi^{-1/6}e^{\pm i\xi}\frac{O(1)}{\lambda^{m+1}}, \qquad \textit{when } 0 < x \textit{ and } |\xi| > N.$$

*The functions* $y_1(x)$, $y_2(x)$ *are in this instance those which are given by the formulas* (7.9) *with the coefficients* $(c_{1,1}, c_{1,2}) = (1, 0)$ *and* $(c_{2,1}, c_{2,2}) = (e^{\pi i/3}, 1)$.

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